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Coherency, free inverse monoids and related free algebras

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Abstract

A monoid S is *right coherent* if every finitely generated subact of every finitely presented right S-act is finitely presented. This is the non-additive notion corresponding to that for a ring R stating that every finitely generated submodule of every finitely presented right R-module is finitely presented. For monoids (and rings) right coherency is an important finitary property which determines, amongst other things, the existence of a *model companion* of the class of right S-acts (right R-modules) and hence that the class of existentially closed right S-acts (right R-modules) is axiomatisable.

Choo, Lam and Luft have shown that free rings are right (and left) coherent; the authors, together with Ruškuc, have shown that (free) groups, free commutative monoids and free monoids, have the same properties. It is then natural to ask whether other free algebras in varieties of monoids, possibly with an augmented signature, are right coherent. We demonstrate that free inverse monoids are not.

Munn described the free inverse monoid FIM(Ω) on Ω as consisting of birooted finite connected subgraphs of the Cayley graph of the free group on Ω. Sitting within FIM(Ω) we have free algebras in other varieties and quasi-varieties, in particular the free left ample monoid FLA(Ω) and the free ample monoid FAM(Ω). The former is the free algebra in the variety of unary monoids corresponding to partial maps with distinguished domain; the latter is the two-sided dual. For example, FLA(Ω) is obtained from FIM(Ω) by considering only subgraphs with vertices labelled by elements of the free monoid on Ω.

The main objective of the paper is to show that FLA(Ω) *is* right coherent. Furthermore, by making use of the same techniques we show that FIM(Ω), FLA(Ω) and FAM(Ω) satisfy (**R**), (**r**), (**L**) and (**l**), related conditions arising from the axiomatisability of certain classes of right S-acts and of left S-acts.

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1. *Introduction*

Let S be a monoid. A *right S-act* is a set A together with a map $A \times S \to A$ where $(a, s) \mapsto as$, such that for all $a \in A$ and $s, t \in S$ we have $a1 = a$ and $a(st) = (as)t$. We also have the dual notion of a *left S-act*: where handedness for S-acts is not specified later in this article we will always mean *right* S-acts. The study of S-acts is, effectively, that of representations of the monoid S by mappings of sets.

Clearly S-acts over a monoid S are the non-additive analogue of R-modules over a (unital) ring R. Although the study of the two notions diverges considerably once technicalities set in, one can often begin by forming analogous notions and asking analogous questions. In this article we study coherency for monoids. A monoid S is said to be *right coherent* if every finitely generated subact of every finitely presented right S-act is finitely presented. *Left coherency* is defined dually; S is *coherent* if it is both left and right coherent. These notions correspond to those for a ring R (where, of course, S-acts are replaced by R-modules). Coherency is a finitary condition for rings and monoids, weaker than, for example, the condition that says all finitely generated R-modules or S-acts be finitely presented. In fact, a monoid S is right coherent if and only if every finitely generated subact of any monogenic finitely presented S-act is finitely presented. For rings we can do even better: a ring R is right coherent if and only if every finitely generated right ideal of R is finitely presented. As demonstrated by Eklof and Sabbagh [**6**], coherency is intimately related to the model theory of R-modules. The corresponding results for S-acts appear in [**10**], the latter informed by the more general approach of Wheeler [**19**]. Finally, we mention that right coherency for a ring R is equivalent to the class of flat left R-modules being closed under product [**2**]. Similar results exist for monoids but the correspondence is not quite so exact [**1**, **12**].

Chase [**2**] gave useful internal conditions on a ring R such that R is right coherent. Correspondingly, a monoid S is right coherent if and only if for any finitely generated right congruence ρ on S, and for any $a, b \in S$, the right congruence

$$r(a\rho) = \{(u, v) \in S \times S : au \,\rho\, av\}$$

is finitely generated, and the subact $(a\rho)S \cap (b\rho)S$ of the right S-act S/ρ is finitely generated [**12**].

Choo, Lam and Luft [**3**, Corollary 2.2 and remarks] have shown that free rings are coherent. The first author proved that free commutative monoids are coherent [**12**] and recently the authors, together with Ruškuc [**13**], have shown that free monoids are coherent. The class of coherent inverse monoids contains all semilattices of groups [**12**] and so, in particular, all groups and all semilattices. Certainly then free groups are coherent. It therefore becomes natural to ask whether free inverse monoids are coherent, since, not only are they free objects in a variety of unary algebras corresponding to injective partial maps, they are constructed from free groups acting on semilattices. Moreover, they have a realisation as *Munn trees* [**17**], that is, birooted finite connected subgraphs of the Cayley graph of the free group. As we show at the end of this article, coherency fails for free inverse monoids. This negative result motivates us to ask whether free left ample monoids, which may be thought of as the 'positive' part of free inverse monoids, being constructed from Cayley graphs of free monoids rather than free groups, are coherent. We remark that free left ample monoids are the free algebras in a variety of unary monoids corresponding to partial maps with distinguished domain. In our main result, Theorem 5·7, we show that free left ample monoids are right coherent.

For the convenience of the reader we describe in Section 2 the construction of the free inverse FIM(Ω), free left ample FLA(Ω) and free ample FAM(Ω) monoids on a set Ω. For ease of notation, we do this in terms of (prefix) closed subsets of the free group FG(Ω) - we could equally well use Munn trees. In Section 3 we focus on showing that the finitary properties (**R**),(**r**),(**L**) and (**l**) (defined therein) hold for FIM(Ω) and FLA(Ω). These properties (which arise from considerations of first order axiomatisability of the class of strongly flat right and left S-acts - see [**11**]) are similar in flavour, although easier to handle, than coherency. Our main work is in Section 4, where we make a detailed analysis of finitely generated right congruences on FLA(Ω). This hard work is then put to use in Section 5 where we show that FLA(Ω) is right coherent for any set Ω. In Section 6 we argue that the class of right coherent monoids is closed under retract. As a consequence of this, we have an alternative (albeit rather longer) proof to [**13**] that free monoids are coherent. Finally, in Section 7, we show that FIM(Ω), FLA(Ω) and FAM(Ω) are not coherent (for $|\Omega| \geq 2$).

2. *Preliminaries*

For background on the theory of S-acts and semigroups, we refer the reader to [**15**] and [**14**]. Let Ω be a non-empty set, let Ω^* be the free monoid and let FG(Ω) be the free group on Ω, respectively. We follow standard practice and denote by $l(a)$ the length of a reduced word $a \in \mathrm{FG}(\Omega)$ and so, in particular, of $a \in \Omega^*$. The empty word will be denoted by ϵ. Of course, Ω^* is a submonoid of the free group FG(Ω), and in the sequel, if $a \in \Omega^*$, by a^{-1} we mean the inverse of a in FG(Ω). For any $a \in \mathrm{FG}(\Omega)$ we denote by $a{\downarrow}$ the set of prefixes of the *reduced word* corresponding to a. Thus, if a is reduced and $a = x_1 \dots x_n$ where $x_i \in \Omega \cup \Omega^{-1}$, then

$$a{\downarrow} = \{\epsilon, x_1, x_1x_2, \dots, x_1x_2 \dots x_n\}.$$

The free inverse monoid on Ω is denoted by FIM(Ω). The structure of FIM(Ω) was determined by Munn [**17**] and Scheiblich [**18**]; the description we give below follows that of [**18**], of which further details may be found in [**14**]. However, we keep the equivalent characterisation via Munn trees constantly in mind.

Let $\mathcal{P}_c^f(\Omega)$ be the set of finite prefix closed subsets of FG(Ω). If $A \in \mathcal{P}_c^f(\Omega)$, then - regarding elements of A as reduced words - a *leaf* a of A is a word such that a is not a proper prefix of any other word in A. Note that FG(Ω) acts in the obvious way on its semilattice of subsets under union. Using this action we define

$$\mathrm{FIM}(\Omega) = \{(A, a) : A \in \mathcal{P}_c^f(\Omega), a \in A\}.$$

With binary operation given by

$$(A, a)(B, b) = (A \cup aB, ab),$$

FIM(Ω) is the free inverse monoid generated by Ω. The identity is $(\{\epsilon\}, \epsilon)$, the inverse $(A, a)^{-1}$ of (A, a) is $(a^{-1}A, a^{-1})$ and the natural injection of $\Omega \to \mathrm{FIM}(\Omega)$ is given by

$$x \mapsto (\{\epsilon, x\}, x).$$

We will make use of the fact that the free inverse monoid (in fact, every inverse monoid) possesses a left-right duality, by virtue of the anti-isomorphism given by $x \mapsto x^{-1}$. For future purposes we remark that if $a \in FG(X)$ is reduced, then

$$a^{-1} \cdot a{\downarrow} = (a^{-1}){\downarrow}.$$

Throughout this article we denote elements of FIM(Ω) by boldface letters, elements of $\mathcal{P}_c^f(\Omega)$ by capital letters, and elements of FG(Ω) by lowercase letters. We write a typical element of FIM(Ω) as $\mathbf{a} = (A, a)$; A and a will always denote the first and second coordinate of $\mathbf{a}$, respectively. One exception to this convention is that we denote the identity $(\{\epsilon\}, \epsilon)$ of FIM(Ω) by $\mathbf{1}$.

The free left ample monoid FLA(Ω) on Ω is the submonoid of FIM(Ω) given by

$$\text{FLA}(\Omega) = \{(A, a) \in \text{FIM}(\Omega) : A \subseteq \Omega^*\},$$

note that perforce, $a \in \Omega^*$ and we assume from the outset, when dealing with an element $\mathbf{a} = (A, a) \in \text{FLA}(\Omega)$, that all the words in A are reduced. We remark that FLA(Ω) also possesses a unary operation of $(A, a)^+ = (A, \epsilon) = (A, a)(A, a)^{-1}$ and (as a unary semigroup) is the free algebra on Ω in both the variety of left restriction semigroups and the quasi-varieties of (weakly) left ample semigroups [**7**, **9**, **5**].

Similarly, the free ample semigroup on Ω is the submonoid of FIM(Ω) given by

$$\text{FAM}(\Omega) = \{(A, a) \in \text{FIM}(\Omega) : a \in \Omega^*\}.$$

The free ample monoid possesses another unary operation defined by

$$(A, a)^* = (A, a)^{-1}(A, a) = (a^{-1}A, \epsilon)$$

and (as a biunary semigroup) is the free algebra on Ω in both the variety of restriction semigroups and the quasi-varieties of (weakly) ample semigroups. We remark here that the set of identities and quasi-identities defininWhile the class of ample monoids is left-right dual, so that FAM(Ω) consequently also has a left-right duality.

Note that FLA(Ω) is built from Ω^* (see [**8**]), but to simplify notation we make use of the embedding of Ω^* into FG(Ω). However, when dealing with FLA(Ω), we will use inverses only when we know that the resulting element lies in Ω^*, for example we will write $u^{-1}v$ only if u is a prefix of v.

Let S be a monoid, let $H \subseteq S \times S$ and let $\rho = \langle H \rangle$ be the right congruence generated by H. It is easy to see that if $a, b \in S$, then $a \,\rho\, b$ if and only if $a = b$ or there is an $n \geq 1$ and a sequence

$$(c_1, d_1, t_1; c_2, d_2, t_2; \ldots ; c_n, d_n, t_n)$$

of elements of S, with $(c_i, d_i) \in H$ or $(d_i, c_i) \in H$, such that the following equalities hold:

$$a = c_1t_1,\ d_1t_1 = c_2t_2,\ \ldots\ , d_nt_n = b.$$

Such a sequence will be referred to as an *H-sequence* (of length n) *connecting* a and b; where the factorisations are clear from context, we may simply refer to the sequence of equalities as the H-sequence. Moreover, when H is clear (usually when we are focussing on a specific H-sequence), we may drop the qualifier 'H'. It is convenient to allow $n = 0$ in the above; the empty sequence is interpreted as asserting equality $a = b$.

3. FIM(Ω), FAM(Ω) *and* FLA(Ω) *satisfy* (**R**), (**r**), (**L**) *and* (**l**)

The conditions (**R**) and (**r**) ((**L**) and (**l**)) are connected to the axiomatisability of certain classes of right (left) acts, and were introduced in [**11**]. Connected via axiomatisability to coherency, they are somewhat easier to handle. In this section we show that the free inverse, the free ample and the free left ample monoids satisfy these conditions. In doing so we develop some facility for handling products and factorisations in these monoids.

Definition 3·1. Let S be a monoid. We say that S satisfies Condition (**r**) if for every $s, t \in S$ the right ideal

$$\mathbf{r}^S(s,t) = \{u \in S : su = tu\}$$

is finitely generated.

The monoid S satisfies Condition (**R**) if for every $s, t \in S$ the S-subact

$$\mathbf{R}^S(s,t) = \{(u,v) : su = tv\}$$

of the right S-act $S \times S$ is finitely generated. (Note that we allow $\emptyset$ to be an ideal and an S-act.)

The conditions (**L**) and (**l**) are defined dually.

LEMMA 3·2. *Let A be a prefix closed subset of* $\mathrm{FG}(\Omega)$ *and let $g, h \in A$. Then*

$$g((g^{-1}h)\downarrow) \subseteq A.$$

Proof. Let x be the longest common prefix of the reduced words $g, h \in \mathrm{FG}(\Omega)$. That is, $g = xg'$ and $h = xh'$ where g', h' do not have a common nonempty prefix. Then

$$g((g^{-1}h)\downarrow) = xg'(g'^{-1}h')\downarrow \subseteq (xg')\downarrow \cup (xh')\downarrow = g\downarrow \cup h\downarrow \subseteq A.$$

□

LEMMA 3·3. *Let S denote either* $\mathrm{FIM}(\Omega)$, $\mathrm{FLA}(\Omega)$ *or* $\mathrm{FAM}(\Omega)$, *let* $\mathbf{au} = \mathbf{bv}$ *in S and suppose that there exists a leaf $x \in A \cup aU = B \cup bV$ such that $x \notin A \cup B$. Then there exist* $\mathbf{u}', \mathbf{v}', \mathbf{z} \in S$ *such that* $|A \cup aU'| < |A \cup aU|$,

$$\mathbf{au}' = \mathbf{bv}' \text{ and } (\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')\mathbf{z}.$$

Furthermore, if $\mathbf{u} = \mathbf{v}$ *then* $\mathbf{u}' = \mathbf{v}'$.

Proof. Clearly $\mathbf{u} \neq \mathbf{1}$. If $S = \mathrm{FLA}(\Omega)$ then it is easy to see that $x = ak$ where $k \in \Omega^* \setminus \{\epsilon\}$ is a leaf of U. The statement for S now follows from Lemma 4·3. We therefore consider the case where $S = \mathrm{FIM}(\Omega)$ of $S = \mathrm{FAM}(\Omega)$.

We can suppose that the words x, a, b, u and v are reduced. Note that $x \notin A \cup B$ implies that $x \in aU \cap bV$. We have that $x \notin A$ so in particular, x is not a prefix of a. In this case the last letter of x does not cancel in the product $a^{-1}x$. Now if $a^{-1}x$ is not a leaf of U then there exists $c \in \Omega \cup \Omega^{-1}$, different from the last letter of x, such that $a^{-1}xc \in U$. In this case $xc \in A \cup aU$, contradicting that x is a leaf of $A \cup aU$. So we have shown that $a^{-1}x$ is a leaf of U. Similarly $b^{-1}x$ is a leaf of V. There are two different cases to consider.

Case (i): $x \neq au$. Let $z = (au)^{-1}x$. Note that $u, a^{-1}x \in U$, which is prefix closed, and $z = (au)^{-1}x = u^{-1} \cdot a^{-1}x$. Lemma 3·2 then gives that $u(z\downarrow) \subseteq U$. Since $uz = a^{-1}x$, we have that

$$(U, u) = (U \setminus \{a^{-1}x\}, u)(z\downarrow, 1).$$

Furthermore, $z = (au)^{-1}x = (bv)^{-1}x$, giving in a similar fashion

$$(V, v) = (V \setminus \{b^{-1}x\}, v)(z\downarrow, 1).$$

Also, $A \cup a(U \setminus \{a^{-1}x\}) = B \cup b(V \setminus \{b^{-1}x\}) = (A \cup aU) \setminus \{x\}$, so we have that

$$(A, a)(U \setminus \{a^{-1}x\}, u) = (B, b)(V \setminus \{b^{-1}x\}, v).$$

Consequently, if we let

$$(U', u') = (U \setminus \{a^{-1}x\}, u), (V', v') = (V \setminus \{b^{-1}x\}, v) \text{ and } \mathbf{z} = (z\downarrow, z),$$

then, (noticing that if $(U, u) = (V, v)$ we must have that $a = b$), the statements of the lemma are satisfied.

Case (ii): $x = au = bv$. Since $x \notin A \cup B$, but $a, b \in A \cup B$ we have that $u, v \neq \epsilon$. In case $S = \mathrm{FAM}(\Omega)$, this implies that the last letters of x, u and v are the same which we denote by $z \in \Omega$. Note that $uz^{-1}, vz^{-1} \in \Omega^*$ in this case.

If $S = \mathrm{FIM}(\Omega)$ then let z be the last letter of the reduced word x. If z is not the last letter of u then in the product $x = au$, all letters of u must cancel, so $a = xu^{-1}$ where xu^{-1} is reduced. However, this contradicts the fact that x is a leaf, showing that the last letter of the reduced word u is z. Similarly the last letter of the reduced word v is z.

In both the cases $S = \mathrm{FAM}(\Omega)$ and $S = \mathrm{FIM}(\Omega)$, $u \neq uz^{-1}$ and $u \neq \epsilon$ imply that $uz^{-1} \in U \setminus \{u\}$, and similarly $vz^{-1} \in V \setminus \{v\}$. Now let $\mathbf{u}' = (U \setminus \{u\}, uz^{-1}), \mathbf{v}' = (V \setminus \{v\}, vz^{-1})$ and $\mathbf{z} = (\{1, z\}, z)$. Then

$$(U, u) = (U', u')(\{1, z\}, z),\ (V, v) = (V', v'), (\{1, z\}, z)$$

and

$$(A, a)(U', u') = \big((A \cup aU) \setminus \{au\}, au'\big) = (B, b)(V', v').$$

Furthermore, if $\mathbf{u} = \mathbf{v}$ then clearly $\mathbf{u}' = \mathbf{v}'$, which finishes the proof. $\square$

PROPOSITION 3·4. *The monoids* FIM(Ω)*,* FAM(Ω) *and* FLA(Ω) *satisfy* (**R**) *and* (**r**)*.*

Proof. Let S denote FIM(Ω), FAM(Ω) or FLA(Ω) and let $\mathbf{a}, \mathbf{b} \in S$. We claim that the finite set

$$X = \{(\mathbf{u}, \mathbf{v}) : \mathbf{au} = \mathbf{bv},\ A \cup aU = A \cup B\}$$

generates $\mathbf{R}(\mathbf{a}, \mathbf{b})$. Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{R}(\mathbf{a}, \mathbf{b})$. We prove by induction on the size of $A \cup aU$ that $(\mathbf{u}, \mathbf{v}) \in X \cdot S$. Note that $A \cup aU = B \cup bV$ implies $A \cup B \subseteq A \cup aU$, so that if $|A \cup aU| \leq |A \cup B|$, then necessarily $A \cup aU = B \cup bV = A \cup B$, which shows that $(\mathbf{u}, \mathbf{v}) \in X$.

Suppose now that we have that there exists an $n \geq |A \cup B|$ such that whenever $|A \cup aU| \leq n$ and $(\mathbf{u}, \mathbf{v}) \in \mathbf{R}(\mathbf{a}, \mathbf{b})$, then necessarily $(\mathbf{u}, \mathbf{v}) \in X \cdot S$. Now let $(\mathbf{u}, \mathbf{v}) \in \mathbf{R}(\mathbf{a}, \mathbf{b})$ be such that $|A \cup aU| = n + 1$. Since $(\mathbf{u}, \mathbf{v}) \in \mathbf{R}(\mathbf{a}, \mathbf{b})$ we have that $A \cup B \subseteq A \cup aU = B \cup bV$, and since $n + 1 > |A \cup B|$, there exists $x \in A \cup aU = B \cup bV$ such that $x \notin A \cup B$. This implies that $x \in aU \cap bV$. We can also assume that x is a leaf of $A \cup aU = B \cup bV$. Then Lemma 3·3 implies that there exist elements $\mathbf{u}', \mathbf{v}', \mathbf{z} \in S$ such that $|A \cup aU'| < |A \cup aU|$ and

$$(\mathbf{u}', \mathbf{v}') \in \mathbf{R}(\mathbf{a}, \mathbf{b}),\ (\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')\mathbf{z}.$$

In this case the induction hypothesis implies that $(\mathbf{u}', \mathbf{v}') \in X \cdot S$, so that $(\mathbf{u}, \mathbf{v}) \in X \cdot S$ as required.

For (**r**), the proof is entirely similar. We show that the set

$$Y = \{\mathbf{u} \in S : \mathbf{au} = \mathbf{bu},\ A \cup aU = A \cup B\}$$

generates $\mathbf{r}(s,t)$, making particular use of the final statement of Lemma 3·3. $\square$

The free inverse monoid and the free ample monoid are left-right dual, so from the dual of Lemma 3·3 they satisfy ($\mathbf{L}$) and ($\mathbf{l}$). To show that $\mathrm{FLA}(\Omega)$ satisfies ($\mathbf{L}$) and ($\mathbf{l}$), we first prove a result corresponding to Lemma 3·3.

LEMMA 3·5. *Let* $\mathbf{ua} = \mathbf{vb}$ *in* $\mathrm{FLA}(\Omega)$ *and suppose that there exists* $x \in U \cup uA = V \cup vB$ *such that* x *is either a leaf, or* $x = \epsilon$ *and every element of* $(U \cup uA) \setminus \{\epsilon\}$ *has a common nonempty prefix (this corresponds to a tree having a root with degree* 1*). Furthermore, suppose that* $x \notin uA \cup vB$. *Then there exist* $\mathbf{u}', \mathbf{v}', \mathbf{z} \in \mathrm{FLA}(\Omega)$ *such that* $|U' \cup u'A| < |U \cup uA|$,

$$\mathbf{u}'\mathbf{a} = \mathbf{v}'\mathbf{b} \text{ and } (\mathbf{u}, \mathbf{v}) = \mathbf{z}(\mathbf{u}', \mathbf{v}').$$

Furthermore, if $\mathbf{u} = \mathbf{v}$ *then* $\mathbf{u}' = \mathbf{v}'$.

Proof. Note that as $x \notin uA \cup vB$, $x \neq u$ and $x \neq v$. If x is a leaf, then let $\mathbf{z} = (x{\downarrow}, 1)$, $U' = U \setminus \{x\}, u' = u, V' = V \setminus \{x\}, v' = v$. In this case

$$\mathbf{u}'\mathbf{a} = \big((U \cup uA) \setminus \{x\}, ua\big) = \big((V \cup vB) \setminus \{x\}, vb\big) = \mathbf{v}'\mathbf{b}, \mathbf{z}\mathbf{u}' = \mathbf{u}, \mathbf{z}\mathbf{v}' = \mathbf{v}.$$

Furthermore, if $\mathbf{u} = \mathbf{v}$ then of course $\mathbf{u}' = \mathbf{v}'$.

If $x = \epsilon$ then $x \notin uA \cup vB$ implies $u, v \neq \epsilon$. Let z be the common first letter of elements of $(U \cup uA) \setminus \{\epsilon\}$ and let $\mathbf{z} = (\{\epsilon, z\}, z)$. Then if we set $(U', u') = (z^{-1}(U \setminus \{\epsilon\}), z^{-1}u)$ and $(V', v') = (z^{-1}(V \setminus \{\epsilon\}, z^{-1}v)$ then

$$U' \cup u'A = z^{-1}(U \setminus \{\epsilon\}) \cup z^{-1}uA = z^{-1}\big((U \cup uA) \setminus \{\epsilon\}\big) = \ldots = V' \cup v'B,$$

which shows that $\mathbf{u}'\mathbf{a} = \mathbf{v}'\mathbf{b}$. Also we have

$$Z \cup zU' = \{\epsilon, z\} \cup (U \setminus \{\epsilon\}) = U,$$

because $z \in U$ (being the first letter of u). As a consequence $\mathbf{z}\mathbf{u}' = \mathbf{u}$ and similarly $\mathbf{z}\mathbf{v}' = \mathbf{v}$ also. Lastly, if $\mathbf{u} = \mathbf{v}$ then clearly $\mathbf{u}' = \mathbf{v}'$ which finishes the proof. $\square$

PROPOSITION 3·6. *The free inverse monoid* $\mathrm{FIM}(\Omega)$*, the free ample monoid* $\mathrm{FAM}(\Omega)$ *and the free left ample monoid* $\mathrm{FLA}(\Omega)$ *satisfy* ($\mathbf{L}$) *and* ($\mathbf{l}$).

Proof. We have already mentioned that $\mathrm{FIM}(\Omega)$ and $\mathrm{FAM}(\Omega)$ must satisfy ($\mathbf{L}$) and ($\mathbf{l}$). For $\mathrm{FLA}(\Omega)$, let $\mathbf{a}, \mathbf{b} \in \mathrm{FLA}(\Omega)$. Then either $\mathbf{L}(\mathbf{a}, \mathbf{b})$ is empty or one of a and b is a suffix of the other. Without loss of generality we can assume that $b = ya$ for some $y \in \Omega^*$. In this case we claim that the finite set

$$X = \{(\mathbf{u}, \mathbf{v}) : \mathbf{ua} = \mathbf{vb}, U \cup uA = B \cup yA\}$$

generates $\mathbf{L}(\mathbf{a}, \mathbf{b})$. Note that if $(\mathbf{u}, \mathbf{v}) \in \mathbf{L}(\mathbf{a}, \mathbf{b})$ then necessarily $u = vy$ so from the equation $U \cup vyA = V \cup vB$ we conclude that $v(B \cup yA) \subseteq U \cup uA$. As a consequence we see that if $|U \cup uA| \leq |B \cup yA|$ then $U \cup uA = v(B \cup yA)$, which implies that $v = \epsilon$ so that $U \cup uA = B \cup yA$ and $(\mathbf{u}, \mathbf{v}) \in X$.

Suppose now that there exists an $n \geq |B \cup yA|$ such that whenever $|U \cup uA| \leq n$ and $(\mathbf{u}, \mathbf{v}) \in \mathbf{L}(\mathbf{a}, \mathbf{b})$, then necessarily $(\mathbf{u}, \mathbf{v}) \in \mathrm{FLA}(\Omega) \cdot X$. Now let $(\mathbf{u}, \mathbf{v}) \in \mathbf{L}(\mathbf{a}, \mathbf{b})$ be such that $|U \cup uA| = n+1$. Note that $ua = vya$ implies that $u = vy$. Then $U \cup vyA = V \cup vB$, so $v(B \cup yA) \subseteq U \cup vyA$. However, $|v(B \cup yA)| = |B \cup yA| < |U \cup vyA|$, so $U \cup uA \neq v(B \cup yA) = uA \cup vB$.

If there exists a leaf of $U \cup uA$ which is not contained in $uA \cup vB$ then let x be one such leaf. However, if there is no such leaf then that means that every leaf of $U \cup uA$ is contained in $v(B \cup yA)$. If $v = \epsilon$ then as $y \in B$, $v(B \cup yA)$ is prefix closed so $U \cup uA = v(B \cup yA) = uA \cup vB$, which is a contradiction. So $v \neq \epsilon$, and we have that all leaves of $U \cup uA$ have v as a prefix. This can only happen if $U \cup uA = v{\downarrow} \cup vC$ for some prefix closed set C, which shows that every element of $(U \cup uA) \setminus \{\epsilon\}$ has the same first letter as v. In this case let $x = \epsilon$. Then Lemma 3·5 implies that there exists $\mathbf{u}', \mathbf{v}', \mathbf{z} \in \mathrm{FLA}(\Omega)$ such that $|U' \cup u'A| < |U \cup uA|$,

$$(\mathbf{u}', \mathbf{v}') \in \mathbf{L}(\mathbf{a}, \mathbf{b}) \text{ and } (\mathbf{u}, \mathbf{v}) = \mathbf{z}(\mathbf{u}', \mathbf{v}').$$

In this case the induction hypothesis implies that $(\mathbf{u}', \mathbf{v}') \in \mathrm{FLA}(\Omega) \cdot X$ and so we have $(\mathbf{u}, \mathbf{v}) \in \mathrm{FLA}(\Omega) \cdot X$ as required.

For $(\mathbf{l})$, the proof is entirely similar, namely the finite set

$$Y = \{U \in S : \mathbf{ua} = \mathbf{ub}, U \cup uA = B \cup yA\}$$

generates $\mathbf{l}(\mathbf{a}, \mathbf{b})$ if $b = ya$. □

4. FLA(Ω)*: analysis of* H*-sequences*

In order to show that $\mathrm{FLA}(\Omega)$ is right coherent, we make a careful examination of H-sequences for finite sets $H \subseteq \mathrm{FLA}(\Omega) \times \mathrm{FLA}(\Omega)$.

Definition 4·1. Let $\mathbf{a} \in \mathrm{FLA}(\Omega)$.
- (i) The *weight* $w(\mathbf{a})$ of $\mathbf{a}$ is defined by $w(\mathbf{a}) = |A| - 1 + l(a)$.
- (ii) The *diameter* $d(\mathbf{a})$ of $\mathbf{a}$ is defined by $d(\mathbf{a}) = \max\, \{l(u) : u \in A\}$.

The following lemma states the most important basic properties of the weight function.

LEMMA 4·2. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathrm{FLA}(\Omega)$. *Then*
- (W0) $w(\mathbf{a}) = 0$ *if and only if* $\mathbf{a} = \mathbf{1}$*;*
- (W1) $w(\mathbf{a}), w(\mathbf{b}) \leq w(\mathbf{ab}) \leq w(\mathbf{a}) + w(\mathbf{b})$*;*
- (W2) $w(\mathbf{ab}) = w(\mathbf{a})$ *if and only if* $\mathbf{ab} = \mathbf{a}$*, and this is equivalent to* $\mathbf{b} \in E(\mathrm{FLA}(\Omega))$ *with* $\mathbf{a} \leq_{\mathcal{L}} \mathbf{b}$.

Proof. The proof of (W0) is clear.

For (W1), let $\mathbf{a} = (A, a)$ and $\mathbf{b} = (B, b)$, so that $\mathbf{ab} = (A \cup aB, ab)$. Then

$$w(\mathbf{ab}) = |A \cup aB| - 1 + l(ab)$$

and as $|A \cup aB| \geq |A|, |aB|$ where $|aB| = |B|$ and $l(ab) \geq l(a), l(b)$, we have $w(\mathbf{a}), w(\mathbf{b}) \leq w(\mathbf{ab})$.

On the other hand, the second inequality for (W1) follows from the observation that as $a \in A \cap aB$ we have

$$|A \cup aB| = |A| + |aB \setminus A| \leq |A| + |aB| - 1 = |A| + |B| - 1.$$

Clearly $|A \cup aB| \geq |A|$ and $l(ab) \geq l(a)$, so that if $w(\mathbf{ab}) = w(\mathbf{a})$, we must have $|A \cup aB| = |A|$ and $l(b) = 0$. Hence $b = \epsilon$, $aB \subseteq A$ and so $\mathbf{ab} = \mathbf{a}$.

If $\mathbf{ab} = \mathbf{a}$ (equivalently, $w(\mathbf{ab}) = w(\mathbf{a})$), then we have shown that $\mathbf{b} \in E(FLA(\Omega))$ and clearly $\mathbf{a} \leq_{\mathcal{L}} \mathbf{b}$. The converse is clear. Thus (W2) holds. □

The proof of our main result depends heavily on the fact that certain factorisations can be carried through sequences. The following two lemmas constitute the foundations of this process.

LEMMA 4·3. *Let* $\mathbf{dz} = \mathbf{bv}$, $\mathbf{z} \neq \mathbf{1}$ *and let* x *be a leaf of* Z *such that* $dx \notin B$. *Then there exist elements* $\mathbf{z}', \mathbf{x}, \mathbf{v}' \in \mathrm{FLA}(\Omega)$ *such that*

$$Z' = Z \setminus \{x\}, w(\mathbf{z}') < w(\mathbf{z}),\ \mathbf{z} = \mathbf{z}'\mathbf{x},\ \mathbf{v} = \mathbf{v}'\mathbf{x},\ \mathbf{dz}' = \mathbf{bv}'$$

and

(i) *if* $x \neq z$ *and* $dx \notin D$ *then* $\mathbf{x} = (\tilde{x}{\downarrow} \cup \tilde{z}{\downarrow}, \tilde{z}), \mathbf{v}' = (V \setminus \{b^{-1}dx\}, v\tilde{z}^{-1})$ *where* $\tilde{x}, \tilde{z} \in \Omega^*$ *have no common non-empty prefix,* $x = z'\tilde{x}, z = z'\tilde{z}$ *(so* $dx = dz'\tilde{x} = bv'\tilde{x}$*),*

(ii) *if* $x = z$ *(then necessarily* $x \neq \epsilon$*) and* $dx \notin D$ *then* $\mathbf{z}' = (Z', zx'^{-1}), \mathbf{x} = (\{\epsilon, x'\}, x')$ *and* $\mathbf{v}' = (V \setminus \{v\}, vx'^{-1})$*, where* x' *is the last letter of* x*,*

(iii) *if* $x = z$ *(then necessarily* $x \neq \epsilon$*) and* $dx \in D$ *then* $\mathbf{z}' = (Z', zx'^{-1}), \mathbf{x} = (\{\epsilon, x'\}, x')$ *and* $\mathbf{v}' = (V, vx'^{-1})$*, where* x' *is the last letter of* x*,*

(iv) *if* $x \neq z$ *and* $dx \in D$ *then* $\mathbf{z}' = (Z', z'), \mathbf{x} = (\tilde{x}{\downarrow} \cup \tilde{z}{\downarrow}, \tilde{z}), \mathbf{v}' = (V, v\tilde{z}^{-1})$ *where* $\tilde{x}, \tilde{z} \in \Omega^*$ *have no common non-empty prefix.*

Furthermore, the following are true:

(A) *in Cases (i) and (ii) we have* $|D \cup dZ'| < |D \cup dZ|$ *and that if* $\mathbf{z} = \mathbf{v}$ *then* $\mathbf{z}' = \mathbf{v}'$*,*

(B) *in Cases (i), (ii) and (iii) we have* $w(\mathbf{bv}') = w(\mathbf{dz}') < w(\mathbf{dz}) = w(\mathbf{bv})$.

Proof. We investigate all 4 cases separately:

Case (i): $dx \notin D$ and $x \neq z$. Let z' be the greatest common prefix of z and x, that is, there exist $\tilde{z}$ and $\tilde{x}$ such that $z = z'\tilde{z}$ and $x = z'\tilde{x}$ and $\tilde{z}$ and $\tilde{x}$ have no common non-empty prefix. It is important to note that $\tilde{x} \neq \epsilon$, for x is a leaf different from z. Now let

$$\mathbf{z}' = (Z \setminus \{x\}, z'), \mathbf{x} = (\tilde{x}{\downarrow} \cup \tilde{z}{\downarrow}, \tilde{z}).$$

Then it is easy to check that $\mathbf{z}', \mathbf{x} \in \mathrm{FLA}(\Omega)$ and $\mathbf{z} = \mathbf{z}'\mathbf{x}$. Note that since $dx \notin B$, but $dx \in B \cup bV$, we have that $dx = dz'\tilde{x} \in bV$, and that $bv = dz = dz'\tilde{z} \in bV$ also. Since $\tilde{z}$ and $\tilde{x}$ have no common non-empty prefix, we conclude that b is a prefix of dz'. As a consequence of the fact that $bv = dz'\tilde{z}$, we conclude that $\tilde{z}$ is a suffix of v, so $v\tilde{z}^{-1} \in V$. Furthermore, $bv = dz'\tilde{z}$ implies that $v\tilde{z}^{-1} = b^{-1}dz' \neq b^{-1}dz'\tilde{x} = b^{-1}dx$. Now let

$$\mathbf{v}' = (V \setminus \{b^{-1}dx\}, v\tilde{z}^{-1}).$$

Note that our assumption that $dx \notin D$ implies that dx is a leaf of $B \cup bV$. Then, since $dx \notin B$, we have that $b^{-1}dx$ is a leaf of V, so $\mathbf{v}' \in \mathrm{FLA}(\Omega)$. It is then easy to check that $\mathbf{v} = \mathbf{v}'\mathbf{x}$, since the second coordinates are the same, and $b^{-1}dx = b^{-1}dz'\tilde{x} = v\tilde{z}^{-1}\tilde{x}$. Similarly $\mathbf{dz}' = \mathbf{bv}'$, for the second coordinates are both equal dz', and the first coordinates both equal $(B \cup bV) \setminus \{dx\}$. Also we have that $w(\mathbf{bv}') < w(\mathbf{bv})$, because $dx \in B \cup bV$. Furthermore, if $\mathbf{z} = \mathbf{v}$ then from $\mathbf{dz} = \mathbf{bv}$ we conclude that $d = b$ which implies that $b^{-1}dx = x$. Similarly $v\tilde{z}^{-1} = b^{-1}dz' = z'$, showing that $\mathbf{z}' = \mathbf{v}'$.

Case (ii): $dx \notin D$, and $x = z$. We have that $z \neq \epsilon$, for otherwise $\mathbf{z} = \mathbf{1}$. So let $z = z'x'$ where $x' \in \Omega$, and let

$$\mathbf{z}' = (Z \setminus \{z\}, z'),\ \mathbf{x} = (\{\epsilon, x'\}, x').$$

We have that $\mathbf{z}', \mathbf{x} \in \mathrm{FLA}(\Omega)$, and that $\mathbf{z} = \mathbf{z}'\mathbf{x}$. Note that $dz \notin B$, but it is the second coordinate of $\mathbf{bv}$. Thus, $v \neq \epsilon$, and we have that x' is the last letter of v and as a consequence, $dz' = bv'$, where $v' = v(x')^{-1}$. We see that v is a leaf of V and similarly to the previous case it is easy to show that if we define

$$\mathbf{v}' = (V \setminus \{v\}, v'),$$

then $\mathbf{v}' \in \mathrm{FLA}(\Omega), w(\mathbf{bv}') < w(\mathbf{bv}), \mathbf{v} = \mathbf{v}'\mathbf{x}$ and $\mathbf{dz}' = \mathbf{bv}' = \big((D \cup dZ) \setminus \{dz\}, dz'\big)$. Furthermore, if $\mathbf{z} = \mathbf{v}$ then of course $z = v$ and we conclude that $\mathbf{z}' = \mathbf{v}'$, so the statements of the lemma are true.

Case (iii): $dx \in D$, and $x = z$. This case is similar to Case (ii), the only difference being that we have to define

$$\mathbf{v}' = (V, v').$$

Since the second coordinate of $\mathbf{bv}'$ is one letter shorter than bv, we have that $w(\mathbf{bv}') < w(\mathbf{bv})$.

Case (iv): $dx \in D$ and $x \neq z$. Put

$$\mathbf{z}' = (Z \setminus \{x\}, z'),\ \ \mathbf{x} = (\tilde{x}{\downarrow} \cup \tilde{z}{\downarrow}, \tilde{z}) \text{ and } \mathbf{v}' = (V, v\tilde{z}^{-1})$$

where $z', \tilde{z}$ and $\tilde{x}$ are defined as in Case (i). It is easy to check (using the same argument as in Case (i)) that $b^{-1}dx = v\tilde{z}^{-1}\tilde{x}$ is a leaf in V, $\mathbf{z}', \mathbf{x}, \mathbf{v}' \in \mathrm{FLA}(\Omega)$, $w(\mathbf{z}') < w(\mathbf{z})$ and

$$\mathbf{z} = \mathbf{z}'\mathbf{x},\ \ \mathbf{v} = \mathbf{v}'\mathbf{x} \text{ and } \mathbf{dz}' = \mathbf{bv}',$$

so that again, the statements of the lemma are true. □

LEMMA 4·4. *Let* $\mathbf{ab} = \mathbf{cd}$ *such that* $\mathbf{b} = (x{\downarrow} \cup b{\downarrow}, b)$ *for some* $b, x \in \Omega^*, x \neq \epsilon$, *having no common non-empty prefix. If* $ax \notin A \cup C$ *and* $A = (A \cup aB) \setminus \{ax\}$*, then* $\mathbf{d} = \mathbf{d}'\mathbf{b}$ *for some* $\mathbf{d}' = (D \setminus \{d'x\}, d')$ *such that* $\mathbf{a} = \mathbf{cd}'$.

Proof. First remark that our hypotheses guarantee that ax is a leaf of $A \cup aB = C \cup cD$.

Since $ab = cd$, c is a prefix of ab. However, since $ax \in C \cup cD$, but $ax \notin C$, we have that c is also a prefix of ax. Since b and x have no common non-empty prefix, this implies that c is a prefix of a.

Let $d' \in \Omega^*$ be such that $a = cd'$. We have that $ax = cd'x \in cD$, so $d'x \in D$. From $cd'b = ab = cd$ we deduce that $d'b = d \in D$. From $d'b, d'x \in D$, the prefix closure of D gives that $d'B \subseteq D$. Observe now that $d'x$ is a leaf of D and $d'x \neq d'$, so that $\mathbf{d}' = (D \setminus \{d'x\}, d') \in \mathrm{FLA}(\Omega)$ and clearly, $cd'x \notin C \cup cD'$. Moreover, it is easy to check that

$$\mathbf{a} = \mathbf{cd}' \text{ and } \mathbf{d} = \mathbf{d}'\mathbf{b}.$$

□

Let ρ be a finitely generated right congruence on $\mathrm{FLA}(\Omega)$. Without loss of generality we may suppose that $\rho = \langle H \rangle$ for some finite $H \subseteq \mathrm{FLA}(\Omega) \times \mathrm{FLA}(\Omega)$ with $H^{-1} = H$. Let us denote by $\mathcal{D}$ the maximum of the diameters of the components of the elements of H. In the following definition, we abuse terminology a little, along the lines of that for H-sequences at the end of Section 2. The elements $\mathbf{a}, \mathbf{u}, \mathbf{b}$ and $\mathbf{v}$ play a special role, but are only distinguished by the very notation from the products $\mathbf{au}$ and $\mathbf{bv}$. We employ similar conventions in other circumstances.

Definition 4·5. Suppose that we have an H-sequence

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \mathbf{d}_1\mathbf{t}_1 = \mathbf{c}_2\mathbf{t}_2, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv}$$

connecting $\mathbf{au}$ and $\mathbf{bv}$. Then we say that the H-sequence is *reducible* if there exist elements $\mathbf{y}, \mathbf{u}', \mathbf{t}_1', \ldots, \mathbf{t}_n', \mathbf{v}'$ such that

(Red1) $w(\mathbf{au}') < w(\mathbf{au})$, $w(\mathbf{bv}') < w(\mathbf{bv})$ or $w(\mathbf{t}_i') < w(\mathbf{t}_i)$ for some i;

(Red2) $\mathbf{u} = \mathbf{u}'\mathbf{y}, \mathbf{t}_1 = \mathbf{t}'_1\mathbf{y}, \ldots, \mathbf{t}_n = \mathbf{t}'_n\mathbf{y}, \mathbf{v} = \mathbf{v}'\mathbf{y}$;
(Red3) $\mathbf{a}\mathbf{u}' = \mathbf{c}_1\mathbf{t}'_1, \mathbf{d}_1\mathbf{t}'_1 = \mathbf{c}_2\mathbf{t}'_2, \ldots, \mathbf{d}_n\mathbf{t}'_n = \mathbf{b}\mathbf{v}'$.

If an H-sequence is not reducible, we call it *irreducible.*

From the above definition, a length-0 H-sequence $\mathbf{au} = \mathbf{bv}$ is reducible if and only if there exist elements $\mathbf{y}, \mathbf{u}', \mathbf{v}' \in \mathrm{FLA}(\Omega)$ such that $\mathbf{u} = \mathbf{u}'\mathbf{y}, \mathbf{v} = \mathbf{v}'\mathbf{y}, \mathbf{au}' = \mathbf{bv}'$ and $w(\mathbf{au}') = w(\mathbf{bv}') < w(\mathbf{au}) = w(\mathbf{bv})$.

Note that if (Red2) holds, then in view of (W2) in Lemma 4·2, (Red1) is equivalent to saying that $\mathbf{au}' \neq \mathbf{au}$, $\mathbf{bv}' \neq \mathbf{bv}$ or $\mathbf{t}'_i \neq \mathbf{t}_i$ for some i - we are going to make use of this fact in the sequel. We are going to show that every irreducible H-sequence has an element with diameter less than or equal to $2\max(\mathcal{D}, d(\mathbf{a}), d(\mathbf{b}))$.

LEMMA 4·6. *If the H-sequence* $\mathbf{au} = \mathbf{bv}$ *is irreducible then* $d(\mathbf{u}) \leq \max(d(\mathbf{a}), d(\mathbf{b}))$.

Proof. Suppose that $d(\mathbf{u}) > d(\mathbf{a}), d(\mathbf{b})$. Then there exists a leaf $x \in U$ such that $l(x) > d(\mathbf{a}), d(\mathbf{b})$. As a consequence we have $ax \notin A \cup B$, so by Cases (1) and (2) of Lemma 4·3 there exist $\mathbf{u}', \mathbf{v}', \mathbf{x} \in \mathrm{FLA}(\Omega)$ such that $\mathbf{au}' = \mathbf{bv}', \mathbf{u} = \mathbf{u}'\mathbf{x}, \mathbf{v} = \mathbf{v}'\mathbf{x}$ and $w(\mathbf{bv}') < w(\mathbf{bv})$, contradicting the irreducibility of the sequence $\mathbf{au} = \mathbf{bv}$. ☐

The following Lemma shows that elements of FLA(Ω) which are connected by an irreducible sequence are 'lean' - the length of their second component limits their diameter. In fact, much more is true, but this statement will suffice for our proof. Furthermore, it is worth noting that this lemma is one (the other one is Statement (4·4) of Lemma 4·3) which is not dualisable - it fails if we swap from right congruences to left congruences.

LEMMA 4·7. *If*

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \mathbf{d}_1\mathbf{t}_1 = \mathbf{c}_2\mathbf{t}_2, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv} \tag{4·1}$$

is an irreducible H-sequence, then $d(\mathbf{au}) \leq 2\max(l(au), d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$.

Proof. Let $\mathcal{M} = \max(l(au), d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$. For brevity let $\mathbf{c}_{n+1} = \mathbf{b}$ and $\mathbf{t}_{n+1} = \mathbf{v}$. Suppose that $d(\mathbf{au}) > 2\mathcal{M}$, which clearly implies that $\mathbf{u} \neq \mathbf{1}$. Let y be a leaf of $A \cup aU$ with $l(y) = d(\mathbf{au}) > 2\mathcal{M}$. Then clearly $y \notin A$, so $y = ax$ for some leaf $x \in U$. Notice that since $l(a) \leq d(\mathbf{a})$, we have that $l(x) > \mathcal{M} \geq d(\mathbf{a}), d(\mathbf{c}_1)$, so $ax \notin A \cup C_1$. Also, $l(ax) > l(au)$ implies that $x \neq u$. Then if we apply Lemma 4·3 to the equality $\mathbf{au} = \mathbf{c}_1\mathbf{t}_1$ and the leaf $x \in U$, we obtain by Case (1) that there exist elements $\mathbf{x}, \mathbf{u}', \mathbf{t}'_1 \in \mathrm{FLA}(\Omega)$ such that

$$w(\mathbf{au}') < w(\mathbf{au}),\ \mathbf{u} = \mathbf{u}'\mathbf{x},\ \mathbf{t}_1 = \mathbf{t}'_1\mathbf{x}, \mathbf{au}' = \mathbf{c}_1\mathbf{t}'_1,$$

$$\mathbf{x} = (\tilde{x}{\downarrow} \cup \tilde{u}{\downarrow}, \tilde{u}) \text{ and } \mathbf{t}'_1 = (T_1 \setminus \{t'_1\tilde{x}\}, t'_1)$$

with $\tilde{x}, \tilde{u} \in \Omega^*$ having no common non-empty prefix and $x = u'\tilde{x}$. Note that $ax = au'\tilde{x}$, $l(ax) > 2\mathcal{M} \geq \mathcal{M} + l(au)$ and au' is a prefix of au, so we have that $l(\tilde{x}) > \mathcal{M}$. Further, $C_1 \cup c_1T'_1 = (C_1 \cup c_1T_1) \setminus \{c_1t'_1\tilde{x}\}$.

Note that if $n = 0$ then we have already contradicted the irreducibility of the sequence (4·1), so in the sequel we suppose that $n > 0$.

Suppose for induction that we have constructed elements $\mathbf{u}', \mathbf{t}'_1, \ldots, \mathbf{t}'_m \in \mathrm{FLA}(\Omega)$ satisfying $\mathbf{u} = \mathbf{u}'\mathbf{x}$, $\mathbf{t}_i = \mathbf{t}'_i\mathbf{x}$ for all $1 \leq i \leq m$, $T'_m = T_m \setminus \{t'_m\tilde{x}\}$ and $C_m \cup c_mT'_m = (C_m \cup c_mT_m) \setminus \{c_mt'_m\tilde{x}\}$.

Since $l(\tilde{x}) > \mathcal{M}$, we have that $d_mt'_m\tilde{x} \notin (D_m \cup d_mT'_m) \cup C_{m+1}$, so $D_m \cup d_mT'_m =$

$(D_m \cup d_m T_m) \setminus \{d_m t'_m \tilde{x}\}$. We can therefore apply Lemma 4·4 to the equality $\mathbf{d}_m \mathbf{t}'_m \cdot \mathbf{x} = \mathbf{c}_{m+1}\mathbf{t}_{m+1}$ and obtain that $\mathbf{t}_{m+1} = \mathbf{t}'_{m+1}\mathbf{x}$ for some $\mathbf{t}'_{m+1}$ with $T'_{m+1} = T_{m+1} \setminus \{t_{m+1}\tilde{x}\}$ and $\mathbf{d}_m \mathbf{t}'_m = \mathbf{c}_{m+1}\mathbf{t}'_{m+1}$, so that $C_{m+1} \cup c_{m+1}T'_{m+1} = (C_{m+1} \cup c_{m+1}T_{m+1}) \setminus \{c_{m+1}t_{m+1}\tilde{x}\}$.

Applying induction (note that $\mathcal{M} \geq d(\mathbf{b})$ is required at the last step), there exist elements $\mathbf{u}', \mathbf{t}'_1, \ldots, \mathbf{t}'_n, \mathbf{v}'$ such that $\mathbf{u} = \mathbf{u}'\mathbf{x}, \mathbf{t}_1 = \mathbf{t}'_1\mathbf{x}, \ldots, \mathbf{t}_n = \mathbf{t}'_n\mathbf{x}, \mathbf{v} = \mathbf{v}'\mathbf{x}, w(\mathbf{au}') < w(\mathbf{au})$ and

$$\mathbf{au}' = \mathbf{c}_1\mathbf{t}'_1, \mathbf{d}_1\mathbf{t}'_1 = \mathbf{c}_2\mathbf{t}'_2, \ldots, \mathbf{d}_n\mathbf{t}'_n = \mathbf{bv}'.$$

This contradicts the irreducibility of the sequence (4·1) and so we conclude that $d(\mathbf{au}) \leq 2\mathcal{M}$. □

Definition 4·8. We say that the pair $(\mathbf{au}, \mathbf{bv})$ is *irreducible* if $\mathbf{au}$ and $\mathbf{bv}$ can be connected by an irreducible H-sequence.

We are again a little cavalier in the above; more properly, we should talk of the quadruple $(\mathbf{a}, \mathbf{u}, \mathbf{b}, \mathbf{v})$ as being irreducible. However, clarity is always given in the context.

Definition 4·9. Let $\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \mathbf{d}_1\mathbf{t}_1 = \mathbf{c}_2\mathbf{t}_2, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv}$ be an H-sequence $\mathcal{S}$. We define the *weight* w of $\mathcal{S}$ to be $w(\mathbf{au}) + w(\mathbf{t}_1) + \ldots + w(\mathbf{t}_n) + w(\mathbf{bv})$.

LEMMA 4·10. *Let*

$$\mathcal{S} : \mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \mathbf{d}_1\mathbf{t}_1 = \mathbf{c}_2\mathbf{t}_2, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv}$$

be an H-sequence. Then there exist elements $\mathbf{y}, \mathbf{u}', \mathbf{t}'_1, \ldots, \mathbf{t}'_n, \mathbf{v}'$ such that

$$\mathbf{u} = \mathbf{u}'\mathbf{y}, \mathbf{t}_1 = \mathbf{t}'_1\mathbf{y}, \ldots, \mathbf{t}_n = \mathbf{t}'_n\mathbf{y}, \mathbf{v} = \mathbf{v}'\mathbf{y},$$

and

$$\mathbf{au}' = \mathbf{c}_1\mathbf{t}'_1, \mathbf{d}_1\mathbf{t}'_1 = \mathbf{c}_2\mathbf{t}'_2, \ldots, \mathbf{d}_n\mathbf{t}'_n = \mathbf{bv}'$$

is an irreducible H-sequence.

Proof. We use induction on the weight of $\mathcal{S}$. First note that by Lemma 4·2, $w(\mathcal{S}) \geq w(\mathbf{a}) + w(\mathbf{b})$.

If $w(\mathcal{S}) = w(\mathbf{a}) + w(\mathbf{b})$, then again by Lemma 4·2 we have that $\mathbf{au} = \mathbf{a}$, $\mathbf{bv} = \mathbf{b}$ and $w(\mathbf{t}_1) = \ldots = w(\mathbf{t}_n) = 0$, so that $\mathbf{t}_1 = \ldots = \mathbf{t}_n = \mathbf{1}$ and our H-sequence is irreducible in view of (Red1).

Suppose now that $w(\mathcal{S}) > w(\mathbf{a}) + w(\mathbf{b})$ and the H-sequence

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \mathbf{d}_1\mathbf{t}_1 = \mathbf{c}_2\mathbf{t}_2, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv}$$

is reducible. Then there exist elements $\tilde{\mathbf{y}}, \tilde{\mathbf{u}}, \tilde{\mathbf{t}}_1, \ldots, \tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}$ satisfying conditions (Red1)-(Red3), that is, $\mathbf{u} = \tilde{\mathbf{u}}\tilde{\mathbf{y}}, \mathbf{t}_i = \tilde{\mathbf{t}}_i\tilde{\mathbf{y}}$ for all $1 \leq i \leq n$, $\mathbf{v} = \tilde{\mathbf{v}}\tilde{\mathbf{y}}$,

$$\mathbf{a}\tilde{\mathbf{u}} = \mathbf{c}_1\tilde{\mathbf{t}}_1, \mathbf{d}_1\tilde{\mathbf{t}}_1 = \mathbf{c}_2\tilde{\mathbf{t}}_2, \ldots, \mathbf{d}_n\tilde{\mathbf{t}}_n = \mathbf{b}\tilde{\mathbf{v}} \tag{4·2}$$

and

$$w(\mathbf{a}\tilde{\mathbf{u}}) + w(\tilde{\mathbf{t}}_1) + \ldots + w(\tilde{\mathbf{t}}_n) + w(\mathbf{b}\tilde{\mathbf{v}}) < w(\mathbf{au}) + w(\mathbf{t}_1) + \ldots + w(\mathbf{t}_n) + w(\mathbf{bv}).$$

This inequality shows that we can apply the inductive hypothesis to the H-sequence (4·2). Thus there exists an irreducible sequence

$$\mathbf{au}' = \mathbf{c}_1\mathbf{t}'_1, \ldots, \mathbf{d}_n\mathbf{t}'_n = \mathbf{bv}'$$

and an element $\mathbf{y}'$ such that $\tilde{\mathbf{u}} = \mathbf{u}'\mathbf{y}', \tilde{\mathbf{t}}_i = \mathbf{t}_i'\mathbf{y}'$ and $\tilde{\mathbf{v}} = \mathbf{v}'\mathbf{y}'$. In this case let $\mathbf{y} = \mathbf{y}'\tilde{\mathbf{y}}$, and the lemma is proved. □

This lemma shows that if $(\mathbf{au}, \mathbf{bv})$ is not irreducible, then it is a 'direct consequence' of an irreducible pair $(\mathbf{au}', \mathbf{bv}')$. The following lemma will be used to 'dismantle' irreducible sequences, and to show that they always contain a 'small' element.

LEMMA 4·11. *Let*

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \dots, \mathbf{d}_{n-1}\mathbf{t}_{n-1} = \mathbf{c}_n\mathbf{t}_n, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv} \tag{4·3}$$

be an irreducible H-sequence. Then there exist $\mathbf{z}, \mathbf{u}', \mathbf{t}_1', \dots, \mathbf{t}_n' \in FLA(\Omega)$ *such that*

$$d(\mathbf{z}) \leq \max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D}), \tag{4·4}$$

$$\mathbf{u} = \mathbf{u}'\mathbf{z}, \mathbf{t}_1 = \mathbf{t}_1'\mathbf{z}, \dots, \mathbf{t}_n = \mathbf{t}_n'\mathbf{z}, \tag{4·5}$$

and such that the H-sequence

$$\mathbf{au}' = \mathbf{c}_1\mathbf{t}_1', \dots, \mathbf{d}_{n-1}\mathbf{t}_{n-1}' = \mathbf{c}_n\mathbf{t}_n' \tag{4·6}$$

is irreducible. Furthermore, if $\mathbf{z} \neq \mathbf{1}$, *then*

$$min(d(\mathbf{au}), d(\mathbf{bv})) \leq 2\max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D}). \tag{4·7}$$

Proof. If the sequence

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \dots, \mathbf{d}_{n-1}\mathbf{t}_{n-1} = \mathbf{c}_n\mathbf{t}_n \tag{4·8}$$

is irreducible then $\mathbf{z} = \mathbf{1}, \mathbf{u} = \mathbf{u}', \mathbf{t}_i' = \mathbf{t}_i$ for $1 \leq i \leq n$ satisfy the requirements of the lemma. Let us therefore suppose that the sequence (4·8) is reducible. Then by Lemma 4·10 there exist $\mathbf{z} \neq \mathbf{1}, \mathbf{u}', \mathbf{t}_1', \dots, \mathbf{t}_n' \in \mathrm{FLA}(\Omega)$ such that (4·5) and (4·6) are satisfied.

Let us fix $\mathbf{u}', \mathbf{t}_1', \dots, \mathbf{t}_n'$, and choose a $\mathbf{z}$ such that its weight is minimal amongst those satisfying the equalities (4·5). We claim that this particular $\mathbf{z}$ satisfies (4·4) by first showing that $Z \subseteq (au')^{-1}A \cup (d_nt_n')^{-1}B$ where

$$g^{-1}X = \{y \in \Omega^* : gy \in X\}.$$

Note that if X is prefix closed then so is $g^{-1}X$. Therefore it is enough to show that the leaves of Z are contained in $(au')^{-1}A \cup (d_nt_n')^{-1}B$. Let x be a leaf of Z, and suppose that $d_nt_n'x \notin B$.

Then by applying Lemma 4·3 to the equation $\mathbf{d}_n\mathbf{t}_n' \cdot \mathbf{z} = \mathbf{b} \cdot \mathbf{v}$, there exist elements $\mathbf{z}', \mathbf{v}', \mathbf{x} \in \mathrm{FLA}(\Omega)$ such that $\mathbf{z} = \mathbf{z}'\mathbf{x}, w(\mathbf{z}') < w(\mathbf{z}), \mathbf{v} = \mathbf{v}'\mathbf{x}$ and $\mathbf{d}_n\mathbf{t}_n'\mathbf{z}' = \mathbf{bv}'$. If we multiply the sequence (4·6) by $\mathbf{z}'$ and combine it with the equality $\mathbf{d}_n\mathbf{t}_n'\mathbf{z}' = \mathbf{bv}'$ we obtain the H-sequence

$$\mathbf{au}'\mathbf{z}' = \mathbf{c}_1\mathbf{t}_1'\mathbf{z}', \dots, \mathbf{d}_{n-1}'\mathbf{t}_{n-1}'\mathbf{z}' = \mathbf{c}_n\mathbf{t}_n'\mathbf{z}', \mathbf{d}_n\mathbf{t}_n'\mathbf{z}' = \mathbf{bv}'. \tag{4·9}$$

Note that if we multiply the sequence (4·9) by the element $\mathbf{x}$ we obtain the sequence (4·3).

If $x = z$ or $d_nt_n'x \notin D_n \cup d_nT_n'$, then we also have that $w(\mathbf{bv}') < w(\mathbf{bv})$, contradicting the irreducibility of sequence (4·3).

We therefore conclude that $x \neq z$ and $d_nt_n'x \in D_n \cup d_nT_n'$. Since sequence (4·3) is irreducible, this can only happen if $\mathbf{au}'\mathbf{z}' = \mathbf{au}, \mathbf{t}_1'\mathbf{z}' = \mathbf{t}_1, \dots \mathbf{t}_n'\mathbf{z}' = \mathbf{t}_n$ and $\mathbf{bv}' = \mathbf{bv}$. Note that $w(\mathbf{z}') < w(\mathbf{z})$, so by the minimality of $w(\mathbf{z})$, one of the equations of (4·5) must fail for $\mathbf{z}'$, and since we have just shown that $\mathbf{t}_i = \mathbf{t}'_i\mathbf{z}'$ for all i, we have that $\mathbf{u} \neq \mathbf{u}'\mathbf{z}'$. Notice that $\mathbf{a}\mathbf{u}'\mathbf{z}' = \mathbf{a}\mathbf{u}$ implies that the second coordinates of $\mathbf{u}$ and $\mathbf{u}'\mathbf{z}'$ are the same and so the first coordinates of $\mathbf{u}$ and $\mathbf{u}'\mathbf{z}'$ are different. Since $\mathbf{z}' = (Z \setminus \{x\}, z')$, the first coordinate of $\mathbf{u}'\mathbf{z}'$ can differ from the first coordinate of $\mathbf{u} = \mathbf{u}'\mathbf{z}$ only in the element $u'x$. That is, $u'x \notin U' \cup u'Z'$. However, $\mathbf{a}\mathbf{u} = \mathbf{a}\mathbf{u}'\mathbf{z}'$ and $au'x \in A \cup aU$, so $au'x \in A \cup a(U' \cup u'Z')$, that is, $au'x \in A$.

So far we have shown that for every leaf x of Z, if $d_nt'_nx \notin B$, then $au'x \in A$. This shows that every leaf x of Z is contained in the prefix closed set $(au')^{-1}A \cup (d_nt'_n)^{-1}B$, so $Z \subseteq (au')^{-1}A \cup (d_nt'_n)^{-1}B$. Since $d(g^{-1}X) \leq d(X)$ for every $g \in \Omega^*$ and finite $X \subseteq \Omega^*$, we conclude that $d(\mathbf{z}) \leq \max(d(\mathbf{a}), d(\mathbf{b})) \leq \max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$.

We have observed that $\mathbf{z} \neq \mathbf{1}$. Either $au'z \in A$ or $d_nt'_nz \in B$. If $d_nt'_nz \in B$ then $l(bv) = l(d_nt_n) = l(d_nt'_nz) \leq d(\mathbf{b})$, whilst if $au'z \in A$, then $l(au) = l(au'z) \leq d(\mathbf{a})$. Lemma 4·7 implies in the first case that $d(\mathbf{bv}) \leq 2\max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$, whilst in the second case $d(\mathbf{au}) \leq 2\max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$. ☐

As a consequence of this lemma we can show that every irreducible sequence contains a 'small' element.

Lemma 4·12. *Let*

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv} \tag{4·10}$$

be an irreducible H-sequence. Then there exists an element in the sequence having diameter less than or equal to $2\max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$.

Proof. Let $\mathcal{D}' = \max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$. If $d(\mathbf{au}) \leq 2\mathcal{D}'$, then the statement is true, so let us suppose that $d(\mathbf{au}) > 2\mathcal{D}'$.

Apply Lemma 4·11 to the sequence (4·10). Note that $\mathbf{z} \neq 1$ if and only if the shortened sequence

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_{m-1}\mathbf{t}_{m-1} = \mathbf{c}_m\mathbf{t}_m$$

is also irreducible. In this case we can apply Lemma 4·11 to this shortened sequence, and repeat the procedure until $\mathbf{z} \neq \mathbf{1}$. Note that such a $\mathbf{z}$ exists, for otherwise we would have that the sequence $\mathbf{au} = \mathbf{c}_1\mathbf{t}_1$ is irreducible, which by Lemma 4·6 contradicts our assumption that $d(\mathbf{au}) > 2\mathcal{D}'$. That is, there exists $2 \leq i \leq n+1$ such that

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_{j-1}\mathbf{t}_{j-1} = \mathbf{c}_j\mathbf{t}_j$$

is irreducible for all $i \leq j \leq n+1$ (where we denote $\mathbf{b}$ by $\mathbf{c}_{n+1}$ and $\mathbf{v}$ by $\mathbf{t}_{n+1}$), but

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_{i-2}\mathbf{t}_{i-2} = \mathbf{c}_{i-1}\mathbf{t}_{i-1}$$

is reducible. In this case if we apply Lemma 4·11 to the first sequence with $j = i$, then the acquired element $\mathbf{z}$ will be different from $\mathbf{1}$, and as a consequence the lemma implies that $\min(d(\mathbf{au}), d(\mathbf{c}_i\mathbf{t}_i)) \leq 2\mathcal{D}'$. ☐

Now let

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_{n-1}\mathbf{t}_{n-1} = \mathbf{c}_n\mathbf{t}_n, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv} \tag{4·11}$$

be an irreducible H-sequence with $n \geq 1$ and let $\mathcal{D}' = \max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$. Then by Lemma 4·11 there exist $\mathbf{z}, \mathbf{u}', \mathbf{t}'_1, \ldots, \mathbf{t}'_n \in \mathrm{FLA}(\Omega)$, $d(\mathbf{z}) \leq \mathcal{D}'$ such that $\mathbf{u} = \mathbf{u}'\mathbf{z}$ and $\mathbf{t}_i = \mathbf{t}'_i\mathbf{z}$ for every $1 \leq i \leq n$, and such that the sequence

$$\mathbf{au}' = \mathbf{c}_1\mathbf{t}'_1, \ldots, \mathbf{d}_{n-1}\mathbf{t}'_{n-1} = \mathbf{c}_n\mathbf{t}'_n$$

is irreducible. Now let us apply Lemma 4·11 to this sequence. Thus, there exist elements $\mathbf{y}^{(n)}, \mathbf{u}^{(n)}, \mathbf{t}_1^{(n)}, \ldots, \mathbf{t}_{n-1}^{(n)} \in \mathrm{FLA}(\Omega)$, $d(\mathbf{y}^{(n)}) \leq \mathcal{D}'$ satisfying $\mathbf{u}' = \mathbf{u}^{(n)}\mathbf{y}^{(n)}$, $\mathbf{t}_i' = \mathbf{t}_i^{(n)}\mathbf{y}^{(n)}$ for every $1 \leq i \leq n-1$ and such that the H-sequence

$$\mathbf{a}\mathbf{u}^{(n)} = \mathbf{c}_1\mathbf{t}_1^{(n)}, \ldots, \mathbf{d}_{n-2}\mathbf{t}_{n-2}^{(n)} = \mathbf{c}_{n-1}\mathbf{t}_{n-1}^{(n)} \tag{4·12}$$

is irreducible.

Note that $\mathbf{u} = \mathbf{u}^{(n)}\mathbf{y}^{(n)}\mathbf{z}$ and $\mathbf{t}_i = \mathbf{t}_i^{(n)}\mathbf{y}^{(n)}\mathbf{z}$ for every $1 \leq i \leq n-1$. Inductively, for every $2 \leq k \leq n$ we can define the elements $\mathbf{u}^{(k)}, \mathbf{y}^{(k)}$ and $\mathbf{t}_i^{(k)}$ where $1 \leq i \leq k-1$ satisfying $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)}\mathbf{y}^{(k)}$ and $\mathbf{t}_i^{(k+1)} = \mathbf{t}_i^{(k)}\mathbf{y}^{(k)}$ for every $1 \leq i \leq k-1$ such that the H-sequence

$$\mathbf{a}\mathbf{u}^{(k)} = \mathbf{c}_1\mathbf{t}_1^{(k)}, \ldots, \mathbf{d}_{k-2}\mathbf{t}_{k-2}^{(k)} = \mathbf{c}_{k-1}\mathbf{t}_{k-1}^{(k)} \tag{4·13}$$

is irreducible, and $d(\mathbf{y}^{(k)}) \leq \mathcal{D}'$.

The last step is to define $\mathbf{y}^{(1)}$: at this point we have that the H-sequence

$$\mathbf{a}\mathbf{u}^{(2)} = \mathbf{c}_1\mathbf{t}_1^{(2)} \tag{4·14}$$

is irreducible. By Lemma 4·6, we have that $d(\mathbf{u}^{(2)}) \leq \max(d(\mathbf{a}), d(\mathbf{c}_1)) \leq \mathcal{D}'$. So if we define $\mathbf{y}^{(1)} = \mathbf{u}^{(2)}$ then $d(\mathbf{y}^{(1)}) \leq \mathcal{D}'$. For later reference, we summarise the properties of the elements $\mathbf{y}_j^{(i)}$ in the following lemma.

LEMMA 4·13. *If*

$$\mathbf{a}\mathbf{u} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_{n-1}\mathbf{t}_{n-1} = \mathbf{c}_n\mathbf{t}_n, \mathbf{d}_n\mathbf{t}_n = \mathbf{b}\mathbf{v}$$

is an irreducible H-sequence with $n \geq 1$, then there exist elements $\mathbf{z}, \mathbf{u}^{(i)}, \mathbf{y}^{(i)}$ and $\mathbf{t}_j^{(i)}$ where $1 \leq j < i \leq n$ such that

(Y1) $\mathbf{u} = \mathbf{y}^{(1)} \ldots \mathbf{y}^{(n)}\mathbf{z}$, $\mathbf{u}^{(i)} = \mathbf{y}^{(1)} \ldots \mathbf{y}^{(i-1)}$ *for every* $2 \leq i \leq n$,
(Y2) $\mathbf{t}_i^{(j)} = \mathbf{t}_i^{(j-1)}\mathbf{y}^{(j-1)}$,
(Y3) *the H-sequence*

$$\mathbf{a}\mathbf{u}^{(j)} = \mathbf{c}_1\mathbf{t}_1^{(j)}, \ldots, \mathbf{d}_{j-2}\mathbf{t}_{j-2}^{(j)} = \mathbf{c}_{j-1}\mathbf{t}_{j-1}^{(j)}$$

is irreducible for every $2 \leq j \leq n$,
(Y4) $d(\mathbf{z}), d(\mathbf{y}^{(i)}) \leq \max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$ *for all* $1 \leq i \leq n$.

Notice that for every $1 \leq i \leq n$ we have that either $\mathbf{a}\mathbf{y}^{(1)} \ldots \mathbf{y}^{(i)} \neq \mathbf{a}\mathbf{y}^{(1)} \ldots \mathbf{y}^{(i+1)}$ or $\mathbf{y}^{(i+1)}$ is an idempotent (here we assume that $\mathbf{y}^{(n+1)} = \mathbf{z}$).

5. *The free left ample monoid and right coherency*

We are now in a position to show that $\mathrm{FLA}(\Omega)$ is right coherent. Assume first that Ω is finite. Continuing from Lemma 4·13, let $\mathcal{W}$ be the maximal weight of elements of $\mathrm{FLA}(\Omega)$ having diameter less than or equal to $\mathcal{D}'$. Since Ω is finite, so $\mathcal{W}$ exists. If we multiply any number of idempotents having diameter less than or equal to $\mathcal{D}'$, then the diameter of the resulting element will be less than or equal to $\mathcal{D}'$, so the weight of the product will be less than or equal to $\mathcal{W}$.

Now let us 'merge' the consecutive idempotents of the sequence $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}, \mathbf{z}$ with the succeeding non-idempotent elements. That is, if $\mathbf{y}^{(1)}$ is not idempotent, then let $\mathbf{y}_1 = \mathbf{y}^{(1)}$. Otherwise, let $\mathbf{y}^{(1)} \ldots \mathbf{y}^{(i)}$ be the first maximal idempotent subsequence, and let $\mathbf{y}_1 = \mathbf{y}^{(1)} \ldots \mathbf{y}^{(i)}\mathbf{y}^{(i+1)}$, and so on: if the next element is not idempotent, it will be $\mathbf{y}_2$, otherwise $\mathbf{y}_2$ will be the product of the following maximal subsequence of idempotents multiplied by the next non-idempotent. In case $\mathbf{z}$ is idempotent, the last element of the sequence $\mathbf{y}_1, \ldots, \mathbf{y}_m$ will be idempotent, but all the others are non-idempotent. Notice that for every $1 \leq i \leq m$, $\mathbf{y}_i$ is a product of idempotents followed by a non-idempotent except (possibly) in the case $i = m$. All factors of $\mathbf{y}_i$ have diameter less than or equal to $\mathcal{D}'$, so the product of their diameters also has this property. This implies that $w(\mathbf{y}_i) \leq \mathcal{W}$. Notice that by merging the sequence of $y^{(j)}$s in this way we have $a\mathbf{y}_1 \ldots \mathbf{y}_i = a\mathbf{y}^{(1)} \ldots y^{(j-1)}$ and we define $\tilde{\mathbf{t}}_i$ to be $t^{(j)}_{j-1}$. The properties of the sequence $\mathbf{y}_1, \ldots, \mathbf{y}_m$ are summarised in the following lemma.

LEMMA 5·1. *If*

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv}$$

is an irreducible H-sequence, then there exist elements $\mathbf{y}_1, \ldots, \mathbf{y}_m$ such that

(C1) $\mathbf{u} = \mathbf{y}_1\mathbf{y}_2 \ldots \mathbf{y}_m$,

(C2) $w(\mathbf{y}_i) \leq \mathcal{W}$ *for every* $1 \leq i \leq m$, *where* $\mathcal{W}$ *denotes the maximal weight of elements of FLA*(Ω) *having diameter less than or equal to* $\max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$,

(C3) $\mathbf{y}_i$ *is not an idempotent for all* $1 \leq i \leq m-1$,

(C4) *For every* $1 \leq i \leq m-1$, *there exists an irreducible H-sequence connecting* $\mathbf{ay}_1\mathbf{y}_2 \ldots \mathbf{y}_i$ *with an element of the form* $\mathbf{c}_i\tilde{\mathbf{t}}_i$ *where* $(\mathbf{c}_i, \mathbf{d}_i) \in H$.

Recall that $\rho = \langle H \rangle$ where $H \subseteq \mathrm{FLA}(\Omega) \times \mathrm{FLA}(\Omega)$ is our given symmetric set of generators. We aim to show that the right annihilator congruence

$$r(\mathbf{a}\rho) = \{(\mathbf{u}, \mathbf{v}) \in \mathrm{FLA}(\Omega) \times \mathrm{FLA}(\Omega) : \mathbf{au}\ \rho\ \mathbf{av}\}$$

is finitely generated for all $\mathbf{a} \in \mathrm{FLA}(\Omega)$. To show this, let $\mathbf{a} \in \mathrm{FLA}(\Omega)$ be fixed. Now let

$$\mathbb{K} = \{\mathbf{au}\rho : \exists\, \mathbf{bv} \in \mathrm{FLA}(\Omega) \text{ with } d(\mathbf{b}) \leq \max(d(\mathbf{a}), \mathcal{D}) \text{ and } (\mathbf{au}, \mathbf{bv}) \text{ irreducible}\}.$$

LEMMA 5·2. *The set* $\mathbb{K}$ *is finite.*

Proof. Let $\mathbf{au}\rho \in \mathbb{K}$ and let

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv}$$

be an irreducible H-sequence connecting $\mathbf{au}$ to an element $\mathbf{bv} \in \mathrm{FLA}(\Omega)$ testifying $\mathbf{au}\rho \in \mathbb{K}$. Then by Lemma 4·12 there exists an element in the sequence having diameter less than or equal to $2\max(d(\mathbf{a}), \mathcal{D})$. Since there are only finitely many such elements of $\mathrm{FLA}(\Omega)$, we have that $\mathbb{K}$ is finite. □

Now let $\mathcal{K} = |\mathbb{K}|$, and let us define the set

$$H' = \{(\mathbf{u}, \mathbf{v}) : \mathbf{au}\ \rho\ \mathbf{av} \text{ and } w(\mathbf{au}), w(\mathbf{av}) \leq (\mathcal{K}+3)\mathcal{W}'\},$$

where $\mathcal{W}'$ is the maximum of the weights of elements of $\mathrm{FLA}(\Omega)$ having diameter less than or equal to $2\max(d(\mathbf{a}), \mathcal{D})$.

LEMMA 5·3. *The finite set* H' *generates the right annihilator congruence of* $\mathbf{a}\rho$.

Proof. Denote the right annihilator congruence of $\mathbf{a}\rho$ by τ. By definition, $H' \subseteq \tau$. Now let $(\mathbf{u}, \mathbf{v}) \in \tau$. We are going to show that $(\mathbf{u}, \mathbf{v}) \in \langle H' \rangle$. Without loss of generality we can suppose that $w(\mathbf{au}) \geq w(\mathbf{av})$. If the pair $(\mathbf{au}, \mathbf{av})$ is reducible, then by Lemma 4·10 there exist elements $\mathbf{u}', \mathbf{v}'$ and $\mathbf{y}$ such that the pair $(\mathbf{au}', \mathbf{av}')$ is irreducible and

$(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')\mathbf{y}$. We therefore suppose that the pair $(\mathbf{au}, \mathbf{av})$ is irreducible and prove by induction on $l(au) + l(av)$ that $(\mathbf{u}, \mathbf{v}) \in \langle H' \rangle$. If $l(au) + l(av) \leq \max(d(\mathbf{a}), \mathcal{D})$ then certainly $l(au) \leq \max(d(\mathbf{a}), \mathcal{D})$, so by Lemma 4·7, $d(\mathbf{au}) \leq 2\max(d(\mathbf{a}), \mathcal{D})$, thus $w(\mathbf{av}) \leq w(\mathbf{au}) \leq \mathcal{W}'$, so $(\mathbf{au}, \mathbf{av}) \in H'$.

Suppose now that whenever $(\mathbf{au}', \mathbf{av}') \in \tau$ is any irreducible pair such that $l(au') + l(av') \leq M$ for some $M \geq \max(d(\mathbf{a}), \mathcal{D})$, then $(\mathbf{au}', \mathbf{av}') \in \langle H' \rangle$. Let $(\mathbf{au}, \mathbf{av}) \in \tau$ be an irreducible pair such that $l(au) + l(av) = M + 1$. We are going to show that $(\mathbf{au}, \mathbf{av}) \in \langle H' \rangle$. If $w(\mathbf{au}) \leq (\mathcal{K} + 3)\mathcal{W}'$, then by definition $(\mathbf{au}, \mathbf{av}) \in H'$, so we can suppose that $w(\mathbf{au}) > (\mathcal{K} + 3)\mathcal{W}'$. Of course, this implies that $d(\mathbf{au}) > 2\max(d(\mathbf{a}), \mathcal{D})$.

Now let

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \ldots, \mathbf{d}_n\mathbf{t}_n = \mathbf{av}$$

be an irreducible H-sequence connecting $\mathbf{au}$ and $\mathbf{av}$. Note that $n \geq 1$, for otherwise $\mathbf{au} = \mathbf{av}$ is an irreducible H-sequence such that $d(\mathbf{au}) > 2\max(d(\mathbf{a}), \mathcal{D})$, which contradicts Lemma 4·6. By Lemma 5·1 we have that there exist elements $\mathbf{y}_1, \ldots, \mathbf{y}_m$ satisfying Conditions (C1)-(C4). Of course, $\mathcal{W} < \mathcal{W}'$, for the latter corresponds to a doubled diameter. Furthermore, since $w(\mathbf{a}), w(\mathbf{y}_i) \leq \mathcal{W}'$ for every i, we have that $w(\mathbf{ay}_1 \ldots \mathbf{y}_m) \leq (m+1)\mathcal{W}'$. However, $w(\mathbf{ay}_1 \ldots \mathbf{y}_m) > (\mathcal{K} + 3)\mathcal{W}'$, so that making use of Lemma 4·2, we see that $m > \mathcal{K} + 2$. By Condition (C4), $(\mathbf{ay}_1 \ldots \mathbf{y}_i)\rho \in \mathbb{K}$ for all $1 \leq i \leq m - 1$, so we have that there exist $1 \leq i < j \leq \mathcal{K} + 1$ such that

$$\mathbf{ay}_1 \ldots \mathbf{y}_i \; \rho \; \mathbf{ay}_1 \ldots \mathbf{y}_j.$$

Note that $w(\mathbf{ay}_1 \ldots \mathbf{y}_i), w(\mathbf{ay}_1 \ldots \mathbf{y}_j) \leq (\mathcal{K} + 2)\mathcal{W}'$, so we have that the pair

$$(\mathbf{y}_1 \ldots \mathbf{y}_i, \mathbf{y}_1 \ldots \mathbf{y}_j) \tag{5·1}$$

is contained in H'. For brevity, denote the product $\mathbf{y}_1 \ldots \mathbf{y}_i\mathbf{y}_{j+1} \ldots \mathbf{y}_m$ by $\mathbf{t}$. If we multiply the pair (5·1) by $\mathbf{y}_{j+1} \ldots \mathbf{y}_m$, we conclude that

$$(\mathbf{t}, \mathbf{u}) \in \langle H' \rangle,$$

so $\mathbf{at} \; \rho \; \mathbf{av}$. Note that $l(at) < l(au)$, because $\mathbf{t}$ lacks at least one non-idempotent factor (namely $\mathbf{y}_j$). As a consequence $l(at) + l(av) < l(au) + l(av) = M + 1$, so by the induction hypotheses we have that

$$(\mathbf{t}, \mathbf{v}) \in \langle H' \rangle.$$

That is, $(\mathbf{t}, \mathbf{u}), (\mathbf{t}, \mathbf{v}) \in \langle H' \rangle$, so by transitivity we have that $(\mathbf{u}, \mathbf{v}) \in \langle H' \rangle$, and the lemma is proved. □

Lemma 5·4. *Let* $\mathbf{a}, \mathbf{b} \in FLA(\Omega)$, $H \subseteq FLA(\Omega) \times FLA(\Omega)$ *be finite and let* $\rho = \langle H \rangle$ *be a finitely generated right congruence. Then*

$$\mathbf{a}\rho \cdot S \cap \mathbf{b}\rho \cdot S = \{\mathbf{c}\rho : \mathbf{c} \; \rho \; \mathbf{au} \; \rho \; \mathbf{bv} \textit{ for some } \mathbf{u}, \mathbf{v} \in FLA(\Omega)\}$$

is either empty or finitely generated as a right S*-act.*

Proof. Suppose that $\mathbf{a}\rho \cdot S \cap \mathbf{b}\rho \cdot S \neq \emptyset$. Let

$$\mathbb{K}' = \{\mathbf{au}\rho : \text{ there exists } \mathbf{v} \in \mathrm{FLA}(\Omega), \text{ such that } (\mathbf{au}, \mathbf{bv}) \text{ is irreducible}\}.$$

Note that similarly to the set $\mathbb{K}$ defined before Lemma 5·2, $\mathbb{K}'$ is also finite, because by Lemma 4·12, if $(\mathbf{au}, \mathbf{bv})$ is irreducible then $\mathbf{au}$ is ρ-related to an element of $\mathrm{FLA}(\Omega)$

having diameter less than or equal to $\max(d(\mathbf{a}), d(\mathbf{b}), \mathcal{D})$. We claim that $\mathbb{K}'$ generates $\mathbf{a}\rho \cdot S \cap \mathbf{b}\rho \cdot S$. Let $\mathbf{au}\rho = \mathbf{bv}\rho \in \mathbf{a}\rho \cdot S \cap \mathbf{b}\rho \cdot S$. Then there exists an H-sequence

$$\mathbf{au} = \mathbf{c}_1\mathbf{t}_1, \dots, \mathbf{d}_n\mathbf{t}_n = \mathbf{bv}$$

connecting $\mathbf{au}$ and $\mathbf{bv}$. By Lemma 4·10, there exist an irreducible pair $(\mathbf{au}', \mathbf{bv}')$ and $\mathbf{y} \in \mathrm{FLA}(\Omega)$ such that $(\mathbf{au}, \mathbf{bv}) = (\mathbf{au}', \mathbf{bv}')\mathbf{y}$. In this case $\mathbf{au}'\rho \in \mathbb{K}'$, so $\mathbf{au}\rho \in \mathbb{K}'S$, thus $\mathbb{K}'$ generates $\mathbf{a}\rho \cdot S \cap \mathbf{b}\rho \cdot S$. ☐

As a consequence of Lemmas 5·3 and 5·4 we have our first main result.

THEOREM 5·5. *If Ω is finite, then the free left ample monoid FLA(Ω) is right coherent.*

To show Theorem 5·5 is true for arbitrary Ω we need a simple consequence of Lemma 4·3.

LEMMA 5·6. *Let $\mathbf{dz} = \mathbf{bv}$ and let Π be a subset of Ω containing all letters appearing in D and B. Then there exists $\mathbf{z}', \mathbf{v}' \in \mathrm{FLA}(\Pi)$ and $\mathbf{x} \in \mathrm{FLA}(\Omega)$ such that $\mathbf{dz}' = \mathbf{bv}'$ and $(\mathbf{z}, \mathbf{v}) = (\mathbf{z}', \mathbf{v}')\mathbf{x}$.*

Proof. Let $\mathbf{z}', \mathbf{v}'$ be minimal (with respect to $w(\mathbf{z}') + w(\mathbf{v}')$) in FLA($\Omega$) satisfying that there exists $\mathbf{x} \in \mathrm{FLA}(\Omega)$ such that $\mathbf{dz}' = \mathbf{bv}', \mathbf{z} = \mathbf{z}'\mathbf{x}$ and $\mathbf{v} = \mathbf{v}'\mathbf{x}$. We claim that $\mathbf{z}', \mathbf{v}' \in \mathrm{FLA}(\Pi)$. Suppose on the contrary that either $\mathbf{z}' \notin \mathrm{FLA}(\Pi)$ or $\mathbf{v}' \notin \mathrm{FLA}(\Pi)$. We can suppose without loss of generality that $\mathbf{z}' \notin \mathrm{FLA}(\Pi)$. Then there exists a leaf $x \in Z'$ such that x contains a letter which is not in Π. In this case clearly $dx \notin D \cup B$, so Lemma 4·3 implies that there exist elements $\mathbf{z}'', \mathbf{v}'', \mathbf{x}'$ such that $\mathbf{dz}'' = \mathbf{bv}'', \mathbf{z}' = \mathbf{z}''\mathbf{x}', \mathbf{v}' = \mathbf{v}''\mathbf{x}'$ and $w(\mathbf{z}'') < w(\mathbf{z}')$. However, these facts together with the observations $\mathbf{z} = \mathbf{z}''(\mathbf{x}'\mathbf{x}), \mathbf{v} = \mathbf{v}''(\mathbf{x}'\mathbf{x})$ contradict the minimality of $\mathbf{z}'$ and $\mathbf{v}'$. This shows that $\mathbf{z}', \mathbf{v}' \in \mathrm{FLA}(\Pi)$, finishing the proof. ☐

THEOREM 5·7. *For any set Ω, we have that* FLA(Ω) *is right coherent.*

Proof. Let ρ be a right congruence on FLA(Ω) with finite set of generators H, denoted by $\rho = \langle H \rangle_{\mathrm{FLA}(\Omega)}$, and let $\mathbf{b}, \mathbf{c} \in \mathrm{FLA}(\Omega)$. Let Π be the finite set of letters occuring in $\mathbf{b}, \mathbf{c}$ or in components of H and put $\rho' = \langle H \rangle_{\mathrm{FLA}(\Pi)}$.

We claim that for any $\mathbf{u}, \mathbf{v} \in \mathrm{FLA}(\Omega)$ with $\mathbf{bu}\, \rho\, \mathbf{cv}$ via an H-sequence

$$\mathbf{bu} = \mathbf{c}_1\mathbf{t}_1, \mathbf{d}_1\mathbf{t}_1 = \mathbf{c}_2\mathbf{t}_2, \dots, \mathbf{d}_n\mathbf{t}_n = \mathbf{cv}$$

in FLA(Ω), there exist

$$\mathbf{u}', \mathbf{t}_i' \, (1 \le i \le n), \mathbf{v}' \in \mathrm{FLA}(\Pi), \mathbf{x} \in \mathrm{FLA}(\Omega)$$

such that

$$\mathbf{u} = \mathbf{u}'\mathbf{x}, \mathbf{t}_i = \mathbf{t}_i'\mathbf{x} \, (1 \le i \le n), \mathbf{v} = \mathbf{v}'\mathbf{x}$$

and

$$\mathbf{bu}' = \mathbf{c}_1\mathbf{t}_1', \mathbf{d}_1\mathbf{t}_1' = \mathbf{c}_2\mathbf{t}_2', \dots, \mathbf{d}_n\mathbf{t}_n' = \mathbf{cv}'.$$

If $n = 0$, then $\mathbf{bu} = \mathbf{cv}$ so by Lemma 5·6 we have that $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')\mathbf{x}$ and $\mathbf{bu}' = \mathbf{cv}'$ for some $\mathbf{u}', \mathbf{v}' \in \mathrm{FLA}(\Pi)$ and $\mathbf{x} \in \mathrm{FLA}(\Omega)$ as required.

Suppose now that $n > 0$ and the result holds for all sequences of length $n-1$. Consider the H-sequence

$$\mathbf{bu} = \mathbf{c}_1\mathbf{t}_1, \mathbf{d}_1\mathbf{t}_1 = \mathbf{c}_2\mathbf{t}_2, \dots, \mathbf{d}_n\mathbf{t}_n = \mathbf{cv}.$$

From the first equality, and the fact that $\mathbf{c}_1 \in \mathrm{FLA}(\Pi)$, we deduce that there exists $\mathbf{u}', \mathbf{t}_1' \in \mathrm{FLA}(\Pi)$ and $\mathbf{x} \in \mathrm{FLA}(\Omega)$ such that

$$\mathbf{u} = \mathbf{u}'\mathbf{x}, \mathbf{t}_1 = \mathbf{t}_1'\mathbf{x} \text{ and } \mathbf{b}\mathbf{u}' = \mathbf{c}_1\mathbf{t}_1'.$$

From the remaining part of the sequence, the fact that $\mathbf{d}_1 \in \mathrm{FLA}(\Pi)$ and our inductive hypothesis, we deduce that there exists $\mathbf{v}'', \mathbf{t}_i''\,(1 \leq i \leq n) \in \mathrm{FLA}(\Pi)$ and $\mathbf{z} \in \mathrm{FLA}(\Omega)$ such that

$$\mathbf{t}_i = \mathbf{t}_i''\mathbf{z}, \mathbf{v} = \mathbf{v}''\mathbf{z} \text{ and } \mathbf{d}_1\mathbf{t}_1'' = \mathbf{c}_2\mathbf{t}_2'', \ldots, \mathbf{d}_n\mathbf{t}_n'' = \mathbf{c}\mathbf{v}''.$$

We now examine the equality

$$\mathbf{t}_1 = \mathbf{t}_1'\mathbf{x} = \mathbf{t}_1''\mathbf{z}.$$

Again by Lemma 5·6 we have that $(\mathbf{x}, \mathbf{z}) = (\mathbf{x}', \mathbf{z}')\mathbf{w}$ for some $\mathbf{x}', \mathbf{z}' \in \mathrm{FLA}(\Pi)$ and $\mathbf{w} \in \mathrm{FLA}(\Omega)$ with $\mathbf{t}_1'\mathbf{x}' = \mathbf{t}_1''\mathbf{z}'$. Now let

$$\tilde{\mathbf{u}} = \mathbf{u}'\mathbf{x}', \tilde{\mathbf{t}}_i = \mathbf{t}_i''\mathbf{z}'\,(1 \leq i \leq n) \text{ and } \tilde{\mathbf{v}} = \mathbf{v}''\mathbf{z}'.$$

Then it is easy to check that

$$\mathbf{u} = \tilde{\mathbf{u}}\mathbf{w}, \mathbf{t}_i = \tilde{\mathbf{t}}_i\mathbf{w}\,(1 \leq i \leq n), \mathbf{v} = \tilde{\mathbf{v}}\mathbf{w}$$

and

$$\mathbf{b}\tilde{\mathbf{u}} = \mathbf{c}_1\tilde{\mathbf{t}}_1, \mathbf{d}_1\tilde{\mathbf{t}}_1 = \mathbf{c}_2\tilde{\mathbf{t}}_2, \ldots, \mathbf{d}_n\tilde{\mathbf{t}}_n = \mathbf{c}\tilde{\mathbf{v}}.$$

Hence our claim holds by induction.

Since $\mathrm{FLA}(\Pi)$ is right coherent, the right congruence $r(\mathbf{a}\rho')$ on $\mathrm{FLA}(\Pi)$ has a finite set of generators K. Clearly $K \subseteq r(\mathbf{a}\rho)$. Conversely, if $(\mathbf{u}, \mathbf{v}) \in r(\mathbf{a}\rho)$, then as $\mathbf{au}$ is connected to $\mathbf{av}$ via an H-sequence, we can apply the above claim to obtain that $\mathbf{au}'\,\rho'\,\mathbf{av}'$ for some $\mathbf{u}', \mathbf{v}' \in \mathrm{FLA}(\Pi)$ such that $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')\mathbf{x}$ for some $\mathbf{x} \in \mathrm{FLA}(\Omega)$. Thus $(\mathbf{u}', \mathbf{v}') \in \langle K\rangle_{\mathrm{FLA}(\Pi)} \subseteq \langle K\rangle_{\mathrm{FLA}(\Omega)}$, and it follows that $\langle K\rangle_{\mathrm{FLA}(\Omega)} = r(\mathbf{a}\rho)$.

Now take $\mathbf{b} = \mathbf{a}$ and $\mathbf{c} = \mathbf{a}'$ and suppose that $\mathbf{a}\rho \cdot \mathrm{FLA}(\Omega) \cap \mathbf{a}'\rho \cdot \mathrm{FLA}(\Omega) \neq \emptyset$. Then $\mathbf{au}\,\rho\,\mathbf{a}'\mathbf{v}$ for some $\mathbf{u}, \mathbf{v} \in \mathrm{FLA}(\Omega)$ and we have that $\mathbf{au}'\,\rho'\,\mathbf{a}'\mathbf{v}'$ for some $\mathbf{u}', \mathbf{v}' \in \mathrm{FLA}(\Pi)$ such that $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')\mathbf{x}$ for some $\mathbf{x} \in \mathrm{FLA}(\Omega)$. Since $\mathbf{a}\rho' \cdot \mathrm{FLA}(\Pi) \cap \mathbf{a}'\rho' \cdot \mathrm{FLA}(\Pi) \neq \emptyset$ and $\mathrm{FLA}(\Pi)$ is right coherent, we have that $\mathbf{a}\rho' \cdot \mathrm{FLA}(\Pi) \cap \mathbf{a}'\rho' \cdot \mathrm{FLA}(\Pi) = L \cdot \mathrm{FLA}(\Pi)$ for some finite set $L = \{\mathbf{u}_i\rho' : 1 \leq i \leq n\}$, where the $\mathbf{u}_i$ are fixed representatives of their ρ'-classes.

For each $i \in \{1, \ldots, n\}$ we therefore have that

$$\mathbf{aw}_i\,\rho'\,\mathbf{u}_i\mathbf{x}_i\,\rho'\,\mathbf{a}'\mathbf{z}_i$$

for some $\mathbf{w}_i, \mathbf{x}_i, \mathbf{z}_i \in \mathrm{FLA}(\Pi)$, so that clearly

$$\mathbf{aw}_i\,\rho\,\mathbf{u}_i\mathbf{x}_i\,\rho\,\mathbf{a}'\mathbf{z}_i$$

and so

$$L' = \{\mathbf{u}_i\rho : 1 \leq i \leq n\} \subseteq \mathbf{a}\rho \cdot \mathrm{FLA}(\Omega) \cap \mathbf{a}'\rho \cdot \mathrm{FLA}(\Omega).$$

Conversely, if $\mathbf{ab}\,\rho\,\mathbf{a}'\mathbf{c}$ then as above we have that $(\mathbf{b}, \mathbf{c}) = (\mathbf{b}', \mathbf{c}')\mathbf{t}$ for some $\mathbf{b}', \mathbf{c}' \in \mathrm{FLA}(\Pi)$ and $\mathbf{t} \in \mathrm{FLA}(\Omega)$ with $\mathbf{ab}'\,\rho'\,\mathbf{a}'\mathbf{c}'$. Now $(\mathbf{ab}')\rho' = (\mathbf{u}_i\rho')\mathbf{w}$ for some $i \in \{1, \ldots, n\}$ and $\mathbf{w} \in \mathrm{FLA}(\Pi)$ so that $(\mathbf{ab}')\rho = (\mathbf{u}_i\rho)\mathbf{w}$ and hence $(\mathbf{ab})\rho = (\mathbf{u}_i\rho)\mathbf{wt} \in L' \cdot \mathrm{FLA}(\Omega)$. Thus $\mathbf{a}\rho \cdot \mathrm{FLA}(\Omega) \cap \mathbf{a}'\rho \cdot \mathrm{FLA}(\Omega) = L' \cdot \mathrm{FLA}(\Omega)$ as required. □

6. *Coherency and retracts*

Investigations of how coherency behaves with respect to certain constructions will be the subject of a future paper, however, to show how the coherency of the free monoid follows from our result, we show that retracts of (right) coherent monoids are (right) coherent.

Definition 6·1. Let S be a monoid. Then $T \subseteq S$ is a *retract* of S if there exists a homomorphism $\phi\colon S \to S$ such that $\phi^2 = \phi$ and $\mathrm{Im}\ \phi = T$.

Note that any retract is a subsemigroup and a monoid.

Lemma 6·2. *Let S be a monoid and let T be a retract of S. Let ρ be a right congruence on T, and let ρ' be the right congruence on S generated by ρ. Then the restriction of ρ' to T coincides with ρ.*

Proof. Let $a, b \in T$ such that $a\ \rho'\ b$. Since ρ' is generated by ρ, there exist elements $c_1, \ldots, c_n, d_1, \ldots, d_n \in T$ and $t_1, \ldots, t_n \in S$ such that $c_i\ \rho\ d_i$ for every $1 \leq i \leq n$, and such that

$$a = c_1 t_1, \ldots, d_n t_n = b.$$

If we take the image of this sequence under ϕ we obtain the ρ-sequence

$$a = c_1(t_1\phi), \ldots, d_n(t_n\phi) = b$$

connecting a and b in T, so $a\ \rho\ b$. □

Theorem 6·3. *Let S be a right coherent monoid and let T be a retract of S. Then T is right coherent.*

Proof. Let ρ be a finitely generated right congruence on T, so that $\rho = \langle H \rangle_T$ for some finite set $H \subseteq T \times T$. Denote by ρ' the right congruence on S generated by ρ. Clearly, $\rho' = \langle H \rangle_S$.

First we show that if $a, b \in S$ and $a\ \rho'\ b$, then $a\phi\ \rho\ b\phi$. For this, let

$$a = c_1 t_1, \ldots, d_n t_n = b$$

be an H-sequence connecting a and b in S. Since $H \subseteq T \times T$, if we take the image of this sequence under ϕ we obtain the H-sequence

$$a\phi = c_1(t_1\phi), \ldots, d_n(t_n\phi) = b\phi$$

connecting $a\phi$ and $b\phi$ in T, so that $a\phi\ \rho\ b\phi$.

Now let $a \in T$ be fixed. Note that $r(a\rho')$ is a right congruence on S, and $r(a\rho)$ is a right congruence on T. Since S is right coherent, we have that $r(a\rho') = \langle X \rangle_S$ for some finite $X \subseteq S \times S$. We claim that the finite set

$$X\phi = \{(u\phi, v\phi) : (u, v) \in X\} \subseteq T \times T$$

generates $r(a\rho)$.

First note that if $(u, v) \in X$, then $au\ \rho'\ av$, so we have that

$$a(u\phi) = (au)\phi\ \rho\ (av)\phi = a(v\phi),$$

that is, $(u\phi, v\phi) \in r(a\rho)$. Thus we have shown that $X\phi \subseteq r(a\rho)$.

On the other hand, if $(u, v) \in r(a\rho)$, then necessarily $(u, v) \in r(a\rho')$, so there exists an X-sequence

$$u = c_1 t_1, \ldots, d_n t_n = v$$

connecting u and v in S. If we take the image of this sequence under ϕ (and remember that $u, v \in T$), then we obtain the $X\phi$-sequence

$$u = (c_1\phi)(t_1\phi), \ldots, (d_n\phi)(t_n\phi) = v$$

connecting u and v. That is, $(u, v) \in \langle X\phi \rangle_T$, and we have shown that $r(a\rho)$ is finitely generated.

Now suppose that $a, b \in T$ are such that $a\rho{\cdot}T \cap b\rho{\cdot}T \neq \emptyset$. Then clearly $a\rho'{\cdot}S \cap b\rho'{\cdot}S \neq \emptyset$, so there exists a finite set $Y \subseteq S$ such that $a\rho' \cdot S \cap b\rho' \cdot S = Y \cdot S$. We claim that $a\rho \cdot T \cap b\rho \cdot T = Y\phi \cdot T$ where

$$Y\phi = \{(x\phi)\rho : x\rho' \in Y\} \subseteq T \times T.$$

Notice that $Y\phi$ is well defined, for if $x \;\rho'\; y$, then $x\phi \;\rho\; y\phi$.

First note that if $x\rho' \in Y$, then $au \;\rho'\; x \;\rho'\; bv$ for some $u, v \in S$. By an earlier comment, this implies that $a(u\phi) \;\rho\; x\phi \;\rho\; b(v\phi)$, so $(x\phi)\rho \in a\rho{\cdot}T \cap b\rho{\cdot}T$, and so $Y\phi{\cdot}T \subseteq a\rho{\cdot}T \cap b\rho{\cdot}T$.

Conversely, let $w\rho \in a\rho \cdot T \cap b\rho \cdot T$ for some $w \in T$. Then clearly $w\rho' \in a\rho' \cdot S \cap b\rho' \cdot S$, so there exist an $x\rho' \in Y$ and $s \in S$ such that $w\rho' = x\rho' \cdot s$, that is, $w \;\rho'\; xs$. Applying ϕ we see that $w = w\phi \;\rho\; (x\phi)(s\phi)$, that is, $w\rho = (x\phi)\rho \cdot s\phi \in Y\phi \cdot T$. Consequently, $a\rho \cdot T \cap b\rho \cdot T \subseteq Y\phi \cdot T$ as required. □

COROLLARY 6·4. [**13**] *The free monoid* Ω^* *is right coherent.*

Proof. Note that the idempotent map

$$\phi\colon \mathrm{FLA}(\Omega) \to \Omega^*, \mathbf{a} \mapsto (a{\downarrow}, a)$$

is a homomorphism, so Ω^* is a retract of $\mathrm{FLA}(\Omega)$. Then Theorem 6·3 implies that Ω^* is right coherent. □

Note that the free monoid is (right) coherent, however, there exist non-coherent monoids, so the class of (right) coherent monoids is not closed under homomorphic images.

7. *The negative results*

In this section, we show that the free inverse monoid is not left coherent. By duality, neither can it be right coherent. A few simple remarks then yield that the free left ample monoid is not left coherent and that the free ample monoid is neither left nor right coherent.

Let $\Omega = \{x, y\}$, $\mathbf{a} = (\{\epsilon, x\}, x) \in \mathrm{FIM}(\Omega)$ and $\mathbf{b} = (\{\epsilon, y\}, y) \in \mathrm{FIM}(\Omega)$. Denote by ρ the left congruence generated by the pair $(\mathbf{a}, \mathbf{1})$, and by τ the left annihilator of $\mathbf{b}\rho$, that is,

$$\tau = \{(\mathbf{u}, \mathbf{v}) : \mathbf{ub} \;\rho\; \mathbf{vb}\} \subseteq \mathrm{FIM}(\Omega) \times \mathrm{FIM}(\Omega).$$

It is easy to see that τ is a left congruence on $\mathrm{FIM}(\Omega)$. We claim that it is not finitely generated.

The following lemma is effectively folklore, but we prove it here for completeness.

LEMMA 7·1. *For every* $\mathbf{u}, \mathbf{v} \in \mathrm{FIM}(\Omega)$, *we have that* $\mathbf{u} \;\rho\; \mathbf{v}$ *if and only if there exist* $m, n \in \mathbb{N}^0$ *such that* $\mathbf{ua}^n = \mathbf{va}^m$.

Proof. It is straightforward that if such n and m exist, then $\mathbf{u}$ and $\mathbf{v}$ are ρ-related. For the converse part, suppose that $\mathbf{u}\ \rho\ \mathbf{v}$. Thus, since ρ is generated by $(\mathbf{a},\mathbf{1})$, there exist elements $\mathbf{c}_1,\dots,\mathbf{c}_p,\mathbf{d}_1,\dots,\mathbf{d}_p,\mathbf{t}_1,\dots,\mathbf{t}_p \in \mathrm{FIM}(\Omega)$ such that for any $1\le i\le p$, $(\mathbf{c}_i,\mathbf{d}_i)=(\mathbf{a},\mathbf{1})$ or $(\mathbf{c}_i,\mathbf{d}_i)=(\mathbf{1},\mathbf{a})$, satisfying

$$\mathbf{u}=\mathbf{t}_1\mathbf{c}_1,\mathbf{t}_1\mathbf{d}_1=\mathbf{t}_2\mathbf{c}_2,\dots,\mathbf{t}_{p-1}\mathbf{d}_{p-1}=\mathbf{t}_p\mathbf{c}_p,\mathbf{t}_p\mathbf{d}_p=\mathbf{v}.$$

Note that for all $1\le i\le p$, we have that either $\mathbf{t}_i\mathbf{c}_i=\mathbf{t}_i\mathbf{d}_i\mathbf{a}$ (exactly when $(\mathbf{c}_i,\mathbf{d}_i)=(\mathbf{a},\mathbf{1})$) or $\mathbf{t}_i\mathbf{c}_i\mathbf{a}=\mathbf{t}_i\mathbf{d}_i$ (exactly when $(\mathbf{c}_i,\mathbf{d}_i)=(\mathbf{1},\mathbf{a})$). Applying this argument successively to $i=1,2,\dots,p$, we obtain the result of the lemma (actually, we also see that n and m are just the number of the pairs $(\mathbf{1},\mathbf{a})$ and $(\mathbf{a},\mathbf{1})$, respectively, in the sequence $(\mathbf{c}_1,\mathbf{d}_1),\dots,(\mathbf{c}_p,\mathbf{t}_p)$). □

As a direct consequence, we have the following lemma:

LEMMA 7·2. *For every* $\mathbf{u},\mathbf{v}\in\mathrm{FIM}(\Omega)$*,* $\mathbf{u}\ \tau\ \mathbf{v}$ *if and only if there exist* $m,n\in\mathbb{N}^0$ *such that* $\mathbf{u}\mathbf{b}\mathbf{a}^n=\mathbf{v}\mathbf{b}\mathbf{a}^m$*.*

For any $0\le i$, let

$$U_i=\{\epsilon,y,yx,\dots,yx^i\}.$$

LEMMA 7·3. *We have that* $(U_i,\epsilon)\ \tau\ (U_1,\epsilon)$ *for any* $1\le i$*.*

Proof. Since

$$(U_i,\epsilon)\mathbf{b}\mathbf{a}^i=(U_1,\epsilon)\mathbf{b}\mathbf{a}^i=(\{\epsilon,y,yx,yx^2,\dots,yx^i)$$

we have by Lemma 7·2 that $(U_i,\epsilon)\ \tau\ (U_1,\epsilon)$. □

LEMMA 7·4. *The left annihilator congruence* $\tau=l(\mathbf{b}\rho)$ *is not finitely generated.*

Proof. Suppose for contradiction that H is a finite symmetric subset of τ generating τ and let k be a natural number such that for every $((S,s),(T,t))\in H$ we have that $k>|S|$.

Now suppose that $(U_k,\epsilon)=\mathbf{t}\mathbf{c}$ where $(\mathbf{c},\mathbf{d})\in H$ and $\mathbf{t}\in\mathrm{FIM}(\Omega)$. Then $c^{-1}=t\in U_k$ and $c^{-1}C\subseteq U_k$. Note that since $c\in C$, $c^{-1}C$ is also prefix closed. The facts that U_k is a single path and $|C|<k$ imply that $c^{-1}C\subseteq\{\epsilon,y,yx,\dots,yx^{k-1}\}$. However, $U_k=T\cup c^{-1}C$, and as a consequence we have that $yx^k\in T$, so $T=U_k$.

We also have $\mathbf{c}\ \tau\ \mathbf{d}$, so there exist i,j such that $\mathbf{c}\mathbf{b}\mathbf{a}^i=\mathbf{d}\mathbf{b}\mathbf{a}^j$. By multiplying this equality from the right by an appropriate power of $\mathbf{a}$ we can ensure that $i,j>k$. Note that since $C\subseteq cU_k$, the first component of $\mathbf{c}\mathbf{b}\mathbf{a}^i$ is $\{c,cy,cyx,\dots,cyx^i\}$, whereas the first component of $\mathbf{d}\mathbf{b}\mathbf{a}^j$ contains the vertices $\{d,dy,dyx,dyx^2,\dots,dyx^j\}$. Given that $c^{-1}\in U_k$, a brief analysis shows this can only happen if $d=c$, and then $c^{-1}D\subseteq\{\epsilon,y,\dots,yx^{k-1}\}$ follows from the facts that

$$c^{-1}D\subseteq c^{-1}\{c,cy,cyx,\dots,cyx^i\}=\{\epsilon,y,yx,\dots,yx^i\},$$

$c^{-1}D$ is prefix closed and $\left|c^{-1}D\right|<k$. So altogether we obtain that $T=U_k$ and $tD=c^{-1}D\subseteq\{\epsilon,y,yx,\dots,yx^{k-1}\}\subseteq U_k$, so $T\cup tD=U_k$ and as a consequence we conclude that $\mathbf{t}\mathbf{d}=(U_k,\epsilon)$. That is, applying elements of H to right factors of (U_k,ϵ) does not change (U_k,ϵ), so the τ-class of (U_k,ϵ) is singleton, that is, $(U_k,\epsilon)\ \not\tau\ (U_{k+1},\epsilon)$, contradicting Lemma 7·3. □

THEOREM 7·5. *Let* $|\Omega| > 1$. *Then the free inverse monoid* $\mathrm{FIM}(\Omega)$ *and the free ample monoid* $\mathrm{FAM}(\Omega)$ *are neither left nor right coherent. The free left ample monoid* $\mathrm{FLA}(\Omega)$ *is right coherent, but not left coherent.*

Proof. Lemma 7·4 shows that $\mathrm{FIM}(\Omega)$ is not left coherent. Exactly the same argument applies to show that $\mathrm{FLA}(\Omega)$ and $\mathrm{FAM}(\Omega)$ are not left coherent, simplifying further, since $c = t = \epsilon$. By duality, $\mathrm{FIM}(\Omega)$ and $\mathrm{FAM}(\Omega)$ cannot be right coherent. □

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