MIURA-TYPE TRANSFORMATIONS FOR LATTICE EQUATIONS AND LIE GROUP ACTIONS ASSOCIATED WITH DARBOUX-LAX REPRESENTATIONS

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Abstract. Miura-type transformations (MTs) are an essential tool in the theory of integrable nonlinear partial differential and difference equations. We present a geometric method to construct MTs for differential-difference (lattice) equations from Darboux-Lax representations (DLRs) of such equations.

The method is applicable to parameter-dependent DLRs satisfying certain conditions. We construct MTs and modified lattice equations from invariants of some Lie group actions on manifolds associated with such DLRs.

Using this construction, from a given suitable DLR one can obtain many MTs of different orders. The main idea behind this method is closely related to the results of Drinfeld and Sokolov on MTs for the partial differential KdV equation.

Considered examples include the Volterra, Narita-Itoh-Bogoyavlensky, Toda, and Adler-Postnikov lattices. Some of the constructed MTs and modified lattice equations seem to be new.

Keywords: Miura-type transformations, differential-difference equations, Lie group actions, Darboux-Lax representations, Narita-Itoh-Bogoyavlensky lattice, Toda lattice, Adler-Postnikov lattices
PACS numbers: 02.30.Ik, 02.40.-k
Mathematics Subject Classification: 37K35, 53C30

1. Introduction

1.1. **An overview of the results.** It is well known that Miura-type transformations (MTs) play an essential role in the theory of integrable nonlinear partial differential and difference equations. (For partial differential equations, MTs are also called differential substitutions.)

In particular, when one tries to classify a certain class of integrable equations, one often finds a few basic equations such that all other equations from the considered class can be obtained from the basic ones by means of MTs (see, e.g., [11, 20, 5, 10] and references therein). Applications of MTs to construction of conservation laws [13, 18] and auto-Bäcklund transformations are also well known.

Therefore, it is highly desirable to develop systematic methods for constructing MTs. In this paper, we focus on MTs of differential-difference (lattice) equations.

Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be integers such that $\boldsymbol{\alpha} \le \boldsymbol{\beta}$. We study differential-difference equations of the form

$$u_t = \mathbf{F}(u_{\boldsymbol{\alpha}}, u_{\boldsymbol{\alpha}+1}, \dots, u_{\boldsymbol{\beta}}), \tag{1}$$

where $u = u(n,t)$ is a vector-function of an integer variable n and a real or complex variable t. We use the standard notation $u_t = \partial_t(u)$ and $u_l = u(n+l,t)$ for $l \in \mathbb{Z}$. In particular, $u_0 = u$.

Equation (1) must be valid for all $n \in \mathbb{Z}$, so (1) encodes an infinite sequence of differential equations

$$\partial_t\big(u(n,t)\big) = \mathbf{F}\big(u(n+\boldsymbol{\alpha},t), u(n+\boldsymbol{\alpha}+1,t), \dots, u(n+\boldsymbol{\beta},t)\big), \qquad n \in \mathbb{Z}.$$

Consider another differential-difference equation of similar type

$$v_t = \tilde{\mathbf{F}}(v_{\tilde{\boldsymbol{\alpha}}}, v_{\tilde{\boldsymbol{\alpha}}+1}, \dots, v_{\tilde{\boldsymbol{\beta}}}) \tag{2}$$

for a vector-function $v = v(n,t)$ and some integers $\tilde{\boldsymbol{\alpha}} \le \tilde{\boldsymbol{\beta}}$.

A *Miura-type transformation* (MT) from equation (2) to equation (1) is determined by an expression of the form

$$u = f(v_p, v_{p+1}, \dots, v_r), \qquad p, r \in \mathbb{Z}, \qquad p \le r, \tag{3}$$

such that if v satisfies (2) then u given by (3) satisfies (1). A more precise definition of MTs is presented in Section 2.1.

Equation (2) is called the *modified equation* corresponding to the described MT.

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Let $p, r \in \mathbb{Z}$, $p \leq r$, be such that the function f in (3) may depend only on $v_p, v_{p+1}, \dots, v_r$ and depends nontrivially on v_p, v_r. Then the number $r - p$ is called the *order* of the MT (3).

Example 1. Let u and v be scalar functions. It is known that the formula $u = vv_1$ determines an MT from the modified Volterra equation $v_t = v^2(v_1 - v_{-1})$ to the Volterra equation $u_t = u(u_1 - u_{-1})$.

According to our notation, $v = v_0$, so $vv_1 = v_0v_1$. Therefore, the MT $u = vv_1$ is of order 1.

We present a method to construct equations (2) and MTs (3) of different orders for a given equation (1) possessing a matrix Darboux-Lax representation (DLR). Note that the order of the obtained MTs may be higher than the size of the matrices in the DLR. (The definition of DLRs is given in Remark 1 below.)

The method uses invariants of some Lie group actions on manifolds associated with DLRs and is applicable to parameter-dependent DLRs satisfying certain conditions, see Section 1.3 and Section 2 for more details. The main idea behind the method is closely related to the results of Drinfeld, Sokolov [3] and Igonin [6] on MTs for evolution partial differential equations (PDEs).

The papers [3, 6] construct MTs for evolution PDEs from zero-curvature representations of evolution PDEs. We construct MTs for differential-difference equations from Darboux-Lax representations of differential-difference equations. Since the structure of Darboux-Lax representations of differential-difference equations is quite different from the structure of zero-curvature representations of PDEs, translation of ideas of [3, 6] to the case of differential-difference equations is a challenging problem.

Note that the papers [3, 6] study properly only the case of scalar equations. More precisely, the paper [3] presents (without proof) a construction of MTs for the (scalar) partial differential KdV equation from the standard zero-curvature representation of KdV. A similar (but more general) construction of MTs for scalar $(1 + 1)$-dimensional evolution PDEs is given in [6]. The multicomponent case is not discussed at all in [3]. In addition to scalar evolution PDEs, the paper [6] considers a small class of MTs for some multicomponent evolution PDEs, but not the general multicomponent case.

We present a method to construct MTs for differential-difference equations in the general multicomponent case. To clarify the main idea, we first consider the scalar case in Theorem 1 in Section 2, and then the general multicomponent case in Theorem 2. Note that we give detailed proofs of these results.

A detailed exposition of the method is given in Section 2. The main ideas are briefly outlined in Section 1.3.

To illustrate the method, in Section 2 we include a derivation of some MTs for the Volterra equation. These MTs (up to a change of variables) can be found also in [20].

Using the described method, in Sections 3, 4, 5 we construct a number of MTs for the Narita-Itoh-Bogoyavlensky, Toda lattices and Adler-Postnikov lattices from [1]. Some of the constructed MTs and modified equations seem to be new.

Some abbreviations, conventions, and notation used in the paper are presented in Section 1.2.

Remark 1. Let $\mathcal{S}$ be the shift operator which replaces n by $n+1$ and u_l by u_{l+1} in all considered functions. (A more detailed definition of $\mathcal{S}$ is given in Section 2.1.)

Let d be a positive integer. Let $M = M(u_l, \dots, \lambda)$ and $\mathcal{U} = \mathcal{U}(u_l, \dots, \lambda)$ be $d \times d$ matrix-functions depending on a finite number of the variables u_l and a complex parameter λ. Suppose that the matrix M is invertible, and the equation

$$\partial_t(M) = \mathcal{S}(\mathcal{U})M - M\mathcal{U} \tag{4}$$

holds as a consequence of (1). It is well known that such a pair $(M, \mathcal{U})$ is an analogue of a zero-curvature (Lax) representation for differential-difference equations. Also, it is known that such $(M, \mathcal{U})$ often arise from Darboux transformations of PDEs (see, e.g., [8]).

Motivated by these facts, following [8], we say that the pair $(M, \mathcal{U})$ is a *Darboux-Lax representation* (DLR) for equation (1). This implies that the auxiliary linear system

$$\begin{aligned} \mathcal{S}(\Psi) &= M\Psi, \\ \partial_t(\Psi) &= \mathcal{U}\Psi \end{aligned} \tag{5}$$

is compatible modulo (1). Here $\Psi = \Psi(n, t)$ is an invertible $d \times d$ matrix-function. (A more precise definition of DLRs is given in Section 2.1.)

Some authors say that a pair $(M, \mathcal{U})$ (with invertible M) is a DLR for (1) if equation (4) is equivalent to equation (1). For our purposes, it is sufficient to assume that (4) holds as a consequence of (1).

Remark 2. A different method to construct MTs for differential-difference equations is described by Yamilov [19]. Yamilov's method uses the following diagram, where (F), (G), (T), (H) are differential-difference equations and m_1, m_2, m_3, m_4 are MTs.

$$\begin{CD} (F) @<{m_3}<< (H) \\ @V{m_2}VV @VV{m_4}V \\ (T) @<{m_1}<< (G) \end{CD} \tag{6}$$

For given equations (F), (G), (T) and MTs m_1, m_2 of order 1 satisfying certain conditions, the paper [19] constructs another equation (H) and MTs m_3, m_4 of order 1 so that the diagram (6) is commutative.

Yamilov's method is very different from ours. Indeed, we construct MTs of arbitrary order from certain Lie group actions associated with a given DLR. Yamilov [19] constructs new MTs of order 1 from given MTs of order 1, not using any Lie group actions. (Compositions of such MTs of order 1 can produce MTs of higher order [19].) It would be interesting to determine if Yamilov's MTs can be obtained from some DLRs by our method. (But we do not study Yamilov's MTs in the present paper.)

Remark 3. As has been said above, in this paper we study the following problem: how to construct MTs (3) and equations (2) for a given equation (1)?

Sokolov [16] studied the inverse problem: how to construct MTs (3) and equations (1) for a given equation (2)? (More precisely, Sokolov [16] studied this for $(1+1)$-dimensional evolution PDEs and more general MTs which may change the space variable.) According to [16], the study of the inverse problem does not require Lax representations.

1.2. **Abbreviations, conventions, and notation.** The following abbreviations, conventions, and notation are used in the paper. MT = Miura-type transformation, DLR = Darboux-Lax representation.

The symbols $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote the sets of positive and nonnegative integers respectively.

Each considered function is supposed to be analytic on its domain of definition. (This includes meromorphic functions. By our convention, the poles of a meromorphic function do not belong to its domain of definition, so a meromorphic function is analytic on its domain of definition.)

Unless otherwise specified, all scalar variables and functions are assumed to be $\mathbb{C}$-valued. The symbol n denotes an integer variable. All considered Lie groups and Lie subgroups are supposed to be complex-analytic.

For every $d \in \mathbb{Z}_{>0}$, we denote by $\mathrm{Mat}_d(\mathbb{C})$ the algebra of $d \times d$ matrices with entries from $\mathbb{C}$ and by $\mathrm{GL}_d(\mathbb{C})$ the group of invertible $d \times d$ matrices with entries from $\mathbb{C}$.

For $i \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$, the symbols c_i denote complex constants, and $c_i^k = (c_i)^k$ is the kth power of c_i.

Also, we use the notation $u_k^i = \mathcal{S}^k(u^i)$ and $v_k^i = \mathcal{S}^k(v^i)$, where $\mathcal{S}^k$ is the kth power of the operator $\mathcal{S}$, and u^i, v^i are dependent variables in differential-difference equations.

1.3. **The main ideas.** In this subsection we outline the main ideas of our method to construct MTs from DLRs. Let $d \in \mathbb{Z}_{>0}$. Let $(M, \mathcal{U})$ be a $d \times d$ matrix DLR for equation (1), as described in Remark 1. So the matrix M is invertible, and system (5) is compatible modulo (1).

Our method to construct MTs is applicable to the case when the matrix $M = M(u, \lambda)$ depends only on u and λ. (That is, M does not depend on u_l for $l \neq 0$.) So we consider the case when $M, \mathcal{U}$ are of the form

$$M = M(u, \lambda), \qquad \mathcal{U} = \mathcal{U}(u_a, u_{a+1}, \dots, u_b, \lambda) \tag{7}$$

for some integers $a \leq b$. Without loss of generality, we can assume $a < 0$. (Indeed, $\mathcal{U}(u_a, u_{a+1}, \dots, u_b, \lambda)$ does not have to depend on u_a nontrivially, so we can always take $a < 0$.) Note that some DLRs of other types can be transformed to the form (7) by a change of variables, see Remark 7 in Section 2.1.

Remark 4. In some examples of DLRs, it may happen that $M(u, \lambda)$ is invertible for almost all (but not all) values of u and λ. The exceptional points (u, λ) where $M(u, \lambda)$ is not invertible are excluded from further consideration.

Consider the algebra $\mathbb{C}[\lambda]$ of polynomials in λ. Fix $c \in \mathbb{C}$ and $k \in \mathbb{Z}_{>0}$. We denote by $((\lambda - c)^k) \subset \mathbb{C}[\lambda]$ the ideal generated by the polynomial $(\lambda - c)^k \in \mathbb{C}[\lambda]$. Clearly, the quotient algebra $\mathbb{C}[\lambda]/((\lambda - c)^k)$ is k-dimensional.

Denote by $\mathcal{G}(d, c, k)$ the group of invertible $d \times d$ matrices with entries from $\mathbb{C}[\lambda]/((\lambda - c)^k)$. Equivalently, an element of $\mathcal{G}(d, c, k)$ can be described as a sum of the form

$$\sum_{q=0}^{k-1} (\lambda - c)^q W^q, \qquad W^0 \in \mathrm{GL}_d(\mathbb{C}), \qquad W^1, \dots, W^{k-1} \in \mathrm{Mat}_d(\mathbb{C}). \tag{8}$$

This shows that $\mathcal{G}(d,c,k)$ is a kd^2-dimensional Lie group. When we multiply two elements of the form (8) in the group $\mathcal{G}(d,c,k)$, we use the fact that in the algebra $\mathbb{C}[\lambda]/((\lambda-c)^k)$ one has $(\lambda-c)^k=0$.

For $i,j=1,\dots,d$ and $q=0,\dots,k-1$, we denote by $w^q_{ij}=(W^q)_{ij}$ the entries of the matrix W^q. Then w^q_{ij} can be regarded as coordinates on the Lie group $\mathcal{G}(d,c,k)$. Set $G=\mathcal{G}(d,c,k)$.

Now assume that $w^q_{ij}=w^q_{ij}(n,t)$ are functions of the variables n, t. This means that W^q are $d\times d$ matrix-functions of the variables n, t. Substituting $\Psi=\sum_{q=0}^{k-1}(\lambda-c)^qW^q$ in (5), one obtains the system

$$\sum_{q=0}^{k-1}(\lambda-c)^q\mathcal{S}\big(W^q\big)=M(u,\lambda)\Big(\sum_{q=0}^{k-1}(\lambda-c)^qW^q\Big),\qquad(\lambda-c)^k=0,\tag{9}$$

$$\sum_{q=0}^{k-1}(\lambda-c)^q\partial_t\big(W^q\big)=\mathcal{U}(u_a,\dots,u_b,\lambda)\Big(\sum_{q=0}^{k-1}(\lambda-c)^qW^q\Big),\qquad(\lambda-c)^k=0.\tag{10}$$

To compute the right-hand sides of (9) and (10), we take the corresponding Taylor series with respect to λ and truncate these series at the term $(\lambda-c)^{k-1}$. (Recall that in the algebra $\mathbb{C}[\lambda]/((\lambda-c)^k)$ one has $(\lambda-c)^k=0$, which we use here.)

Since $w^q_{ij}=(W^q)_{ij}$ are the entries of the matrix W^q, equation (9) allows us to express $\mathcal{S}(w^q_{ij})$ in terms of u and $w^{\tilde q}_{\tilde i\tilde j}$ for $\tilde q=0,\dots,k-1$, $\tilde i,\tilde j=1,\dots,d$. Similarly, equation (10) allows us to express $\partial_t(w^q_{ij})$ in terms of $u_a,\dots,u_b$ and $w^{\tilde q}_{\tilde i\tilde j}$. That is, system (9), (10) can be rewritten as

$$\mathcal{S}(w^q_{ij})=A^q_{ij}(u,w^{\tilde q}_{\tilde i\tilde j}),\qquad \partial_t(w^q_{ij})=B^q_{ij}(u_a,\dots,u_b,w^{\tilde q}_{\tilde i\tilde j})\tag{11}$$

for some scalar functions A^q_{ij}, B^q_{ij}.

Recall that w^q_{ij} are coordinates on the Lie group $G=\mathcal{G}(d,c,k)$. In what follows, by a function on G we mean an analytic function defined on an open dense subset of G.

Let $z=z(w^q_{ij})$ be a function on G. Using formulas (11) and the property $\mathcal{S}(u_l)=u_{l+1}$, one can compute $\partial_t(z)$ and $\mathcal{S}^i(z)$ for any $i\in\mathbb{Z}_{>0}$. Here and below $\mathcal{S}^i$ denotes the ith power of the operator $\mathcal{S}$.

To clarify the main idea of the method to construct MTs, suppose that u is a scalar function. In the scalar case, our goal is to find a function $z=z(w^q_{ij})$ such that there is $r\in\mathbb{Z}_{>0}$ satisfying the following conditions.

(12) $\mathcal{S}^r(z)$ can be expressed in terms of $z,\ \mathcal{S}(z),\ \mathcal{S}^2(z),\ \dots,\ \mathcal{S}^{r-1}(z),\ u$ so that the obtained expression $\mathcal{S}^r(z)=F\big(z,\mathcal{S}(z),\mathcal{S}^2(z),\dots,\mathcal{S}^{r-1}(z),u\big)$ depends nontrivially on u.

(13) $\partial_t(z)$ can be expressed in terms of $z,\ \mathcal{S}(z),\ \mathcal{S}^2(z),\ \dots,\ \mathcal{S}^{r-1}(z),\ u_a,\ \dots,\ u_b$, so $\partial_t(z)=Q\big(z,\mathcal{S}(z),\mathcal{S}^2(z),\dots,\mathcal{S}^{r-1}(z),u_a,\dots,u_b\big)$ for some function Q.

Using a function z satisfying (12), (13), one obtains an MT for equation (1) as follows. Set $z_0=z$ and $z_i=\mathcal{S}^i(z)$ for $i=1,\dots,r$. Then the above formulas

$$\mathcal{S}^r(z)=F\big(z,\mathcal{S}(z),\mathcal{S}^2(z),\dots,\mathcal{S}^{r-1}(z),u\big),$$
$$\partial_t(z)=Q\big(z,\mathcal{S}(z),\mathcal{S}^2(z),\dots,\mathcal{S}^{r-1}(z),u_a,\dots,u_b\big)$$

become

$$z_r=F(z_0,z_1,z_2,\dots,z_{r-1},u),\tag{14}$$

$$\partial_t(z)=Q\big(z_0,z_1,z_2,\dots,z_{r-1},u_a,\dots,u_b\big).\tag{15}$$

Since, according to (12), the function F depends nontrivially on u, locally from equation (14) one can express u in terms of $z_0,z_1,\dots,z_r$

$$u=f(z_0,z_1,\dots,z_r).\tag{16}$$

We introduce new variables v_l for $l\in\mathbb{Z}$. Using the function $f(z_0,z_1,\dots,z_r)$ from (16), for each $j\in\mathbb{Z}$ one can consider the function $f(v_j,v_{j+1},\dots,v_{j+r})$. Let $P(v_a,\dots,v_{b+r})$ be the function obtained from $Q(z_0,\dots,z_{r-1},u_a,\dots,u_b)$ by replacing z_i with v_i and u_j with $f(v_j,v_{j+1},\dots,v_{j+r})$. That is,

$$P(v_a,\dots,v_{b+r})=Q\big(v_0,\dots,v_{r-1},f(v_a,\dots,v_{a+r}),\dots,f(v_b,\dots,v_{b+r})\big).$$

We introduce the formula

$$v_t=P(v_a,\dots,v_{b+r})\tag{17}$$

and regard (17) as a differential-difference equation for v. Then the formula

$$u = f(v_0, v_1, \dots, v_r) \tag{18}$$

determines an MT from equation (17) to equation (1). As usual, we can use the identification $v_0 = v$. Then (18) becomes $u = f(v, v_1, \dots, v_r)$.

So we have shown that a function $z = z(w^q_{ij})$ satisfying (12), (13) gives an MT. Now let us explain how to construct such z. It turns out that one can construct such functions z as invariants with respect to the left and right actions of certain subgroups of G. (Subgroups of G act on G by left and right multiplication, which induces actions on the algebra of functions on G.)

In Section 2.2, for a given DLR (7) we define a sequence of groups

$$\mathbb{H}_0 \subset \mathbb{H}_1 \subset \mathbb{H}_2 \subset \dots$$

For each $p \in \mathbb{Z}_{\geq 0}$, the group $\mathbb{H}_p$ consists of certain invertible $d \times d$ matrix-functions of λ. The groups $\mathbb{H}_p$ are defined by induction on p as follows.

We set $\mathbb{H}_0 = \mathbf{1}$, where $\mathbf{1}$ is the identity matrix of size d. The group $\mathbb{H}_1$ is generated by the matrix-functions $M(\tilde{u}, \lambda) \cdot M(u, \lambda)^{-1}$ for all values of u, $\tilde{u}$.

For each $p \in \mathbb{Z}_{>0}$, the group $\mathbb{H}_{p+1}$ is generated by the elements $h_p \in \mathbb{H}_p$ and the elements

$$M(u, \lambda) \cdot h_p \cdot M(u, \lambda)^{-1}$$

for all $h_p \in \mathbb{H}_p$ and all values of u.

Since the elements of $\mathbb{H}_p$ are invertible $d \times d$ matrix-functions of λ, the group $\mathbb{H}_p$ acts on the group $G = \mathcal{G}(d, c, k)$ by left multiplication. (Performing this multiplication, we work modulo the relation $(\lambda - c)^k = 0$, as has been said above.) A function $\mathbf{f}$ on G is called $\mathbb{H}_p$*-left-invariant* if $\mathbf{f}$ is invariant with respect to this left action of $\mathbb{H}_p$.

Let $\mathcal{H} \subset G$ be a subgroup of G. A function $\mathbf{f}$ on G is called $\mathcal{H}$*-right-invariant* if $\mathbf{f}$ is invariant with respect to the action of $\mathcal{H}$ on G by right multiplication. (See Section 2.2 for a detailed definition of left-invariant and right-invariant functions.)

For an arbitrary function $z = z(w^q_{ij})$ on G, the functions $\mathcal{S}^\gamma(z)$ for $\gamma \in \mathbb{Z}_{>0}$ may depend on w^q_{ij} and u_l. In Section 2 we prove the following statements.

If a function z on G is $\mathbb{H}_p$-left-invariant for some $p \in \mathbb{Z}_{>0}$ then the functions

$$\mathcal{S}(z), \quad \mathcal{S}^2(z), \quad \dots, \quad \mathcal{S}^p(z) \tag{19}$$

do not depend on u_l for any $l \in \mathbb{Z}$. So $\mathcal{S}(z)$, $\mathcal{S}^2(z)$, $\dots$, $\mathcal{S}^p(z)$ depend only on w^q_{ij} and can be regarded as functions on G. This implies that $\mathcal{S}^{p+1}(z)$ may depend only on w^q_{ij} and $u = u_0$.

Furthermore, for any closed connected Lie subgroup $H \subset G$, if a function z on G is H-right-invariant and is $\mathbb{H}_p$-left-invariant for some $p \in \mathbb{Z}_{>0}$ then the functions (19) are H-right-invariant as well.

These statements (along with some other considerations) allow us to prove the following. Suppose that a function z on G satisfies the following conditions:

(20) there is a closed connected Lie subgroup $H \subset G$ of codimension $r > 0$ such that z is H-right-invariant and $\mathbb{H}_{r-1}$-left-invariant,

(21) the differentials of the functions z, $\mathcal{S}(z)$, $\mathcal{S}^2(z)$, $\dots$, $\mathcal{S}^{r-1}(z)$ are linearly independent on an open dense subset of G,

(22) the function $\mathcal{S}^r(z)$ depends nontrivially on u.

Then z obeys (12), (13).

So (12), (13) follow from (20), (21), (22). As has been shown above, a function z obeying (12), (13) gives an MT.

Note that (21), (22) can be regarded as non-degeneracy conditions. In all examples known to us, if a nonconstant function z satisfies (20) then (21), (22) are also satisfied. Therefore, to construct an MT, one needs to find a nonconstant function z obeying (20). This can often be done as follows.

Note that H-right-invariant functions on G can be identified with functions on the quotient manifold G/H. For any $p \in \mathbb{Z}_{\geq 0}$, the above-mentioned left action of $\mathbb{H}_p$ on G induces a left action of $\mathbb{H}_p$ on G/H.

Therefore, we need to choose a positive integer r and a closed connected Lie subgroup $H \subset G$ of codimension $r > 0$ such that the left action of $\mathbb{H}_{r-1}$ on G/H possesses a nonconstant invariant z. (That is, z is a nonconstant $\mathbb{H}_{r-1}$-left-invariant function on G/H.) Then z can be viewed as an H-right-invariant function on G and obeys (20).

A detailed description of this theory is given in Section 2. Actually, in Section 2 we consider a more general Lie group G which is equal to the product of several groups of the form $\mathcal{G}(d, c, k)$. (See formula (42).) Because of this, some notation in Section 2 is different from the notation in the present section.

In the above discussion we have assumed that u is a scalar function. In Section 2 we develop this theory in the case when $u = \big(u^1(n,t), \dots, u^N(n,t)\big)$ is an N-component vector-function for arbitrary $N \geq 1$. To construct an MT in the case $N > 1$, we use several invariants with respect to the left actions of the groups $\mathbb{H}_p$ and the right action of a Lie subgroup $H \subset G$, which must satisfy certain conditions (see Theorem 2 for details).

Remark 5. Let $\mathbf{g}(\lambda)$ be a $\mathrm{GL}_d(\mathbb{C})$-valued function of λ. Using the DLR (7), consider the matrix-functions

$$\tilde{M} = \mathbf{g}(\lambda) \cdot M \cdot \mathbf{g}(\lambda)^{-1}, \qquad \tilde{\mathcal{U}} = \mathbf{g}(\lambda) \cdot \mathcal{U} \cdot \mathbf{g}(\lambda)^{-1}. \tag{23}$$

Then (23) is a DLR as well, which is said to be *gauge equivalent* to (7). Replacing (7) by (23), one can sometimes simplify the form of the groups $\mathbb{H}_p$. Examples of this procedure are presented in Section 3.

Remark 6. As has been discussed above, if a function z on G is $\mathbb{H}_p$-left-invariant for some $p \in \mathbb{Z}_{>0}$ then the functions (19) do not depend on u_l for any $l \in \mathbb{Z}$.

Recall that by a function on G we mean an analytic function defined on an open dense subset of G. So z is defined on an open dense subset $\mathbb{U} \subset G$. Then the definition of the operator $\mathcal{S}$ implies that for each $l = 1, \dots, p$ the function $\mathcal{S}^l(z)$ is defined on an open dense subset $\mathbb{U}^l \subset G$, and it may happen that $\mathbb{U}^l \neq \mathbb{U}$.

Since $\mathbb{U}$, $\mathbb{U}^1, \dots, \mathbb{U}^p$ are open dense subsets of G, the intersection $\mathbb{U} \cap \bigcap_{l=1}^p \mathbb{U}^l$ is also open and dense in G. Then we can regard (19) as functions on $\mathbb{U} \cap \bigcap_{l=1}^p \mathbb{U}^l$.

2. General theory

2.1. **Differential-difference equations and Darboux-Lax representations.** Fix $N \in \mathbb{Z}_{>0}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}$ such that $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$. Consider a differential-difference equation of the form

$$u_t = \mathbf{F}(u_{\boldsymbol{\alpha}}, u_{\boldsymbol{\alpha}+1}, \dots, u_{\boldsymbol{\beta}}) \tag{24}$$

for an N-component vector-function $u = \big(u^1(n,t), \dots, u^N(n,t)\big)$ of an integer variable n and a real or complex variable t. We use the standard notation $u_t = \partial_t(u)$ and $u_l = u(n+l, t)$ for $l \in \mathbb{Z}$. In particular, $u_0 = u$.

In other words, equation (24) encodes an infinite sequence of differential equations

$$\partial_t\big(u(n,t)\big) = \mathbf{F}\big(u(n+\boldsymbol{\alpha}, t), u(n+\boldsymbol{\alpha}+1, t), \dots, u(n+\boldsymbol{\beta}, t)\big), \qquad n \in \mathbb{Z}.$$

Here $\mathbf{F}$ is also an N-component vector-function $\mathbf{F} = (\mathbf{F}^1, \dots, \mathbf{F}^N)$. So in components equation (24) reads

$$u^i_t = \mathbf{F}^i(u^\gamma_{\boldsymbol{\alpha}}, u^\gamma_{\boldsymbol{\alpha}+1}, \dots, u^\gamma_{\boldsymbol{\beta}}), \qquad i = 1, \dots, N. \tag{25}$$

As usual in the formal theory of differential-difference equations, one regards

$$u_l = (u^1_l, \dots, u^N_l), \qquad l \in \mathbb{Z},$$

as independent quantities (which are sometimes called *dynamical variables* in the literature). When we write $f = f(u_l, \dots)$, we mean that f may depend on any finite number of the variables u^γ_l for $l \in \mathbb{Z}$ and $\gamma = 1, \dots, N$.

When we write $f = f(u_p, \dots, u_q)$ or $f = f(u_p, u_{p+1}, \dots, u_q)$ for some integers $p \leq q$, we mean that f may depend on u^γ_l for $l = p, \dots, q$ and $\gamma = 1, \dots, N$.

Let $\mathcal{S}$ be the *shift operator* with respect to the integer variable n. That is, for any function $g = g(n,t)$ one has $\mathcal{S}(g) = g(n+1, t)$.

Since u_l corresponds to $u(n+l, t)$, the operator $\mathcal{S}$ and its powers $\mathcal{S}^k$ for $k \in \mathbb{Z}$ act on functions of u_l as follows

$$\mathcal{S}(u_l) = u_{l+1}, \qquad \mathcal{S}^k(u_l) = u_{l+k}, \qquad \mathcal{S}^k\big(f(u_l, \dots)\big) = f(\mathcal{S}^k(u_l), \dots). \tag{26}$$

That is, applying $\mathcal{S}^k$ to a function $f = f(u_l, \dots)$, we replace u^γ_l by u^γ_{l+k} in f for all l, γ.

The *total derivative operator* D_t corresponding to (24) is given by the formula

$$D_t(f) = \sum_{l,\gamma} \mathcal{S}^l(\mathbf{F}^\gamma) \cdot \frac{\partial f}{\partial u^\gamma_l}. \tag{27}$$

Let $d \in \mathbb{Z}_{>0}$. Let $M = M(u_l, \dots, \lambda)$ and $\mathcal{U} = \mathcal{U}(u_l, \dots, \lambda)$ be $d \times d$ matrix-functions depending on the variables u_l and a complex parameter λ. Suppose that M is invertible and

$$D_t(M) = \mathcal{S}(\mathcal{U})M - M\mathcal{U}, \tag{28}$$

where D_t is given by (27). Then the pair $(M, \mathcal{U})$ is called a *(matrix) Darboux-Lax representation* (DLR) for equation (24). This implies that the auxiliary linear system

$$\mathcal{S}(\Psi) = M\Psi, \qquad \partial_t(\Psi) = \mathcal{U}\Psi \tag{29}$$

is compatible modulo (24). Here $\Psi = \Psi(n,t)$ is an invertible $d \times d$ matrix-function.

Our method to construct MTs is applicable to the case when

$$M = M(u, \lambda) = M(u^1, \dots, u^N, \lambda)$$

depends only on $u = (u^1, \dots, u^N)$ and λ. (That is, M does not depend on u_l for $l \neq 0$.)

So we will mainly consider the case when $M, \mathcal{U}$ are of the form

$$M = M(u, \lambda), \qquad \mathcal{U} = \mathcal{U}(u_a, u_{a+1}, \dots, u_b, \lambda) \tag{30}$$

for some integers $a \le b$. Without loss of generality, we can assume $a < 0$.

We will use also Remark 4 on invertibility of the matrix $M(u, \lambda)$.

Remark 7. If $M = M(u^1_{l_1}, \dots, u^N_{l_N}, \lambda)$ depends on λ and $u^1_{l_1}, \dots, u^N_{l_N}$ for some fixed integers $l_1, \dots, l_N$, then one can relabel

$$u^1 := u^1_{l_1}, \quad \dots \quad , \quad u^N := u^N_{l_N},$$

which reduces $M(u^1_{l_1}, \dots, u^N_{l_N}, \lambda)$ to $M(u^1, \dots, u^N, \lambda)$. An example of such relabelling is used in Section 4.

Similarly to (24), consider a differential-difference equation

$$v_t = \tilde{\mathbf{F}}(v_{\tilde{\boldsymbol{\alpha}}}, v_{\tilde{\boldsymbol{\alpha}}+1}, \dots, v_{\tilde{\boldsymbol{\beta}}}) \tag{31}$$

for an N-component vector-function $v = \big(v^1(n,t), \dots, v^N(n,t)\big)$. Here $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}} \in \mathbb{Z}$, $\tilde{\boldsymbol{\alpha}} \le \tilde{\boldsymbol{\beta}}$, and $v_l = v(n+l, t)$ for $l \in \mathbb{Z}$. In particular, $v_0 = v$.

Similarly to (26), (27), the operators $\mathcal{S}$ and D_t act on functions of the variables $v_l = (v^1_l, \dots, v^N_l)$ as follows

$$\mathcal{S}(v_l) = v_{l+1}, \qquad \mathcal{S}^k(v_l) = v_{l+k}, \qquad \mathcal{S}^k\big(f(v_l, \dots)\big) = f(\mathcal{S}^k(v_l), \dots), \qquad k \in \mathbb{Z},$$

$$D_t\big(f(v_l, \dots)\big) = \sum_{l,\gamma} \mathcal{S}^l(\tilde{\mathbf{F}}^\gamma) \cdot \frac{\partial f}{\partial v^\gamma_l},$$

where $\tilde{\mathbf{F}}^\gamma$ are the components of the vector-function $\tilde{\mathbf{F}} = (\tilde{\mathbf{F}}^1, \dots, \tilde{\mathbf{F}}^N)$ from (31).

Definition 1. A *Miura-type transformation* (MT) from equation (31) to equation (24) is determined by an expression of the form

$$u = f(v_l, \dots) \tag{32}$$

(where f may depend on any finite number of the variables $v_l = (v^1_l, \dots, v^N_l)$, $l \in \mathbb{Z}$,) such that if v satisfies (31) then u given by (32) satisfies (24).

More precisely, this means the following. In components formula (32) reads

$$u^i = f^i(v^\gamma_l, \dots), \qquad i = 1, \dots, N, \tag{33}$$

where f^i are the components of the vector-function $f = (f^1, \dots, f^N)$. If we substitute the right-hand side of (33) in place of u^i in (25), we get

$$D_t\big(f^i(v^\gamma_l, \dots)\big) = \mathbf{F}^i\big(\mathcal{S}^{\boldsymbol{\alpha}}(f^\gamma), \mathcal{S}^{\boldsymbol{\alpha}+1}(f^\gamma), \dots, \mathcal{S}^{\boldsymbol{\beta}}(f^\gamma)\big), \qquad i = 1, \dots, N,$$

which must be an identity in the variables v^γ_l.

Remark 8. Consider an MT $u = f(v_l, \dots)$ from equation (31) to equation (24). Suppose that equation (24) possesses a Darboux-Lax representation (DLR) of the form (30) such that the equation $D_t(M) = \mathcal{S}(\mathcal{U})M - M\mathcal{U}$ is equivalent to (24).

Let $\hat{M} = \hat{M}(v_l, \dots, \lambda)$ and $\hat{\mathcal{U}} = \hat{\mathcal{U}}(v_l, \dots, \lambda)$ be the matrix-functions obtained from (30) by substituting $f(v_l, \dots)$ in place of u. So $\hat{M} = M\big(f(v_l, \dots), \lambda\big)$, and $\hat{\mathcal{U}}$ is obtained from $\mathcal{U}(u_a, \dots, u_b, \lambda)$ by substituting $\mathcal{S}^i\big(f(v_l, \dots)\big)$ in place of u_i for all $i \in \mathbb{Z}$.

Since $(M, \mathcal{U})$ is a DLR for (24) and $u = f(v_l, \dots)$ is an MT from (31) to (24), the obtained matrices $\hat{M}$, $\hat{\mathcal{U}}$ form a DLR for equation (31) in the sense that the equation $D_t(\hat{M}) = \mathcal{S}(\hat{\mathcal{U}})\hat{M} - \hat{M}\hat{\mathcal{U}}$ is equivalent to a consequence of (31).

Then for any invertible $d \times d$ matrix $\mathbf{g} = \mathbf{g}(v_l, \dots, \lambda)$ the matrices

$$\check{M} = \mathcal{S}(\mathbf{g}) \cdot \hat{M} \cdot \mathbf{g}^{-1}, \qquad \check{\mathcal{U}} = D_t(\mathbf{g}) \cdot \mathbf{g}^{-1} + \mathbf{g} \cdot \hat{\mathcal{U}} \cdot \mathbf{g}^{-1} \tag{34}$$

form a DLR for (31) as well. The DLR $(\check{M}, \check{\mathcal{U}})$ is *gauge equivalent* to the DLR $(\hat{M}, \hat{\mathcal{U}})$ with respect to the *gauge transformation* $\mathbf{g}$.

Very often, one can find a matrix $\mathbf{g} = \mathbf{g}(v_l, \dots, \lambda)$ such that the equation $D_t(\check{M}) = \mathcal{S}(\check{\mathcal{U}})\check{M} - \check{M}\check{\mathcal{U}}$ with $\check{M}, \check{\mathcal{U}}$ given by (34) is equivalent to (31). (So $D_t(\check{M}) = \mathcal{S}(\check{\mathcal{U}})\check{M} - \check{M}\check{\mathcal{U}}$ is equivalent to equation (31) itself, while $D_t(\hat{M}) = \mathcal{S}(\hat{\mathcal{U}})\hat{M} - \hat{M}\hat{\mathcal{U}}$ is equivalent to a consequence of (31).)

For instance, consider the MT $u = vv_1$ from the modified Volterra equation $v_t = v^2(v_1 - v_{-1})$ to the Volterra equation $u_t = u(u_1 - u_{-1})$. As we discuss in Example 2 below, the matrices (40) form a DLR for the Volterra equation. Substituting vv_1 in place of u in (40), one obtains the matrices

$$\hat{M} = \begin{pmatrix} 0 & vv_1 \\ -1 & \lambda \end{pmatrix}, \qquad \hat{\mathcal{U}} = \begin{pmatrix} vv_1 & \lambda v_{-1}v \\ -\lambda & \lambda^2 + v_{-1}v \end{pmatrix}. \tag{35}$$

The equation $D_t(\hat{M}) = \mathcal{S}(\hat{\mathcal{U}})\hat{M} - \hat{M}\hat{\mathcal{U}}$ is equivalent to $(vv_1)_t = vv_2(v_1)^2 - v_1v_{-1}v^2$, which is a consequence of $v_t = v^2(v_1 - v_{-1})$.

Consider the gauge transformation $\mathbf{g} = \begin{pmatrix} 1/v & 0 \\ 0 & 1 \end{pmatrix}$. Computing (34) for (35), we get

$$\check{M} = \begin{pmatrix} 0 & v \\ -v & \lambda \end{pmatrix}, \qquad \check{\mathcal{U}} = \begin{pmatrix} vv_{-1} & \lambda v_{-1} \\ -\lambda v & \lambda^2 + vv_{-1} \end{pmatrix}. \tag{36}$$

The equation $D_t(\check{M}) = \mathcal{S}(\check{\mathcal{U}})\check{M} - \check{M}\check{\mathcal{U}}$ is equivalent to $v_t = v^2(v_1 - v_{-1})$. The DLR (36) is well known.

2.2. **Some algebraic and geometric structures associated with Darboux-Lax representations.** Fix $d \in \mathbb{Z}_{>0}$. Consider a DLR of the form (30), where $M, \mathcal{U}$ are $d \times d$ matrix-functions satisfying (28), and M is invertible.

Let $\mathbf{G}$ be the group of $\mathrm{GL}_d(\mathbb{C})$-valued functions of λ. Since $M(u, \lambda)$ in (30) is supposed to be invertible, for every fixed value of u we have $M(u, \lambda) \in \mathbf{G}$.

Remark 9. The group $\mathbf{G}$ consists of $\mathrm{GL}_d(\mathbb{C})$-valued functions of λ defined on some connected open subset of $\mathbb{C}$. Essentially, the only requirement is that $M(u, \lambda) \in \mathbf{G}$ for every fixed value of u, where u runs through some connected open subset of $\mathbb{C}^N$. To simplify notation, we do not mention these open subsets explicitly.

As has been said in Section 1.2, each considered function is supposed to be analytic on its domain of definition.

We define a sequence of subgroups

$$\mathbb{H}_0 \subset \mathbb{H}_1 \subset \mathbb{H}_2 \subset \dots \subset \mathbf{G},$$

associated with the DLR (30) as follows.

We set $\mathbb{H}_0 = \mathbf{1}$, where $\mathbf{1} \in \mathbf{G}$ is the identity element. The subgroup $\mathbb{H}_1 \subset \mathbf{G}$ is generated by the elements

$$M(\tilde{u}, \lambda) \cdot M(u, \lambda)^{-1} \in \mathbf{G} \tag{37}$$

for all values of u, $\tilde{u}$.

Now we define $\mathbb{H}_k \subset \mathbf{G}$ by induction on $k \in \mathbb{Z}_{>0}$ as follows. For each $k \in \mathbb{Z}_{>0}$, the subgroup $\mathbb{H}_{k+1} \subset \mathbf{G}$ is generated by the elements $h_k \in \mathbb{H}_k$ and the elements

$$M(u, \lambda) \cdot h_k \cdot M(u, \lambda)^{-1} \in \mathbf{G} \tag{38}$$

for all $h_k \in \mathbb{H}_k$ and all values of u.

Example 2. For the Volterra equation

$$u_t = u(u_1 - u_{-1}), \tag{39}$$

one has $N = 1$, and we can take

$$M(u, \lambda) = \begin{pmatrix} 0 & u \\ -1 & \lambda \end{pmatrix}, \qquad \mathcal{U}(u_{-1}, u, \lambda) = \begin{pmatrix} u & \lambda u_{-1} \\ -\lambda & \lambda^2 + u_{-1} \end{pmatrix}. \tag{40}$$

Note that $M(u, \lambda) = \begin{pmatrix} 0 & u \\ -1 & \lambda \end{pmatrix}$ is not invertible for $u = 0$. In agreement with Remark 4, we assume $u \neq 0$.

In this example we have $d = 2$, so $\mathbf{G}$ consists of $\mathrm{GL}_2(\mathbb{C})$-valued functions of λ. The subgroup $\mathbb{H}_1 \subset \mathbf{G}$ is generated by the elements

$$M(\tilde{u}, \lambda) \cdot M(u, \lambda)^{-1} = \begin{pmatrix} 0 & \tilde{u} \\ -1 & \lambda \end{pmatrix} \cdot \begin{pmatrix} \frac{\lambda}{u} & -1 \\ \frac{1}{u} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\tilde{u}}{u} & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $\mathbb{H}_1$ consists of the constant matrix-functions $\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}$, where $a_1 \in \mathbb{C}$ is an arbitrary nonzero constant.

The subgroup $\mathbb{H}_2 \subset \mathbf{G}$ is generated by the elements $h_1 = \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{H}_1$ and the elements

$$M(u,\lambda) \cdot h_1 \cdot M(u,\lambda)^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{(1-a_1)\lambda}{u} & a_1 \end{pmatrix}$$

for all nonzero $a_1, u \in \mathbb{C}$. Therefore, $\mathbb{H}_2$ consists of the matrix-functions $\begin{pmatrix} a_1 & 0 \\ a_3\lambda & a_2 \end{pmatrix}$ for all $a_1, a_2, a_3 \in \mathbb{C}$, $a_1 \neq 0$, $a_2 \neq 0$.

Consider the algebra $\mathbb{C}[\lambda]$ of polynomials in λ. For every $c \in \mathbb{C}$ and $k \in \mathbb{Z}_{>0}$, we denote by $((\lambda-c)^k) \subset \mathbb{C}[\lambda]$ the ideal generated by the polynomial $(\lambda - c)^k \in \mathbb{C}[\lambda]$. Clearly, the quotient algebra $\mathbb{C}[\lambda]/((\lambda - c)^k)$ is k-dimensional.

Denote by $\mathcal{G}(d, c, k)$ the group of invertible $d \times d$ matrices with entries from $\mathbb{C}[\lambda]/((\lambda-c)^k)$. Equivalently, an element of $\mathcal{G}(d, c, k)$ can be described as a sum of the form

$$\sum_{q=0}^{k-1} (\lambda - c)^q W^q, \qquad W^0 \in \mathrm{GL}_d(\mathbb{C}), \qquad W^1, \ldots, W^{k-1} \in \mathrm{Mat}_d(\mathbb{C}). \tag{41}$$

This shows that $\mathcal{G}(d, c, k)$ is a kd^2-dimensional Lie group. When we multiply two elements of the form (41) in the group $\mathcal{G}(d, c, k)$, we use the fact that in the algebra $\mathbb{C}[\lambda]/((\lambda - c)^k)$ one has $(\lambda - c)^k = 0$.

Fix

$$m \in \mathbb{Z}_{>0}, \qquad c_1, \ldots, c_m \in \mathbb{C}, \qquad k_1, \ldots, k_m \in \mathbb{Z}_{>0}.$$

Remark 10. According to Remark 9, the group $\mathbf{G}$ consists of $\mathrm{GL}_d(\mathbb{C})$-valued functions of λ defined on some connected open subset of $\mathbb{C}$. We assume that $c_1, \ldots, c_m$ belong to this open subset.

Consider the following Lie group

$$G = \mathcal{G}(d, c_1, k_1) \times \cdots \times \mathcal{G}(d, c_m, k_m). \tag{42}$$

An element $g \in G$ is an m-tuple $g = (g^1, \ldots, g^m)$, where $g^p \in \mathcal{G}(d, c_p, k_p)$, $p = 1, \ldots, m$, can be described as a sum of the form

$$g^p = \sum_{q=0}^{k_p-1} (\lambda - c_p)^q W^{p,q}, \qquad W^{p,0} \in \mathrm{GL}_d(\mathbb{C}), \qquad W^{p,1}, \ldots, W^{p,k_p-1} \in \mathrm{Mat}_d(\mathbb{C}).$$

For $i, j = 1, \ldots, d$, we denote by $w_{ij}^{pq} = (W^{p,q})_{ij}$ the entries of the matrix $W^{p,q}$. Then w_{ij}^{pq} can be regarded as coordinates on the Lie group G.

Consider functions of the form $f(w_{ij}^{pq}, u_l, \ldots)$ which may depend on the coordinates w_{ij}^{pq} and a finite number of the variables $u_l = (u_l^1, \ldots, u_l^N)$, $l \in \mathbb{Z}$. Using the DLR (30), we extend the operators $\mathcal{S}$, D_t to such functions as follows.

To define $\mathcal{S}(w_{ij}^{pq})$ and $D_t(w_{ij}^{pq})$, we use the formulas

$$\sum_{q=0}^{k_p-1} (\lambda - c_p)^q \mathcal{S}\big(W^{p,q}\big) = M(u,\lambda)\bigg(\sum_{q=0}^{k_p-1} (\lambda - c_p)^q W^{p,q}\bigg), \qquad (\lambda - c_p)^{k_p} = 0, \tag{43}$$

$$\sum_{q=0}^{k_p-1} (\lambda - c_p)^q D_t\big(W^{p,q}\big) = \mathcal{U}(u_a, \ldots, u_b, \lambda)\bigg(\sum_{q=0}^{k_p-1} (\lambda - c_p)^q W^{p,q}\bigg), \qquad (\lambda - c_p)^{k_p} = 0. \tag{44}$$

To compute the right-hand sides of (43) and (44), we take the corresponding Taylor series with respect to λ and truncate these series at the term $(\lambda - c_p)^{k_p-1}$. (Recall that in the algebra $\mathbb{C}[\lambda]/((\lambda - c_p)^{k_p})$ one has $(\lambda - c_p)^{k_p} = 0$, which we use here.)

Since $\mathcal{S}(w_{ij}^{pq})$ and $D_t(w_{ij}^{pq})$ appear on the left-hand sides of (43) and (44) respectively, formulas (43), (44) define $\mathcal{S}(w_{ij}^{pq})$ and $D_t(w_{ij}^{pq})$. Now for a function $f(w_{ij}^{pq}, u_l, \ldots)$ we set

$$\mathcal{S}\big(f(w_{ij}^{pq}, u_l, \ldots)\big) = f(\mathcal{S}(w_{ij}^{pq}), \mathcal{S}(u_l), \ldots),$$

$$D_t\big(f(w_{ij}^{pq}, u_l, \ldots)\big) = \sum_{p,q,i,j} \frac{\partial f}{\partial w_{ij}^{pq}} D_t(w_{ij}^{pq}) + \sum_{l,\gamma} \frac{\partial f}{\partial u_l^\gamma} D_t(u_l^\gamma),$$

where we use the fact that $\mathcal{S}(w_{ij}^{pq})$, $\mathcal{S}(u_l)$, $D_t(w_{ij}^{pq})$, $D_t(u_l^\gamma)$ have already been defined.

The presented definition of the operator $\mathcal{S}$ implies that the inverse $\mathcal{S}^{-1}$ is given by

$$\sum_{q=0}^{k_p-1}(\lambda-c_p)^q\mathcal{S}^{-1}\big(W^{p,q}\big)=M(u_{-1},\lambda)^{-1}\bigg(\sum_{q=0}^{k_p-1}(\lambda-c_p)^qW^{p,q}\bigg),\qquad(\lambda-c_p)^{k_p}=0,\tag{45}$$

$$\mathcal{S}^{-1}\big(f(w^{pq}_{ij},u_l,\dots)\big)=f(\mathcal{S}^{-1}(w^{pq}_{ij}),\mathcal{S}^{-1}(u_l),\dots),$$

where $\mathcal{S}^{-1}(u_l)=u_{l-1}$ and $\mathcal{S}^{-1}(w^{pq}_{ij})$ is determined by (45).

Lemma 1. *One has $\mathcal{S}\circ D_t=D_t\circ\mathcal{S}$.*

Proof. It is sufficient to prove $\mathcal{S}^{-1}\circ D_t\circ\mathcal{S}=D_t$. Note that D_t is a derivation of the algebra of functions of the form $f=f(w^{pq}_{ij},u_l,\dots)$, and $\mathcal{S}$ is an automorphism of this algebra. Hence $\mathcal{S}^{-1}\circ D_t\circ\mathcal{S}$ is also a derivation of this algebra.

Since D_t and $\mathcal{S}^{-1}\circ D_t\circ\mathcal{S}$ are derivations, to prove that $\mathcal{S}^{-1}\circ D_t\circ\mathcal{S}=D_t$, it is sufficient to show

$$(\mathcal{S}^{-1}\circ D_t\circ\mathcal{S})(u_l)=D_t(u_l)\qquad\forall\, l,\tag{46}$$

$$(\mathcal{S}^{-1}\circ D_t\circ\mathcal{S})(w^{pq}_{ij})=D_t(w^{pq}_{ij})\qquad\forall\, p,q,i,j.\tag{47}$$

Equation (46) follows immediately from the definition of $\mathcal{S}$ and D_t. Equation (47) is equivalent to

$$(D_t\circ\mathcal{S})(w^{pq}_{ij})=(\mathcal{S}\circ D_t)(w^{pq}_{ij})\qquad\forall\, p,q,i,j.\tag{48}$$

Applying D_t to (43) and $\mathcal{S}$ to (44), and using (28), we obtain (48). ☐

For any $h\in G$, we define the operator $\mathcal{R}_h$ on functions of the form $f(w^{pq}_{ij},u_l,\dots)$ as follows. We set $\mathcal{R}_h(u_l)=u_l$ for all $l\in\mathbb{Z}$. For a function y on an open subset $\mathbb{U}\subset G$, the function $\mathcal{R}_h(y)$ is defined on the open subset $\mathbb{U}h^{-1}\subset G$ by the formula

$$\mathcal{R}_h(y)(g)=y(gh),\qquad g\in\mathbb{U}h^{-1}.$$

Since w^{pq}_{ij} are functions on G, we see that $\mathcal{R}_h(w^{pq}_{ij})$ is well defined. Then for any function $f=f(w^{pq}_{ij},u_l,\dots)$ we set $\mathcal{R}_h(f)=f(\mathcal{R}_h(w^{pq}_{ij}),u_l,\dots)$.

Note that the operator $\mathcal{R}_h$ is invertible, and one has $\mathcal{R}_h^{-1}=\mathcal{R}_{h^{-1}}$.

Lemma 2. *We have $\mathcal{S}\circ\mathcal{R}_h=\mathcal{R}_h\circ\mathcal{S}$ and $\mathcal{R}_h\circ D_t=D_t\circ\mathcal{R}_h$ for all $h\in G$.*

Proof. To prove $\mathcal{S}\circ\mathcal{R}_h=\mathcal{R}_h\circ\mathcal{S}$, it is sufficient to show that $(\mathcal{S}\circ\mathcal{R}_h)(w^{pq}_{ij})=(\mathcal{R}_h\circ\mathcal{S})(w^{pq}_{ij})$, which follows easily from (43) and the definition of $\mathcal{R}_h$. The main idea is the following. According to (43), the action of $\mathcal{S}$ is given by the left multiplication by $M(u,\lambda)$. The action of $\mathcal{R}_h$ is given by the right multiplication by h. Since the left and right multiplications commute, we have $\mathcal{S}\circ\mathcal{R}_h=\mathcal{R}_h\circ\mathcal{S}$.

Let us prove $\mathcal{R}_h\circ D_t=D_t\circ\mathcal{R}_h$, which is equivalent to $\mathcal{R}_h^{-1}\circ D_t\circ\mathcal{R}_h=D_t$. Note that D_t is a derivation of the algebra of functions of the form $f=f(w^{pq}_{ij},u_l,\dots)$, and $\mathcal{R}_h$ is an automorphism of this algebra. Therefore, $\mathcal{R}_h^{-1}\circ D_t\circ\mathcal{R}_h$ is also a derivation of this algebra.

Since D_t and $\mathcal{R}_h^{-1}\circ D_t\circ\mathcal{R}_h$ are derivations, to prove that $\mathcal{R}_h^{-1}\circ D_t\circ\mathcal{R}_h=D_t$, it is sufficient to show

$$(\mathcal{R}_h^{-1}\circ D_t\circ\mathcal{R}_h)(u_l)=D_t(u_l)\qquad\forall\, l,\tag{49}$$

$$(\mathcal{R}_h^{-1}\circ D_t\circ\mathcal{R}_h)(w^{pq}_{ij})=D_t(w^{pq}_{ij})\qquad\forall\, p,q,i,j.\tag{50}$$

Equation (49) is obvious, because $\mathcal{R}_h(u_l)=u_l$ for all l. Equation (50) is equivalent to

$$(D_t\circ\mathcal{R}_h)(w^{pq}_{ij})=(\mathcal{R}_h\circ D_t)(w^{pq}_{ij})\qquad\forall\, p,q,i,j,$$

which follows easily from (44) and the definition of $\mathcal{R}_h$. ☐

Example 3. Consider the case $d=2$, $m=1$, $k_1=2$ and the DLR (40) of the Volterra equation (39). Then $G=\mathcal{G}(2,c_1,2)$, and formulas (43), (44) become

$$\mathcal{S}\begin{pmatrix}w^{10}_{11}&w^{10}_{12}\\w^{10}_{21}&w^{10}_{22}\end{pmatrix}+(\lambda-c_1)\mathcal{S}\begin{pmatrix}w^{11}_{11}&w^{11}_{12}\\w^{11}_{21}&w^{11}_{22}\end{pmatrix}=$$
$$=\begin{pmatrix}0&u\\-1&\lambda\end{pmatrix}\left(\begin{pmatrix}w^{10}_{11}&w^{10}_{12}\\w^{10}_{21}&w^{10}_{22}\end{pmatrix}+(\lambda-c_1)\begin{pmatrix}w^{11}_{11}&w^{11}_{12}\\w^{11}_{21}&w^{11}_{22}\end{pmatrix}\right),\qquad(\lambda-c_1)^2=0,\tag{51}$$

$$D_t\begin{pmatrix}w^{10}_{11}&w^{10}_{12}\\w^{10}_{21}&w^{10}_{22}\end{pmatrix}+(\lambda-c_1)D_t\begin{pmatrix}w^{11}_{11}&w^{11}_{12}\\w^{11}_{21}&w^{11}_{22}\end{pmatrix}=$$
$$=\begin{pmatrix}u&\lambda u_{-1}\\-\lambda&\lambda^2+u_{-1}\end{pmatrix}\left(\begin{pmatrix}w^{10}_{11}&w^{10}_{12}\\w^{10}_{21}&w^{10}_{22}\end{pmatrix}+(\lambda-c_1)\begin{pmatrix}w^{11}_{11}&w^{11}_{12}\\w^{11}_{21}&w^{11}_{22}\end{pmatrix}\right),\qquad(\lambda-c_1)^2=0.\tag{52}$$

Let $\mathcal{H} \subset G$ be a subgroup of G and $\mathbb{U} \subset G$ be an open subset. A function f on $\mathbb{U}$ is called *$\mathcal{H}$-right-invariant* if for any $g_1 \in \mathcal{H}$ and $g_2 \in \mathbb{U}$ such that $g_2 g_1 \in \mathbb{U}$ we have $f(g_2 g_1) = f(g_2)$.

Similarly, f is called *$\mathcal{H}$-left-invariant* if for any $g_1 \in \mathcal{H}$ and $g_2 \in \mathbb{U}$ such that $g_1 g_2 \in \mathbb{U}$ we have $f(g_1 g_2) = f(g_2)$.

Let $H \subset G$ be a closed connected Lie subgroup of G. Consider the manifold G/H and the natural projection

$$\pi\colon G \to G/H, \qquad \pi(g) = gH \in G/H, \qquad g \in G.$$

Here we use the fact that the points of G/H correspond to the left cosets gH of H in G.

Remark 11. Let $Y \subset G/H$ be an open subset. For each function f on Y, one defines the function $\pi^*(f)$ on the open subset $\pi^{-1}(Y) \subset G$ as follows

$$\pi^*(f)(g) = f(\pi(g)) \qquad \forall\, g \in \pi^{-1}(Y).$$

A function y on $\pi^{-1}(Y)$ is of the form $y = \pi^*(f)$ for some function f on Y iff y is H-right-invariant. Therefore, functions on $Y \subset G/H$ can be regarded as H-right-invariant functions on $\pi^{-1}(Y) \subset G$.

Note that a function y on $\pi^{-1}(Y)$ is H-right-invariant iff $\mathcal{R}_h(y) = y$ for all $h \in H$.

Let $H \subset G$ be a closed connected Lie subgroup of codimension $r > 0$. Then $\dim G/H = r$. Let $y^1, \dots, y^r$ be functions defined on an open dense subset of the manifold G/H. We say that $y^1, \dots, y^r$ *form a system of local coordinates almost everywhere on* G/H if there is an open dense subset $\mathcal{Y} \subset G/H$ such that $y^1, \dots, y^r$ form a system of local coordinates on a neighborhood of each point of $\mathcal{Y}$.

This is equivalent to the fact that the differentials of the functions $y^1, \dots, y^r$ are linearly independent almost everywhere on G/H.

Remark 12. In this remark, by a function on G/H we mean a function defined on an open subset of G/H, and similarly for functions on G. Suppose that $y^1, \dots, y^r$ form a system of local coordinates almost everywhere on G/H. Then any H-right-invariant function φ on G can locally be expressed as $\varphi = \psi(y^1, \dots, y^r)$ for some function ψ. (Such an expression for φ exists locally almost everywhere, i.e., on neighborhoods of the points where $y^1, \dots, y^r$ form a system of local coordinates.)

Let y be a function on G/H. According to Remark 11, we can regard y as an H-right-invariant function on G, so $y = y(w^{pq}_{ij})$ depends on the coordinates w^{pq}_{ij} of G. Then, according to the definition of the operators $\mathcal{S}$ and D_t,

$$\mathcal{S}(y) \text{ may depend on } w^{pq}_{ij} \text{ and } u, \tag{53}$$

$$D_t(y) \text{ may depend on } w^{pq}_{ij} \text{ and } u_a, \dots, u_b. \tag{54}$$

Since y is H-right-invariant and $\mathcal{S} \circ \mathcal{R}_h = \mathcal{R}_h \circ \mathcal{S}$, $\mathcal{R}_h \circ D_t = D_t \circ \mathcal{R}_h$ for all $h \in H$, the following property holds.

If we fix the values of the variables u_l for all l (including the variable $u = u_0$), then $\mathcal{S}(y)$, $D_t(y)$ become H-right-invariant functions on G and can locally be expressed as functions of $y^1, \dots, y^r$. Combining this observation with (53) and (54), we see that locally $\mathcal{S}(y)$, $D_t(y)$ can be written in the form

$$\mathcal{S}(y) = \mu(y^1, \dots, y^r, u), \qquad D_t(y) = \xi(y^1, \dots, y^r, u_a, \dots, u_b)$$

for some functions μ, ξ.

Recall that $\mathbf{G}$ is the group of $\mathrm{GL}_d(\mathbb{C})$-valued functions of λ, and the group G is defined by (42). Consider the natural homomorphism $\rho\colon \mathbf{G} \to G$ given by

$$\rho(\mathbf{g}) = (g^1, \dots, g^m) \in \mathcal{G}(d, c_1, k_1) \times \dots \times \mathcal{G}(d, c_m, k_m) = G, \qquad \mathbf{g} \in \mathbf{G}. \tag{55}$$

Here $\mathbf{g} = \mathbf{g}(\lambda)$ is a $\mathrm{GL}_d(\mathbb{C})$-valued function of λ, and for each $p = 1, \dots, m$ the element

$$g^p = \sum_{q=0}^{k_p-1} (\lambda - c_p)^q g^{p,q} \in \mathcal{G}(d, c_p, k_p), \qquad g^{p,0} \in \mathrm{GL}_d(\mathbb{C}), \qquad g^{p,1}, \dots, g^{p,k_p-1} \in \mathrm{Mat}_d(\mathbb{C}), \tag{56}$$

is determined by the Taylor expansion of $\mathbf{g}(\lambda)$ at the point $\lambda = c_p$.

Let $H \subset G$ be a closed connected Lie subgroup. For any subgroup $\mathcal{H} \subset G$, we have defined the notion of $\mathcal{H}$-left-invariant functions on open subsets of G. Since G acts by left multiplication on the manifold G/H, one can define similarly the notion of $\mathcal{H}$-left-invariant functions on open subsets of G/H.

Recall that $\mathbb{H}_k \subset \mathbf{G}$ for each $k \in \mathbb{Z}_{\geq 0}$. Using the embedding $\rho(\mathbb{H}_k) \subset G$, we can speak about $\mathbb{H}_k$-left-invariant functions on open subsets of G and G/H. So, by definition, a function is *$\mathbb{H}_k$-left-invariant* if it is $\rho(\mathbb{H}_k)$-left-invariant.

Lemma 3. *Let $y = y(w^{pq}_{ij})$ be a function on an open dense subset of G. According to* (43)*, the function $\mathcal{S}(y) = y(\mathcal{S}(w^{pq}_{ij}))$ may depend on w^{pq}_{ij}, u. If y is $\mathbb{H}_k$-left-invariant for some $k > 0$, then $\mathcal{S}(y)$ does not depend on u and is $\mathbb{H}_{k-1}$-left-invariant.*

Proof. In what follows, we use the symbols g and $\hat{g}$ as arguments of some functions defined on open dense subsets of G. When we write $g \in G$ (or $\hat{g} \in G$), we mean that g (or $\hat{g}$) belongs to an open subset where the considered function is defined.

For example, since y is defined on an open dense subset of G, for any fixed values of $\tilde{u}$, u there is an open dense subset $\mathbb{U} \subset G$ such that $y(\hat{g})$ and $y\big(\rho(M(\tilde{u},\lambda)\cdot M(u,\lambda)^{-1})\hat{g}\big)$ are defined for all $\hat{g} \in \mathbb{U}$. In equation (58) below we assume that $\hat{g}$ belongs to such an open dense subset, so that the left-hand side and right-hand side of (58) are well defined. Similar considerations apply also to other equations in this proof.

Since $\mathcal{S}(y) = y(\mathcal{S}(w^{pq}_{ij}))$ may depend on w^{pq}_{ij} and u, for each fixed value of u we can regard $\mathcal{S}(y)$ as a function on an open dense subset of G. Formula (43) implies that this interpretation of $\mathcal{S}(y)$ can be written as

$$\mathcal{S}(y)(g) = y\big(\rho(M(u,\lambda))g\big), \qquad g \in G. \tag{57}$$

Recall that $\mathbb{H}_1$ is generated by the elements (37), and we have $\mathbb{H}_1 \subset \mathbb{H}_k$ for $k > 0$. Therefore, the elements (37) belong to $\mathbb{H}_k$. Since y is defined on an open dense subset of G and is $\mathbb{H}_k$-left-invariant, for any fixed values of $\tilde{u}$, u we have

$$y\big(\rho(M(\tilde{u},\lambda)\cdot M(u,\lambda)^{-1})\hat{g}\big) = y(\hat{g}), \qquad \hat{g} \in G. \tag{58}$$

Taking $g = \rho(M(u,\lambda)^{-1})\hat{g}$, from (58) we obtain

$$y\big(\rho(M(\tilde{u},\lambda))g\big) = y\big(\rho(M(u,\lambda))g\big), \qquad g \in G. \tag{59}$$

Combining (59) with (57), we see that $\mathcal{S}(y)$ does not depend on u.

Let us show that $\mathcal{S}(y)$ is $\mathbb{H}_{k-1}$-left-invariant. Recall that for any $h \in \mathbb{H}_{k-1}$ and any fixed value of u one has $M(u,\lambda)\cdot h\cdot M(u,\lambda)^{-1} \in \mathbb{H}_k$. Since y is defined on an open dense subset of G and is $\mathbb{H}_k$-left-invariant, we have

$$y\big(\rho(M(u,\lambda)\cdot h\cdot M(u,\lambda)^{-1})\hat{g}\big) = y(\hat{g}), \qquad \hat{g} \in G. \tag{60}$$

Taking $g = \rho(M(u,\lambda)^{-1})\hat{g}$, from (60) we get

$$y\big(\rho(M(u,\lambda)\cdot h)g\big) = y\big(\rho(M(u,\lambda))g\big), \qquad g \in G. \tag{61}$$

Combining (61) with (57), one obtains

$$\mathcal{S}(y)\big(\rho(h)g\big) = \mathcal{S}(y)(g), \qquad g \in G, \qquad h \in \mathbb{H}_{k-1},$$

which says that $\mathcal{S}(y)$ is $\mathbb{H}_{k-1}$-left-invariant. □

Lemma 4. *Let $H \subset G$ be a closed connected Lie subgroup of G. Let z be a function on an open dense subset of G/H. According to Remark 11, the function z can also be regarded as an H-right-invariant function on an open dense subset of G. Therefore, we can consider the functions $\mathcal{S}^k(z)$, $k \in \mathbb{Z}_{>0}$, which may depend on w^{pq}_{ij} and u_l.*

Suppose that z is $\mathbb{H}_\alpha$-left-invariant for some $\alpha \in \mathbb{Z}_{>0}$. Then the functions

$$\mathcal{S}^k(z), \qquad k = 1,\dots,\alpha, \tag{62}$$

do not depend on u_l for any $l \in \mathbb{Z}$. Furthermore, the functions (62) *are H-right-invariant and can be viewed as functions on an open dense subset of G/H.*

Proof. Applying Lemma 3 to $y = z$, we see that $\mathcal{S}(z)$ does not depend on u_l for any $l \in \mathbb{Z}$ and is $\mathbb{H}_{\alpha-1}$-left-invariant. If $\alpha \geq 2$, then, applying Lemma 3 to $y = \mathcal{S}(z)$, one obtains that the function $\mathcal{S}^2(z) = \mathcal{S}(\mathcal{S}(z))$ does not depend on u_l for any $l \in \mathbb{Z}$ and is $\mathbb{H}_{\alpha-2}$-left-invariant.

Similarly, using Lemma 3, by induction on $k = 1,\dots,\alpha$ we prove that the function $\mathcal{S}^k(z)$ does not depend on u_l for any $l \in \mathbb{Z}$ and is $\mathbb{H}_{\alpha-k}$-left-invariant.

Since z is H-right-invariant and is defined on an open dense subset of G/H, we have also the same property for the functions (62), because $\mathcal{S}\circ\mathcal{R}_h = \mathcal{R}_h\circ\mathcal{S}$ for all $h \in H$. □

2.3. **The method to construct MTs.** Recall that the Lie group G is given by (42). We are going to describe how to construct MTs from some functions on open subsets of G/H, where $H \subset G$ is a closed connected Lie subgroup.

Recall that $u_l = (u_l^1, \dots, u_l^N)$ is an N-component vector for each $l \in \mathbb{Z}$. To clarify the main idea, we consider first the case $N = 1$. (The case of arbitrary $N \in \mathbb{Z}_{>0}$ will be described in Theorem 2.)

Note that Theorems 1, 2 do not mention the DLR (30) explicitly, but the matrices $M, \mathcal{U}$ from (30) appear in the definition (43), (44) of the operators $\mathcal{S}$, D_t, which are used in Theorems 1, 2.

In the case $N = 1$ the vector u_l has only one component u_l^1, which we denote by the same symbol u_l. According to our notation, $u = u_0$.

Theorem 1. *Suppose that $N = 1$. Recall that the Lie group G is defined by (42). Let $H \subset G$ be a closed connected Lie subgroup of codimension $r > 0$. Let z be an $\mathbb{H}_{r-1}$-left-invariant function on an open dense subset of the manifold G/H. (According to Remark 11, the function z can also be regarded as an H-right-invariant and $\mathbb{H}_{r-1}$-left-invariant function on an open dense subset of G.)*

By Lemma 4, the functions $\mathcal{S}^k(z)$, $k = 1, \dots, r-1$, do not depend on u_l, and we can view $\mathcal{S}^k(z)$, $k = 1, \dots, r-1$, as functions on an open dense subset of G/H. Suppose that

$$\begin{gathered}\text{the functions } z,\ \mathcal{S}(z),\ \mathcal{S}^2(z),\ \dots,\ \mathcal{S}^{r-1}(z) \text{ form}\\ \text{a system of local coordinates almost everywhere on } G/H\\ (\text{i.e., the differentials of the functions are linearly independent almost everywhere}),\end{gathered} \tag{63}$$

$$\text{the function } \frac{\partial}{\partial u}\big(\mathcal{S}^r(z)\big) \text{ is not identically zero.} \tag{64}$$

Set $z_0 = z$ and $z_k = \mathcal{S}^k(z)$ for $k = 1, \dots, r$. Condition (63) says that $z_0, z_1, \dots, z_{r-1}$ form a system of local coordinates almost everywhere on G/H. Applying Remark 12 to this system of local coordinates, we see that locally one has

$$\begin{gathered} z_r = \mathcal{S}^r(z) = \mathcal{S}(z_{r-1}) = F(z_0, z_1, \dots, z_{r-1}, u),\\ D_t(z) = Q(z_0, \dots, z_{r-1}, u_a, \dots, u_b)\end{gathered} \tag{65}$$

for some functions F and Q. By the implicit function theorem, condition (64) implies that locally from equation (65) we can express u in terms of $z_0, z_1, \dots, z_r$

$$u = f(z_0, z_1, \dots, z_r). \tag{66}$$

(We can do this on a neighborhood of a point where $\frac{\partial}{\partial u}\big(\mathcal{S}^r(z)\big) \neq 0$.)

Then one obtains an MT for equation (24) as follows. We introduce new variables v_l for $l \in \mathbb{Z}$. Using the function $f(z_0, z_1, \dots, z_r)$ from (66), for each $j \in \mathbb{Z}$ we can consider the function $f(v_j, v_{j+1}, \dots, v_{j+r})$. Let $P(v_a, \dots, v_{b+r})$ be the function obtained from $Q(z_0, \dots, z_{r-1}, u_a, \dots, u_b)$ by replacing z_i with v_i and u_j with $f(v_j, v_{j+1}, \dots, v_{j+r})$. That is,

$$P(v_a, \dots, v_{b+r}) = Q\big(v_0, \dots, v_{r-1}, f(v_a, \dots, v_{a+r}), \dots, f(v_b, \dots, v_{b+r})\big). \tag{67}$$

We introduce the formula

$$v_t = P(v_a, \dots, v_{b+r}) \tag{68}$$

and regard (68) as a differential-difference equation for the variable v. Then the formula

$$u = f(v_0, v_1, \dots, v_r) \tag{69}$$

determines an MT from equation (68) to equation (24).

As usual, we can use the identification $v_0 = v$. Then (69) becomes $u = f(v, v_1, \dots, v_r)$.

Proof. As has been said above, we can regard z as a function on an open dense subset of G, so $z = z(w_{ij}^{pq})$ is a function of the coordinates w_{ij}^{pq} on G. Set $z_k = \mathcal{S}^k(z)$ for all $k \in \mathbb{Z}$. The functions z_k may depend on w_{ij}^{pq} and u_l.

Lemma 5. *The functions z_k, $k \in \mathbb{Z}$, are functionally independent.*

Proof. Suppose that z_k, $k \in \mathbb{Z}$, are not functionally independent. This means that there are $p, q \in \mathbb{Z}$, $p \leq q$, and a nontrivial relation of the form

$$R(z_p, z_{p+1}, \dots, z_q) = 0. \tag{70}$$

Applying $\mathcal{S}^{-p}$ to equation (70) and using the identities $\mathcal{S}^j(z_k) = z_{k+j}$ for $j \in \mathbb{Z}$, we obtain

$$R(z_0, z_1, \dots, z_{q-p}) = 0,$$

which implies that

$$z_0, \quad z_1, \quad \dots, \quad z_{q-p} \tag{71}$$

are not functionally independent. The functions $z_0, \dots, z_{r-1}$ form a system of local coordinates almost everywhere on G/H, hence $z_0, \dots, z_{r-1}$ are functionally independent. Therefore, $q-p \geq r$.

For any $j \in \mathbb{Z}$, applying $\mathcal{S}^j$ to equations (65), (66) and using the identities $\mathcal{S}^j(z_k) = z_{k+j}$, $\mathcal{S}^j(u) = u_j$, one gets

$$z_{r+j} = F(z_j, z_{j+1}, \dots, z_{j+r-1}, u_j), \tag{72}$$

$$u_j = f(z_j, z_{j+1}, \dots, z_{j+r}) \qquad \forall\, j \in \mathbb{Z}. \tag{73}$$

Equations (72), (73) imply that the functions (71) can be expressed in terms of

$$z_0, \quad z_1, \quad \dots, \quad z_{r-1}, \quad u, \quad u_1, \quad \dots, \quad u_{q-p-r}, \tag{74}$$

and the functions (74) can be expressed in terms of (71). Since (74) are functionally independent, we obtain that (71) are functionally independent as well. □

For any $j, i_1, \dots, i_l \in \mathbb{Z}$ and any function of the form $h = h(v_{i_1}, \dots, v_{i_l})$, we set

$$\mathcal{S}^j(h) = h(v_{i_1+j}, \dots, v_{i_l+j}).$$

According to Definition 1 of MTs, to prove the theorem, we need to show that

$$\sum_{i=0}^{r} \frac{\partial}{\partial v_i}\big(f(v_0, v_1, \dots, v_r)\big) \cdot \mathcal{S}^i\big(P(v_a, \dots, v_{b+r})\big) = \\ = \mathbf{F}\big(f(v_{\boldsymbol{\alpha}}, \dots, v_{\boldsymbol{\alpha}+r}), f(v_{\boldsymbol{\alpha}+1}, \dots, v_{\boldsymbol{\alpha}+1+r}), \dots, f(v_{\boldsymbol{\beta}}, \dots, v_{\boldsymbol{\beta}+r})\big). \tag{75}$$

Applying D_t to equation (66) and using the identity $D_t(u) = \mathbf{F}(u_{\boldsymbol{\alpha}}, u_{\boldsymbol{\alpha}+1}, \dots, u_{\boldsymbol{\beta}})$, one obtains

$$\mathbf{F}(u_{\boldsymbol{\alpha}}, u_{\boldsymbol{\alpha}+1}, \dots, u_{\boldsymbol{\beta}}) = \sum_{i=0}^{r} \frac{\partial}{\partial z_i}\big(f(z_0, z_1, \dots, z_r)\big) \cdot D_t(z_i). \tag{76}$$

Since $\mathcal{S} \circ D_t = D_t \circ \mathcal{S}$, we have

$$D_t(z_i) = D_t(\mathcal{S}^i(z)) = \mathcal{S}^i(D_t(z)) = \mathcal{S}^i(Q(z_0, \dots, z_{r-1}, u_a, \dots, u_b)). \tag{77}$$

Substituting (73) and (77) in (76), we obtain

$$\mathbf{F}\big(f(z_{\boldsymbol{\alpha}}, \dots, z_{\boldsymbol{\alpha}+r}), f(z_{\boldsymbol{\alpha}+1}, \dots, z_{\boldsymbol{\alpha}+1+r}), \dots, f(z_{\boldsymbol{\beta}}, \dots, z_{\boldsymbol{\beta}+r})\big) = \\ = \sum_{i=0}^{r} \frac{\partial}{\partial z_i}\big(f(z_0, z_1, \dots, z_r)\big) \cdot \mathcal{S}^i\Big(Q\big(z_0, \dots, z_{r-1}, f(z_a, \dots, z_{a+r}), \dots, f(z_b, \dots, z_{b+r})\big)\Big). \tag{78}$$

Since z_j, $j \in \mathbb{Z}$, are functionally independent, the identity (78) will remain valid if we replace z_j by v_j for all j. Replacing z_j by v_j in (78) and using (67), we get (75). □

Remark 13. Informally speaking, (63) and (64) can be regarded as non-degeneracy conditions. Condition (63) says that the differentials of the functions z, $\mathcal{S}(z)$, $\mathcal{S}^2(z), \dots, \mathcal{S}^{r-1}(z)$ are linearly independent almost everywhere on the r-dimensional manifold G/H. Condition (64) says that the function $\mathcal{S}^r(z) = F(z_0, z_1, \dots, z_{r-1}, u)$ depends nontrivially on u.

In constructing an MT by the method described in Theorem 1, the most important step is to find a nonconstant $\mathbb{H}_{r-1}$-left-invariant function z on an open dense subset of G/H. For such a function z, one can expect that conditions (63), (64) are usually satisfied. This is the case in all examples known to us.

Example 4. Consider the case $d = 2$, $m = 1$, $k_1 = 2$ and the DLR (40) of the Volterra equation (39). The group $G = \mathcal{G}(2, c_1, 2)$ can be described as follows

$$G = \mathcal{G}(2, c_1, 2) = \left\{ \begin{pmatrix} w^{10}_{11} & w^{10}_{12} \\ w^{10}_{21} & w^{10}_{22} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w^{11}_{11} & w^{11}_{12} \\ w^{11}_{21} & w^{11}_{22} \end{pmatrix} \;\middle|\; w^{10}_{11} w^{10}_{22} - w^{10}_{21} w^{10}_{12} \neq 0 \right\}, \tag{79}$$

where $c_1 \in \mathbb{C}$ is a fixed constant, and w^{10}_{11}, w^{10}_{12}, w^{10}_{21}, w^{10}_{22}, w^{11}_{11}, w^{11}_{12}, w^{11}_{21}, w^{11}_{22} are coordinates on G. Since $k_1 = 2$, we work modulo the relation $(\lambda - c_1)^2 = 0$.

Consider the subgroup

$$H = \left\{ \begin{pmatrix} 1 & w^{10}_{12} \\ 0 & w^{10}_{22} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w^{11}_{11} & w^{11}_{12} \\ w^{11}_{21} & w^{11}_{22} \end{pmatrix} \;\middle|\; w^{10}_{22} \neq 0 \right\} \subset G,$$

which is of codimension $r = 2$. As has been shown in Example 2, for the DLR (40), the group $\mathbb{H}_1$ consists of the constant matrix-functions $\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}$.

For any $h \in H$ and $h_1 \in \mathbb{H}_1$, in the product

$$h_1 \cdot \left(\begin{pmatrix} w_{11}^{10} & w_{12}^{10} \\ w_{21}^{10} & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w_{11}^{11} & w_{12}^{11} \\ w_{21}^{11} & w_{22}^{11} \end{pmatrix} \right) \cdot h$$

the w_{21}^{10} component does not change. Therefore, the function $z = w_{21}^{10}$ is H-right-invariant and $\mathbb{H}_1$-left-invariant.

Using formula (51) and the notation $z_k = \mathcal{S}^k(z)$ for $k \in \mathbb{Z}$, we obtain

$$z_0 = z = w_{21}^{10}, \qquad z_1 = \mathcal{S}(z) = \mathcal{S}(w_{21}^{10}) = c_1 w_{21}^{10} - w_{11}^{10}, \tag{80}$$

$$z_2 = \mathcal{S}^2(z) = \mathcal{S}(z_1) = c_1 \mathcal{S}(w_{21}^{10}) - \mathcal{S}(w_{11}^{10}) = (c_1^2 - u) w_{21}^{10} - c_1 w_{11}^{10}. \tag{81}$$

Recall that in this example we have $r = 2$. Since the differentials of the functions z and $\mathcal{S}(z)$ are linearly independent, condition (63) is valid. Equation (81) shows that condition (64) is valid as well.

Therefore, using Theorem 1, we obtain an MT as follows. Equations (80), (81) imply $z_2 = c_1 z_1 - u z_0$, which yields

$$u = \frac{c_1 z_1 - z_2}{z_0}. \tag{82}$$

Using (52), (80), (81), we get

$$D_t(z) = D_t(w_{21}^{10}) = c_1^2 w_{21}^{10} - c_1 w_{11}^{10} + u_{-1} w_{21}^{10} = c_1 z_1 + u_{-1} z_0. \tag{83}$$

Equation (82) implies $u_{-1} = \mathcal{S}^{-1}((c_1 z_1 - z_2)/z_0) = (c_1 z_0 - z_1)/z_{-1}$. Substituting this in (83), one obtains

$$D_t(z) = \frac{c_1 z_0^2 + c_1 z_{-1} z_1 - z_1 z_0}{z_{-1}}. \tag{84}$$

According to Theorem 1, to obtain an MT, we need to replace z_k by v_k for all $k \in \mathbb{Z}$ in (82), (84). (And we can use the identification $v_0 = v$.)

Thus we get the following result. For any $c_1 \in \mathbb{C}$, the formula

$$u = \frac{c_1 v_1 - v_2}{v} \tag{85}$$

determines an MT from the equation

$$v_t = \frac{c_1 v^2 + c_1 v_{-1} v_1 - v_1 v}{v_{-1}} \tag{86}$$

to the Volterra equation (39). As we show in Remark 14, this MT can be found in [20], after some change of variables.

Note that the same MT can be obtained also in the case $d = 2$, $m = 1$, $k_1 = 1$. In this case, the group (42) is equal to

$$\mathcal{G}(2, c_1, 1) = \left\{ \begin{pmatrix} w_{11}^{10} & w_{12}^{10} \\ w_{21}^{10} & w_{22}^{10} \end{pmatrix} \;\middle|\; w_{11}^{10} w_{22}^{10} - w_{21}^{10} w_{12}^{10} \neq 0 \right\}.$$

Consider the subgroup $\tilde{H} = \left\{ \begin{pmatrix} 1 & w_{12}^{10} \\ 0 & w_{22}^{10} \end{pmatrix} \;\middle|\; w_{22}^{10} \neq 0 \right\} \subset \mathcal{G}(2, c_1, 1)$. The function $z = w_{21}^{10}$ is $\tilde{H}$-right-invariant and $\mathbb{H}_1$-left-invariant, and it gives the same MT.

Remark 14. After the change of variables $v = \exp \tilde{v}$, $t = -\tilde{t}$, equation (86) becomes

$$\partial_{\tilde{t}}(\tilde{v}) = \big(\exp(\tilde{v}_1 - \tilde{v}) - c_1\big)\big(\exp(\tilde{v} - \tilde{v}_{-1}) - c_1\big) - c_1^2. \tag{87}$$

Equation (87) is a particular case of equation (V6) from the list of Volterra-type equations in [20]. The paper [20] presents also an MT reducing this equation to the Volterra equation. The MT (85) coincides with the MT from [20], up to the above change of variables.

Example 5. Let $d = 2$, $m = 1$, $k_1 = 2$. Consider the group $G = \mathcal{G}(2, c_1, 2)$ given by (79), the subgroup

$$H = \left\{ \begin{pmatrix} w_{11}^{10} & w_{12}^{10} \\ 0 & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w_{11}^{11} & w_{12}^{11} \\ 0 & w_{22}^{11} \end{pmatrix} \;\middle|\; w_{11}^{10} \neq 0,\ w_{22}^{10} \neq 0 \right\} \subset G, \tag{88}$$

and the DLR (40) of the Volterra equation (39). Clearly, the subgroup $H \subset G$ is of codimension $r = 2$.

It is straightforward to check that the functions

$$y^1 = \frac{w_{11}^{10}}{w_{21}^{10}}, \qquad y^2 = \frac{w_{21}^{10} w_{11}^{11} - w_{11}^{10} w_{21}^{11}}{(w_{21}^{10})^2} \tag{89}$$

are H-right-invariant. Therefore, any function of the form $\alpha(y^1, y^2)$ is also H-right-invariant.

According to Example 2, for the DLR (40), the group $\mathbb{H}_1$ consists of the constant matrix-functions $\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}$. To apply Theorem 1, we need to find a function $z = \alpha(y^1, y^2)$ such that z is $\mathbb{H}_1$-left-invariant.

The group $\mathbb{H}_1$ acts on G by left multiplication as follows

$$\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} w_{11}^{10} & w_{12}^{10} \\ w_{21}^{10} & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w_{11}^{11} & w_{12}^{11} \\ w_{21}^{11} & w_{22}^{11} \end{pmatrix} \right) = $$
$$= \begin{pmatrix} a_1 w_{11}^{10} & a_1 w_{12}^{10} \\ w_{21}^{10} & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} a_1 w_{11}^{11} & a_1 w_{12}^{11} \\ w_{21}^{11} & w_{22}^{11} \end{pmatrix},$$

which implies that $z = y^1/y^2$ is $\mathbb{H}_1$-left-invariant. From (51), (89) it follows that

$$\mathcal{S}(y^1) = \mathcal{S}\begin{pmatrix} \frac{w_{11}^{10}}{w_{21}^{10}} \end{pmatrix} = \frac{\mathcal{S}(w_{11}^{10})}{\mathcal{S}(w_{21}^{10})} = \frac{u w_{21}^{10}}{c_1 w_{21}^{10} - w_{11}^{10}} = \frac{u}{c_1 - y^1}, \tag{90}$$

$$\mathcal{S}(y^2) = \frac{\mathcal{S}(w_{21}^{10})\mathcal{S}(w_{11}^{11}) - \mathcal{S}(w_{11}^{10})\mathcal{S}(w_{21}^{11})}{\mathcal{S}(w_{21}^{10})^2} = \frac{u(w_{21}^{10}w_{11}^{11} - w_{11}^{10}w_{21}^{11} - w_{21}^{10}w_{21}^{10})}{(c_1 w_{21}^{10} - w_{11}^{10})^2} = \frac{u(y^2-1)}{(c_1 - y^1)^2}. \tag{91}$$

Using (90), (91) and the notation $z_k = \mathcal{S}^k(z)$ for $k \in \mathbb{Z}$, we obtain

$$z_0 = z = \frac{y^1}{y^2}, \qquad z_1 = \mathcal{S}(z) = \frac{\mathcal{S}(y^1)}{\mathcal{S}(y^2)} = \frac{c_1 - y^1}{y^2 - 1}, \tag{92}$$

$$z_2 = \mathcal{S}^2(z) = \mathcal{S}(z_1) = \frac{c_1 - \mathcal{S}(y^1)}{\mathcal{S}(y^2) - 1} = \frac{(y^1 - c_1)(c_1(c_1 - y^1) - u)}{(c_1 - y^1)^2 - (y^2 - 1)u}. \tag{93}$$

Recall that in this example we have $r = 2$. Since the differentials of the functions z and $\mathcal{S}(z)$ are linearly independent, condition (63) is valid. Equation (93) shows that condition (64) is valid as well.

From (92), (93) one gets

$$y^1 = \frac{z_0(c_1 + z_1)}{z_0 + z_1}, \qquad y^2 = \frac{c_1 + z_1}{z_0 + z_1}, \tag{94}$$

$$u = \frac{z_1^2(c_1 - z_0)(c_1 + z_2)}{(z_0 + z_1)(z_1 + z_2)}. \tag{95}$$

Since $z = y^1/y^2$, we have

$$D_t(z) = \frac{D_t(y^1)y^2 - D_t(y^2)y^1}{(y^2)^2}. \tag{96}$$

To compute $D_t(y^1)$ and $D_t(y^2)$ in (96), one can use formulas (89), (52). This gives

$$D_t(z) = \frac{u_{-1}(c_1 y^2 - y^1) - (y^1)^2(c_1(y^2 - 2) + y^1)}{(y^2)^2}. \tag{97}$$

Equation (95) implies $u_{-1} = \mathcal{S}^{-1}\left(\frac{z_1^2(c_1 - z_0)(c_1 + z_2)}{(z_0 + z_1)(z_1 + z_2)}\right) = \frac{z_0^2(c_1 - z_{-1})(c_1 + z_1)}{(z_{-1} + z_0)(z_0 + z_1)}$. Substituting this and (94) in (97), we obtain

$$D_t(z) = \frac{z_0^2(z_{-1} - z_1)(z_0 - c_1)(z_0 + c_1)}{(z_0 + z_{-1})(z_0 + z_1)}. \tag{98}$$

According to Theorem 1, to obtain an MT, we need to replace z_k by v_k for all $k \in \mathbb{Z}$ in (95), (98). Thus we get the following result. For any $c_1 \in \mathbb{C}$, the formula

$$u = \frac{v_1^2(c_1 - v)(c_1 + v_2)}{(v + v_1)(v_1 + v_2)} \tag{99}$$

determines an MT from the equation

$$v_t = \frac{v^2(v_{-1} - v_1)(v - c_1)(v + c_1)}{(v + v_{-1})(v + v_1)} \tag{100}$$

to the Volterra equation (39).

Equation (100) is a particular case of equation (V2) from the list of Volterra-type equations in [20]. On page 599 of [20] Yamilov presents an MT connecting equation (V2) with the Volterra equation. Formula (99) is a particular case of the MT from [20] (up to a shift of u).

Now consider the case of arbitrary $N \in \mathbb{Z}_{>0}$. Recall that $u_l = (u_l^1, \dots, u_l^N)$ for each $l \in \mathbb{Z}$ and $u^j = u_0^j$ for each $j = 1, \dots, N$.

Theorem 2. *Recall that the Lie group G is defined by (42). Let $H \subset G$ be a closed connected Lie subgroup of codimension $r > 0$. Suppose that there are nonnegative integers $\boldsymbol{d}_1, \dots, \boldsymbol{d}_N$ and functions $z^1, \dots, z^N$ on an open dense subset of G/H such that $\boldsymbol{d}_1 + \dots + \boldsymbol{d}_N + N = r$, the function z^i is $\mathbb{H}_{\boldsymbol{d}_i}$-left-invariant for each $i = 1, \dots, N$, and the following conditions hold*

$$\text{(101)}\qquad \begin{gathered}\textit{the functions } \mathcal{S}^k(z^i),\ i = 1, \dots, N,\ k = 0, \dots, \boldsymbol{d}_i, \textit{ form}\\ \textit{a system of local coordinates almost everywhere on } G/H\\ (\textit{i.e., the differentials of the functions are linearly independent almost everywhere}),\end{gathered}$$

$$\text{(102)}\qquad \begin{gathered}\textit{the determinant of the } N \times N \textit{ matrix-function } \left(\frac{\partial}{\partial u^j}\big(\mathcal{S}^{\boldsymbol{d}_i+1}(z^i)\big)\right)\\ \textit{is not identically zero.}\end{gathered}$$

(Note that, by Lemma 4, since z^i is $\mathbb{H}_{\boldsymbol{d}_i}$-left-invariant, the functions $\mathcal{S}^k(z^i)$, $i = 1, \dots, N$, $k = 0, \dots, \boldsymbol{d}_i$, do not depend on u_l and are well defined on an open dense subset of G/H, so condition (101) makes sense.)

Set $z_0^i = z^i$ and $z_k^i = \mathcal{S}^k(z^i)$ for $i = 1, \dots, N$ and $k = 1, \dots, \boldsymbol{d}_i + 1$. Condition (101) says that z_k^i, $i = 1, \dots, N$, $k = 0, \dots, \boldsymbol{d}_i$, form a system of local coordinates almost everywhere on G/H. Applying Remark 12 to this system of local coordinates, we see that locally one has

$$\text{(103)}\qquad \begin{aligned} z^i_{\boldsymbol{d}_i+1} = \mathcal{S}^{\boldsymbol{d}_i+1}(z^i) = \mathcal{S}(z^i_{\boldsymbol{d}_i}) = F^i(z^{i'}_{k'}, u^{i'}), &\qquad i, i' = 1, \dots, N, \qquad k' = 0, \dots, \boldsymbol{d}_{i'},\\ D_t(z^i) = Q^i(z^{i'}_{k'}, u^{i'}_a, \dots, u^{i'}_b), &\qquad i, i' = 1, \dots, N, \qquad k' = 0, \dots, \boldsymbol{d}_{i'},\end{aligned}$$

for some functions F^i and Q^i. By the implicit function theorem, condition (102) implies that locally from equations (103) we can express $u^1, \dots, u^N$ in terms of $z^{\tilde{\imath}}_{\tilde{k}}$, $\tilde{\imath} = 1, \dots, N$, $\tilde{k} = 0, \dots, \boldsymbol{d}_{\tilde{\imath}} + 1$,

$$\text{(104)}\qquad u^{i'} = f^{i'}(z^{\tilde{\imath}}_{\tilde{k}}), \qquad i' = 1, \dots, N.$$

(We can do this on a neighborhood of a point where $\det\left(\frac{\partial}{\partial u^j}\big(\mathcal{S}^{\boldsymbol{d}_i+1}(z^i)\big)\right) \neq 0$.)

Then one obtains an MT for equation (24) as follows. We introduce new variables v^i_k for $i = 1, \dots, N$ and $k \in \mathbb{Z}$. For each $j \in \mathbb{Z}$, we define the operator $\mathcal{S}^j$ on functions $h(v^i_k)$ by the usual rule

$$\mathcal{S}^j\big(h(v^i_k)\big) = h(\mathcal{S}^j(v^i_k)), \qquad \mathcal{S}^j(v^i_k) = v^i_{k+j}.$$

That is, applying $\mathcal{S}^j$ to a function $h = h(v^i_k)$, we replace v^i_k by v^i_{k+j} in h for all i, k.

Using the function $f^{i'}(z^{\tilde{\imath}}_{\tilde{k}})$ from (104), we can consider the function $f^{i'}(v^{\tilde{\imath}}_{\tilde{k}})$. Let $P^i(v^{\hat{\imath}}_{\hat{k}})$ be the function obtained from $Q^i(z^{i'}_{k'}, u^{i'}_a, \dots, u^{i'}_b)$ by replacing $z^{i'}_{k'}$ with $v^{i'}_{k'}$ and $u^{i'}_j$ with $\mathcal{S}^j\big(f^{i'}(v^{\tilde{\imath}}_{\tilde{k}})\big)$. That is,

$$\text{(105)}\qquad P^i(v^{\hat{\imath}}_{\hat{k}}) = Q^i\Big(v^{i'}_{k'}, \mathcal{S}^a\big(f^{i'}(v^{\tilde{\imath}}_{\tilde{k}})\big), \dots, \mathcal{S}^b\big(f^{i'}(v^{\tilde{\imath}}_{\tilde{k}})\big)\Big), \qquad i = 1, \dots, N.$$

We introduce the formula

$$\text{(106)}\qquad v^i_t = P^i(v^{\hat{\imath}}_{\hat{k}}), \qquad i = 1, \dots, N,$$

and regard (106) as a differential-difference equation for the variables $v^1, \dots, v^N$.

Then the formulas $u^{i'} = f^{i'}(v^{\tilde{\imath}}_{\tilde{k}})$, $i' = 1, \dots, N$, determine an MT from equation (106) to equation (25), where (25) is the component form of (24).

Proof. According to Remark 11, one can regard $z^1, \dots, z^N$ as functions on an open dense subset of G. Since for any $k \in \mathbb{Z}$ we can apply the operator $\mathcal{S}^k$ to such functions, we can consider $z^i_k = \mathcal{S}^k(z^i)$ for all $k \in \mathbb{Z}$.

The functions z^i_k may depend on the coordinates of G and the variables u^γ_l. Similarly to Lemma 5, one shows that z^i_k, $k \in \mathbb{Z}$, $i = 1, \dots, N$, are functionally independent.

According to Definition 1 of MTs, to prove the theorem, we need to show that

$$\text{(107)}\qquad \sum_{\check{\imath}, \check{k}} \frac{\partial}{\partial v^{\check{\imath}}_{\check{k}}}\big(f^{i'}(v^{\tilde{\imath}}_{\tilde{k}})\big) \cdot \mathcal{S}^{\check{k}}\big(P^{\check{\imath}}(v^{\hat{\imath}}_{\hat{k}})\big) = \mathbf{F}^{i'}\Big(\mathcal{S}^{\boldsymbol{\alpha}}\big(f^\gamma(v^{\tilde{\imath}}_{\tilde{k}})\big), \mathcal{S}^{\boldsymbol{\alpha}+1}\big(f^\gamma(v^{\tilde{\imath}}_{\tilde{k}})\big), \dots, \mathcal{S}^{\boldsymbol{\beta}}\big(f^\gamma(v^{\tilde{\imath}}_{\tilde{k}})\big)\Big),$$
$$i' = 1, \dots, N.$$

Applying D_t to equation (104) and using the identity $D_t(u^{i'}) = \mathbf{F}^{i'}(u^\gamma_{\boldsymbol{\alpha}}, u^\gamma_{\boldsymbol{\alpha}+1}, \dots, u^\gamma_{\boldsymbol{\beta}})$, one obtains

$$\text{(108)}\qquad \mathbf{F}^{i'}(u^\gamma_{\boldsymbol{\alpha}}, u^\gamma_{\boldsymbol{\alpha}+1}, \dots, u^\gamma_{\boldsymbol{\beta}}) = \sum_{\check{\imath}, \check{k}} \frac{\partial}{\partial z^{\check{\imath}}_{\check{k}}}\big(f^{i'}(z^{\tilde{\imath}}_{\tilde{k}})\big) \cdot D_t(z^{\check{\imath}}_{\check{k}}), \qquad i' = 1, \dots, N.$$

Since $\mathcal{S}\circ D_t = D_t\circ\mathcal{S}$, we have

$$D_t(z^{\check{\imath}}_{\check{k}}) = D_t(\mathcal{S}^{\check{k}}(z^{\check{\imath}})) = \mathcal{S}^{\check{k}}(D_t(z^{\check{\imath}})) = \mathcal{S}^{\check{k}}\big(Q^{\check{\imath}}(z^{i'}_{k'}, u^{i'}_a, \dots, u^{i'}_b)\big). \tag{109}$$

For any $j\in\mathbb{Z}$, applying $\mathcal{S}^j$ to equation (104), one gets

$$u^{i'}_j = \mathcal{S}^j(f^{i'}(z^{\tilde{\imath}}_{\tilde{k}})), \qquad i' = 1,\dots,N, \qquad j\in\mathbb{Z}. \tag{110}$$

Substituting (110) and (109) in (108), we obtain

$$\begin{aligned} \mathbf{F}^{i'}\big(\mathcal{S}^{\boldsymbol{\alpha}}\big(f^{\gamma}(z^{\tilde{\imath}}_{\tilde{k}})\big), \mathcal{S}^{\boldsymbol{\alpha}+1}\big(f^{\gamma}(z^{\tilde{\imath}}_{\tilde{k}})\big),\dots,\mathcal{S}^{\boldsymbol{\beta}}\big(f^{\gamma}(z^{\tilde{\imath}}_{\tilde{k}})\big)\big) = \\ = \sum_{\check{\imath},\check{k}} \frac{\partial}{\partial z^{\check{\imath}}_{\check{k}}}\big(f^{i'}(z^{\tilde{\imath}}_{\tilde{k}})\big)\cdot \mathcal{S}^{\check{k}}\Big(Q^{\check{\imath}}\big(z^{i'}_{k'}, \mathcal{S}^a\big(f^{i'}(z^{\tilde{\imath}}_{\tilde{k}})\big),\dots,\mathcal{S}^b\big(f^{i'}(z^{\tilde{\imath}}_{\tilde{k}})\big)\big)\Big), \qquad i'=1,\dots,N. \end{aligned} \tag{111}$$

Since z^i_k, $k\in\mathbb{Z}$, $i=1,\dots,N$, are functionally independent, the identity (111) will remain valid if we replace z^i_k by v^i_k for all i,k. Replacing z^i_k by v^i_k in (111) and using (105), we get (107). □

Examples of MTs in the case $N=2$ are constructed in Sections 4, 5.

Remark 15. In Remark 13 we have discussed why (63), (64) can be regarded as non-degeneracy conditions. The same applies to (101), (102).

Remark 16. Let $\mathbb{C}^d\setminus\{0\}$ be the space $\mathbb{C}^d$ without 0. Recall that the $(d-1)$-dimensional complex projective space $\mathbb{CP}^{d-1}$ can be identified with the set of one-dimensional subspaces in $\mathbb{C}^d$. The standard action of the group $\mathrm{GL}_d(\mathbb{C})$ on $\mathbb{C}^d\setminus\{0\}$ is transitive, which gives a transitive action of $\mathrm{GL}_d(\mathbb{C})$ on $\mathbb{CP}^{d-1}$.

For $i=1,\dots,m$, let M_i be either $\mathbb{C}^d\setminus\{0\}$ or $\mathbb{CP}^{d-1}$. As has been shown above, one has the standard transitive action of $\mathrm{GL}_d(\mathbb{C})$ on M_i. Set $\mathbb{M} = M_1\times\dots\times M_m$.

Recall that G is defined by (42). Consider the case $k_1=\dots=k_m=1$, so

$$G = \mathcal{G}(d,c_1,1)\times\dots\times\mathcal{G}(d,c_m,1). \tag{112}$$

Using the isomorphism $\mathcal{G}(d,c_i,1)\cong\mathrm{GL}_d(\mathbb{C})$ and the transitive action of $\mathrm{GL}_d(\mathbb{C})$ on M_i for $i=1,\dots,m$, we obtain a transitive action of G on $\mathbb{M}=M_1\times\dots\times M_m$.

Let $x\in\mathbb{M}$. Let $H\subset G$ be the stabilizer subgroup of the point x. That is,

$$H = \{g\in G \mid gx = x\}. \tag{113}$$

Then we have the analytic diffeomorphism

$$G/H \to \mathbb{M} = M_1\times\dots\times M_m, \qquad gH\mapsto gx, \qquad \forall\, g\in G.$$

Note that $\operatorname{codim} H = \dim G/H = \dim M_1+\dots+\dim M_m$, because G/H is diffeomorphic to $\mathbb{M}=M_1\times\dots\times M_m$.

Therefore, when we apply Theorems 1 or 2 to such G and H, we can replace G/H by $\mathbb{M}$ and consider functions on open dense subsets of $\mathbb{M}$. Examples of MTs arising from such G and H are given in the next sections.

Since $\mathcal{G}(d,c_i,1)\cong\mathrm{GL}_d(\mathbb{C})$, the structure of the group (112) does not depend on $c_1,\dots,c_m$. However, the homomorphism $\rho\colon \mathbf{G}\to G$ given by (55), (56) depends on $c_1,\dots,c_m$. Hence the subgroups $\rho(\mathbb{H}_k)\subset G$ for $k\in\mathbb{Z}_{\ge 0}$ and the set of $\mathbb{H}_k$-left-invariant functions depend on $c_1,\dots,c_m$ as well.

3. MTs for the Narita-Itoh-Bogoyavlensky lattice

For each $p\in\mathbb{Z}_{>0}$, the Narita-Itoh-Bogoyavlensky lattice [2, 7, 14] is the following differential-difference equation

$$u_t = u\Big(\sum_{k=1}^{p} u_k - \sum_{k=1}^{p} u_{-k}\Big). \tag{114}$$

It possesses the operator Lax pair [2, 8]

$$L = \mathcal{S} + u\mathcal{S}^{-p}, \qquad A = (L^{(p+1)})_{\ge 0},$$

where ≥ 0 means taking the terms with nonnegative power of $\mathcal{S}$ in $L^{(p+1)}$. The corresponding Lax equation $\partial_t(L) = [A,L]$ is equivalent to (114).

This implies that equation (114) is equivalent to the compatibility of the linear system

$$L\phi = \lambda\phi, \tag{115}$$

$$\phi_t = A\phi, \tag{116}$$

where $\phi=\phi(n,t)$ is a scalar function and λ is a parameter.

3.1. **The case p=2.** For $p = 2$ equation (114) reads

$$u_t = u(u_2 + u_1 - u_{-1} - u_{-2}). \tag{117}$$

In this case $L = \mathcal{S} + u\mathcal{S}^{-2}$, hence

$$A = (L^3)_{\geq 0} = \big((\mathcal{S}^2 + (u_1 + u)\mathcal{S}^{-1} + uu_{-2}\mathcal{S}^{-4})(\mathcal{S} + u\mathcal{S}^{-2})\big)_{\geq 0} = \mathcal{S}^3 + u + u_1 + u_2. \tag{118}$$

To apply the theory described in Section 2, we need to find a DLR of the form (30), (28). Recall that equation (117) is equivalent to the compatibility of system (115), (116). To obtain a DLR of the form (30), (28), we are going to rewrite system (115), (116) in the form (29) for some vector Ψ and matrices $M, \mathcal{U}$.

For $L = \mathcal{S} + u\mathcal{S}^{-2}$ equation (115) reads

$$\mathcal{S}(\phi) + u\mathcal{S}^{-2}(\phi) = \lambda\phi. \tag{119}$$

Set

$$\psi^1 = \mathcal{S}^{-2}(\phi), \qquad \psi^2 = \mathcal{S}^{-1}(\phi), \qquad \psi^3 = \phi. \tag{120}$$

Then equation (119) is equivalent to

$$\mathcal{S}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -u & 0 & \lambda\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}. \tag{121}$$

Set

$$\Psi = \begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}, \qquad M(u,\lambda) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -u & 0 & \lambda\end{pmatrix}.$$

Then (121) reads $\mathcal{S}(\Psi) = M(u,\lambda)\Psi$. Applying $\mathcal{S}^{-1}$ to this equation, we obtain

$$\Psi = \mathcal{S}^{-1}\big(M(u,\lambda)\big)\mathcal{S}^{-1}(\Psi) = M(u_{-1},\lambda)\mathcal{S}^{-1}(\Psi),$$

which implies $\mathcal{S}^{-1}(\Psi) = M(u_{-1},\lambda)^{-1}\Psi$, that is,

$$\mathcal{S}^{-1}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}0 & \frac{\lambda}{u_{-1}} & -\frac{1}{u_{-1}}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}. \tag{122}$$

According to (118), we have $A = \mathcal{S}^3 + u + u_1 + u_2$, so equation (116) reads

$$\phi_t = \mathcal{S}^3(\phi) + (u + u_1 + u_2)\phi. \tag{123}$$

Using (120), (121), we can rewrite (123) as

$$\psi^3_t = \mathcal{S}^3(\psi^3) + (u + u_1 + u_2)\psi^3 = -\lambda^2 u\psi^1 - \lambda u_1\psi^2 + (\lambda^3 + u + u_1)\psi^3, \tag{124}$$

where we have used (121) to compute $\mathcal{S}^3(\psi^3)$. According to (120), one has $\psi^2 = S^{-1}(\psi^3)$ and $\psi^1 = S^{-1}(\psi^2)$. Combining this with (122) and (124), we get

$$\begin{aligned}\psi^2_t = S^{-1}(\psi^3_t) &= -\lambda^2 u_{-1}S^{-1}(\psi^1) - \lambda u S^{-1}(\psi^2) + (\lambda^3 + u_{-1} + u)S^{-1}(\psi^3) =\\ &= -\lambda u\psi^1 + (u_{-1} + u)\psi^2 + \lambda^2\psi^3,\end{aligned} \tag{125}$$

$$\begin{aligned}\psi^1_t = S^{-1}(\psi^2_t) &= -\lambda u_{-1}S^{-1}(\psi^1) + (u_{-2} + u_{-1})S^{-1}(\psi^2) + \lambda^2 S^{-1}(\psi^3) =\\ &= (u_{-1} + u_{-2})\psi^1 + \lambda\psi^3.\end{aligned} \tag{126}$$

Equations (124), (125), (126) can be written in matrix form as follows

$$\partial_t\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix} = \begin{pmatrix}u_{-2} + u_{-1} & 0 & \lambda\\ -\lambda u & u_{-1} + u & \lambda^2\\ -\lambda^2 u & -\lambda u_1 & \lambda^3 + u_1 + u\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}. \tag{127}$$

Since the compatibility of system (121), (127) is equivalent to equation (117), we obtain the following DLR for (117)

$$M(u,\lambda) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -u & 0 & \lambda\end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix}u_{-2} + u_{-1} & 0 & \lambda\\ -\lambda u & u_{-1} + u & \lambda^2\\ -\lambda^2 u & -\lambda u_1 & \lambda^3 + u_1 + u\end{pmatrix}. \tag{128}$$

Note that $M(u,\lambda)$ is not invertible for $u = 0$. In agreement with Remark 4, we assume $u \neq 0$.

Let us compute the groups $\mathbb{H}_1$, $\mathbb{H}_2$, $\mathbb{H}_3$ for this DLR. In the course of computation, we will see that it is convenient to replace (128) by a gauge-equivalent DLR, in agreement with Remark 5.

According to (37), the group $\mathbb{H}_1$ is generated by the matrix-functions

$$M(\tilde{u},\lambda)\cdot M(u,\lambda)^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&\lambda\big(1-\frac{\tilde{u}}{u}\big)&\frac{\tilde{u}}{u}\end{pmatrix}. \tag{129}$$

Note that the right-hand side of (129) can be simplified by the following conjugation

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-\lambda&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&\lambda\big(1-\frac{\tilde{u}}{u}\big)&\frac{\tilde{u}}{u}\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-\lambda&1\end{pmatrix}^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{\tilde{u}}{u}\end{pmatrix}.$$

Therefore, it makes sense to consider the matrix $\mathbf{g}^1(\lambda)=\begin{pmatrix}1&0&0\\0&1&0\\0&-\lambda&1\end{pmatrix}$ and the following gauge-equivalent DLR

$$\hat{M}(u,\lambda)=\mathbf{g}^1(\lambda)\cdot M(u,\lambda)\cdot\mathbf{g}^1(\lambda)^{-1}=\begin{pmatrix}0&1&0\\0&\lambda&1\\-u&0&0\end{pmatrix},\qquad \hat{\mathcal{U}}=\mathbf{g}^1(\lambda)\cdot\mathcal{U}\cdot\mathbf{g}^1(\lambda)^{-1}.$$

For this DLR, the group $\mathbb{H}_1$ is generated by the matrix-functions

$$\hat{M}(\tilde{u},\lambda)\cdot\hat{M}(u,\lambda)^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{\tilde{u}}{u}\end{pmatrix},$$

so $\mathbb{H}_1$ consists of the constant matrix-functions $\begin{pmatrix}1&0&0\\0&1&0\\0&0&a_1\end{pmatrix}$, where $a_1\in\mathbb{C}$ is an arbitrary nonzero constant.

According to (38), the group $\mathbb{H}_2$ is generated by $h_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&a_1\end{pmatrix}\in\mathbb{H}_1$ and

$$\hat{M}(u,\lambda)\cdot h_1\cdot\hat{M}(u,\lambda)^{-1}=\begin{pmatrix}1&0&0\\\lambda(1-a_1)&a_1&0\\0&0&1\end{pmatrix} \tag{130}$$

for all nonzero $a_1\in\mathbb{C}$. The right-hand side of (130) can be simplified by the following conjugation

$$\begin{pmatrix}1&0&0\\-\lambda&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\\lambda(1-a_1)&a_1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-\lambda&1&0\\0&0&1\end{pmatrix}^{-1}=\begin{pmatrix}1&0&0\\0&a_1&0\\0&0&1\end{pmatrix}.$$

Hence, it makes sense to consider the matrix $\mathbf{g}^2(\lambda)=\begin{pmatrix}1&0&0\\-\lambda&1&0\\0&0&1\end{pmatrix}$ and the following gauge-equivalent DLR

$$\check{M}(u,\lambda)=\mathbf{g}^2(\lambda)\cdot\hat{M}(u,\lambda)\cdot\mathbf{g}^2(\lambda)^{-1}=\begin{pmatrix}\lambda&1&0\\0&0&1\\-u&0&0\end{pmatrix}, \tag{131}$$

$$\check{\mathcal{U}}=\mathbf{g}^2(\lambda)\cdot\hat{\mathcal{U}}\cdot\mathbf{g}^2(\lambda)^{-1}=\begin{pmatrix}\lambda^3+u_{-2}+u_{-1}&\lambda^2&\lambda\\-\lambda u_{-2}&u_{-1}+u&0\\-\lambda^2u_{-1}&-\lambda u_{-1}&u+u_1\end{pmatrix}. \tag{132}$$

For this DLR, the groups $\mathbb{H}_1$, $\mathbb{H}_2$ are as follows

$$\mathbb{H}_1=\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&a_1\end{pmatrix}\;\middle|\;a_1\in\mathbb{C},\ a_1\neq0\right\}, \tag{133}$$

$$\mathbb{H}_2=\left\{\begin{pmatrix}1&0&0\\0&a_2&0\\0&0&a_1\end{pmatrix}\;\middle|\;a_1,a_2\in\mathbb{C},\ a_1,a_2\neq0\right\}. \tag{134}$$

According to (38), the group $\mathbb{H}_3$ is generated by $h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \in \mathbb{H}_2$ and

$$\check{M}(u,\lambda) \cdot h_2 \cdot \check{M}(u,\lambda)^{-1} = \begin{pmatrix} a_2 & 0 & \frac{(a_2-1)\lambda}{u} \\ 0 & a_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for all nonzero $a_1, a_2, u \in \mathbb{C}$. Therefore, the group $\mathbb{H}_3$ for the DLR (131), (132) consists of the matrix-functions $\begin{pmatrix} a_3 & 0 & a_4\lambda \\ 0 & a_2 & 0 \\ 0 & 0 & a_1 \end{pmatrix}$ for all $a_1, a_2, a_3, a_4 \in \mathbb{C}$, $a_1, a_2, a_3 \neq 0$.

Let us construct some MTs for equation (117), using Theorem 1 and Remark 16. To clarify the construction, we begin with simple examples (Examples 6 and 7), which give some known MTs. Example 8 is more interesting and gives an MT which seems to be new.

Following Remark 16, in Examples 6, 7, 8 we consider a group G of the form (112) and a transitive action of G on a manifold $\mathbb{M}$. We take $H \subset G$ to be the stabilizer subgroup of a point $x \in \mathbb{M}$, so H is given by (113). Then $G/H \cong \mathbb{M}$. Since the results do not depend on the choice of $x \in \mathbb{M}$, in Examples 6, 7, 8 we do not mention x explicitly.

Example 6. Using the notation from Section 2.2, consider the case $d = 3$, $m = 1$, $k_1 = 1$, $c_1 \in \mathbb{C}$. Then $G = \mathcal{G}(3, c_1, 1) \cong \mathrm{GL}_3(\mathbb{C})$.

Consider the space $\mathbb{C}^3$ with coordinates α^1, α^2, α^3 and the standard transitive action of the group $G \cong \mathrm{GL}_3(\mathbb{C})$ on the manifold $\mathbb{M} = \mathbb{C}^3 \setminus \{0\}$.

According to Remark 16, to apply Theorem 1, we need to find an $\mathbb{H}_2$-left-invariant function z on an open dense subset of $\mathbb{C}^3 \setminus \{0\}$ such that conditions (63), (64) hold for $G/H \cong \mathbb{C}^3 \setminus \{0\}$ and $r = \dim G/H = 3$.

Since we are using the DLR (131), (132) and the standard action of $G \cong \mathrm{GL}_3(\mathbb{C})$ on $\mathbb{C}^3 \setminus \{0\}$, one has the formulas

$$\mathcal{S}\begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{pmatrix} = \begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 1 \\ -u & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{pmatrix}, \tag{135}$$

$$D_t\begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{pmatrix} = \begin{pmatrix} c_1^3 + u_{-2} + u_{-1} & c_1^2 & c_1 \\ -c_1 u_{-2} & u_{-1} + u & 0 \\ -c_1^2 u_{-1} & -c_1 u_{-1} & u + u_1 \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{pmatrix}. \tag{136}$$

In the right-hand side of (135) we use the matrix $\check{M}(u,\lambda)$ given by (131), but we substitute c_1 in place of λ, for the following reason. In the general definition (43) of $\mathcal{S}$ we work over the algebra $\mathbb{C}[\lambda]/((\lambda - c_p)^{k_p})$, where one has $(\lambda - c_p)^{k_p} = 0$. In the present example we have $k_1 = 1$, so we need to impose the relation $\lambda - c_1 = 0$.

The right-hand side of (136) is obtained in the same way from the matrix $\check{\mathcal{U}}$ given by (132).

As the group $\mathbb{H}_2$ is of the form (134), the function $z = \alpha^1$ is $\mathbb{H}_2$-left-invariant. Using formula (135) and the notation $z_k = \mathcal{S}^k(z)$ for $k \in \mathbb{Z}$, we obtain

$$z_0 = z = \alpha^1, \qquad z_1 = \mathcal{S}(z) = \mathcal{S}(\alpha^1) = c_1\alpha^1 + \alpha^2, \tag{137}$$

$$z_2 = \mathcal{S}(z_1) = c_1\mathcal{S}(\alpha^1) + \mathcal{S}(\alpha^2) = c_1^2\alpha^1 + c_1\alpha^2 + \alpha^3, \tag{138}$$

$$z_3 = \mathcal{S}^3(z) = \mathcal{S}(z_2) = c_1^2\mathcal{S}(\alpha^1) + c_1\mathcal{S}(\alpha^2) + \mathcal{S}(\alpha^3) = (c_1^3 - u)\alpha^1 + c_1^2\alpha^2 + c_1\alpha^3. \tag{139}$$

Recall that in this example we have $r = 3$. From (137), (138) we see that the functions z, $\mathcal{S}(z)$, $\mathcal{S}^2(z)$ form a system of coordinates on the manifold $\mathbb{C}^3 \setminus \{0\}$, so condition (63) is valid. Equation (139) shows that condition (64) is valid as well.

Therefore, using Theorem 1, we obtain an MT as follows. Equations (137), (138), (139) imply $z_3 = -uz_0 + c_1 z_2$, which yields

$$u = \frac{c_1 z_2 - z_3}{z_0}. \tag{140}$$

Using (136), (137), (138), we get

$$D_t(z) = D_t(\alpha^1) = (c_1^3 + u_{-2} + u_{-1})\alpha^1 + c_1^2\alpha^2 + c_1\alpha^3 = (u_{-2} + u_{-1})z_0 + c_1 z_2. \tag{141}$$

Equation (140) implies

$$u_{-1} = \mathcal{S}^{-1}\left(\frac{c_1 z_2 - z_3}{z_0}\right) = \frac{c_1 z_1 - z_2}{z_{-1}}, \qquad u_{-2} = \mathcal{S}^{-2}\left(\frac{c_1 z_2 - z_3}{z_0}\right) = \frac{c_1 z_0 - z_1}{z_{-2}}.$$

Substituting this in (141), one obtains

$$D_t(z)=\left(\frac{c_1z_0-z_1}{z_{-2}}+\frac{c_1z_1-z_2}{z_{-1}}\right)z_0+c_1z_2. \tag{142}$$

According to Theorem 1, to obtain an MT, we need to replace z_k by v_k for all $k\in\mathbb{Z}$ in (140), (142). Here and below we use also the identification $v_0=v$.

Thus we get the following result. For any $c_1\in\mathbb{C}$, the formula

$$u=\frac{c_1v_2-v_3}{v} \tag{143}$$

determines an MT from the equation

$$v_t=\left(\frac{c_1v-v_1}{v_{-2}}+\frac{c_1v_1-v_2}{v_{-1}}\right)v+c_1v_2. \tag{144}$$

to equation (117). As is shown in Remark 17 below, this MT can also be obtained immediately from system (115), (116). We have included this simple example, in order to demonstrate how the method described in Theorem 1 and Remark 16 works.

Remark 17. The MT (143) can be obtained immediately from system (115), (116) as follows. Recall that in this subsection we consider the case $p=2$. Since for $p=2$ we have $L=\mathcal{S}+u\mathcal{S}^{-2}$, the equation $L\phi=\lambda\phi$ is equivalent to

$$u=\frac{\lambda\phi-\mathcal{S}(\phi)}{\mathcal{S}^{-2}(\phi)}. \tag{145}$$

Setting $v=\mathcal{S}^{-2}(\phi)$ and replacing λ by c_1 in (145), one gets (143). Substituting (143) and $\phi=\mathcal{S}^2(v)$ in (116) for $p=2$, one obtains an equation equivalent to (144).

Example 7. Consider the case

$$d=3,\quad m=1,\quad k_1=1,\quad G=\mathcal{G}(3,c_1,1)\cong\mathrm{GL}_3(\mathbb{C}),\quad c_1\in\mathbb{C},$$

and the standard transitive action of the group $G\cong\mathrm{GL}_3(\mathbb{C})$ on the 2-dimensional complex projective space $\mathbb{CP}^2$. So $\mathbb{M}=\mathbb{CP}^2$ in this example.

According to Remark 16, to apply Theorem 1, one needs to find an $\mathbb{H}_1$-left-invariant function z on an open dense subset of $\mathbb{CP}^2$ such that conditions (63), (64) hold for $G/H\cong\mathbb{CP}^2$ and $r=\dim G/H=2$.

Recall that in Example 6 we have considered the space $\mathbb{C}^3$ with coordinates α^1, α^2, α^3. Now we regard α^1, α^2, α^3 as homogeneous coordinates for the projective space $\mathbb{CP}^2$.

Then $y^1=\alpha^2/\alpha^1$, $y^2=\alpha^3/\alpha^1$ are affine coordinates on the open dense subset

$$\mathbb{V}=\big\{(\alpha^1:\alpha^2:\alpha^3)\in\mathbb{CP}^2\ \big|\ \alpha^1\neq0\big\}\subset\mathbb{CP}^2.$$

As the group $\mathbb{H}_1$ is of the form (133), the function $z=y^1=\alpha^2/\alpha^1$ is $\mathbb{H}_1$-left-invariant. Using formula (135) and the notation $z_k=\mathcal{S}^k(z)$ for $k\in\mathbb{Z}$, we obtain

$$z_0=z=y^1=\frac{\alpha^2}{\alpha^1},\qquad z_1=\mathcal{S}(z)=\mathcal{S}\Big(\frac{\alpha^2}{\alpha^1}\Big)=\frac{\alpha^3}{c_1\alpha^1+\alpha^2}=\frac{y^2}{c_1+y^1}, \tag{146}$$

$$z_2=\mathcal{S}^2(z)=\mathcal{S}(z_1)=\frac{\mathcal{S}(\alpha^3)}{c_1\mathcal{S}(\alpha^1)+\mathcal{S}(\alpha^2)}=\frac{-u\alpha^1}{c_1(c_1\alpha^1+\alpha^2)+\alpha^3}=\frac{-u}{(c_1+z_0)(c_1+z_1)}. \tag{147}$$

Recall that in this example we have $r=2$. Since the differentials of the functions z and $\mathcal{S}(z)$ are linearly independent, condition (63) is valid. Equation (147) shows that condition (64) is valid as well. Therefore, using Theorem 1, we obtain an MT as follows. From (147) one gets

$$u=-z_2(c_1+z_0)(c_1+z_1). \tag{148}$$

Using (136) and (146), we obtain

$$\begin{aligned} D_t(z)&=D_t\Big(\frac{\alpha^2}{\alpha^1}\Big)=\frac{D_t(\alpha^2)\alpha^1-\alpha^2D_t(\alpha^1)}{(\alpha^1)^2}=\\ &=\frac{\big(-c_1u_{-2}\alpha^1+(u_{-1}+u)\alpha^2\big)\alpha^1-\alpha^2\big((c_1^3+u_{-2}+u_{-1})\alpha^1+c_1^2\alpha^2+c_1\alpha^3\big)}{(\alpha^1)^2}=\\ &=uz_0-u_{-2}(c_1+z_0)-z_0c_1(c_1+z_0)(c_1+z_1). \end{aligned} \tag{149}$$

Equation (148) yields $u_{-2} = \mathcal{S}^{-2}\big(-z_2(c_1+z_0)(c_1+z_1)\big) = -z_0(c_1+z_{-2})(c_1+z_{-1})$. Substituting this and (148) in (149), we get

$$D_t(z) = z_0(c_1+z_0)\big((c_1+z_{-2})(c_1+z_{-1}) - (c_1+z_1)(c_1+z_2)\big). \tag{150}$$

According to Theorem 1, to obtain an MT, we need to replace z_k by v_k for all $k \in \mathbb{Z}$ in (148), (150). Thus we get the following result. For any $c_1 \in \mathbb{C}$, the formula

$$u = -v_2(c_1+v)(c_1+v_1) \tag{151}$$

determines an MT from the equation

$$v_t = v(c_1+v)\big((c_1+v_{-2})(c_1+v_{-1}) - (c_1+v_1)(c_1+v_2)\big) \tag{152}$$

to equation (117). This MT is equivalent to a known MT. Indeed, set $\tilde{v} = v + c_1$. Then formulas (151), (152) can be written as

$$u = -(\tilde{v}_2 - c_1)\tilde{v}\tilde{v}_1, \tag{153}$$

$$\tilde{v}_t = (\tilde{v} - c_1)\tilde{v}(\tilde{v}_{-2}\tilde{v}_{-1} - \tilde{v}_1\tilde{v}_2). \tag{154}$$

As Yu. B. Suris told us, equation (154) and the MT (153) are well known.

Example 8. Now consider the case

$$d = 3,\ m = 2,\ k_1 = k_2 = 1,\ \ G = \mathcal{G}(3, c_1, 1) \times \mathcal{G}(3, c_2, 1) \cong \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C}),\ \ c_1, c_2 \in \mathbb{C}.$$

Using the standard transitive action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{CP}^2$, we obtain a transitive action of the group $G \cong \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ on $\mathbb{M} = \mathbb{CP}^2 \times \mathbb{CP}^2$.

According to Remark 16, to apply Theorem 1, we need to find an $\mathbb{H}_3$-left-invariant function z on an open dense subset of $\mathbb{CP}^2 \times \mathbb{CP}^2$ such that conditions (63), (64) hold for $G/H \cong \mathbb{CP}^2 \times \mathbb{CP}^2$ and $r = \dim G/H = 4$.

As has been said in Example 7, we regard α^1, α^2, α^3 as homogeneous coordinates for the 2-dimensional projective space $\mathbb{CP}^2$. We introduce also homogeneous coordinates β^1, β^2, β^3 for another copy of $\mathbb{CP}^2$. Similarly to (135), (136), one has the formulas

$$\mathcal{S}\begin{pmatrix}\beta^1\\ \beta^2\\ \beta^3\end{pmatrix} = \begin{pmatrix} c_1 & 1 & 0\\ 0 & 0 & 1\\ -u & 0 & 0\end{pmatrix}\begin{pmatrix}\beta^1\\ \beta^2\\ \beta^3\end{pmatrix}, \tag{155}$$

$$D_t\begin{pmatrix}\beta^1\\ \beta^2\\ \beta^3\end{pmatrix} = \begin{pmatrix} c_1^3 + u_{-2} + u_{-1} & c_1^2 & c_1\\ -c_1u_{-2} & u_{-1} + u & 0\\ -c_1^2u_{-1} & -c_1u_{-1} & u + u_1\end{pmatrix}\begin{pmatrix}\beta^1\\ \beta^2\\ \beta^3\end{pmatrix}. \tag{156}$$

Recall that, according to Section 2.2, the group $\mathbf{G}$ consists of $\mathrm{GL}_d(\mathbb{C})$-valued functions of λ, and in the present example we have $d = 3$.

As has been computed above, the subgroup $\mathbb{H}_3 \subset \mathbf{G}$ for the DLR (131), (132) consists of the matrix-functions $\begin{pmatrix} a_3 & 0 & a_4\lambda\\ 0 & a_2 & 0\\ 0 & 0 & a_1\end{pmatrix} \in \mathbf{G}$ for all $a_1, a_2, a_3, a_4 \in \mathbb{C}$, $a_1, a_2, a_3 \neq 0$.

When we speak about $\mathbb{H}_3$-left-invariant functions on open subsets of $\mathbb{CP}^2 \times \mathbb{CP}^2$, we need to consider the action of $\mathbb{H}_3$ on $\mathbb{CP}^2 \times \mathbb{CP}^2$ that is determined by the embedding $\rho(\mathbb{H}_3) \subset G$ and the action of G on $\mathbb{CP}^2 \times \mathbb{CP}^2$, where the homomorphism $\rho\colon \mathbf{G} \to G$ is defined by (55), (56).

According to (55), (56), one has

$$\rho\left(\begin{pmatrix} a_3 & 0 & a_4\lambda\\ 0 & a_2 & 0\\ 0 & 0 & a_1\end{pmatrix}\right) = \left(\begin{pmatrix} a_3 & 0 & a_4c_1\\ 0 & a_2 & 0\\ 0 & 0 & a_1\end{pmatrix}, \begin{pmatrix} a_3 & 0 & a_4c_2\\ 0 & a_2 & 0\\ 0 & 0 & a_1\end{pmatrix}\right) \in \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C}) \cong G.$$

Therefore, the action of $\mathbb{H}_3$ on $\mathbb{CP}^2 \times \mathbb{CP}^2$ is as follows

$$\begin{pmatrix} a_3 & 0 & a_4\lambda\\ 0 & a_2 & 0\\ 0 & 0 & a_1\end{pmatrix} \cdot \left(\begin{pmatrix}\alpha^1\\ \alpha^2\\ \alpha^3\end{pmatrix}, \begin{pmatrix}\beta^1\\ \beta^2\\ \beta^3\end{pmatrix}\right) = \left(\begin{pmatrix} a_3\alpha^1 + a_4c_1\alpha^3\\ a_2\alpha^2\\ a_1\alpha^3\end{pmatrix}, \begin{pmatrix} a_3\beta^1 + a_4c_2\beta^3\\ a_2\beta^2\\ a_1\beta^3\end{pmatrix}\right).$$

This formula implies that the function $z = \dfrac{\alpha^2\beta^3}{\alpha^3\beta^2}$ is $\mathbb{H}_3$-left-invariant.

Consider the affine coordinates

$$q^1 = \frac{\alpha^1}{\alpha^3}, \qquad q^2 = \frac{\alpha^2}{\alpha^3}, \qquad \hat{q}^1 = \frac{\beta^1}{\beta^3}, \qquad \hat{q}^2 = \frac{\beta^2}{\beta^3} \tag{157}$$

on the open dense subset $\mathbb{W} \subset \mathbb{CP}^2 \times \mathbb{CP}^2$ where $\alpha^3, \beta^3 \neq 0$.

Using (135), (155), (157), and the formula $z = \dfrac{\alpha^2\beta^3}{\alpha^3\beta^2} = \dfrac{q^2}{\hat{q}^2}$, one can express the functions $\mathcal{S}^k(z)$, $k = 0, 1, 2, 3$, in terms of q^1, q^2, $\hat{q}^1$, $\hat{q}^2$. These expressions show that condition (63) is valid.

Set $z_k = \mathcal{S}^k(z)$ for $k \in \mathbb{Z}$. Similarly to the previous examples, one obtains a formula of the form $z_4 = \mathcal{S}^4(z) = F(z_0, z_1, z_2, z_3, u)$ with some rational function F, which depends nontrivially on u, so condition (64) is valid as well. From the equation $z_4 = F(z_0, z_1, z_2, z_3, u)$ one can express u in terms of z_0, z_1, z_2, z_3, z_4 as follows

$$u = \frac{z_2z_3(c_1z_4 - c_2)(c_2z_0z_1 - c_1)(c_2z_1z_2 - c_1)}{(z_0z_1z_2 - 1)(z_1z_2z_3 - 1)(z_2z_3z_4 - 1)}. \tag{158}$$

Similarly to the computation of $D_t(z)$ in Example 7, using formulas (136), (156), (158), one can obtain the following

$$D_t(z) = \frac{z_0(z_{-2}z_{-1} - z_1z_2)(c_2 - c_1z_0)(c_2z_0z_{-1} - c_1)(c_2z_0z_1 - c_1)}{(z_0z_{-2}z_{-1} - 1)(z_0z_{-1}z_1 - 1)(z_0z_1z_2 - 1)}. \tag{159}$$

According to Theorem 1, to obtain an MT, we need to replace z_k by v_k for all $k \in \mathbb{Z}$ in (158), (159), which gives the following result. For any $c_1, c_2 \in \mathbb{C}$, the formula

$$u = \frac{v_2v_3(c_1v_4 - c_2)(c_2vv_1 - c_1)(c_2v_1v_2 - c_1)}{(vv_1v_2 - 1)(v_1v_2v_3 - 1)(v_2v_3v_4 - 1)}. \tag{160}$$

determines an MT from the equation

$$v_t = \frac{v(v_{-2}v_{-1} - v_1v_2)(c_2 - c_1v)(c_2vv_{-1} - c_1)(c_2vv_1 - c_1)}{(vv_{-2}v_{-1} - 1)(vv_{-1}v_1 - 1)(vv_1v_2 - 1)} \tag{161}$$

to equation (117).

Equation (161) and the MT (160) with arbitrary $c_1, c_2 \in \mathbb{C}$ seem to be new.

In the case $c_2 = 0$ equation (161) is equivalent (via scaling) to a particular case of equation (17.8.24) from [18, Section 17] and to equation (43a) from [12, Section 4]. The MT (160) for $c_2 = 0$ can be extracted from the results of [18, Section 17] and [12, Section 4] as well.

In the case $c_1 = c_2 \neq 0$ equation (161) can be transformed (by scaling of t and a linear-fractional transformation of v) to equation (3.4) from the arxiv version of [15]. Note that equation (3.3b) from the journal version of [15] is supposed to be the same as equation (3.4) from the arxiv version of [15], but there is a misprint in equation (3.3b) in the journal version of [15].

Recall that the MT (160) has been constructed from the DLR (128). Note that the order of the MT (160) is higher than the size of the matrices in the DLR (128). Indeed, the MT (160) is of order 4, while the matrices (128) are of size 3.

Recall that in this example we consider the case $p = 2$. See Section 3.2 for a discussion of an analogue of (161), (160) for arbitrary $p \in \mathbb{Z}_{>0}$ and its relations with some formulas of Yu. B. Suris.

3.2. **The case of arbitrary** p. Recall that we have obtained the 3×3 matrix DLR (128) for equation (117) from system (115), (116) for $p = 2$. Analogously, one can obtain a $(p+1) \times (p+1)$ matrix DLR for equation (114) from system (115), (116) for arbitrary $p \in \mathbb{Z}_{>0}$.

Then one can construct MTs for equation (114), using this DLR, Theorem 1, and Remark 16. An MT constructed in this way is discussed below.

In Example 8 we have studied the case

$$p = 2, \qquad d = 3, \qquad m = 2, \qquad k_1 = k_2 = 1,$$
$$G = \mathcal{G}(3, c_1, 1) \times \mathcal{G}(3, c_2, 1) \cong \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C}), \qquad c_1, c_2 \in \mathbb{C}.$$

Using the standard transitive action of $G \cong \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ on $\mathbb{CP}^2 \times \mathbb{CP}^2$, in Example 8 we have obtained the MT (160), (161).

Let us generalize this to the case of arbitrary $p \in \mathbb{Z}_{>0}$. Since the corresponding computations are very similar to those in Section 3.1, we present only the final result. Let

$$p \in \mathbb{Z}_{>0}, \qquad d = p + 1, \qquad m = 2, \qquad k_1 = k_2 = 1,$$
$$G = \mathcal{G}(p+1, c_1, 1) \times \mathcal{G}(p+1, c_2, 1) \cong \mathrm{GL}_{p+1}(\mathbb{C}) \times \mathrm{GL}_{p+1}(\mathbb{C}), \qquad c_1, c_2 \in \mathbb{C}.$$

Similarly to Example 8, using the standard transitive action of $G \cong \mathrm{GL}_{p+1}(\mathbb{C}) \times \mathrm{GL}_{p+1}(\mathbb{C})$ on $\mathbb{CP}^p \times \mathbb{CP}^p$, one can obtain the following. For any $c_1, c_2 \in \mathbb{C}$, the formula

$$u = \frac{(c_1 v_{2p} - c_2)\big(\prod_{j=p}^{2p-1} v_j\big)\prod_{i=0}^{p-1}\Big(c_2\big(\prod_{j=i}^{i+p-1} v_j\big) - c_1\Big)}{\prod_{i=0}^{p}\big(-1 + \prod_{j=i}^{i+p} v_j\big)} \tag{162}$$

determines an MT from the equation

$$v_t = \frac{v(c_2 - c_1 v)\big(\prod_{i=1}^{p} v_{-i} - \prod_{i=1}^{p} v_i\big)\prod_{i=0}^{p-1}\Big(c_2\big(\prod_{j=i+1-p}^{i} v_j\big) - c_1\Big)}{\prod_{i=0}^{p}\big(-1 + \prod_{j=i-p}^{i} v_j\big)} \tag{163}$$

to equation (114). Formulas (162), (163) are a generalization of (160), (161) to the case of arbitrary $p \in \mathbb{Z}_{>0}$. The fact that (162) is indeed an MT from (163) to (114) can also be checked by a straightforward computation.

Equation (163) and the MT (162) with arbitrary $c_1, c_2 \in \mathbb{C}$ seem to be new. In the case $c_2 = 0$ equation (163) is equivalent to equation (17.8.24) from [18, Section 17] via scaling of the variables.

We showed (162), (163) to Yu. B. Suris, and he told us that this MT can be written in a more symmetric form as follows. Let α, β be complex parameters. Consider the following differential-difference equations for scalar functions $w = w(n,t)$ and $v = v(n,t)$

$$w_t = w(\alpha + \beta w)\Big(\prod_{i=1}^{p} w_i - \prod_{i=1}^{p} w_{-i}\Big), \tag{164}$$

$$v_t = v(\alpha + \beta v)\Big(\prod_{i=1}^{p} v_i - \prod_{i=1}^{p} v_{-i}\Big)\frac{\prod_{j=1}^{p}(1 + \alpha\prod_{i=1}^{p} v_{j-i})}{\prod_{j=0}^{p}(1 - \beta\prod_{i=0}^{p} v_{j-i})}. \tag{165}$$

Equation (164) is connected with (114) by the MT

$$u = (\alpha + \beta w_p)\prod_{i=0}^{p-1} w_i. \tag{166}$$

It is easy to check that equations (164), (165) are connected by the MT

$$w = v\frac{1 + \alpha\prod_{i=1}^{p} v_{-i}}{1 - \beta\prod_{i=0}^{p} v_{-i}}. \tag{167}$$

In the case $\beta \neq 0$, equation (165) is equivalent to (163) via scaling, and the composition of the MTs (166), (167) is equivalent (up to a shift and scaling) to the MT (162).

As Yu. B. Suris told us, equation (164) and the MT (166) are well known. Formulas (165), (167) with arbitrary $\alpha, \beta \in \mathbb{C}$ seem to be new. In the cases $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$ formulas equivalent to (165), (167) can be found in [18, Section 17].

Remark 18. Svinin [17] constructed the equation

$$v_t = \prod_{j=1}^{p}\frac{1}{v_{-j} - v_{-j+p+1}} \tag{168}$$

and the MT

$$u = \prod_{j=p}^{2p}\frac{1}{v_{-j} - v_{-j+p+1}} \tag{169}$$

connecting (168) with (114). It would be interesting to find out whether the MT (169) can be constructed by the method of the present paper.

4. MTs for the Toda lattice in the Flaschka-Manakov coordinates

The Toda lattice equation for a scalar function $q = q(n,t)$

$$q_{tt} = \exp(q_1 - q) - \exp(q - q_{-1}), \qquad q_1 = q(n+1,t), \qquad q_{-1} = q(n-1,t), \tag{170}$$

can be rewritten in the evolution form (24) for a two-component vector-function $u = (u^1, u^2)$ as follows [4, 9]. Set $u^1 = \exp(q_1 - q)$ and $u^2 = q_t$. Then (170) implies

$$u^1_t = u^1(u^2_1 - u^2), \qquad u^2_t = u^1 - u^1_{-1}. \tag{171}$$

System (171) is called the Toda lattice in the Flaschka-Manakov coordinates.

It is known (see, e.g., [8]) that the following matrices form a DLR for (171)

$$M=\begin{pmatrix}\lambda+u^2_1 & u^1\\ -1 & 0\end{pmatrix},\qquad \mathcal{U}=\begin{pmatrix}0 & -u^1\\ 1 & \lambda+u^2\end{pmatrix}. \tag{172}$$

The equation $\partial_t(M)=\mathcal{S}(\mathcal{U})M-M\mathcal{U}$ is equivalent to (171).

To construct MTs by means of the method described in Section 2, we need a DLR $M,\mathcal{U}$ such that M depends only on u^i and λ. The matrix M in (172) is not of this type, because it depends on u^2_1. But this can be easily overcome as follows.

We relabel $u^2:=u^2_1$ and $u^1:=u^1$. After this change of variables, system (171) and the DLR (172) become

$$u^1_t=u^1(u^2-u^2_{-1}),\qquad u^2_t=u^1_1-u^1, \tag{173}$$

$$M(u^1,u^2,\lambda)=\begin{pmatrix}\lambda+u^2 & u^1\\ -1 & 0\end{pmatrix},\qquad \mathcal{U}=\begin{pmatrix}0 & -u^1\\ 1 & \lambda+u^2_{-1}\end{pmatrix}. \tag{174}$$

According to (37), for the DLR (174), the group $\mathbb{H}_1$ is generated by the matrix-functions

$$M(\tilde u^1,\tilde u^2,\lambda)\cdot M(u^1,u^2,\lambda)^{-1}=\begin{pmatrix}\frac{\tilde u^1}{u^1} & \frac{\tilde u^1u^2}{u^1}-\tilde u^2+\lambda\big(\frac{\tilde u^1}{u^1}-1\big)\\ 0 & 1\end{pmatrix}. \tag{175}$$

Note that the right-hand side of (175) can be simplified by the following conjugation

$$\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\tilde u^1}{u^1} & \frac{\tilde u^1u^2}{u^1}-\tilde u^2+\lambda\big(\frac{\tilde u^1}{u^1}-1\big)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}\frac{\tilde u^1}{u^1} & \frac{\tilde u^1u^2}{u^1}-\tilde u^2\\ 0 & 1\end{pmatrix}.$$

Therefore, it makes sense to consider the following gauge-equivalent DLR

$$\hat M=\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}\cdot M(u^1,u^2,\lambda)\cdot\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}u^2 & u^1-u^2\lambda\\ -1 & \lambda\end{pmatrix}, \tag{176}$$

$$\hat{\mathcal{U}}=\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}\cdot\mathcal{U}\cdot\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}\lambda & \lambda u^2_{-1}-u^1\\ 1 & u^2_{-1}\end{pmatrix}. \tag{177}$$

For this DLR, the group $\mathbb{H}_1$ consists of the constant matrix-functions $\begin{pmatrix}a_1 & a_2\\ 0 & 1\end{pmatrix}$ for all $a_1,a_2\in\mathbb{C}$, $a_1\neq 0$.

Let us construct some MTs for (173), using Theorem 2. Recall that G is given by (42). Using the notation from Section 2.2, consider the case $d=2$, $m=1$, $k_1=1$, $c_1\in\mathbb{C}$. Then

$$G=\mathcal{G}(2,c_1,1)=\left\{\begin{pmatrix}w^{10}_{11} & w^{10}_{12}\\ w^{10}_{21} & w^{10}_{22}\end{pmatrix}\;\middle|\; w^{10}_{11}w^{10}_{22}-w^{10}_{21}w^{10}_{12}\neq 0\right\}\cong \mathrm{GL}_2(\mathbb{C}). \tag{178}$$

We set also $H=\mathbf{1}$, where $\mathbf{1}\in G$ is the identity element. Since in the product

$$\begin{pmatrix}a_1 & a_2\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}w^{10}_{11} & w^{10}_{12}\\ w^{10}_{21} & w^{10}_{22}\end{pmatrix}$$

the w^{10}_{21} and w^{10}_{22} components do not change, the functions $z^1=w^{10}_{21}$ and $z^2=w^{10}_{22}$ are $\mathbb{H}_1$-left-invariant. As the subgroup $H\subset G$ is trivial, these functions are also H-right-invariant.

For the DLR (176), (177) and the group (178), formulas (43), (44) become

$$\mathcal{S}\begin{pmatrix}w^{10}_{11} & w^{10}_{12}\\ w^{10}_{21} & w^{10}_{22}\end{pmatrix}=\begin{pmatrix}u^2 & u^1-u^2\lambda\\ -1 & \lambda\end{pmatrix}\begin{pmatrix}w^{10}_{11} & w^{10}_{12}\\ w^{10}_{21} & w^{10}_{22}\end{pmatrix},\qquad \lambda-c_1=0, \tag{179}$$

$$D_t\begin{pmatrix}w^{10}_{11} & w^{10}_{12}\\ w^{10}_{21} & w^{10}_{22}\end{pmatrix}=\begin{pmatrix}\lambda & \lambda u^2_{-1}-u^1\\ 1 & u^2_{-1}\end{pmatrix}\begin{pmatrix}w^{10}_{11} & w^{10}_{12}\\ w^{10}_{21} & w^{10}_{22}\end{pmatrix},\qquad \lambda-c_1=0. \tag{180}$$

Using formula (179) and the notation $z^1_k=\mathcal{S}^k(z^1)$, $z^2_k=\mathcal{S}^k(z^2)$ for $k\in\mathbb{Z}$, we obtain

$$z^1_0=z^1=w^{10}_{21},\qquad z^1_1=\mathcal{S}(z^1)=\mathcal{S}(w^{10}_{21})=c_1w^{10}_{21}-w^{10}_{11}, \tag{181}$$

$$z^1_2=\mathcal{S}^2(z^1)=\mathcal{S}(z^1_1)=c_1^2w^{10}_{21}-c_1w^{10}_{11}-u^2w^{10}_{11}-u^1w^{10}_{21}+c_1u^2w^{10}_{21}, \tag{182}$$

$$z^2_0=z^2=w^{10}_{22},\qquad z^2_1=\mathcal{S}(z^2)=\mathcal{S}(w^{10}_{22})=c_1w^{10}_{22}-w^{10}_{12}, \tag{183}$$

$$z^2_2=\mathcal{S}^2(z^2)=\mathcal{S}(z^2_1)=c_1^2w^{10}_{22}-c_1w^{10}_{12}-u^2w^{10}_{12}-u^1w^{10}_{22}+c_1u^2w^{10}_{22}. \tag{184}$$

Equations (181), (183) imply that z^1_0, z^1_1, z^2_0, z^2_1 form a system of coordinates for G, so condition (101) is valid for $\mathbf{d}_1=\mathbf{d}_2=1$. Formulas (182), (184) show that condition (102) is valid as well.

From (181), (182), (183), (184) one can express u^1, u^2 in terms of z^1_0, z^1_1, z^2_0, z^2_1, z^1_2, z^2_2 as follows

$$u^1=\frac{z^1_2z^2_1-z^1_1z^2_2}{z^1_1z^2_0-z^1_0z^2_1},\qquad u^2=-c_1+\frac{z^1_0z^2_2-z^1_2z^2_0}{z^1_0z^2_1-z^1_1z^2_0}. \tag{185}$$

Using formulas (180), (185), we can compute $D_t(z^1)$, $D_t(z^2)$ and get

$$D_t(z^1)=\frac{z^1_{-1}(z^1_0z^2_1-z^1_1z^2_0)}{z^1_{-1}z^2_0-z^1_0z^2_{-1}},\qquad D_t(z^2)=\frac{z^2_{-1}(z^1_0z^2_1-z^1_1z^2_0)}{z^1_{-1}z^2_0-z^1_0z^2_{-1}}. \tag{186}$$

According to Theorem 2, to obtain an MT, we need to replace z^i_k by v^i_k for all $i=1,2$ and $k\in\mathbb{Z}$ in (185), (186). (And we can use the identification $v^1_0=v^1$, $v^2_0=v^2$.)

This gives the following result. For any $c_1\in\mathbb{C}$, the formulas

$$u^1=\frac{v^1_2v^2_1-v^1_1v^2_2}{v^1_1v^2-v^1v^2_1},\qquad u^2=-c_1+\frac{v^1v^2_2-v^1_2v^2}{v^1v^2_1-v^1_1v^2} \tag{187}$$

determine an MT from

$$v^1_t=\frac{v^1_{-1}(v^1v^2_1-v^1_1v^2)}{v^1_{-1}v^2-v^1v^2_{-1}},\qquad v^2_t=\frac{v^2_{-1}(v^1v^2_1-v^1_1v^2)}{v^1_{-1}v^2-v^1v^2_{-1}} \tag{188}$$

to (173). We cannot find (187), (188) in the existing literature, so this MT may be new.

According to the above notation, $v^1=z^1=w^{10}_{21}$ and $v^2=z^2=w^{10}_{22}$, where w^{10}_{21}, w^{10}_{22} satisfy (179), (180). Therefore, solutions of system (188) can be obtained from solutions of system (173) as follows. For a given solution $u^1(n,t)$, $u^2(n,t)$ of system (173), one needs to solve the auxiliary linear system (179), (180) for the functions $w^{10}_{ij}(n,t)$, $i,j=1,2$, and then one can take $v^1(n,t)=w^{10}_{21}(n,t)$, $v^2(n,t)=w^{10}_{22}(n,t)$.

For example, consider the constant solution $u^1(n,t)=u^2(n,t)=1$ of (173), and set $\lambda=c_1=1$. Then, solving (179), (180) in the case $u^1(n,t)=u^2(n,t)=\lambda=1$, we obtain

$$w^{10}_{11}=b_1\mathrm{e}^t,\quad w^{10}_{12}=b_2\mathrm{e}^t,\quad w^{10}_{21}=(b_1t+b_3-nb_1)\mathrm{e}^t,\quad w^{10}_{22}=(b_2t+b_4-nb_2)\mathrm{e}^t,$$

where b_1, b_2, b_3, b_4 are arbitrary constants. Then

$$v^1(n,t)=w^{10}_{21}=(b_1t+b_3-nb_1)\mathrm{e}^t,\qquad v^2(n,t)=w^{10}_{22}=(b_2t+b_4-nb_2)\mathrm{e}^t$$

is a solution of system (188).

5. MTs for Adler-Postnikov lattices

In what follows, *difference operators* are polynomials in $\mathcal{S}$, $\mathcal{S}^{-1}$.

Adler and Postnikov [1] studied integrable hierarchies of differential-difference equations associated with spectral problems of the form

$$P\psi=\lambda Q\psi, \tag{189}$$

where P, Q are difference operators and $\psi=\psi(n,t)$ is a scalar function.

Such hierarchies are constructed in [1] as follows. One considers the Lax type equations

$$P_t=BP-PA,\qquad Q_t=BQ-QA \tag{190}$$

for some difference operators P, Q, A, B. Then the equation

$$\psi_t=A\psi \tag{191}$$

is compatible with (189) modulo (190). For certain fixed operators P and Q, Adler and Postnikov [1] find an infinite collection of operators A, B so that (190) becomes a commutative hierarchy of differential-difference equations.

Recall that in Section 2.3 we have described a method to construct MTs for differential-difference equations possessing DLRs of the form (30). This method is applicable to equations presented in [1]. To illustrate this, let us consider the following operators from [1, Section 5]

$$P=u^1\mathcal{S}^3+\mathcal{S},\qquad Q=\mathcal{S}^2+u^2, \tag{192}$$

$$A^-=u^2_{-2}u^2_{-1}S^{-2}+\mathbf{f}_{-3}+\mathbf{f}_{-2},\qquad B^-=u^2_{-1}u^2S^{-2}+\mathbf{f}_{-1}+\mathbf{f}, \tag{193}$$

$$A^+=u^1_{-2}u^1_{-1}S^2+\mathbf{g}_{-1}+\mathbf{g},\qquad B^+=u^1u^1_1S^2+\mathbf{g}+\mathbf{g}_1, \tag{194}$$

where $\mathbf{f}=u^1u^2_1u^2_2$, $\mathbf{g}=u^1_{-2}u^1_{-1}u^2$, $\mathbf{f}_k=\mathcal{S}^k(\mathbf{f})$, $\mathbf{g}_k=\mathcal{S}^k(\mathbf{g})$ for $k\in\mathbb{Z}$.

Following [1], we consider equations (190), (191) in the case $A=A^-$, $B=B^-$ and in the case $A=A^+$, $B=B^+$. In these cases, the variable t is denoted by t^- and t^+ respectively.

The system $P_{t^-}=B^-P-PA^-$, $Q_{t^-}=B^-Q-QA^-$ is equivalent to

$$\begin{aligned} u^1_{t^-}&=u^1(\mathbf{f}_{-1}-\mathbf{f}_1),\\ u^2_{t^-}&=u^2(\mathbf{f}+\mathbf{f}_{-1}-\mathbf{f}_{-2}-\mathbf{f}_{-3}-u^2_1+u^2_{-1}). \end{aligned} \tag{195}$$

The system $P_{t^+} = B^+P - PA^+$, $Q_{t^+} = B^+Q - QA^+$ is equivalent to

$$u^1_{t^+} = u^1(\mathbf{g} + \mathbf{g}_1 - \mathbf{g}_2 - \mathbf{g}_3 - u^1_{-1} + u^1_1),\quad u^2_{t^+} = u^2(\mathbf{g}_1 - \mathbf{g}_{-1}). \tag{196}$$

Let us construct matrix DLRs for the differential-difference equations (195) and (196). Set

$$\varphi^1 = \psi, \qquad \varphi^2 = \mathcal{S}(\psi), \qquad \varphi^3 = \mathcal{S}^2(\psi). \tag{197}$$

Using (189), (192), and (197), one gets

$$\mathcal{S}\begin{pmatrix}\varphi^1\\ \varphi^2\\ \varphi^3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \lambda\frac{u^2}{u^1} & -\frac{1}{u^1} & \frac{\lambda}{u^1}\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\\ \varphi^3\end{pmatrix}. \tag{198}$$

Using formulas (193), (197) and equation (191) for $A = A^-$, $t = t^-$, we obtain

$$\partial_{t^-}(\varphi^1) = \psi_{t^-} = A^-\psi = u^2_{-2}u^2_{-1}\mathcal{S}^{-2}(\psi) + (\mathbf{f}_{-3} + \mathbf{f}_{-2})\psi, \tag{199}$$

$$\partial_{t^-}(\varphi^2) = \mathcal{S}(\psi_{t^-}) = u^2_{-1}u^2\mathcal{S}^{-1}(\psi) + (\mathbf{f}_{-2} + \mathbf{f}_{-1})\mathcal{S}(\psi), \tag{200}$$

$$\partial_{t^-}(\varphi^3) = \mathcal{S}^2(\psi_{t^-}) = u^2u^2_1\psi + (\mathbf{f}_{-1} + \mathbf{f})\mathcal{S}^2(\psi). \tag{201}$$

Applying the operators $\mathcal{S}^{-1}$ and $\mathcal{S}^{-2}$ to (198), one can express the functions $\mathcal{S}^{-1}(\psi) = \mathcal{S}^{-1}(\varphi^1)$ and $\mathcal{S}^{-2}(\psi) = \mathcal{S}^{-2}(\varphi^1)$ in terms of φ^1, φ^2, φ^3. Substituting these expressions and $\psi = \varphi^1$ in the right-hand sides of (199), (200), (201), we get

$$\partial_{t^-}\begin{pmatrix}\varphi^1\\ \varphi^2\\ \varphi^3\end{pmatrix} = \begin{pmatrix}\mathbf{f}_{-3} + \mathbf{f}_{-2} - u^2_{-1} + \frac{1}{\lambda^2} & \frac{u^1_{-2}u^2_{-1}-1}{\lambda} & \frac{u^1_{-1}}{\lambda^2}\\ \frac{u^2}{\lambda} & -u^2 + \mathbf{f}_{-2} + \mathbf{f}_{-1} & \frac{u^2u^1_{-1}}{\lambda}\\ u^2u^2_1 & 0 & \mathbf{f} + \mathbf{f}_{-1}\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\\ \varphi^3\end{pmatrix}. \tag{202}$$

The equations $P\psi = \lambda Q\psi$ and $\psi_{t^-} = A^-\psi$ are compatible modulo the system

$$P_{t^-} = B^-P - PA^-, \qquad Q_{t^-} = B^-Q - QA^-,$$

which is equivalent to (195). Since (198) and (202) have been obtained from the equations $P\psi = \lambda Q\psi$, $\psi_{t^-} = A^-\psi$, we see that system (198), (202) is compatible modulo (195).

Therefore, the matrices

$$M = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \lambda\frac{u^2}{u^1} & -\frac{1}{u^1} & \frac{\lambda}{u^1}\end{pmatrix},\quad \mathcal{U}^- = \begin{pmatrix}\mathbf{f}_{-3} + \mathbf{f}_{-2} - u^2_{-1} + \frac{1}{\lambda^2} & \frac{u^1_{-2}u^2_{-1}-1}{\lambda} & \frac{u^1_{-1}}{\lambda^2}\\ \frac{u^2}{\lambda} & -u^2 + \mathbf{f}_{-2} + \mathbf{f}_{-1} & \frac{u^2u^1_{-1}}{\lambda}\\ u^2u^2_1 & 0 & \mathbf{f} + \mathbf{f}_{-1}\end{pmatrix} \tag{203}$$

form a DLR for (195). The equation $\partial_{t^-}(M) = \mathcal{S}(\mathcal{U}^-)M - M\mathcal{U}^-$ is equivalent to (195).

Similarly, using the equations $P\psi = \lambda Q\psi$ and $\psi_{t^+} = A^+\psi$, one gets the following DLR for (196)

$$M = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \lambda\frac{u^2}{u^1} & -\frac{1}{u^1} & \frac{\lambda}{u^1}\end{pmatrix}, \qquad \mathcal{U}^+ = \begin{pmatrix}\mathbf{g} + \mathbf{g}_{-1} & 0 & u^1_{-2}u^1_{-1}\\ \lambda u^2u^1_{-1} & \mathbf{g} + \mathbf{g}_1 - u^1_{-1} & \lambda u^1_{-1}\\ \lambda^2u^2 & \lambda\left(u^1u^2_1 - 1\right) & \lambda^2 - u^1 + \mathbf{g}_1 + \mathbf{g}_2\end{pmatrix}. \tag{204}$$

Recall that the groups $\mathbb{H}_k$, $k \in \mathbb{Z}_{\geq 0}$, corresponding to a DLR (30) are determined by the matrix M. Using M from (203), (204), one obtains the following results on the corresponding groups $\mathbb{H}_1$, $\mathbb{H}_2$.

- The group $\mathbb{H}_1$ consists of the matrix-functions $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a_1 & -a_1\lambda & a_2\end{pmatrix}$ for all $a_1, a_2 \in \mathbb{C}$, $a_2 \neq 0$.
- The group $\mathbb{H}_2$ consists of the matrix-functions $\begin{pmatrix}1 & 0 & 0\\ \alpha_{21}(\lambda) & \alpha_{22}(\lambda) & \alpha_{23}(\lambda)\\ \alpha_{31}(\lambda) & \alpha_{32}(\lambda) & \alpha_{33}(\lambda)\end{pmatrix}$ for some scalar functions $\alpha_{ij}(\lambda)$, $i = 2, 3$, $j = 1, 2, 3$. (We do not need the explicit form of these functions.)

Let us construct some MTs for systems (195) and (196), using Theorem 2 and Remark 16.

Since (195) and (196) are two-component systems, we have $N = 2$. Using the notation from Section 2.2, consider the case

$$d = 3,\ m = 2,\ k_1 = k_2 = 1,\ G = \mathcal{G}(3, c_1, 1) \times \mathcal{G}(3, c_2, 1) \cong \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C}),\ c_1, c_2 \in \mathbb{C}.$$

The standard transitive actions of $\mathrm{GL}_3(\mathbb{C})$ on the manifolds $\mathbb{C}^3 \setminus \{0\}$ and $\mathbb{CP}^2$ give a transitive action of $G \cong \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ on $\mathbb{M} = (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{CP}^2$. We take $H \subset G$ to be the stabilizer subgroup of a point $x \in \mathbb{M}$, so H is given by (113). Then $G/H \cong \mathbb{M}$.

We are going to apply Theorem 2 in the case $\mathbf{d}_1 = 2$, $\mathbf{d}_2 = 1$. According to Remark 16 and Theorem 2, we need to find an $\mathbb{H}_2$-left-invariant function z^1 and an $\mathbb{H}_1$-left-invariant function z^2 on an open dense subset of the manifold $(\mathbb{C}^3 \setminus \{0\}) \times \mathbb{CP}^2$ such that conditions (101), (102) hold for $G/H \cong (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{CP}^2$ and $r = \dim G/H = 5$.

Let α^1, α^2, α^3 be coordinates on $\mathbb{C}^3 \setminus \{0\}$ and β^1, β^2, β^3 be homogeneous coordinates on the projective space $\mathbb{CP}^2$. The above results on the structure of $\mathbb{H}_1$ and $\mathbb{H}_2$ imply that the function $z^1 = \alpha^1$ is $\mathbb{H}_2$-left-invariant and the function $z^2 = \beta^1/\beta^2$ is $\mathbb{H}_1$-left-invariant.

A straightforward computation shows that conditions (101), (102) are satisfied, so we can apply Theorem 2. This gives the following result. For any $c_1, c_2 \in \mathbb{C}$, the formulas

$$u^1 = \frac{v^2_2\left(c_2 v^2 v^2_1\left(v^1_1 - c_1 v^1_2\right) + c_1 v^1\left(c_2 - v^2_1\right)\right)}{c_1 v^1 - c_2 v^1_3 v^2 v^2_1 v^2_2}, \qquad u^2 = \frac{-c_1 v^1_2 + v^1_3 v^2_2\left(c_2 - v^2_1\right) + v^1_1}{c_1 v^1 - c_2 v^1_3 v^2 v^2_1 v^2_2} \tag{205}$$

determine an MT from the equation

$$\begin{aligned} v^1_{t^-} &= v^1\left(u^1_{-2}u^1_{-1}u^2 + u^1_{-3}u^1_{-2}u^2_{-1}\right) + v^1_2 u^1_{-2}u^1_{-1}, \\ v^2_{t^-} &= \frac{-c_2 v^2_1 (v^2)^2 u^1_{-1}u^2 + v^2\left(v^2_1 u^1_{-3}u^1_{-2}u^2_{-1} - u^1_{-1}\left(c_2 + v^2_1\left(u^1 u^2_1 - 1\right)\right)\right) + u^1_{-2}u^1_{-1}}{v^2_1} \end{aligned} \tag{206}$$

to (195), and from the equation

$$\begin{aligned} v^1_{t^+} &= \frac{c_1 v^1_1\left(u^1_{-2}u^2_{-1} - 1\right) + v^1\left(c^2_1 u^2_{-1}\left(u^1_{-2}u^2 + u^1_{-3}u^2_{-2} - 1\right) + 1\right) + v^1_2 u^1_{-1}}{c^2_1}, \\ v^2_{t^+} &= \frac{-c_2 (v^2)^2 u^2 + v^2\left(c^2_2 u^2_{-1}\left(u^1_{-3}u^2_{-2} - 1\right) + c^2_2 u^2\left(1 - u^1_{-1}u^2_1\right) + 1\right)}{c^2_2} + \\ &\quad + \frac{v^2_1 c_2\left(u^1_{-2}u^2_{-1} - 1\right) + u^1_{-1}\left(1 - c_2 v^2 u^2\right)}{c^2_2 v^2_1} \end{aligned} \tag{207}$$

to (196). In formulas (206), (207), one has $u^1_k = \mathcal{S}^k(u^1)$ and $u^2_k = \mathcal{S}^k(u^2)$ for $k \in \mathbb{Z}$, where u^1, u^2 are given by (205). For example, according to (205),

$$u^2_{-1} = \mathcal{S}^{-1}(u^2) = \mathcal{S}^{-1}\Big(\frac{-c_1 v^1_2 + v^1_3 v^2_2\left(c_2 - v^2_1\right) + v^1_1}{c_1 v^1 - c_2 v^1_3 v^2 v^2_1 v^2_2}\Big) = \frac{-c_1 v^1_1 + v^1_2 v^2_1\left(c_2 - v^2\right) + v^1}{c_1 v^1_{-1} - c_2 v^1_2 v^2_{-1} v^2 v^2_1}.$$

6. Conclusion

In this paper, we have presented a geometric method to construct Miura-type transformations (MTs) and modified equations for differential-difference (lattice) equations, including multicomponent ones. We construct MTs and modified equations from invariants of certain Lie group actions on manifolds associated with matrix Darboux-Lax representations (DLRs) of differential-difference equations.

Using this construction, from a given suitable DLR one can obtain many MTs of different orders. As has been shown in Example 8, the order of the obtained MTs may be higher than the size of the matrices in the DLR.

Applying this method to DLRs of the Volterra, Narita-Itoh-Bogoyavlensky, Toda lattices, and Adler-Postnikov lattices from [1], we have constructed a number of MTs and modified lattice equations. The MTs (160), (162), (187), (205) and modified equations (161), (163), (188), (206), (207) seem to be new.

The described method to construct MTs can help in solving classification problems for integrable differential-difference equations. Indeed, when one tries to classify a certain class of such equations, one often finds a few basic equations such that all other equations from the considered class can be obtained from the basic ones by means of MTs [20]. Therefore, systematic methods to construct MTs can help in obtaining such classification results.

In Section 2, for any given DLR of the form (30) we have defined an increasing sequence of groups $\mathbb{H}_k$, $k \in \mathbb{Z}_{\geq 0}$. As has been shown in Section 2, these groups play a crucial role in the construction of MTs. In Sections 2, 3, 4, 5, we have computed $\mathbb{H}_k$ for some small values of k for a number of DLRs, and this has allowed us to construct the above-mentioned MTs.

It would be interesting to compute the groups $\mathbb{H}_k$ for higher values of k, because this may give many new kinds of MTs. Also, it would be interesting to try to describe all MTs that can be constructed by the presented method for some well-known DLRs (such as the DLRs of the Narita-Itoh-Bogoyavlensky and Toda lattices).

It is well known that auto-Bäcklund transformations can often be obtained as compositions of MTs. Therefore, it is natural to ask the following question: when can one obtain auto-Bäcklund transformations, using MTs constructed by the presented method?

We leave these problems for future work.

Acknowledgements

The authors would like to thank V. E. Adler, A. V. Mikhailov, Yu. B. Suris, P. Xenitidis, and R. I. Yamilov for useful discussions. The authors gratefully acknowledge support from a Leverhulme Trust grant.

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