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A priori stability results for PFC

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Despite is popularity in industry and obvious efficacy, Predictive Functional Control has few rigorous a priori stability results in the literature. In many cases, common sense and intuition with some trial and error are the main design tools. This paper seeks to tackle that gap by providing some analysis of the control law and showing what forms of stability assurances can be given and how these depend on the user choices of coincidence horizon and desired closed-loop pole. The conditions are separated into necessary, but not sufficient conditions for stability and conversely, sufficient but not necessary conditions. Numerical examples demonstrate the efficacy of these conditions and the ease of use.

Keywords: PFC, guaranteed stability, coincidence horizon, monotonicity

1 Introduction

Predictive functional control (PFC) has been widely used in industry Richalet et al. (1978), Changenet et al. (2008), Fallasohi et al. (2010) but received relatively little attention in the academic literature Mayne et al. (2000). The popularity with practitioners is due to the approach making good intuitive sense and the vendors have worked hard to ensure that the practical implementation builds as much as possible on existing software and a relatively simple understanding of feedback. PFC is a valuable industrial tool because it exploits the benefits of a predictive control approach Richalet and O'Donovan (2009), Haber et al. (2011), Maciejowski (2001), Rossiter (2003) while at the same time requiring relatively trivial coding and very fast computation times; indeed it should be emphasised that PFC is a competitor with classical approaches such as PID on single input single output loops and not with large scale, expensive, multivariable predictive control algorithms such as Dynamic Matrix Control. Nevertheless some of the PFC design rules are not immediately obvious and the stability and convergence results are weak, albeit this is rarely an issue in practice.

The prime purpose of this paper is to explore the extent to which some rigorous *a priori* stability results can be given for PFC and thus to increase confidence in its use; by *a priori* we mean that the stability of the closed-loop can be determined before computing the controller parameters explicitly. It is already known that for a 1st order system one can give explicit stability results (e.g.Richalet and O'Donovan (2009), Rossiter et al. (2016)) hence these results will be summarised briefly, but the more interesting aspect is the extent to which one can give rigorous results for systems with higher order dynamics. The paper does not cover parameter uncertainty explicitly although, as with all control laws, some degree of robustness is implicit. This paper develops several results which provide insight into the choices of coincidence horizon and its effect on stability. After some background in section 2, section 3 summarises the basic stability results available and section 4 then introduces a number of new results and insights. Section 5 demonstrates the efficacy of the results on several examples.

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2 Predictive Functional Control

This paper will use the most common PFC variant where the targets are assumed constant and therefore prediction is based on a constant future input whose magnitude is the degree of freedom at each sample. For simplicity of presentation, non-zero dead-time examples are not presented; the required modification is such that delay does not affect any nominal stability results presented, although clearly it would affect the sensitivity.

2.1 *Target and coincidence point*

The basic objective of PFC is to ensure that the predicted process output y_p matches a specified reference trajectory at a chosen coincidence point n_y samples in the future. The reference trajectory $r(k)$ is taken as the response of a 'desired' first order system from the current point to the steady-state target R, and hence at sample k:

$$r(k+i) = R - (R - y_p(k))\lambda^i, \quad i = 1, 2, ... \tag{1}$$

where λ is the desired closed-loop pole location and $r(k+i)$ is the reference trajectory or target i-steps ahead. Typically $\lambda = e^{\left(\frac{-3t_s}{t_{cl}}\right)}$ where, t_{cl} and t_s are desired closed-loop settling time (to 95%) and sampling time, respectively. PFC is defined by determining the future value of the input $u(k) = u(k+1) = \cdots = u(k+n_y)$ such that the output prediction matches the target trajectory at a single specified point, the so called coincidence point, n_y samples into the future. Thus choose $u(k)$ such that:

$$y_p(k+n_y|k) = R - (R - y_p(k))\lambda^{n_y} = r(k+n_y) \tag{2}$$

The notation $y_p(k+n_y|k)$ means the n_y step ahead prediction made at sample k.

Remark 2.1 PFC designers usually pose the problem slightly different and talk about the change in the output as this has slight advantages in the associated algebra, thus:

$$\delta y_p = y_p(k+n_y|k) - y_p(k) = (R - y_p(k))(1 - \lambda^{n_y}) \tag{3}$$

2.2 *Offset free tracking and independent models*

PFC ensures offset free tracking by basing predictions on an independent model (which is simulated in parallel with the actual process and thus effectively doubles as an observer) and then using the difference between the independent model output y_m and the true process output y_p as a bias term to correct any predictions. A key point is that one can show that the expected change in the model output matches the expected change in the process output and thus control law (3) can be replaced by:

$$\delta y_m = y_m(k+n_y|k) - y_m(k) = (R - y_p(k))(1 - \lambda^{n_y}) \tag{4}$$

These subtleties are not core to the current paper as the stability results to be derived apply for the nominal case and thus one would be assuming that the model and process are identical. Small gain theorems and the like could be used to establish the level of uncertainty the process could cope with, but such analysis is well established elsewhere and therefore not pursued here.

2.3 *The challenge*

A PFC designer has two decisions to make:

Table 1. Dependence of dominant closed-loop pole ρ on coincidence horizon n_y for $\lambda = 0.75$ for examples 1-4.

$G(z)$	$n_y = 1$	$n_y = 3$	$n_y = 5$	$n_y = 7$	$n_y = 9$	$n_y = 12$
G_1	-1+3j	j0.6	0.75	0.76	0.78	0.82
G_2	0.75	0.8+j0.3	0.93+0.5j	0.9+0.7j	.5+.65j	0.7+0.5j
G_3	1.25	1.5	4.2	0.74	0.76	0.81
G_4	0.75	0.74	0.74	0.8	0.82	0.85

(1) How to select the target value of λ (equivalently t_{cl}).
(2) How to select the coincidence horizon n_y.

While there exist sensible practical guidelines Richalet and O'Donovan (2009), Rossiter et al. (2015b) for such selections, these lack a strong theoretical analysis except for 1st order models and a certain amount of trial and error is assumed. Consider an over-damped process G_1 (example 1), an under-damped process G_2 (example 2), a non-minimum phase process G_3 (example 3) and G_4 a slow process (example 4). Compute the dominant closed-loop pole from a PFC design with various choices of n_y (see table 1).

$$\begin{array}{c} G_1 = \frac{0.01z^2+0.02z+0.1}{z^3-2z^2+1.27z-0.252}; \quad G_2 = \frac{-0.4z+0.2}{z^2-1.4z+0.8}; \\ G_3 = \frac{-0.4z+0.5}{z^2-1.5z+0.54}; \quad G_4 = \frac{-0.4z+0.2}{z^2-1.8z+0.81} \end{array} \tag{5}$$

In many cases, a judicious pairing of n_y, λ is very effective in that there exists a pairing which gives the desired closed-loop pole. However, Table 1 also illustrates that the link between the control law of (4) and the actual closed-loop dynamics that results can be weak (Richalet and O'Donovan 2009) if the coincidence horizon (or indeed λ) is inappropriately chosen. Consequently there is a need to establish the extent to which the design of a PFC law can be stated *a priori* to result in good closed-loop behaviour.

3 Simple stability results for PFC

It is possible to establish rigorous closed-loop stability for some elementary cases and thus it is worth summarising those results before moving to the more general case. Hereafter, as we are considering the nominal case only, the distinction between model and process outputs is dropped where not needed to simplify the algebra.

3.1 *First order models*

It is known (Rossiter et al. 2015b) that for a first order system,

$$y(k+1) = ay(k) + bu(k) \tag{6}$$

the best choice of coincidence horizon is $n_y = 1$ and thus the control law (2) is given as:

$$\begin{array}{c} ay(k) + bu(k) - y(k) = (R - y(k))(1-\lambda) \\ \Downarrow \\ u(k) = \frac{R(1-\lambda)-(\lambda+a)y(k)}{b} \end{array} \tag{7}$$

Substituting back into the model (6) one can deduce that:

$$\begin{array}{l} y(k+1) = R(1-\lambda) - (1-\lambda+a)y(k) + ay(k) \\ y(k+1) = R(1-\lambda) + \lambda y(k) \end{array} \tag{8}$$

In other words, the closed-loop pole has moved precisely to λ as desired.

3.2 *Mean-level approaches*

A mean-level approach assumes that the open-loop dynamics are satisfactory (implicitly stable), and therefore the only requirement is to ensure offset free tracking and constraint satisfaction. One can view this like a PFC control law with a coincidence horizon of infinity, that is:

$$\lim_{n_y \to \infty} y_m(k+n_y|k) = R \tag{9}$$

Such a control law reduces to estimating the expected steady-state value for the input:

$$u(k) = u_{ss} = E[u] \ \text{ s.t. } (9) \tag{10}$$

This is calculated from the model steady-state $G(1)$ as:

$$\lim_{n \to \infty} y_m(k+n_y|k) = G(1)u_{ss} = R \tag{11}$$

Remark 3.1 A mean-level approach is guaranteed convergent in the nominal case for stable open-loop processes; the proof of this is obvious. In practice one would modify (11) to include a disturbance estimate correction to cater for uncertainty.

Remark 3.2 Using a mean-level algorithm as a base, one can superimpose a closed-loop response dynamic (pole of $1-\beta$) over the open-loop dynamics by using a control law such as the following:

$$u(k) - u(k-1) = \beta(u_{ss} - u(k-1)) \tag{12}$$

The proof of convergence and pole positioning is obvious.

3.3 *Intuitive stability proof for PFC*

The underlying attraction of PFC is the common sense nature of the approach which works well with certain dynamics. Consider speed control in a car: (i) the driver pushes the accelerator pedal a short distance expecting this to give a desired acceleration towards the desired speed; (ii) after a short time the car has speeded up and the driver decides whether another nudge on the pedal is required; (iii) this process is repeated continuously. The key point is that the decision making process is based on 'an estimated speed profile' over a quite limited horizon (perhaps only 2-3 seconds) and yet, despite the lack of infinite horizons (Mayne et al. 2000) and a relative simple decision, we know it will work, but why?

The basic principle used (implicitly although not explicitly) is that the behaviour is monotonic, that is the speed moves gradually towards the target and does not oscillate. Consequently, when we update our decisions, we are always doing so from a point where the error e_k is smaller than at the previous sample. In other words, there is an implicit constraint that:

$$|e(k+1|k)| < |e(k)| \tag{13}$$

which clearly, if always satisfied, guarantees convergence!

Remark 3.3 One could ensure stability by explicitly enforcing contraction constraints such as (13), but such approaches are outside the remit of this paper although popular in some parts of the literature.

Although the argument above appears intuitively reasonably, embedded within it is the human based reasoning that we consider the entire prediction and asymptotic values, not just the prediction at a single coincidence point. So, our intuition is actually working on the fact that we ensure the asymptotic error reduces, not just the one step ahead (or n-step ahead) error. PFC does not include the asymptotic error explicitly and so, tempting although it is, such an intuitive argument cannot be used rigorously and indeed it is easy to find counter examples. Obviously, the many successful implementations demonstrate that despite this lack of rigour, PFC usually works well and thus the underlying intuition can be an effective argument.

4 New stability results for PFC

This section will develop two more generic convergence proofs for PFC. Both proofs will assume a system with stable open-loop dynamics and an impulse response:

$$G(z) = \sum_{i=1}^{\infty} g_i z^{-i}; \quad \lim_{i\to\infty} g_i = 0 \tag{14}$$

For convenience, and without loss of generality, assume the following:

- The steady-state gain is positive, that is, $\sum_{\infty} g_i > 0$.
- Define monotonicity of the step response if $g_i \geq 0, \forall i$.
- Hereafter the paper takes $y = y_m = y_p$ as the aim is to demonstrate that control law is stabilising in the nominal case, not to discuss sensitivity.

The two approaches considered are described next.

(1) The first assumes that the system has stable zeros.
(2) The second will relax the criteria on the zeros and consider how monotonicity can be exploited and hence give conditions which are slightly more general.

4.1 *Stability for systems with minimum phase step responses*

This section will demonstrate that the absence of non-minimum phase zeros is sufficient to establish a guarantee of stability for PFC (but not necessarily good performance).

Theorem 4.1 For a coincidence horizon of $n_y = 1$ and any choice of λ ($0 < \lambda < 1$), stable zeros are sufficient to guarantee that the PFC law gives a stable closed-loop in the nominal case.

Proof: The control law is defined from:

$$y(k+1|k) - y(k) = (R - y(k))(1 - \lambda) \tag{15}$$

Consequently, by definition, the error is monotonically decreasing in magnitude, which implies convergence of the output and with precisely the desired 1st order response! That is:

$$e(k+i) = \lambda e(k+i-1); \quad e(k+i) = R - y(k+i) \tag{16}$$

It remains only then to demonstrate that the corresponding input is not divergent. Combining (15) with a generic model:

$$\left\{ \begin{array}{l} y(z) = \frac{1-\lambda}{1-\lambda z^{-1}} \frac{R}{1-z^{-1}} \\ a(z)y(z) = b(z)u(z) \end{array} \right\} \Rightarrow u(z) = \frac{a(z)}{b(z)} \frac{1-\lambda}{1-\lambda z^{-1}} \frac{R}{1-z^{-1}} \tag{17}$$

Hence $u(z)$ is stable iff $b(z)$ has no unstable roots. □

Corollary 4.2 If the system numerator has any unstable roots, then the use of $n_y = 1$ is guaranteed to give closed-loop divergence of the inputs. Some readers will recognise the parallels between this result and the minimum variance literature (Clarke and Gawthrop 1975).

Examples 1-4 given earlier are used to demonstrate this.

(1) Models G_2, G_4 from (5) have stable zeros and thus, notwithstanding other aspects of their dynamics, give convergent behaviour when $n_y = 1$ (see table 1).
(2) Models G_1, G_3 in (5) are unstable with PFC when $n_y = 1$ (see table 1). This is the case even though G_1 has a step response which is monotonic and close to that of a 1st order system.

In summary, a choice of $n_y = 1$ will give the desired output dynamic exactly, as long as the system has stable zeros. This is as the expense of whatever input activity is required and thus may not be a good design! Where the implied input is over active, the use of $n_y = 1$ is inadvisable.

4.2 *Exploiting monotonicity*

The previous section demonstrated clearly that for many systems a choice of $n_y = 1$ is insufficient. Conversely, section 3.2 showed that a choice of $n_y = \infty$ is always sufficient to ensure stable closed-loop behaviour. It would be interesting therefore to ask whether one can find a minimum value of n_y beyond which closed-loop convergence can be assured. For convenience, this section will use the impulse response representation of a model dynamics as the PFC law can then be written down by inspection.

The output prediction with model (14) is:

$$y(k+n_y|k) = \sum_{i=1}^{n_y} g_i u(k) + \sum_{i=1}^{\infty} g_{n_y+i} u(k-i) \tag{18}$$

Define the n_y step ahead step response value as:

$$h_{n_y} = \sum_{i=1}^{n_y} g_i \tag{19}$$

Lemma 4.3 The PFC control law with an impulse response model can be expanded as:

$$h_{n_y} u(k) + \sum_{i=1}^{\infty} [g_{n_y+i} - \lambda^{n_y} g_i] u(k-i) = (1-\lambda^{n_y})R \tag{20}$$

Proof: Substitute model predictions from (18) into the nominal control law of (4) and hence:

$$\begin{array}{c} \sum_{i=1}^{n_y} g_i u(k) + \sum_{i=1}^{\infty} g_{n_y+i} u(k-i) = (1-\lambda^{n_y})R \\ +\lambda^{n} \sum_{i=1}^{\infty} g_i u(k-i) \end{array} \tag{21}$$

Next, rearranging to group common terms gives the result. □

Corollary 4.4 The closed-loop pole polynomial for PFC can be represented as:

$$p_c = h_{n_y} + \sum_{i=1}^{\infty} [g_{n_y+i} - \lambda^{n_y} g_i] z^{-i} = \sum_{i=0}^{\infty} p_i z^{-i} \tag{22}$$

This is an obvious corollary of (20).

4.3 *Necessary conditions using Jury tests*

Because the polynomial of (22) has an infinite number of terms, we cannot easily utilise an entire Jury's test, however we deduce conditions which are necessary for p_c to have stable roots; these will allow the user to avoid scenarios where instability is guaranteed! For convenience this section assumes that the first coefficient of a polynomial is positive. Jury tests state that a polynomial $p(z) = p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots$ with roots inside a unit circle must satisfy the following criteria:

$$p_0 > 0, \quad \sum_{i=0}^{\infty} p_i > 0, \quad \sum_{i=0}^{\infty} (-1)^i p_i > 0 \tag{23}$$

Remark 4.5 In classical PFC approach, the coincidence point is chosen large enough to be beyond any negative values in the step response; it is obvious that failure to do so is likely to lead to closed-loop instability. Thus, the condition $p_0 = h_{n_y} = \sum_{i=1}^{n_y} g_i > 0$ is implicitly enforced.

Lemma 4.6 For the polynomial p_c in (22) one can write:

$$\sum_{i=0}^{\infty} p_i = (1 - \lambda^{n_y}) \sum_{i=1}^{\infty} g_i > 0 \tag{24}$$

This follows from the assumption of positive steady-state gain.

Lemma 4.7 For the polynomial p_c in (22) one can write the 3rd Jury condition as:

$$M_{n_y} = \sum_{i=0}^{\infty} (-1)^i p_i = \sum_{i=1}^{n_y} g_i + \sum_{i=1}^{\infty} (-1)^i [g_{n_y+i} - \lambda^{n_y} g_i] \tag{25}$$

Hence a requirement is that $M_{n_y} > 0$ where

$$M_{n_y} = \sum_{i=1}^{n_y} g_i (1 - (-1)^i \lambda^{n_y}) + \sum_{i=1}^{\infty} (-1)^i [g_{n_y+i}(1 - \lambda^{n_y})] \tag{26}$$

Theorem 4.8 For a system controlled with PFC necessary conditions for closed-loop stability given in (23) can be reduced to the satisfaction of (26).

Proof: This evident because the first two conditions of (23) are satisfied by inspection from remark 4.5 and lemma (4.6). □

Algorithm 4.9 For values of n_y s.t. $h_{n_y} > 0$, to a number around the settling time, compute the values M_{n_y} and then establish a minimum plausible value of n_y such that PFC can be stable by requiring that $M_{n_y} > 0$.

4.4 *Sufficient conditions for stability*

The previous subsection gave some necessary conditions for closed-loop stability, but in themselves these are not sufficient. This section introduces sufficient conditions, although ones that are not necessary.

Theorem 4.10 For a polynomial $p(z)$, the following condition gaurantees no roots are outside the unit circle.

$$|p_0| > \sum_{i=1}^{\infty} |p_i| \tag{27}$$

Proof: This follows automatically from a root outside the unit circle being such that $|z| > 1$, and hence this can be a root iff:

$$|p_0| \leq \sum_{i=1}^{\infty} \frac{|p_i|}{|z|^i} \leq \sum_{i=1}^{\infty} |p_i| \tag{28}$$

which thus gives a contradiction. □

Algorithm 4.11 For values of $n_y = 1, 2, \cdots$, define the polynomial $P_c(z)$ using (22) and perform the test of (27). Select n_y large enough so that the test is satisfied.

Remark 4.12 For a 1st order process, $g_i = \alpha^i$ and hence one can write:

$$p_c = \frac{1-\alpha^n}{1-\alpha} + \sum_{i=1}^{\infty} (\alpha^{n_y} - \lambda^{n_y})\alpha^i z^{-i} \tag{29}$$

It is clear in this case, that higher order coefficients are all close to zero, especially if $\alpha \approx \lambda$ and the condition is in fact satisfied for all n_y. Intuitively therefore, for any process whose impulse response is close to a first order response, one would expect the higher order coefficients to be very small and hence the required condition to be satisfied apart from for low n_y where there may be some lag in the step response so that h_{n_y} is small.

4.5 *Results with monotonic step responses*

Many systems have monotonic step responses (here assume $g_i > 0, \ \forall i$ for convenience) and this attribute can be used to determine a less conservative sufficient condition for stability. First rearrange the closed-loop pole polynomial of (22) as:

$$p_c(z) = \sum_{i=1}^{n_y} [g_i(1 - \lambda^{n_y} z^{-i})] + \sum_{i=1}^{\infty} g_{n_y+i} z^{-i}(1 - z^{-n_y}\lambda^{n_y}) \tag{30}$$

The roots of $p_c(z)$ can be solved from:

$$\sum_{i=1}^{n_y} [g_i(1 - \lambda^{n_y} z^{-i})] = -\sum_{i=1}^{\infty} g_{n_y+i} z^{-i}(1 - z^{-n_y}\lambda^{n_y}) \tag{31}$$

Lemma 4.13 Outside the unit circle the lower bound (may not be reachable) of the LHS of (31) is:

$$f_{min} = \min_{|z|>1} \sum_{i=1}^{n_y} [g_i(1 - \lambda^{n_y} z^{-i})] = h_{n_y}(1 - \lambda_y^n) \tag{32}$$

Proof: Using moduli and noting that $g_i > 0, \forall i$ it is clear that:

$$f_{min} > \sum_{i=1}^{n_y} g_i - \lambda^{n_y} \sum_{i=1}^{n_y} g_i |z^{-i}| \quad \Rightarrow \quad f_{min} > h_{n_y} - \lambda^{n_y} \sum_{i=1}^{n_y} g_i |z^{-i}| \tag{33}$$

Hence the minimum is achieved by maximising $\sum_{i=1}^{n_y} g_i |z^{-i}|$. Given that z is outside the unit circle so $|z| > 1$ it is clear that the maximum is achieved with $|z| = 1$ and hence (32) is established. □

Lemma 4.14 The upper bound (may not be reachable) of the RHS of (31)) is:

$$f_{max} = \max_{|z|>1} \sum_{i=1}^{\infty} g_{n_y+i}(z^{-i} - z^{-i-n_y}\lambda^{n_y}) = G(1) - h_{n_y} \tag{34}$$

Proof: Rearrange (34) as follows:

$$f_{max} = \max_{|z|>1} \sum_{i=1}^{\infty} g_{n_y+i} z^{-i}(1 - z^{-n_y}\lambda^{n_y}) \tag{35}$$

Both of the following are clear given $|z| > 1$:

$$|\sum_{i=1}^{\infty} g_{n_y+i} z^{-i}| < G(1) - h_{n_y} \quad \text{and} \quad |(1 - z^{-n_y}\lambda^{n_y})| < 1 \tag{36}$$

Consequently (34) follows immediately. □

One can now use observations (32,34) to determine a simple and clear link between λ and the require coincidence horizon which guarantees closed-loop stability.

Theorem 4.15 Assuming that $G(z)$ has a monotonic step response, then for $p_c(z)$ to have a root outside the unit circle it is necessary that:

$$2 - \frac{G(1)}{h_{n_y}} > \lambda^n \tag{37}$$

Proof: This falls directly out of eqns. (32,34). It is clear that $p(z)$ can have a root outside the unit circle iff the LHS and the RHS of (31) match, which means the maximum of the RHS must exceed the minimum of the LHS.

$$h_{n_y}(1 - \lambda_y^n) < G(1) - h_{n_y} \tag{38}$$

$$h_{n_y}(2 - \lambda^{n_y}) < G(1) \tag{39}$$

□

Corollary 4.16 Obviously, the converse of theorem 4.15 gives a sufficient condition for closed-loop stability, that is one whereby no pole of $p(z)$ can lie outside the unit circle.

$$h_{n_y}(2 - \lambda^{n_y}) > G(1) \quad \text{or} \quad Q_{n_y} = \frac{h_{n_y}(2 - \lambda^{n_y})}{G(1)} > 1 \tag{40}$$

Hence, for a specified λ^{n_y}, one can check whether n_y is large enough by ensuring that $Q_{n_y} > 1$.

Corollary 4.17 The condition of (40) is sufficient for closed-loop stability, but not necessary. However, it gives a neat result which is that a good choice of n_y is such that:

$$h_{n_y} > \frac{G(1)}{2} \tag{41}$$

This is an obvious consequence of corollary 4.16.

Readers may note the similarity between this recommendation and that in Rossiter et al. (2015b) and also the difference with a conventional PFC recommendation Richalet and O'Donovan (2009) of choosing n_y such that g_{n_y} is a maximum.

4.6 *Reflections*

Readers may be disappointed that the results presented are not stronger, especially for processes with monotonic step responses which one would expect to respond well under PFC control. However, given the number of conditions deployed in conventional predictive control (e.g. Mayne et al. (2000)) to ensure stability, it is unsurprising that PFC, which does not meet most of those conditions, has much weaker *a priori* stability results. Nevertheless it is useful to know what it is possible to state with confidence:

(1) For 1st order models stability is automatic, for any n_y and λ, although *this need not imply good performance* and indeed 1st order systems are hardly challenging.
(2) For any order of system, a choice of $n_y = 1$ is guaranteed stabilising with good output behaviour, iff there are no unstable zeros. In this case, PFC reduces to minimum variance control.
(3) More general results are either: (i) necessary, but not sufficient or (ii) sufficient but not necessary. These are useful as they give strong indications of sensible choices for the coincidence horizon.

4.7 *Turpin point*

Within conventional PFC applications, the choice of coincidence horizon is often indicated by the so called Turpin point, see Richalet and O'Donovan (2009). This is defined in this paper so it can be added to the numerical examples for information. The idea is to select the horizon which gives the least aggressive initial control move, assuming zero initial conditions. This input value is given as:

$$u(0) = (1 - \lambda^{n_y})/h_{n_y} \tag{42}$$

Consequently, one could select n_y such that $u(0)$ is minimised although in practice, this is used as information rather than applied rigorously. The key point is consider the extent to which $u(0)$ is larger than the expected steady-state input. One can only speed up dynamics with some over activity, but this should not be excessive.

5 Numerical examples

The examples section will demonstrate the analysis tools available to the PFC designer.

(1) A plot of $T_{n_y} = G(0)u(0)$ against n_y (from eqn.(42)) is an indicator of the aggressiveness of the initial control action with a value of unity implying open-loop characteristics.
(2) A plot $D_{n_y} = |p_0|/\sum_{i=1}^{\infty} |p_i|$. A sufficient condition for closed-loop stability is $D(n_y) < 1$.
(3) A plot M_{n_y} from (26) vs n_y. A necessary condition for stability is $M_{n_y} > 0$.
(4) The maximum modulus P_{n_y} closed-loop pole is superimposed for information as to the efficacy of the tests. Clearly these should be less than zero.
(5) A plot of Q_{n_y} from (40) as $Q_{n_y} > 1$ is a sufficient condition for closed-loop stability, but only where the step response is monotonic.
(6) A line of unity (dashed line) and the open-loop step response (dotted line) are superimposed on the figures.

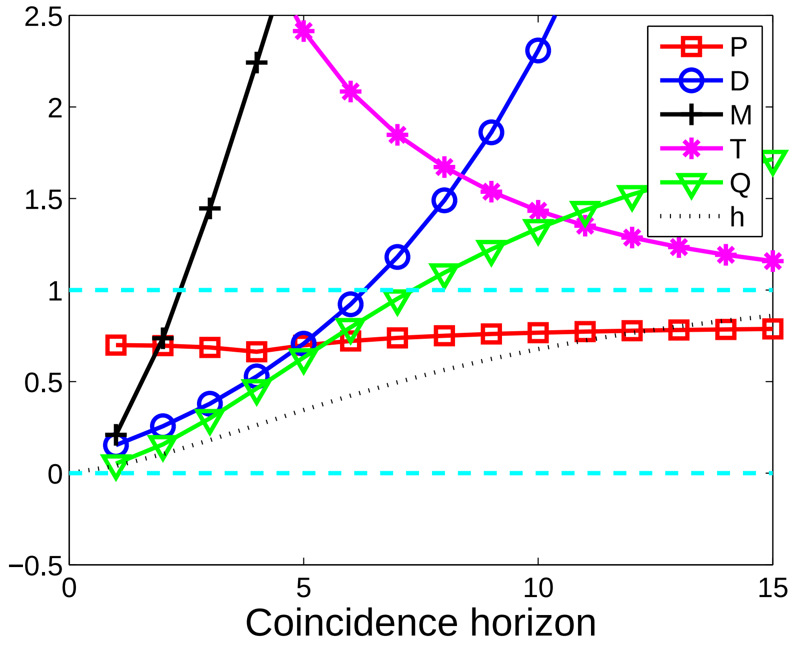

Figure 1. Variation of $P_{n_y}, M_{n_y}, D_{n_y}, T_{n_y}, Q_{n_y}, h_{n_y}$ with n_y for example 5.

5.1 *Example 5*

A simple critically damped 2nd order system is given as:

$$G_5(z) = \frac{z - 0.4}{z^2 - 1.6z + 0.64}; \quad \lambda = 0.7 \tag{43}$$

This system has a simple monotonic step response and thus, intuitively one may expect stable closed-loop behaviour for all n_y, albeit, due to the initial lag, a choice of small n_y would likely cause significant over activity. The associated values of $P_{n_y}, M_{n_y}, D_{n_y}, T_{n_y}, h_{n_y}$ are given in figure 1. It is clear that: (i) PFC is stable for all n_y ; (ii) the Turpin plot indicates that the initial input is significantly over active unless $n_y \geq 10$; (iii) the Jury test is satisfied for all n_y; (iv) the D_{n_y} sufficiency test is satisfied for $n_y > 6$ and the Q_{n_y} sufficiency test for $n_y > 7$; (v) a choice of $n_y \approx 6 - 8$ accords with expectations given the step response Rossiter et al. (2015b) and the target λ being faster than open-loop dynamics.

5.2 *Example 6*

An under-damped 3rd order system is given as:

$$G_6(z) = \frac{z^2 + 1.6z + 0.48}{z^3 - 2.1z^2 + 1.72z - 0.518}; \quad \lambda = 0.75 \tag{44}$$

This system has a slightly oscillatory step response as well as a small initial lag and thus it is less obvious how to choose n_y. The associated values of $P_{n_y}, M_{n_y}, D_{n_y}, T_{n_y}, h_{n_y}$ are given in figure 2. It is clear that: (i) PFC is stable for all $n_y > 1$; (ii) the Jury test fails for $n_y = 1$ as expected (that is $M_1 < 0$) given this is a necessary condition. (ii) the Turpin plot indicates that the initial input is over active unless $n_y \geq 4$; (iv) the D_{n_y} sufficiency test is satisfied for $n_y > 5$ whereas the Q_{n_y} sufficiency test requires $n_y > 3$ and (v) a choice of $n_y \approx 4 - 6$ accords with expectations given the step response Rossiter et al. (2015b). This is validated by closed-loop responses which

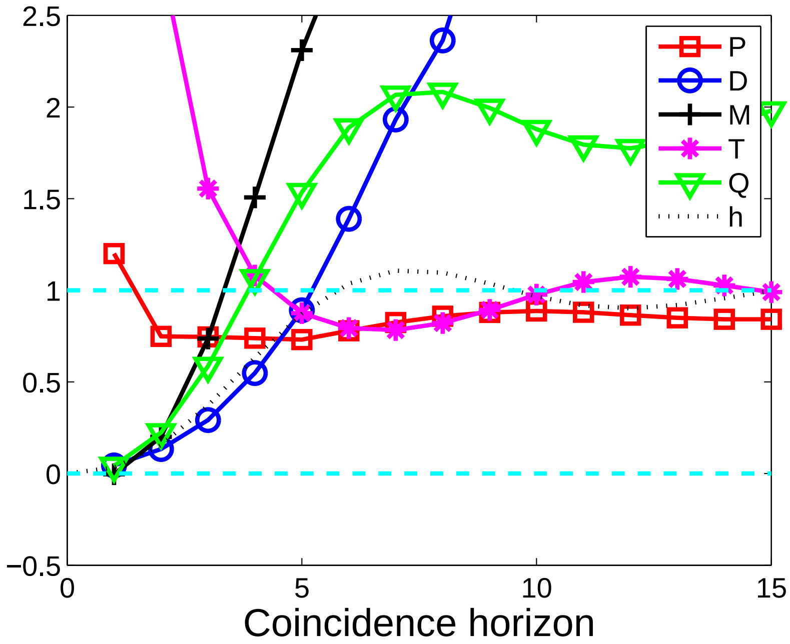

Figure 2. Variation of $P_{n_y}, M_{n_y}, D_{n_y}, T_{n_y}, h_{n_y}$ with n_y for example 6.

are poor if $n_y \leq 4$.

5.3 *Example 7*

An non-minimum phase 2nd order system is given as:

$$G_7(z) = \frac{-0.15z^2 - 0.2z + 0.4}{z^3 - 1.5z^2 + 0.56z}; \quad \lambda = 0.75 \tag{45}$$

This system has a significant non-minimum phase characteristic which means low n_y will be ineffective. The associated values of $P_{n_y}, M_{n_y}, D_{n_y}, T_{n_y}, h_{n_y}$ are given in figure 3: (i) PFC is unstable for all $n_y < 10$; (ii) the Jury test fails for $n_y < 9$ as expected (that is $M_1 < 0$) given this is a necessary (but not sufficient) condition. (ii) the Turpin plot indicates that the initial input is over active unless $n_y \geq 15$; (iv) the sufficiency test is satisfied for $n_y > 11$ and (v) a choice of $n_y \approx 11 - 15$ accords with expectations given the step response Rossiter et al. (2015b); this is validated by closed-loop responses.

6 Conclusions

Despite its obvious success, there are few rigorous *a priori* stability results for PFC in the academic literature. Intuitive arguments such as the PFC strategy implicitly gives control moves that gradually guide one to the target are in fact very weak if analysed in detail, especially where the coincidence horizon is small and small coincidence horizons have often be favoured due to the stronger linking with the desired response time.

This paper has developed some very simple *a priori* conditions that can be used by the designer in order to ascertain reasonable choices for the coincidence horizon in advance. By assuming the steady-state gain is defined as positive (for convenience only), the paper has given two types of conditions:

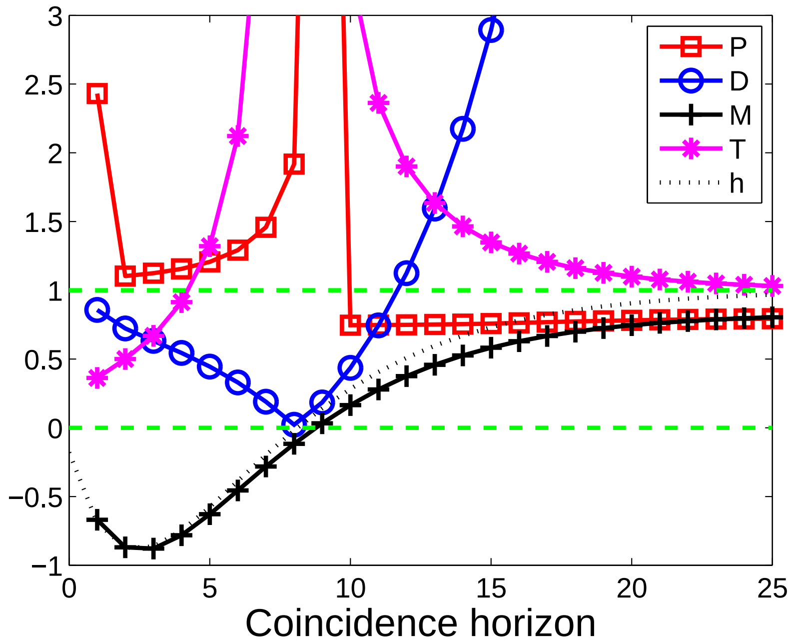

Figure 3. Variation of parameters $P_{n_y}, M_{n_y}, D_{n_y}, T_{n_y}, h_{n_y}$ with n_y for example 7.

(i) Necessary conditions which must be satisfied. These reduce to just one Jury test given in (26).

(ii) Sufficient, but not necessary, conditions which guarantee stability are given in (27) and (40,41). The latter of these can only be applied when the step response is monotonic, but may be less conservative. These tests use only the impulse response coefficients and hence can be determined very quickly.

It is interesting, and perhaps unsurprising, to note that the sufficient conditions in essence reduce to recommending large values of coincidence horizon which in turn means that PFC is closer to a mean-level type of strategy for which a stability guarantee is well known!

It is noted that the analysis of this paper does not apply to open-loop unstable processes for which a slightly modified PFC process is required in order to derive an effective and reliable control law. Indeed such issues are well understood in the traditional MPC literature Rossiter et al. (1998).

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