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A Coevolutionary Particle Swarm Algorithm for Bi-Level Variational Inequalities: Applications to Competition in Highway Transportation Networks

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Abstract A climate of increasing deregulation in traditional highway transportation, where the private sector has an expanded role in the provision of traditional transportation services, provides a background for practical policy issues to be investigated. One of the key issues of interest, and the focus of this chapter, would be the equilibrium decision variables offered by participants in this market. By assuming that the private sector participants play a Nash game, the above problem can be described as a Bi-Level Variational Inequality (BLVI). Our problem differs from the classical Cournot-Nash game because each and every player's actions is constrained by another variational inequality describing the equilibrium route choice of users on the network. In this chapter, we discuss this BLVI and suggest a heuristic coevolutionary particle swarm algorithm for its resolution. Our proposed algorithm is subsequently tested on example problems drawn from the literature. The numerical experiments suggest that the proposed algorithm is a viable solution method for this problem.

1 Introduction

Mathematicians have defined a class of problems known as equilibrium problems with equilibrium constraints (EPECs) [33]. A particular subclass of these problems are Bi-Level Variational Inequalities (BLVI). These are effectively Cournot-Nash games modeled in a hierarchical fashion: The upper level problem is a Cournot-Nash game and the constraints intrinsically define yet another Nash game parameterized by the solutions to former. In this chapter, the proposed algorithm is extended to solve a BLVI that arises from competition in the provision of services within the highway transportation sector.

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Our motivation stems from the observation that in recent years, there has been an increasing amount of private sector participation within areas that are conventionally the privy of the public purse. The driving force behind this change is brought about by the higher efficiency of the private sector coupled with increasing public pressures on governments for accountability and the corresponding need to derive value for money from their various budgetary commitments which are ultimately funded by the tax paying public [30, 53].

In highway transportation, privately operated roads are not novel concepts [56]. However there has been little analysis on this topic in terms of the competition between private sector providers and the equilibrium outcomes, save for theoretical studies by economists (e.g. [39]). In reality, there have already been examples of private sector participation in toll road construction and operation around the world [15]. In return for the private capitalists funding the large initial capital investments for the construction of the road, they are contractually allowed to collect tolls, for some pre-specified duration, on traffic when the road is finally opened to use [12]. In an era when government budgets are becoming increasingly tight and with traffic congestion becoming more of a problem in many major cities, the private sector is recognized as having an increasing role to play in the provision of traditional highway transportation investment. When a private firm is engaged with the provision of such services and in competition with others simultaneously doing the same, the concept of Nash equilibrium [36] can be used to model the equilibrium decision variables offered to the market.

Even though a Cournot-Nash equilibrium problem can be formulated as a variational inequality, the problem that we describe in this chapter differs because the players are constrained by a second variational inequality describing the equilibrium routing behavior of travelers on a transportation network. Hence the problem is formally a "Bi-Level" (denoting the two level hierarchical nature of the problem) Variational Inequality. The objective of our research is to explore the possibilities of employing a multipopulation coevolutionary method, based on the particle swarm algorithm [23] to solve the resulting BLVI.

The rest of this chapter is organized as follows. In the next section, the basic traffic assignment concepts are given to provide sufficient background for readers not familiar with the tools of transportation network analysis. The variational inequality that describes the equilibrium of a transportation system is developed. In Section 3 we discuss the concept of Nash equilibrium in further detail with reference to the behavior of players in a Cournot-Nash game and subsequently formulate the BLVI of the problem at hand. Existing algorithms available to solve this problem are briefly described. In Section 4 we describe the coevolutionary particle swarm algorithm, developed based on an analogy with competition between species in natural systems, to solve the BLVI. The objective of the coevolutionary process is to evolve swarms of strategies for each player that are robust against the strategies evolved by other players while maintaining, at each iteration, the system equilibrium of the transportation network. In Section 5 we give some examples to illustrate the performance of our proposed solution algorithm on several problems. Finally in Section

6 the findings are summarized and we provide further directions for research in this developing subject area.

As the subject matter of this chapter transcends both market structures and game theory, we will use the terms “players” and “firms” interchangeably to refer to the private sector participants in this market who are the players in the Nash game governed by the variational inequality constraint. For the same reasons, “profits” and “payoff” are also used synonymously.

2 Transportation Network Analysis

To facilitate understanding of this chapter, we discuss basic concepts associated with traffic assignment and network analysis which are the key focus areas of this chapter. We discuss the traffic assignment problem (TAP) and the theoretical model explaining the route choice decisions of users on a highway network in a deterministic equilibrium setting and finally show how this can be formulated as a variational inequality. Readers interested in this topic may refer to the texts [37, 49] for further details.

2.1 Traffic Assignment Problem

Transportation network analysis seeks to understand factors affecting the route choice decisions of road users so that control policies affecting it may be formulated. The key to understanding this is facilitated by the traffic assignment problem (TAP) which is the methodology of assigning users desiring to travel between origin destination (OD)[1] pairs onto a traffic network.

In a given transportation network represented by a directed network graph, let:

A: the set of all links in the transportation network

B: the set of all links that are subject to tolls and capacity enhancements $B \subseteq A$,

R: the set of all routes in the network,

H: the set of all OD pairs in the network,

Ω: the set of all feasible flows and demands,

R_h: the set of routes between OD pair $h\,(h \in H)$,

d: the vector of demands between each OD pair $\mathbf{d} = [d_h]\,(h \in H)$,

$\mathbf{D^{-1}}$: the continuous and decreasing inverse demand function giving the travel cost as a function of the number of trips for each OD pair $\mathbf{D}^{-1} = [D_h^{-1}]\ (h \in H)$,

μ: the minimum travel cost between OD pair $\mu = [\mu_h]\,(h \in H)$

$\mathbf{E}$: the route flow on all routes in the network $\mathbf{E} = [E_r]\,(r \in R)$

$\mathbf{C}$: the cost of travel on all routes in the network $\mathbf{C} = [C_r]\,(r \in R)$

[1] Origins and destinations are equivalent to sources and sinks in other fields where analogous concepts of network analysis is employed.

v: the vector of link flows, $\mathbf{v} = [v_a]\ (a \in A)$,
δ_{ar} : 1 if the route $r\,(r \in R)$ uses link $a\,(a \in A)$, 0 otherwise,
x: the vector of tolls, $\mathbf{x} = [x_a]\ (a \in B)$,
K: the vector of capacities, $\mathbf{K} = [k_a]\ (a \in A)$,
y: the vector of capacity enhancements, $\mathbf{y} = [y_a]\ (a \in B)$,
$\mathbf{c}(\mathbf{v},\mathbf{x},\mathbf{y})$: the vector of link travel costs as a function of link flows, $\mathbf{c}(\mathbf{v},\mathbf{x},\mathbf{y}) = [c_a(v_a,0,0)]\ (a \in A, a \notin B); \mathbf{c}(\mathbf{v},\mathbf{x},\mathbf{y}) = [c_a(v_a,x_a,y_a)]\ (a \in B)$
$t_a(v_a,k_a)$: the monotonically non decreasing travel time on the link *excluding* tolls on the link a. Note that we have $\frac{\partial t_a(v_a,k_a)}{\partial v_a} > 0$ and $\frac{\partial t_a(v_a,k_a)}{\partial k_a} < 0$
$t0_a$: a scalar free flow travel time for link $a\ (a \in A)$,
ρ_a: a positive scalar for link $a\ (a \in A)$,
λ_a: a positive scalar for link $a\ (a \in A)$.

Throughout this section we assume that **x** and **y** have already been exogenously specified. We will discuss how these are chosen in Section 3.

2.1.1 From Link Travel Time to Path Travel Costs

The travel time function on a link gives the travel time of the link as a function of traffic flows on the link and a commonly used functional form is depicted in 1.

$$t_a(v_a,k_a) = \begin{cases} t0_a + \rho_a \left(\frac{v_a}{k_a}\right)^{\lambda_a}, \forall a \in A, a \notin B \\ t0_a + \rho_a \left(\frac{v_a}{k_a + y_a}\right)^{\lambda_a}, \forall a \in B \end{cases} \tag{1}$$

The functional form in 1, known as the Bureau of Public Roads (BPR) function [3], includes link capacity as a determinant of the link travel time. When monetary tolls are converted into a time equivalent amount, then 2 allows for an unique map between the link travel time and the travel cost.

$$c_a(v_a,x_a,y_a) = \begin{cases} t_a(v_a,k_a), \forall a \in A, a \notin B \\ t_a(v_a,k_a) + x_a, \forall a \in B \end{cases} \tag{2}$$

Equation 2 states that if the use of the link is not subject to a toll charge, then the travel cost of using that link is given solely by the travel time taken to traverse it. However, if travel on it requires payment of a toll, then the time equivalent of the toll is added to the time cost of travel.

Note that a route or path between an OD pair comprises all links that constitute that route from O to D. In general, there could possibly be several routes available for use by any single OD pair and that in equilibrium not all these routes would necessarily be used.

2.2 TAP as a Variational Inequality

The behavioural principle underlying the highway users' choice of routes in the TAP is founded on Wardrop's Equilibrium Condition [57]. We state this more formally in Lemma 1.

Lemma 1. *Wardrop's Equilibrium Condition of route choice implies that at equilibrium the following conditions are simultaneously satisfied:*

$$\begin{aligned}
&E_{r\in R} \geq 0 \Leftrightarrow C_{r\in R} \leq D_h^{-1}\ \forall h \in H, \forall r \in R;\\
&E_{r\in R} = 0 \Leftrightarrow C_{r\in R} > D_h^{-1}\ \forall h \in H, \forall r \in R;\\
&d_{h\in H} \geq 0 \Leftrightarrow D_h^{-1} \geq \mu_{h\in H}\ \forall h \in H;\\
&d_{h\in H} = 0 \Leftrightarrow D_h^{-1} < \mu_{h\in H}\ \forall h \in H;
\end{aligned}$$

The first two conditions of Lemma 1 state that at equilibrium, all routes used between a given OD pair have equal costs and routes with higher costs will not be used. The next two conditions stipulate that travel occurs between OD pair $h,(h \in H)$ only if the marginal utility derived from travel (given by the inverse demand function) is greater than the minimum travel cost $\mu_h,(h \in H)$. Wardrop's Equilibrium Condition implies therefore that at equilibrium, no user can decrease his OD travel costs by unilaterally changing routes. For a pre-determined vector of tolls $\mathbf{x}$ and capacity enhancement levels $\mathbf{y}$, the following variational inequality (VI) can be used to restate Wardrop's Equilibrium Condition:

Find $(\mathbf{v}^*,\mathbf{d}^*) \in \Omega$ such that:

$$\mathbf{c}(\mathbf{v}^*,\mathbf{x},\mathbf{y})^T(\mathbf{v}-\mathbf{v}^*) - \mathbf{D^{-1}}(\mathbf{d}^*)^T(\mathbf{d}-\mathbf{d}^*) \geq 0, \quad \forall(\mathbf{v},\mathbf{d}) \in \Omega \tag{3}$$

Proposition 1. *The solution of the Variational Inequality defined by 3 results in a vector of link flows and demands $((\mathbf{v}^*,\mathbf{d}^*) \in \Omega)$ that satisfies Wardrop's Equilibrium Condition of route choice given by Lemma 1.*

For a proof of Proposition 1, see [9, 50].

2.3 Convex Optimization Reformulation

In the particular instance (and in the cases considered in this chapter) when the travel cost of using a link is dependent only on its own flow[2], there exists an equivalent convex optimization program for the above VI as given by 4-7 [2].

$$\min_{\mathbf{v},\mathbf{d}} \sum_{a\in A} \int_0^{v_a} c_a(z)\,\mathrm{d}z - \sum_{h\in H} \int_0^{d_h} D_h^{-1}(z)\mathrm{d}z. \tag{4}$$

[2] This is known as the separability assumption.

Subject to: $((\mathbf{v}^*, \mathbf{d}^*) \in \Omega)$ where Ω is a closed and convex set defined by 5 to 7.

$$\sum_{r \in R_h} E_r = d_h, \forall h \in H. \tag{5}$$

$$v_a = \sum_{r \in R} E_r \delta_{ar}, \forall a \in A. \tag{6}$$

$$E_r, d_h \geq 0, \forall r \in R, \forall h \in H. \tag{7}$$

The objective function 4 is a mathematical construct to solve for the equilibrium link and demand flow tuple that satisfies Wardrop's Equilibrium Condition [49]. In this programme, the first constraint states that the flow on each route used by each OD pair is equal to the total demand for that OD pair. The second constraint is a definitional constraint which stipulates that the flow on a link comprises flow on all routes that use that link. The last constraint restricts the equilibrium flows and demands to be non negative. As Ω is closed and convex (Ω is in fact a bounded polyhedron), the equilibrium link flows and demands $((\mathbf{v}^*, \mathbf{d}^*) \in \Omega)$ are unique [2].

3 A Model of Competition

With the above definition of the TAP from Section 2 in place, we are now able to consider the problem of a set of pre-defined P (indexed by $i = 1, 2, \ldots, P$) private firms, individually and non cooperatively, choosing the toll and the capacity enhancement levels on one and only one of P pre-defined links in a highway network.

To simplify our exposition, but without loss of generality, we have implicitly assumed that the cardinality of set P (i.e. $|P|$) is exactly equal to the cardinality of the set of $B, (B \subseteq A)$ network links which are subject to toll charges and capacity enhancements. In the ensuing discussion, we consider a given toll ($x_{i \in P}$) and capacity enhancement combination ($y_{i \in P}$) of a link to be the only strategic variables available to each of these firms. We formulate the optimization problem facing each firm before appealing to the concept of Nash Equilibrium (NE) to determine the strategic combinations offered by these firms in competition. Our problem statement follows that given in [58].

3.1 Optimization Formulation

The profit (or payoff) to firm $i, i \in P$ is the difference between the toll revenue obtained by charging a toll on the link and her investment cost of capacity enhancement, $I(y_i)$. Mathematically, the resulting choice of the strategic variables for each may be represented by the optimization problem in 8:

$$\underset{x_i, y_i}{\text{Max}} \quad \psi_i(\mathbf{x},\mathbf{y}) = v_i(\mathbf{x},\mathbf{y})x_i - \theta I(y_i), \forall i \in P \tag{8}$$

Where v_i is obtained by solving the variational inequality (3) i.e.

$$\mathbf{c}(\mathbf{v}^*,\mathbf{x},\mathbf{y})^T(\mathbf{v}-\mathbf{v}^*) - \mathbf{D}^{-1}(\mathbf{d}^*)^T(\mathbf{d}-\mathbf{d}^*) \geq 0, \quad \forall(\mathbf{v},\mathbf{d}) \in \Omega$$

In 8, the scalar θ, common to all firms, allows for conversion of the enhancement costs from monetary equivalents into time value of money as we work in time units throughout this chapter.

3.1.1 Single Firm Case

If there is only a single firm (i.e. $|P| = 1$), the problem is in fact an instance of a mathematical program with equilibrium constraints (MPEC) [31]. These are hierarchical optimization problems with the key characteristic that the lower level problem describes a variational inequality (such as those in 3) defining an equilibrium in a system. In the parlance of economics, the MPEC is the equivalent of a Stackelberg [51] or "leader follower" game [16] which provides a paradigm for considering the actions of a single leader making decisions in transportation and elsewhere. This hierarchical relationship is illustrated in Figure 1.

This paradigm is developed, in accordance with Stackleberg's proposition, that the sole decision maker (acting as a leader) sets the strategic variables of the system, and the highway users follow by taking into account the firm's decisions in formulating their route choice on the network, manifesting as traffic flows $\mathbf{v}$. As we have shown in 8, the link flows have to satisfy the variational inequality and this serves as a constraint to the leader's optimization problem, fitting therefore into the definition of the MPEC. Evolutionary methods based on Genetic Algorithms [19] and Differential Evolution [52] have been proposed to solve several planning problems formulated as MPECs within transportation network analysis [25, 60].

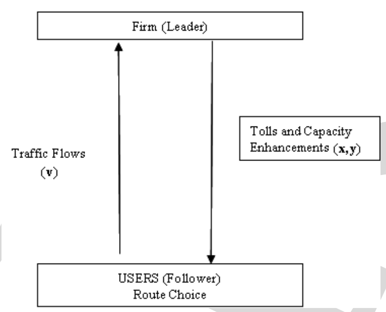

Fig. 1 Single Player Model

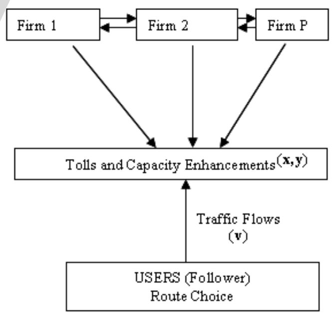

Fig. 2 Multiple Player Model

3.1.2 Extension to Multiple Firms

However, when there are multiple firms, each firm continues to take into consideration the route choices of the users as in the single firm model (i.e. the vertical relationship between leaders and followers in Figure 2) but *additionally* the choices made by other firms as required by the rationality postulate of non-cooperative behavior [18]. The latter is illustrated by the horizontal relationships between the various players at the upper level not present in the single firm model. In other words, whilst these firms play a Cournot-Nash game [8] amongst themselves, the "leader follower"relationship between these firms and the highway users still applies as in the single firm case[3].

3.2 Nash Equilibrium

In our current game context, a firm chooses its strategic combination from a set of feasible toll and capacity enhancement strategies denoted by $X_i \times Y_i$. The common strategy set of tolls and capacity enhancements across all P players may therefore be written as $X (= \prod_{i \in P} X_i)$ and $Y (= \prod_{i \in P} Y_i)$ respectively. When player $i, (i \in P)$ chooses her tolls and capacities, she is faced with the strategic choices of her competitors doing the same simultaneously. Writing her competitors' choices of strategic variables as $\mathbf{x}_{-i} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_P\}$ and $\mathbf{y}_{-i} = \{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_P\}$, then a toll and capacity enhancement combination $(x_i{}^*, y_i{}^*)$ is a Nash Equilibrium if the following in 9 is satisfied:

$$\psi_i \left(x_i^*, y_i^*, \mathbf{x}^*_{-i}, \mathbf{y}^*_{-i} \right) \geq \psi_i \left(x_i, y_i, \mathbf{x}^*_{-i}, \mathbf{y}^*_{-i} \right), \forall \left(x_i, y_i \right) \in X_i \times Y_i, \forall i \in P \qquad (9)$$

Equation 9 states that a NE is attained when no player in the game has an incentive to deviate from her current strategy. She is therefore doing the best she can given what her competitors are doing [43].

3.2.1 Variational Inequality

A Nash Equilibrium can be represented as a VI [18, 35]. In our current context, assume the profit function ψ_i is convex in $X_i \times Y_i$ strategies when viewed as a function of (x_i, y_i) alone, then the first order optimality conditions for the optimization problem facing player i given in 8 are the following:

$$-\frac{\partial \psi_i \left(\mathbf{x}^*, \mathbf{y}^* \right)}{\partial x_i} \left(x_i - x_i^* \right) \geq 0 \qquad (10)$$

and

[3] Thus the model is sometimes known as a Multi-Leader-Follower Game [29].

$$-\frac{\partial \psi_i(\mathbf{x}^*, \mathbf{y}^*)}{\partial y_i}(y_i - y_i^*) \geq 0 \tag{11}$$

Let $f_x(\mathbf{x}^*, \mathbf{y}^*) = -\frac{\partial \psi_i(\mathbf{x}^*, \mathbf{y}^*)}{\partial x_i}, (i \in P)$ and $f_y(\mathbf{x}^*, \mathbf{y}^*) = -\frac{\partial \psi_i(\mathbf{x}^*, \mathbf{y}^*)}{\partial y_i}, (i \in P)$ then combining 10 and 11 we therefore arrive at the following VI

$$(\mathbf{x} - \mathbf{x}^*)^{\mathrm{T}} f_x(\mathbf{x}^*, \mathbf{y}^*) + (\mathbf{y} - \mathbf{y}^*)^{\mathrm{T}} f_y(\mathbf{x}^*, \mathbf{y}^*) \geq 0, \forall \mathbf{x} \in X, \mathbf{y} \in Y \tag{12}$$

Proposition 2. *The solution of the Variational Inequality defined by 12 results in strategies* $(\mathbf{x}^* \in X, \mathbf{y}^* \in Y)$ *satisfying the definition of Nash equilibrium given by 9.*

For a proof of Proposition 2, see [18][4].

3.3 Bi-Level Variational Inequality Representation

Combining Proposition 2 given by 12 and Proposition 1 relating to the equilibrium condition of the TAP as given in 3, we therefore represent the Bi-Level Variational Inequality (BLVI) for this multi-firm game as follows:

Find $\mathbf{x} = \{x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_P^*\}$ and $\mathbf{y} = \{y_1^*, \ldots, y_{i-1}^*, y_i^*, y_{i+1}^*, \ldots, y_P^*\}$ such that

$$(\mathbf{x} - \mathbf{x}^*)^{\mathrm{T}} f_x(\mathbf{x}^*, \mathbf{y}^*, \hat{\mathbf{v}}, \hat{\mathbf{d}}) + (\mathbf{y} - \mathbf{y}^*)^{\mathrm{T}} f_y(\mathbf{x}^*, \mathbf{y}^*, \hat{\mathbf{v}}, \hat{\mathbf{d}}) \geq 0, \forall \mathbf{x} \in X, \mathbf{y} \in Y \tag{13}$$

Where for each $\mathbf{x} \in X$ and $\mathbf{y} \in Y$, $(\hat{\mathbf{v}}, \hat{\mathbf{d}})$ is a unique solution to the following VI in 14

$$\mathbf{c}(\hat{\mathbf{v}}, \mathbf{x}, \mathbf{y})^T (\mathbf{v} - \hat{\mathbf{v}}) - \mathbf{D}^{-1}(\hat{\mathbf{d}})^T (\mathbf{d} - \hat{\mathbf{d}}) \geq 0, \quad \forall (\mathbf{v}, \mathbf{d}) \in \Omega \tag{14}$$

3.3.1 Solution Algorithms for BLVI

The BLVI is not only applicable in transportation network analysis[5], but is also a model encountered in the deregulated electricity transmission markets [4, 22, 44] and elsewhere. The solution method proposed in these references, based primarily on [21], amounts to decomposing the problem into a series of inter-related MPECs i.e. one for each player. An outline of the method is given in Algorithm 1.

Algorithm 1. Gauss-Jacobi Fixed Point Iteration

Step 1: Set iteration counter $\delta = 0$. Select a convergence tolerance parameter, $\varepsilon(\varepsilon > 0)$. Choose a strategy for each player. Let the initial strategy set be $\mathbf{x}^{\delta} = \{x_1^{\delta}, \ldots, x_{i-1}^{\delta}, x_i^{\delta}, x_{i+1}^{\delta} \ldots x_P^{\delta}\}$ and $\mathbf{y}^{\delta} = \{y_1^{\delta}, \ldots, y_{i-1}^{\delta}, y_i^{\delta}, y_{i+1}^{\delta} \ldots y_P^{\delta}\}$. Set $\delta = \delta + 1$ and go to Step 2,

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[4] Proposition 2.2, p276.

[5] This concept has been used implicitly in modeling competition between transit operators, see[62].

Step 2: For the i^{th} player $i \in \{1,2,...,P\}$, solve the following optimization problem:

$$(x_i^{\delta+1}, y_i^{\delta+1}) = \max_{x_i \in X, y_i \in Y} \psi_i(\mathbf{x}^{\delta}, \mathbf{y}^{\delta}) \quad i \in \{1,2,...,P\},$$

subject to

$$\mathbf{c}(\mathbf{v}^*, \mathbf{x}, \mathbf{y})^T(\mathbf{v} - \mathbf{v}^*) - \mathbf{D^{-1}}(\mathbf{d}^*)^T(\mathbf{d} - \mathbf{d}^*) \geq 0, \quad \forall(\mathbf{v}, \mathbf{d}) \in \Omega$$

Step 3: If $(\sum_{i=1}^{P} \left\| x_i^{\delta+1} - x_i^{\delta} \right\|$ and $\sum_{i=1}^{P} \left\| y_i^{\delta+1} - y_i^{\delta} \right\|) \leq \varepsilon$ terminate, else set $\delta = \delta + 1$ and return to Step 2.

The drawback with Algorithm 1 is that it can fall prey to being trapped in local optima [61] and this depends on the starting point assumed in Step 1.

At the time of writing, the study of BLVIs has only just begun to receive the attention of researchers. Aside from the Gauss-Jacobi Algorithm mentioned above, Mordukhovich [34] has applied tools of non smooth analysis (see [5, 46]) in order to solve the BLVI. The proposed method is complex, employing advanced tools of variational analysis, and has thus far not been applied within transportation analysis. In his PhD thesis [54], Su proposed alternatively a sequential method for solving EPECs by iteratively relaxing the complementarity conditions in each player's program and solving a sequence of resulting complementarity problems. These two methods are relatively novel and could provide promise for further algorithmic developments for solving general BLVIs.

However the primary drawback of the above algorithms is their requirement for derivative information. For the specific case of transportation network analysis, it was shown [45] that the equilibrium constraint governing the route choice of users is not continuously differentiable. Additional assumptions made to satisfy the continuous differentiability requirement could possibly limit the practical applications of the algorithms to special cases. Thus methods relying on derivatives for a search direction might be problematic to apply. To overcome these constraints, we propose instead a derivative free coevolutionary particle swarm algorithm as a solution heuristic for the BLVI thus described.

4 A Coevolutionary Particle Swarm Approach

Any proposed solution method must take into account three separate but intertwined elements consistent with the modeling framework of the BLVI viz,

1. optimization for each player as given by equation 8,
2. hierarchy as based on the premise that the firms act as leaders and the highway users at the lower level take the leaders' strategic variables as given in optimizing their route choice decisions within a Stackleberg framework, and

3. intra-firm dependency consistent with the assumptions that each firm makes its decisions taking into account what its competitors are doing.

Thus the proposed algorithm is designed with a view to taking into account these three aforementioned elements. In particular,

1. the Particle Swarm Optimization (PSO) Algorithm is used as the global optimization method,
2. the evaluation process used within PSO to determine the fitness of candidate solutions has to be modified to retain the Stackleberg framework, and
3. the use of coevolution to deal with the intra-firm dependency such that strategies evolved by one player are robust to the strategies played by its competitors.

This section discusses how these elements are integrated within our proposed solution approach. A pseudo code description of our algorithm is then given.

4.1 Mechanics of PSO

Particle Swarm Optimization (PSO), is a member of a class of Swarm Intelligence methods. It was first developed by James Kennedy and Richard Eberhart [23] inspired by the simulation of fish schools and bird flocks. Since 1995, it has gained increasing popularity due to its effectiveness in solving difficult optimization tasks, with practical applications in diverse fields which range from chemistry [7, 38], mechanical engineering [1, 17], to civil engineering [41].

Let $\mathbf{x}_{-i} = \{x_1,...,x_{i-1},x_{i+1},...,x_P\}$ and $\mathbf{y}_{-i} = \{y_1,...,y_{i-1},y_{i+1},...y_P\}$ be predetermined strategies for all players excluding player i and $\psi_i : \Re^{2P} \rightarrow \Re$ as given by the objective function in 8. The global optimization problem for player i is to find x_i^* and y_i^* such that $\psi_i(x_i^*,y_i^*,\mathbf{x}_{-i},\mathbf{y}_{-i}) \geq \psi_i(x_i,y_i,\mathbf{x}_{-i},\mathbf{y}_{-i}), \forall x_i, y_i$.

To solve this problem, the iterative PSO algorithm begins with the initialization (random generation) of the positions of a J particle, 2 dimensional swarm. Each vector of the swarm is defined as: $M_j^i = \{m_{jx}^i, m_{jy}^i\}, \forall j \in J$. Here the subscripts x and y indicate the pertinent toll and capacity enhancements for the link each particle defines, thus each particle represents a real number encoding of the potential solution for the optimization problem. Each particle is associated (also generated randomly initially) with a velocity $S_j^i = \{s_{jx}^i, s_{jy}^i\}, \forall j \in J$. The best position ever encountered by the j^{th} particle at each iteration is denoted $Q_j^i = \{q_{jx}^i, q_{jy}^i\}, \forall j \in J$. Define δ as the iteration counter and let g be the index of the particle that attained the lowest function value ever encountered by the entire swarm up to iteration δ. Each particle is flown through the problem space and has its velocity updated by 15 [6]:

$$S_j(\delta+1) = \chi(S_j(\delta) + \alpha\phi_1(Q_j(\delta) - M_j(\delta)) + \beta\phi_2(Q_g(\delta) - M_j(\delta))), \quad \forall j \in J. \tag{15}$$

[6] We drop i superscripts from this point to reduce notational clutter and only reintroduce it when necessary to avoid confusion.

In 15, χ, the constriction coefficient used to limit velocity explosion and promote convergence, is conventionally set to 0.7298 based on the theoretical analysis of Clerc and Kennedy [6]. The scalars, α and β, are cognitive weights used to represent the attraction of a particle towards the personal and global bests respectively. Finally ϕ_1 and ϕ_2 are pseudo random numbers $\in (0,1)$. Combining the effects of these variables, the velocity update equation given by 15 highlights the fundamental operation of PSO: each of these potential solutions attempt to discover better solutions by "flying" in the search space with a velocity defined as a stochastic combination of its best ever attained position and the entire swarm's best position.

With its new velocity, its particle position can then be updated using 16 :

$$M_j(\delta+1) = M_j(\delta) + S_j(\delta+1), \quad \forall j \in J. \tag{16}$$

Once the particle position is updated, the particle is evaluated again. If the function value is better than the best value encountered by the particle so far, the best ever position for the j^{th} particle is updated and if this value is also better than the highest function value ever encountered by the swarm up to iteration δ, the global best position Q_g is also updated. In other words, when considering player $i, i \in P$ we use 17 to update the personal best particle positions depending on the fitness obtained.

$$Q^i_j(\delta+1) = \begin{cases} Q^i_j(\delta) \text{ if } \psi_i(M^i_j(\delta+1), \mathbf{x}_{-i}, \mathbf{y}_{-i}) \leq \psi_i(Q^i_j(\delta), \mathbf{x}_{-i}, \mathbf{y}_{-i}) \\ M^i_j(\delta+1) \text{ if } \psi_i(M^i_j(\delta+1), \mathbf{x}_{-i}, \mathbf{y}_{-i}) > \psi_i(Q^i_j(\delta), \mathbf{x}_{-i}, \mathbf{y}_{-i}) \end{cases} \tag{17}$$

Now since $Q^i_g(\delta+1) \in \{Q^i_1(\delta+1), ..., Q^i_J(\delta+1)\}$, we choose the global best particle positions from this set such that 18 is satisfied.

$$\psi_i(Q^i_g(\delta), \mathbf{x}_{-i}, \mathbf{y}_{-i}) = \max\{\psi_i(Q^i_1(\delta), \mathbf{x}_{-i}, \mathbf{y}_{-i}), ..., \psi_i(Q^i_J(\delta), \mathbf{x}_{-i}, \mathbf{y}_{-i})\} \tag{18}$$

This process is repeated for a number of iterations until some user specified termination criteria is satisfied, usually the maximum number of iterations.

4.1.1 Global, Local and Unified PSO

The version we have presented above is developed on the assumption that the neighborhood of the particle is defined as the entire swarm [13, 24]. This is often referred to in the literature as the "global best PSO" since in 15, the particle moves towards a combination of its own best position and the *entire* swarm's best position. To elucidate the so called "local best PSO" [13, 24], let l be the index of the particle that attained the best function value ever encountered by the j^{th} particle's *neighborhood* up to the current iteration. The most common neighborhood topology used for this local best variant is known as the ring topology [13] where the neighbors of the j^{th} particle are the $(j-1)^{\text{th}}$ and the $(j+1)^{\text{th}}$ particle. The local best particle position in this case is the best function value attained by either of these 3 particles. In the local best version, the velocity update is performed by 19:

$$S_j(\delta+1) = \chi(S_j(\delta) + \alpha\phi_1(Q_j(\delta) - M_j(\delta)) + \beta\phi_2(Q_l(\delta) - M_j(\delta))), \quad \forall j \in H, \tag{19}$$

with all other terms remaining as previously defined.

In the local best case we therefore have $Q_l^i(\delta+1) \in \{Q_{j-1}^i(\delta+1), Q_j^i(\delta+1), Q_{j+1}^i(\delta+1)\}$ [7] such that neighborhood analogue of 18 is satisfied. It is then trivial to recover the "global best PSO" by defining the neighborhood of the j^{th} particle as the entire swarm [24].

An interesting variation to the basic algorithm was to synergize both the local and global search mechanisms in the Unified Particle Swarm Optimization (UPSO) Algorithm as proposed in [40]. In this model, the velocity update makes use of both the global and local versions simultaneously using 20-22 as follows:

$$G_j(\delta) = \chi(S_j(\delta) + \alpha\phi_1(Q_j(\delta) - M_j(\delta)) + \beta\phi_2(Q_g(\delta) - M_j(\delta))), \quad \forall j \in J \tag{20}$$

$$L_j(\delta) = \chi(S_j(\delta) + \alpha\phi_1(Q_j(\delta) - M_j(\delta)) + \beta\phi_2(Q_l(\delta) - M_j(\delta))), \quad \forall j \in J, \tag{21}$$

$$S_j(\delta+1) = uG_j(\delta) + (1-u)L_j(\delta), \quad \forall j \in J. \tag{22}$$

For purposes of exposition, we have used $G_j(\delta)$ and $L_j(\delta)$ as dummy variables to distinguish the global and local velocity schemes. In 22, $u \in (0,1)$ is a scalar unification factor that combines the global and local updating mechanisms of the PSO search algorithm. When u is 0, the UPSO reverts to "local best PSO" and when u is 1, the UPSO reverts to "global best PSO". Particle position updating then follows as previously i.e. using 16.

4.2 Hierarchical Evaluation Framework

The evaluation process to determine the fitness of a particle has to be developed within the Stackleberg framework (recall our discussion in Section 2) since the firms take into account the traveler's behavior for a given toll and capacity perturbation. To do this, we used the method from [25].

Given $\mathbf{x}_{-i}$ and $\mathbf{y}_{-i}$ (strategies of the competitors), a two stage process is used to evaluate the profit function and hence determine the fitness of a particle. In the first stage, the tolls and capacity enhancement vectors are input into the TAP (equations 4 - 7) to solve for the (unique) vector of link flows such that routing satisfies Wardrop's Equilibrium Condition. These link flows serve as an input into the second stage which determines the actual profits using the leaders' objective specified by 8. A summary of the process to determine the fitness of particle $j, j \in J$ for player $i, i \in P$ is given in Algorithm 2.

[7] Obvious modifications are necessary to deal with the first and last particles in the swarm.

Algorithm 2. Two Stage Fitness Evaluation Process

Step 1: Combine m^i_{jx} and $\mathbf{x}_{-i}$ to obtain $\mathbf{x}$. Combine m^i_{jy} and $\mathbf{y}_{-i}$ to obtain $\mathbf{y}$.

Step 2: Solve the VI (3) to obtain v_i i.e. Find $(\mathbf{v}^*, \mathbf{d}^*) \in \Omega$ such that:

$$\mathbf{c}(\mathbf{v}^*, \mathbf{x}, \mathbf{y})^T(\mathbf{v} - \mathbf{v}^*) - \mathbf{D}^{-1}(\mathbf{d}^*)^T(\mathbf{d} - \mathbf{d}^*) \geq 0, \quad \forall(\mathbf{v}, \mathbf{d}) \in \Omega$$

Step 3: Evaluate $\psi^i_j(\mathbf{x}, \mathbf{y})$ $(\equiv v_i m^i_{jx} - \theta I(m^i_{jy}))$ to determine the profit and hence determine the fitness of the particle j.

4.3 Coevolution

To allow for the players to consider the strategies of their competitors, all strategies are evolved simultaneously. This coevolution can be interpreted as the simultaneous adaptation of fitness from interaction between different species, which in our case corresponds to the different players within the game. The development of coevolutionary algorithms was inspired from an analogy with biological coevolution [11, 55]. In the context of nature inspired optimization algorithms, it is usual to classify coevolutionary algorithms into 2 categories: either cooperative or competitive. Extending the parallel with biological phenomenon in the natural world, the former imitates symbiosis while the latter imitates parasitism. An example of the former class of cooperative type algorithms was exploited in [42] where different species (one species for each problem dimension) were evolved to cooperatively solve an optimization problem using genetic algorithms.

It is however the latter case of competitive coevolution [47, 48] that is the focus of this chapter. In this case, the fitness of a sub-population is “contingent” (i.e. dependent) on the strategies of other species [14, 28]. Hence the key concept of “contingent” fitness evaluation is a necessary ingredient embedded in our proposed search algorithm and it is envisaged that the resulting “evolutionary arms race” [48] would lead to fitter strategies selected by players in response to strategies evolved by their competitors.

4.4 Algorithm Outline

Recall that there are P subpopulations containing J particles each. Combining all the aforementioned elements, the pseudo code describing our PSO based method to tackle the BLVI thus specified is given in Algorithm 3.

Algorithm 3. Coevolutionary Particle Swarm Algorithm

```
Step 1 Initialization: Generate Sub Populations of particles
and velocities randomly for all players decision variables.
Step 2 Randomly select one strategy vector from each player
```

```
as its Nash strategy.
Step 3 Evaluate each sub population given the Nash strategy
of others by the Two Stage Evaluation Process to identify
the global best particle; Set this as the Nash strategy for
the player.
REPEAT
 Step 4 Synchronization:
  Announce Nash Strategy to all players
  For each subpopulation i = 1 to P do
  Step 5 Re-evaluate subpopulation i by the Two Stage
  Evaluation Process given the announced Nash Strategies
  in Step 4 and  obtain personal bests and the global best
   For each particle j = 1 to J do
    Step 6 Fly particle j through problem space using the
    velocity update equation.
    Step 7 Update particle j's position using position
    update equation.
    Step 8 Evaluate new particle fitness by the Two Stage
    Evaluation Process, update personal best if fitter,
    update global best if fitter.
   Next j
  Step 9 Identify new global best particle of
  subpopulation i and set this as the Nash strategy
  for the player.
 Next i
UNTIL Termination Criteria
```

In the initialization phase of the algorithm, we randomly generate particle positions and velocities corresponding to the strategies of each player. One strategy from each subpopulation is then randomly selected as the initial Nash strategy for each player. Subsequently each subpopulation is evaluated separately to determine the profit for each player, given the above initial moves of the other players. From this process, the personal best and global best particle for each player can be identified. When all subpopulations have been evaluated, each individual player's global best particles are announced to the whole group. We term this the synchronization phase of the algorithm so that every participant is cognizant of their competitors' actions.

With the new global best strategy of all players announced, each subpopulation is once more evaluated to determine its own global best strategy in the light of the foregoing announcement. For each subpopulation, particles are then flown through the search space and the particle positions are updated using the particle swarm method. New global best positions are produced and these are announced in the next synchronization phase of the algorithm. The algorithm terminates after a number of user specified maximum number of iterations. Any variant of the PSO algorithm e.g. global, local, UPSO or even other variants (see [13]) can be employed in the velocity and position updates in Steps 6 and 7 of the algorithm. In our examples, we utilize UPSO explained earlier as it integrates the global and local search elements of the swarm.

Whilst the proposed algorithm is primarily a heuristic, our numerical examples in the following section suggest that it provides a potential solution algorithm for the BLVI.

5 Numerical Examples

The proposed coevolutionary particle swarm algorithm is applied on two problems drawn from the literature. The first example assumes that the private firms are only allowed to collect tolls for the operation of the road and ignores the costs associated with capacity enhancement. The objective therefore is one of maximizing revenue from the tolls. The second example is a literature problem given in [58] which explicitly considers capacity provision.

Table 1 PSO Parameters Used for Numerical Examples

Parameter	Value
Maximum Number of Iterations	200
Swarm Size (H)	12
Unification Factor (u)	0.5
Constriction Factor (χ)	0.7298
Cognitive Factor (α)	2.05
Cognitive Factor (β)	2.05

The parameters used in our PSO algorithm are shown in Table 1. Our reported results are based on the average of 30 runs of the algorithm with different random seeds. No attempt was made in this work to seek optimal parameters which might improve the performance of our proposed algorithm.

5.1 Example 1: Network 1

The first example is taken from [27]. The link specific parameters and the specific form of the demand functions can be found therein. This network has 18 one way links with 6 OD pairs ((Node) 1 to 5, 1 to 7, 5 to 1, 5 to 7, 7 to 1 and 7 to 5). As mentioned above, the objective of each player is to maximize toll revenue by modifying the toll level $\mathbf{x}$ and we have ignored capacity enhancement. The problem therefore is a modified form of 8 and adopts that shown in 23.

$$\underset{x_i}{\text{Max}} \quad \psi_i(\mathbf{x}) = v_i(\mathbf{x})x_i, \forall i \in P \tag{23}$$

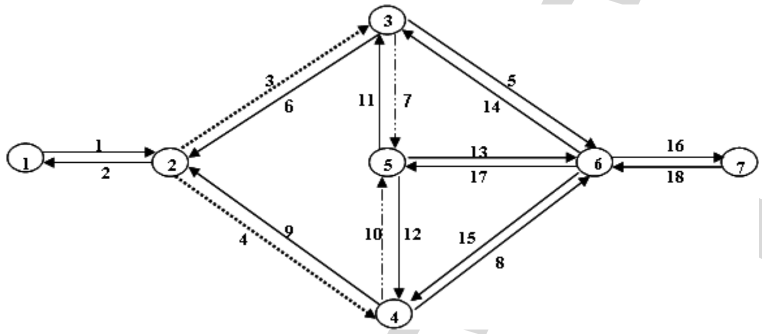

Fig. 3 Highway Network for Example 1, link numbers are indicated on links and travel is in direction indicated by arrows

Where v_i is obtained by solving the variational inequality (3) i.e.

$$\mathbf{c}(\mathbf{v}^*,\mathbf{x})^T(\mathbf{v}-\mathbf{v}^*) - \mathbf{D}^{-\mathbf{1}}(\mathbf{d}^*)^T(\mathbf{d}-\mathbf{d}^*) \geq 0, \quad \forall(\mathbf{v},\mathbf{d}) \in \Omega$$

Two mutually exclusive scenarios are considered for this numerical example. In Scenario A, Links 3 and 4 shown as dashed lines in Figure 3, are the only links in this network that are subject to tolls. In Scenario B, Links 7 and 10, shown in an alternative style of dashed lines, are the only links subject to tolls in the network. In both scenarios, however, there is one private firm operating on each link. The maximum allowable toll was capped at 1000 seconds. Recently, [26] employed a variant of Algorithm 1 for the solution of this problem. We compare the results reported therein with that from application of the Coevolutionary Particle Swarm Algorithm. This is shown in Table 2 and Table 3 for Scenarios A and B respectively.

Table 2 Example 1: Results of PSO algorithm on Scenario A

		Proposed Algorithm		[26]	
Firm	Link	Toll (secs)	Revenue (secs/hr)	Toll (secs)	Revenue (secs/hr)
1	3	530.55	461,861	530.63	461,882
2	4	505.62	420,242	505.65	420,293

For Scenario A, the average revenue over 30 runs for Firm 1 and 2 are 461,861 and 420,242 with standard deviations of 0.0023 and 0.0018 respectively. In the case of Scenario B, the average revenue over 30 runs for Firm 1 and 2 are 105,294 and 100,846 with standard deviations of 0.091 and 0.033 respectively.

From Figure 3 we see that when Links 3 and 4 are both subject to tolls, there are no alternative free routes connecting Origin Node 1 to the rest of the network. On the other hand, even when both Links 7 and 10 are both subject to tolls, Link 17 continues to be free to use. This is the intuitive explanation for the much higher

Table 3 Example 1: Results of PSO algorithm on Scenario B

		Proposed Algorithm		[26]	
Firm	Link	Toll (secs)	Revenue (secs/hr)[1]	Toll (secs)	Revenue (secs/hr)
1	7	141.36	105,294	141.37	105,295
2	10	138.29	100,846	138.29	100,848

toll levels under Scenario A compared to Scenario B. Nevertheless, the proposed Coevolutionary Particle Swarm Algorithm converges to similar solutions to that obtained using Algorithm 1 reported in [26]. The convergence of the algorithm for a typical run under each scenario are shown in Figure 4 and Figure 5.

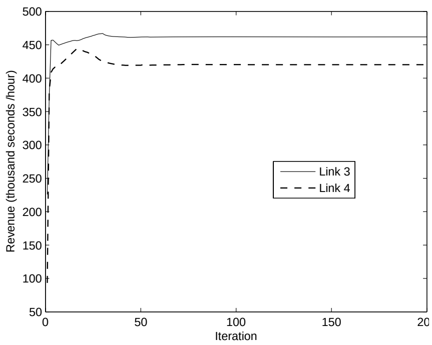

Fig. 4 Convergence of Proposed Algorithm for Scenario A, Example 1

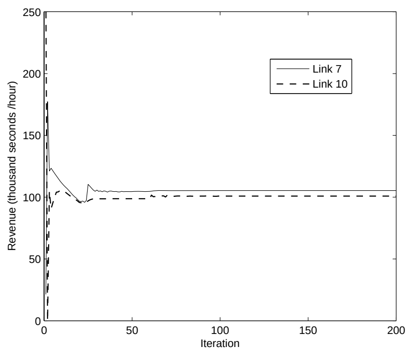

Fig. 5 Convergence of Proposed Algorithm for Scenario B, Example 1

5.2 Example 2: Network 2

The 11 link network for this example is taken from [58] and shown in Figure 6 with 4 OD pairs ((node) 1 to 7, 2 to 7, 3 to 7 and 6 to 7). This model considers capacity enhancement with 3 private operators on this network each optimizing their profits ψ_i as outlined in 8. Specifically, θ, given as 0.114, is common to all players and the enhancement cost functions take the form $I(y_i) = t0_i y_i, \forall i \in P$. In other words, the enhancement cost is proportional to the free flow travel time ($t0_a, a \in B$) for each of these links. The free flow times for the 3 links (9,10 and 11) to be improved (shown as dashed lines in Figure 6) are 11,11 and 15 secs respectively.

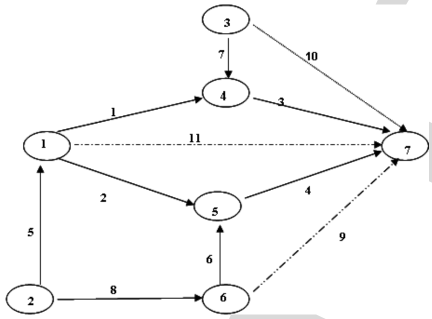

Fig. 6 Highway Network for Example 2, link numbers are indicated on links and travel is in direction indicated by arrows

Details of the link parameters and the functional forms of the travel demand relationships for the 4 OD pairs can be found in [58] where a heuristic gradient based algorithm was applied. We compare the results reported therein with those produced from the Coevolutionary Particle Swarm Algorithm as summarized in Table 4.

Table 4 Example 2: Results of PSO algorithm

Firm	Link	Proposed Algorithm Toll (secs)	Capacity (vehicles)	Profit (secs/hr)	[58] Toll (secs)	Capacity (vehicles)	Profit (secs/hr)
1	9	4.52	151.64	301.93	4.52	151.60	301.43
2	10	4.76	192.90	417.53	4.76	193.04	417.14
3	11	2.97	61.42	25.98	2.97	61.88	25.92

The average profit for firms 1, 2 and 3 over 30 runs (in seconds per hour) are 301.93, 417.53 and 29.58 with standard deviations 0.0033,0.0041 and 0.0017 respectively. The convergence of the proposed PSO based algorithm is shown in Figure 7.

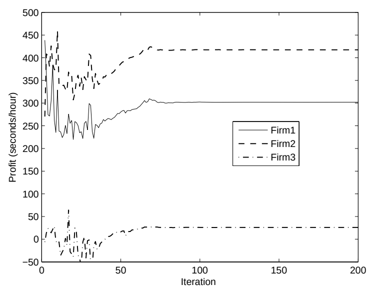

Fig. 7 Convergence of Proposed Algorithm for Example 2

6 Conclusions and Directions for Further Research

In this chapter, we presented a model of competition that arises in a highway network with private sector participants. The private firms on a road network are assumed to act as players in a Cournot-Nash game in choosing their decision variables and their actions are constrained by a variational inequality describing the equilibrium routing pattern of the highway users. The Nash equilibrium paradigm was used to model the behavior of the players in the resulting Bi-Level Variational Inequality (BLVI).

To solve the BLVI and determine the optimal choice of strategic variables of these firms, we employed a coevolutionary particle swarm algorithm to evolve strategies for each player that are robust against strategies evolved by their competitors. The traffic equilibrium constraint is achieved by solving the traffic assignment problem with a view to determining the global best strategy and during the synchronization phase of our proposed algorithm, the Nash strategy is revealed by all players simultaneously. Our algorithm embodies the principle that the fitness of a strategy for one player is contingent on the strategies revealed by all other players which is consistent with the rationality postulate of non-cooperative behavior in games. The proposed method was applied to two problems from the literature. In the examples, we have illustrated that our proposed algorithm easily obtained the solutions provided by others in their earlier research. Thus the proposed algorithm is a potentially useful method for this class of intrinsically non smooth optimization problems.

In our application of the proposed coevolutionary particle swarm algorithm, we utilized the unified particle swarm method which combined both the local and global search elements. There is no compelling reason however to prevent other variants of particle swarm optimization to be employed. Further work will study in greater effect the implications of swarm size and such parameters on convergence of the algorithm. Further research could additionally investigate the impact of these parameters on the convergence of the algorithm to the Nash equilibrium. This includes the possibility of employing fuzzy logic or other algorithms to adaptively tune the parameters of PSO as well as exploring alternative neighborhood topologies in addition to the ring topology that was adopted here.

Implicit in this chapter was the assumption that the private firms did not collude or that regulatory mechanisms prevented them from doing so. If an explicit collusion was possible, then the problem becomes one of monopoly optimization subject to the equilibrium constraint and this reverts to the familiar form of the MPEC. However if collusion was implicit, then the problem is in fact a multiobjective (i.e. vector valued) optimization problem with equilibrium constraints [32],[59]. Further research studying the application of evolutionary multiobjective optimization algorithms [10] to solve this latter category of problems is underway.

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References

1. Abido M(2002) Optimal design of power system stabilizers using particle swarm optimization. IEEE Transactions on Energy Conversion 17(4):406-413
2. Beckmann M, McGuire C, Winsten C(1956) Studies in the Economics of Transportation. Yale University Press, New Haven, Connecticut
3. Bureau of Public Roads(1964) Traffic Assignment Manual. Urban Planning Division, US Dept. of Commerce, Washington DC
4. Cardell J, Hitt C, Hogan W(1997) Market power and strategic interaction in electricity networks. Resource and Energy Economics 19(1-2):109-137
5. Clarke FH(1983) Optimization and Nonsmooth Analysis. Wiley, New York
6. Clerc M, Kennedy J(2002) The particle swarm - explosion,stability, and convergence in a multidimensional complex space. IEEE Transactions on Evolutionary Computation 6(1):58-73
7. Cockshott A, Hartman B(2001) Improving the fermentation medium for Echinocandin B production, Part II: Particle swarm optimization. Process Biochemistry 36(7):661-669
8. Cournot AA(1838) Recherches sur les Principes Mathematiques de la Theorie des Richesses. Hachette, Paris, English trans In: Researches into the Mathematical Principles of the Theory of Wealth. A M Kelley, New York 1971
9. Dafermos SC(1980) Traffic Equilibrium and Variational Inequalities. Transportation Science 14(1):42-54
10. Deb K(2001) Multi-objective Optimization using Evolutionary Algorithms. John Wiley, Chichester
11. Ehrlich PR, Raven PH(1964) Butterflies and plants: a study in coevolution. Evolution-International Journal of Organic Evolution 18(4):586-608
12. Engel E, Fischer R, Galetovic A(2002) A New Approach to Private Roads. Regulation 25(3):18-22
13. Engelbrecht AP(2002) Computational Intelligence: An Introduction. John Wiley, New York
14. Ficici SG, Pollack JB(2000) Game theory and the Simple Coevolutionary Algorithm. In: Schoenauer M, Deb K, Rudolph G, Yao X, Lutton E, Merelo JJ, Schwefel HP(eds) Parallel Problem Solving from Nature VI. Paris, France, September 18-20,LNCS 1917, Springer Verlag, Berlin:467-476
15. Fisher G, Babbar S(1996) Private Financing of Toll Roads. RMC Discussion Paper Series 117, World Bank, Washington DC
16. Fisk C(1984) Game theory and transportation systems modeling. Transportation Research Part B 18(4/5):301-313
17. Fourie P, Groenwold A(2002) The particle swarm optimization algorithm in size and shape optimization. Structural and Multidisciplinary Optimization 23(4):259-267
18. Gabay D, Moulin H(1980) On the uniqueness and stability of Nash-equilibria in non cooperative games. In: Bensoussan A, Dorfer PK, Tapiero C(eds) Applied Stochastic Control in Econometrics and Management Science: Contributions to Economic Analysis Vol 130, North Holland, Amsterdam: 271-293

19. Goldberg, DE(1989) Genetic Algorithms in Search, Optimization, and Machine Learning. Addison-Wesley,Boston
20. Henderson JM, Quandt RE(1980) Microeconomic Theory: A Mathematical Approach. McGraw Hill, Auckland
21. Harker PT(1984)A variational inequality approach for the determination of Oligopolistic Market Equilibrium. Mathematical Programmming 30(1):105-111
22. Hobbs BF, Metzler CB, Pang JS(2000) Strategic gaming analysis for electric power networks: An MPEC approach. IEEE Transactions on Power Systems 15(2):638-645
23. Kennedy J, Eberhart RC(1995) Particle swarm optimization. Proceedings of the IEEE International Conference on Neural Networks:1942-1948
24. Kennedy J, Eberhart RC, Shi Y(2001) Swarm Intelligence. Morgan Kaufmann Series in Artificial Intelligence, San Francisco
25. Koh A(2007) Solving Transportation bi-level programs with Differential Evolution. Proceedings of the IEEE Congress on Evolutionary Computation: 2243-2250
26. Koh A, Shepherd SP(2008) Tolling, Capacity Selection and Equilibrium Problems with Equilibrium Constraints. Proceedings 3rd Kuhmo-Nectar Conference, Free University of Amsterdam, July 3-4
27. Koh A, Shepherd SP, Sumalee A(2009) Second Best Toll and Capacity Optimisation in Networks: Solution Algorithm and Policy Implications. Transportation 36(2):147-165
28. Koza JR(1990) Genetic evolution and coevolution of computer programs. In: Langton CG, Taylor C, Farmer JD, Rasmussen S(eds) Artificial Life II. Addison-Wesley, Redwood City, CA:603-629
29. Leyffer S, Munson T(2005) Solving Multi-Leader-Follower Games. Preprint Argonne National Laboratory/MCS-P1243-0405
30. Lockwood S(1995) Public private partnerships in U.S. highway finance: ISTEA and beyond. Transportation Quarterly 49(l):5-26
31. Luo ZQ, Pang JS, Ralph D(1996) Mathematical Programs with Equilibrium Constraints. Cambridge University Press, Cambridge,England
32. Mordukhovich BS(2004) Equilibrium problems with equilibrium constraints via multiobjective optimization. Optimization Methods and Software 19(5):479-492
33. Mordukhovich BS(2005) Optimization and equilibrium problems with equilibrium constraints. Omega-International Journal of Management Science 33(5):379-384.
34. Mordukhovich BS(2006) Variational Analysis and Generalized Differentiation. II: Applications, Grundlehren Series in Fundamental Principles of Mathematical Sciences, Vol 331, Springer-Verlag, Berlin
35. Nagurney A(1999) Network Economics: a Variational Inequality Approach. Kluwer, Boston
36. Nash J(1950) Equilibrium points in N-person games. Proceedings of the National Academy of Science 36(1):48-49
37. Ortuzar JD, Willumsen L(1998) Modelling Transport. Wiley, Chichester
38. Ourique C, Biscala E, Pinto J(2002) The use of particle swarm optimization for dynamical analysis in chemical processes. Computers and Chemical Engineering 26(12):1783-1793
39. de Palma A, Lindsey R(2000) Private roads: competition under various ownership regimes. Annals of Regional Science 34(1):13-35
40. Parsopoulos KE, Vrahatis MN(2007) Parameter selection and adaptation in Unified Particle Swarm Optimization. Mathematical and Computer Modelling 46(1-2):198-213
41. Perez R, Behdinan K(2007) Particle swarm approach for structural design optimization. Computers and Structures 85(19-20):1579-1588
42. Potter MA, De Jong K(1994) A cooperative coevolutionary approach for function optimization. In: Davidor Y, Schwefel HP, Männer R(eds) Parallel Problem Solving from Nature III, Jerusalem, Israel, October 9-14,LNCS 866, Springer Verlag, Berlin:249-257
43. Pyndyck R, Rubinfeld S(1992) Microeconomics. Macmillan, New York
44. Ralph D, Smeers Y(2006) EPECs as models for electricity markets. Proceedings of IEEE Power Systems Conference and Exposition:74-80
45. Robinson SM(2006) Strong regularity and the sensitivity analysis of traffic equilibria: A comment. Transportation Science 40(4):540-542

46. Rockafellar RT, Wets JB(1998) Variational Analysis. Grundlehren Series in Fundamental Principles of Mathematical Sciences Vol 317, Springer-Verlag, Berlin
47. Rosin CD, Belew RK(1995) Methods for Competitive Co-Evolution: Finding Opponents Worth Beating, In: Eshelman L J(ed) Proceedings of the 6th International Conference on Genetic Algorithms, Pittsburgh, USA, July 15-19:373-381
48. Rosin CD, Belew RK(1997) New methods for competitive coevolution. Evolutionary Computation 5(1):1-29
49. Sheffi Y(1985) Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods. Prentice Hall, Englewood Cliffs, New Jersey
50. Smith MJ(1979) The existence, uniqueness and stability of traffic equilibria. Transportation Research Part B 13(4):295-304
51. von Stackelberg H(1934) The Theory of the Market Economy; English trans In: Peacock A, Oxford University Press, New York, 1952
52. Storn R, Price K(1997) Differential Evolution A Simple and Efficient Heuristic for Global Optimization over Continuous Spaces. Journal of Global Optimization, 11(4):341-359
53. Stough R, Rietveld P(1997) Institutional issues in transport systems. Journal of Transport Geography 5(3):207-214
54. Su C(2005) Equilibrium Problems with Equilibrium Constraints: Stationarities, Algorithms, and Applications. PhD Thesis, Department of Management Science and Engineering, Stanford University
55. Thompson JN(1994) The Coevolutionary Process. University of Chicago Press, Chicago
56. Viton P(1995) Private Roads. Journal of Urban Economics 37(3):260-289
57. Wardrop J G(1952) Some theoretical aspects of road traffic research. Proceedings of the Institution of Civil Engineers - Transport 1(2):325-378
58. Yang H, Xiao F, Huang H(2009) Private Road Competition and Equilibrium with Traffic Equilibrium Constraints. Journal of Advanced Transportation 43(1):21-45
59. Ye JJ, Zhu QJ(2003) Multiobjective optimization problems with variational inequality constraints. Mathematical Programming Series A 96(1):139-160
60. Yin Y(2000) Genetic Algorithms-Based Approach for Bilevel Programming Models. ASCE Journal of Transportation Engineering 126(2):115-120
61. You S, Baldick R(2004) Hybrid coevolutionary programming for Nash equilibrium search in games with local optima. IEEE Transactions on Evolutionary Computation 8(4):305-315
62. Zubeita L(1998) A network equilibrium model for oligopolistic competition in city bus services. Transportation Research Part B 32(6):413-422