Modelling Photospheric Magnetoconvection

S.M. Blanchflower, A.M. Rucklidge and N.O. Weiss

Department of Applied Mathematics & Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, UK

ABSTRACT

The increasing power of computers makes it possible to model the nonlinear inter-
action between magnetic fields and convection at the surfaces of solar-type stars in
ever greater detail. We present the results of idealized numerical experiments on two-
dimensional magnetoconvection in a fully compressible perfect gas. We first vary the
aspect ratio λ of the computational box and show that the system runs through a
sequence of convective patterns, and that it is only for a sufficiently wide box ($\lambda \geq 6$)
that the flow becomes insensitive to further increases in λ. Next, setting $\lambda = 6$, we
decrease the field strength from a value strong enough to halt convection and find
transitions to small-scale steady convection, next to spatially modulated oscillations
(first periodic, then chaotic) and then to a new regime of flux separation, with regions
of strong field (where convection is almost completely suppressed) separated by broad
convective plumes. We also explore the effects of altering the boundary conditions and
show that this sequence of transitions is robust. Finally, we relate these model cal-
culations to recent high-resolution observations of solar magnetoconvection, in plage
regions as well as in light bridges and the umbrae of sunspots.

Key words: convection – MHD – Sun: granulation – Sun: magnetic fields – sunspots
– stars: magnetic fields.

1 INTRODUCTION

Magnetic fields interfere with convective transport in the
photospheres of late-type stars. This interaction can be ob-
served in detail at the surface of the Sun, where features
that are only a few hundred kilometres across can now be
resolved, revealing a variety of fine structure that depends
on the local strength of the magnetic field. At the same
time, rapid advances in computing power have made it possi-
ble to model nonlinear magnetoconvection in regimes where
numerical experiments can be contrasted with solar obser-
vations. In this paper we study the effects of varying the
geometry and boundary conditions in idealized models, and
identify different patterns of behaviour when the fields are
weak or strong. These regimes are then related to convective
structures on the Sun.

In the solar photosphere, the strongest vertical fields
are found in pores and sunspot umbrae, where convective
plumes show up as 'umbral dots' (Danielson 1964). These
small bright features are present in all sunspots, though
large spots contain isolated regions (dark nuclei) that are
free of them (Muller 1992; Sobotka, Bonet & Vázquez 1993;
Sobotka 1997). Until very recently it was thought that um-
bral dots had diameters of 180–300 km and a filling factor of
3–10%. With improved resolution (Sobotka 1997; Sobotka,
Brandt & Simon 1997a,b) it is now clear that there is no
typical diameter; rather, the number density of umbral dots
increases with decreasing size, down to the limit of resolution

at $0''.28$ (200 km). An average specimen has a diameter of
300 km and a lifetime of 14 minutes but the lifetimes range
from 2 hr for the largest bright dots to a few minutes for the
smallest. 'Light bridges' are bright linear features that cut
across sunspots and exhibit a fine granular structure (Muller
1992; Sobotka *et al.* 1993; Sobotka 1997). The field within a
light bridge is weaker than in the surrounding umbra, so the
bridge resembles a slot, contained between magnetic walls
that resist distortion, within which more vigorous convec-
tion can occur. Rimmele (1997) has followed the evolution
of convective plumes within such a slot for a full hour and
confirmed that bright granules are associated with upward
motion. He also found that the velocity and intensity varied
in a manner consistent with oscillatory convection.

Outside sunspots, there is a distinction between plage
regions (with average field strengths greater than 150 G)
and quiet Sun. Plages are characterized by abnormal gran-
ulation, corresponding to convection with a smaller hori-
zontal scale. The fields form a perforated network, includ-
ing fine magnetic structures that give rise to isolated bright
points (Title *et al.* 1992; Muller 1994; Sobotka, Bonet &
Vázquez 1994). These appear most strikingly in the CH G-
band and their dynamic behaviour indicates that magnetic
flux moves rapidly through the intergranular network, form-
ing ephemeral concentrations rather than isolated flux tubes
(Berger *et al.* 1995; Berger & Title 1996; Berger *et al.* 1998).
Within the photospheric network, magnetic structures are
smaller and more nearly isolated, with diameters less than

1 000 km and fields of 1–2 kG (Muller 1994).

There are two approaches to modelling the nonlinear interaction between convection and magnetic fields at the surface of a star like the Sun (Weiss 1997). The first attempts to simulate photospheric magnetoconvection in as much detail as possible; this approach was pioneered by Nordlund (1984), using the anelastic approximation, and later carried through to a fully compressible calculation that demonstrates the formation of the intergranular magnetic network (Nordlund & Stein 1989, 1990). The dynamical evolution of an isolated flux element has also been simulated in the same style, though so far only in a more restricted two-dimensional (2D) geometry (Steiner, Knölker & Schüssler 1994; Steiner *et al.* 1996, 1998). The second approach is less ambitious but more systematic: it relies on idealized models, where different physical processes can be isolated and the key parameters can be varied. The combined effects of stratification and compressibility were first studied in a 2D model which showed that convection in a slot should indeed exhibit spatially modulated oscillations (Weiss *et al.* 1990). Subsequently, three-dimensional (3D) computations revealed the changes that occur in both the scale and the pattern of convection as the overall field strength is varied (Weiss *et al.* 1996). At any stage such calculations are limited by the computing power that is available; in practice this means that turbulent motion cannot be faithfully represented, and that the normalized width of the computational box (its aspect ratio) is limited. Experience shows that the pattern of convection may be drastically altered if the aspect ratio is too small (Weiss *et al.* 1996).

Unfortunately, these idealized 2D and 3D models have not so far reproduced the range of scales that is now known to exist for umbral dots. This discrepancy might be ascribed to various causes. First of all, the aspect ratio may be too narrow to allow a sufficient range of variation. Secondly, the choice of boundary conditions for the idealized models may be inappropriate. In these models the lower boundary is closed and kept at a fixed temperature, with a magnetic field that may vary but is constrained to be vertical. It is known that the return flow has a significant effect (Nordlund, Galsgaard & Stein 1994). Furthermore, the same conditions were applied at the upper boundary, where radiative transport is important and it would be more appropriate to match the internal field to a force-free (or even a potential) field in the upper atmosphere.

The aim of this paper is to explore the effects of varying the aspect ratio and of introducing more realistic boundary conditions at the top of the convecting layer. In order to do so it is necessary to carry out a large number of numerical experiments with different choices of parameters and boundary conditions. With three space dimensions the computational requirements would become prohibitive and our calculations are therefore restricted to two dimensions. In the next section we briefly summarize the model problem and formulate our modified boundary conditions. Heat is conducted to the upper boundary but is then emitted as black body radiation (according to Stefan's law), so the temperature can now vary on that boundary, while the internal magnetic field is matched to an external field that is force-free (and hence current-free in this geometry).

For this survey we adopt a standard polytropic atmosphere as the basic state and fix the relative values of the

magnetic, viscous and thermal diffusivities. The remaining parameters are the Rayleigh number R, which measures the superadiabatic temperature gradient, the Chandrasekhar number $Q \propto |\mathbf{B}_0|^2$, where $\mathbf{B}_0$ is the imposed vertical magnetic field, and the aspect ratio λ; these quantities are defined below. All results here are for a prescribed Rayleigh number $R = 100\,000$, which is high enough to produce vigorous convection with a range of spatial scales. We first study the effects of varying λ for a fixed value of Q. The results in §3, for $Q = 700$ confirm that the pattern of convection is severely constrained in very narrow boxes. As λ is increased there are transitions from steady convection to travelling waves and then to spatially modulated oscillations. In wide boxes ($\lambda \geq 6$) a totally new phenomenon appears: the magnetic field separates from the motion, leaving broad, dynamically active convective plumes and strong isolated concentrations of magnetic flux.

Next, in §4, we take a box that is wide enough ($\lambda = 6$) to allow sufficient freedom, and vary the field strength to reveal new patterns of behaviour. For very strong fields convection is completely suppressed but as Q is decreased there are again transitions to steady and oscillatory convection, culminating in flux separation for $Q < 750$. The effects of choosing different boundary conditions are explored in §5 and the implications of our results for 3D models, as well as for photospheric convection in the Sun and other late-type stars, are discussed in the final section.

2   THE MODEL PROBLEM

Our basic configuration has already been used for both 2D (Hurlburt & Toomre 1988; Weiss *et al.* 1990) and 3D (Weiss *et al.* 1996) investigations. Once again, we introduce cartesian co-ordinates with the z-axis pointing downwards and two-dimensional fields such that the velocity $\mathbf{u}$ and the magnetic field $\mathbf{B}$ lie in the xz-plane and are independent of y. Then we cast the equations into dimensionless form and consider a perfect gas occupying the region $\{0 \leq x \leq \lambda; z_0 \leq z \leq z_0 + 1\}$. In the absence of any motion there is a uniformly stratified equilibrium solution corresponding to a polytrope with index m, for which the temperature $T = z$ and the density $\rho = (\theta z)^m$, where $\theta = 1/z_0$. This stratification is superadiabatic if $m < 1/(\gamma - 1)$, where γ is the ratio of the specific heat at constant pressure to that at constant volume.

The nonlinear partial differential equations that govern the evolution of $\rho(x, z, t)$, $T(x, z, t)$, $\mathbf{B}(x, z, t)$ and $\mathbf{u}(x, z, t)$ with time t are as given by Weiss *et al.* (1996) and need not be repeated here. These equations have to be solved subject to appropriate boundary conditions. The lateral boundaries are straightforward: we assume that all quantities are periodic in x with period λ, so that $T(0, z, t) = T(\lambda, z, t)$ etc. At the top and bottom of the layer there are supposed to be impermeable but slippery boundaries, on which the normal component of the velocity and the tangential component of the viscous stress both vanish; thus

$$u_z = \partial u_x/\partial z = 0 \quad \text{at } z = z_0,\, z_0 + 1. \tag{1}$$

The standard idealized boundary conditions for $\mathbf{B}$ and T require that the field is vertical and the temperature is constant, so that

$$B_x = \partial B_z/\partial z = 0, \quad T = z_0 + 1 \quad \text{at } z = z_0 + 1, \tag{2}$$

(uniform) thermal conductivity and $\bar{\sigma}$ is Stefan's constant. In our formulation, the ratio $\bar{\sigma}/K$ is determined by requiring that the static polytropic atmosphere should be in thermal equilibrium, whence it follows that

$$\partial T/\partial z = (\theta T)^4 \quad \text{at } z = z_0. \tag{7}$$

The numerical experiments described in §§3 and 4 below were all carried out with the 'realistic' boundary conditions (5)–(7) at the upper boundary, while retaining the idealized boundary conditions (2) at the base of the layer. In §5, however, we compare these results with solutions obtained using different combinations of the boundary conditions.

For the calculations in this paper we adopt a standard 'deep layer' model with the polytropic reference atmosphere that has been used in various previous studies (Hurlburt, Toomre & Massaguer 1984; Hurlburt & Toomre 1988; Weiss *et al.* 1990, 1996; Cattaneo *et al.* 1991; Brummell, Hurlburt & Toomre 1996) and set $m = 1$, $\theta = 10$ (or $z_0 = \frac{1}{10}$) and $\gamma = \frac{5}{3}$, as for a monatomic gas; in addition, the Prandtl number (ratio of viscous to thermal diffusivity) $\sigma = 1$. The ratio ζ of the magnetic to the thermal diffusivity is a crucial parameter, for it determines whether convection sets in at an oscillatory or a stationary bifurcation when $Q \gg 1$. Although $\zeta \ll 1$ in the photospheres of cool stars, the ionization of hydrogen increases the opacity and reduces the thermal diffusivity, so that $\zeta > 1$ at depths below $2\,000$ km in the Sun. Since $\zeta \propto \rho$ in our model (with the thermal and electrical conductivities both constant) and $\rho(z)$ increases with depth in our reference atmosphere, the corresponding value of the diffusivity ratio $\zeta(z) = \zeta_0 \rho(z)$. To mimic the variation of ρ in the Sun we choose $\zeta_0 = 0.2$, so that $0.2 \leq \zeta(z) \leq 2.2$ and the value at the midpoint is given by $\hat{\zeta} = \zeta(z_0 + \frac{1}{2}) = 1.2$. The Rayleigh number

$$R(z) = (m+1)^2 \left(\frac{1}{m+1} - \frac{\gamma-1}{\gamma}\right) \frac{z^{2m-1}}{\sigma \bar{K}^2 z_0^{2m}}, \tag{8}$$

where $\bar{K}$ is the dimensionless thermal conductivity, also varies with depth and is specified by its midpoint value $\hat{R} = R(z_0 + \frac{1}{2})$. Throughout this paper we shall take $\hat{R} = 100\,000$ (cf. Weiss *et al.* 1996). The strength of the imposed magnetic field can be measured either by the ratio $\beta(z)$ of the gas pressure to the magnetic pressure in the static reference atmosphere or by the Chandrasekhar number

$$Q = \frac{2z_0}{\sigma \zeta_0 \bar{K}^2 \beta(z_0)}, \tag{9}$$

which is independent of the depth z. With these parameter values, $\beta(z) = 5 \times 10^6 (z^2/12Q)$ and its midpoint value $\hat{\beta} = \beta(z_0 + \frac{1}{2}) = 1.5 \times 10^5/Q$. We shall only consider a regime in which $Q \leq 5\,000$ or $\hat{\beta} \geq 30$, so effects of compressibility will be limited.

In the absence of any magnetic field the reference atmosphere will be stable to perturbations with wavelength $\bar{\lambda}$ if $\bar{\lambda}$ is either very large or very small, so with R fixed convective instability only occurs for $\lambda_1 < \bar{\lambda} < \lambda_2$. When Q is very large convection will be completely suppressed but as Q is reduced there will be a critical value $Q_c(\bar{\lambda})$, for $\lambda_1 < \bar{\lambda} < \lambda_2$, at which the atmosphere becomes unstable. For our choice of parameters we expect instability to set in at a supercritical pitchfork bifurcation (cf. Weiss *et al.* 1990), followed by steady overturning convection. Fig. 1 shows Q_c as a function

Figure 1. Linear stability: the critical Chandrasekhar number Q_c as a function of the wavelength $\bar{\lambda}$. Below the curve the reference atmosphere is unstable.

and

$$B_x = \partial B_z/\partial z = 0, \quad T = z_0 \quad \text{at } z = z_0. \tag{3}$$

A more realistic alternative is to match the field at the upper boundary to a potential field in the half-space $z < z_0$. If the field components are expanded in Fourier series as -

$$B_x(x, z, t) = \sum_{-L}^{L} \hat{B}_{x,l}(z, t) \exp{(2\pi \mathrm{i} l x/\lambda)},$$

$$B_z(x, z, t) = \sum_{-L}^{L} \hat{B}_{z,l}(z, t) \exp{(2\pi \mathrm{i} l x/\lambda)}, \tag{4}$$

with $\hat{B}_{x,-l} = \hat{B}_{x,l}^*$, $\hat{B}_{z,-l} = \hat{B}_{z,l}^*$, $\hat{B}_{x,0} = 0$, $\hat{B}_{z,0} = 1$, then

$$\hat{B}_{x,l}(z_0, t) = \mathrm{i}(l/|l|) \hat{B}_{z,l}(z_0, t) \qquad (l \neq 0) \tag{5}$$

and, since **B** is solenoidal,

$$\frac{\partial \hat{B}_{z,l}}{\partial z} = -\frac{2\pi \mathrm{i} l}{\lambda} \hat{B}_{x,l} = \frac{2\pi l^2}{\lambda |l|} \hat{B}_{z,l} \qquad (l \neq 0) \tag{6}$$

at $z = z_0$.

It is also appropriate to relax the thermal constraints by allowing a variable temperature $T(x, z_0, t)$ on the upper boundary. This can be done by matching the heat conducted to the boundary to black-body radiation from the surface so that, in dimensional terms, $K\partial T/\partial z = \bar{\sigma}T^4$, where K is the

of $\bar{\lambda}$ for the linearized equations with the idealized boundary conditions (1)–(3); replacing (3) by the 'realistic' boundary conditions (5)–(7) has only a small stabilizing effect. With our values $\lambda_1 \approx 0.35$ and $\lambda_2 \approx 45$, and from the figure we see that Q_c attains its maximum value ($Q_c \approx 4\,200$) for $\bar{\lambda} = \bar{\lambda}_c \approx 0.72$. Note, however, that a box with aspect ratio λ only allows discrete wavelengths λ/n, $n = 1, 2, ...,$ corresponding to n pairs of rolls and that for large λ we might expect to find $n \approx \lambda/\bar{\lambda}_c$.

Fully nonlinear solutions have to be obtained numerically. We use a 2D version of the 3D code described by Matthews *et al.* (1995; see also Weiss *et al.* 1996), which is related to codes developed for other nonmagnetic problems (Cattaneo *et al.* 1991; Brummell, Cattaneo & Toomre 1995; Brummell *et al.* 1996). This code uses a pseudospectral method in the x-direction, coupled with fourth-order finite differences in the z-direction. Thus the boundary conditions (5) and (6) for the magnetic field can readily be satisfied. Timestepping is by a second-order Adams-Bashforth scheme, limited by the Courant condition and its diffusive analogue. We typically obtain sufficient accuracy with 32 mesh intervals in the z-direction and 256 points in the x-direction for $\lambda = 6$. Such 2D computations can readily be carried out on Hewlett–Packard, Silicon Graphics and DEC Alpha workstations – but the corresponding 3D calculations require a massively parallel machine.

3   VARYING THE ASPECT RATIO

The results of any numerical experiment depend critically on the choice of boundary conditions. In our runs we impose periodicity in the lateral direction, so the solution will depend on the aspect ratio, λ. In this section we show that the width of the box can have a profound effect on the resulting flow. To do this, we fix the values of all the physical parameters, setting $\hat{R} = 100\,000$ and taking a particular value for Q, and hence for the strength of the magnetic field. As already stated, we adopt the idealized boundary conditions (2) at the base of the box, where the field is held vertical and the temperature is fixed, but introduce the 'realistic' conditions (5)–(7) at the top, where the field is matched continuously to an external potential field and energy is lost by radiation.

With weak fields ($Q \leq 200$) behaviour is effectively kinematic, while strong fields ($Q \geq 2\,000$) allow only steady convection to occur. We need a value of Q that allows different types of dynamically interesting behaviour and hence we choose $Q = 700$. The results of our runs for $\frac{1}{3} \leq \lambda \leq 16$ are summarized in Table 1 where we give the number n of cells in the box, the rms velocity $u_{\rm rms} = \sqrt{\langle\langle\langle|\mathbf{u}|^2\rangle_x\rangle_z\rangle_t}$, the normalized mean surface temperature $\bar{T}$ and the normalized energy transport $\overline{T^4}$, where

$$\bar{T} = \langle\langle\theta T(x, z_0, t)\rangle_x\rangle_t \quad \text{and} \quad \overline{T^4} = \langle\langle[\theta T(x, z_0, t)]^4\rangle_x\rangle_t, \quad (10)$$

and the average maximum field strength $B_{\rm max} = \langle \max|\mathbf{B}|\rangle_t$ (where the initial uniform field strength is unity).

3.1   Weak convection in narrow boxes ($\lambda \leq 2$)

From Fig. 1 we anticipate that convection will be completely suppressed for $\lambda < 0.4$. Indeed, initial perturbations soon die away when $\lambda = \frac{1}{3}$ but with $\lambda = \frac{1}{2}$ we obtain the weakly

Table 1. Varying the aspect ratio, λ, for a fixed field strength ($Q = 700$). We give the pattern of convection; the number of cells in the box, n; the rms velocity, $u_{\rm rms}$; the normalized mean surface temperature, $\bar{T}$; the normalized energy transport, $\overline{T^4}$; and the maximum field strength, $B_{\rm max}$, as the box size is varied.

λ	pattern	n	$u_{\rm rms}$	$\bar{T}$	$\overline{T^4}$	$B_{\rm max}$
$\frac{1}{3}$	*no convection*	0	–	1.000	1.000	1.0
$\frac{1}{2}$	*steady convection*	1	0.16	1.050	1.215	3.7
1	*travelling wave*	1	0.45	1.097	1.465	14.6
2	*travelling wave*	2	0.45	1.098	1.470	15.4
$\frac{8}{3}$	*aperiodic oscillation*	3	0.44	1.098	1.468	17.1
4	*transitional*	4	0.46	1.095	1.454	18.3
6	*flux separation*	1	0.68	1.092	1.445	24.0
8	*flux separation*	1	0.80	1.093	1.449	25.9
16	*flux separation*	2	0.71	1.093	1.448	27.5

convecting steady solution that is shown in Fig. 2(a). Here the pattern is symmetric about vertical planes through the centres of the plumes. As expected, the hot rising plume expands and the highest velocities are in the narrow sinking plumes. The field lines are weakly distorted and flux is concentrated in regions of converging flow.

Reflection symmetry is broken in a bifurcation at $\lambda \approx 0.6$, which gives rise to travelling waves. Fig. 2(b) shows such a solution for $\lambda = 1$. The plumes drift leftward without change of form but are markedly asymmetric, being tilted by the streaming flow. The field is weakened by expansion in the rising plume but there are stronger flux concentrations where the horizontal in-flows meet. When $\lambda = 2$ the box is wide enough to accommodate two pairs of plumes, and the waves have become weakly modulated, so that they oscillate aperiodically as they drift.

3.2   Spatially modulated oscillations in medium-sized boxes ($\frac{8}{3} \leq \lambda \leq 4$)

Once $\lambda > 2$ the solutions are no longer severely constrained by the width of the box, and can pick a length scale that is close to the preferred wavelength, $\bar{\lambda}_c$, for linear instability. Convection consequently becomes much more vigorous. When $\lambda = \frac{8}{3}$ the box contains three rising plumes that oscillate chaotically, and much more vigorously than was the case for $\lambda = 2$. Two opposite phases of a spatially modulated oscillation are displayed in Fig. 2(c). Apart from a slow drift to one side, the positions of the plumes at the base of the layer (where $\zeta > 1$) are virtually unaltered, while there are large changes in velocity at the top (where $\zeta < 1$) as adjacent plumes alternate in strength. This pattern of behaviour is characteristic of convection in strong fields.

Results for the run with $\lambda = 4$ are initially similar to those for $\lambda = \frac{8}{3}$, except that there are now four plumes that oscillate chaotically. However, there is a slow secular change as a particular plume grows in size and oscillates less, while the other plumes contract accordingly. At the same time, practically all magnetic flux is cleared from within the largest plume and remains clustered at its edge. However, the growth of such a plume is soon stopped and it sometimes

Figure 2. The effects of increasing the aspect ratio λ for a fixed field strength ($Q = 700$): (a) shows $\lambda = \frac{1}{2}$, (b) $\lambda = 1$, (c) $\lambda = \frac{8}{3}$, (d) $\lambda = 4$. In each pair of images the left panel shows the relative temperature fluctuations, with velocity arrows, and the right panel gives the field strength $|\mathbf{B}|^2$ with field lines. Bright regions indicate higher temperatures and field strengths.

shrinks again, while another plume is allowed to grow. This aperiodic cycle continues, as illustrated in Fig. 2(d). Note that at different times there may be 2, 3 or 4 plumes in the box.

3.3   Separation of the magnetic field in wide boxes ($\lambda \geq 6$)

The behaviour just described is a prelude to the appearance of an entirely new regime. Increasing the box size from $\lambda = 4$ to $\lambda = 6$ allows the development of a qualitatively different pattern. Initially, there are seven plumes in the box, undergoing spatially modulated oscillations – as would be predicted from the results that have just been described. Just as for $\lambda = 4$, some of these plumes begin to grow, while others shrink and vanish. This evolutionary process can be visualized by displaying the temperature, $T(x, z_0, t)$, on the upper surface – an advantage of using the boundary condition (7) rather than (3). The grey-scale image in Fig. 3(a) shows spatially modulated oscillations for $t < 50$. Subsequently, some plumes (like that at $x \approx 2$) expand and swallow up an immediate neighbour, leaving a single plume of twice the size where two were previously. The horizontal outflow of the resulting plume is then strong enough for it to resist being torn apart again. This process continues and the total number of plumes declines until only two are left. At this stage, the plumes themselves contain virtually no magnetic flux and the magnetic field is confined to the regions between them, where convection is inhibited. The two plumes then engage in a battle for domination as the horizontal outflow at the top of each exerts a lateral force on the magnetic flux between them. Slowly, one of the plumes – usually, but not always, the one that was initially the smaller – is reduced in size, until the surrounding magnetic field is able to suppress it. This results in a single convective plume occupying approximately two-thirds of the box.

This plume can settle to what appears to be a steady state but, as is apparent in Fig. 3(a), it is liable to split. Splitting can occur in either of two ways, as may be seen in Fig. 3(a); it may happen rapidly, for example at $t = 650$, and subsequently several times at around $t = 1100$, or slowly, as seen between $t = 720$ and $t = 1000$. Careful analysis of both situations has demonstrated that they are essentially manifestations of the same process.

Fig. 4 shows a time-sequence of a rapid split that occurs at around $t = 1170$. Initially, there is a broad, symmetric convection cell with a strong upflow in its centre. The hot rising gas then moves horizontally outwards until it meets the wall of high field strength, before sinking suddenly again. The plume itself is practically field-free – in places the field strength is reduced to one-millionth of its original value – except for a narrow flux concentration at its base, where the field is pinned down by the horizontal inflow. The remainder of the box, on the other hand, has a strong and fairly uniform vertical field throughout, with narrow current sheets at its edges. The high field strength halts convection here, and the region is held at a fairly cool and steady temperature.

At the centre of the plume, buoyancy braking (Spruit *et al.* 1990) inhibits upward motion. From time to time this causes local cooling to occur and the plume effectively splits into two as the cool material falls. Some of the magnetic flux accumulated near the upper stagnation point is caught by

the horizontal outflow and rapidly expelled, while the plume recovers its symmetric state. The process takes place on a dynamic timescale of around 25 dimensionless time units and repeats indefinitely at irregular intervals.

If the blob of cool material drops sufficiently close to the centre of the plume then small countercells may be formed before the horizontal outflow can expel it. In this case the split is maintained for around $100 - 300$ time units before the asymmetry of the two resulting plumes causes it to drift to one edge and the single symmetric plume is reformed. If we were actually to impose mirror symmetry about the centre of the plume then the studies of Steffen, Ludwig & Krüss (1989) imply that the sinking plume would remain there indefinitely, as was found in their model of axisymmetric convection.

This new pattern of behaviour persists thereafter as the aspect ratio is increased. When $\lambda = 8$ the main plume takes a little longer to form, and is narrower compared to the box size. Apparently this width of box is on the borderline between being able to hold one stable plume, and two; indeed, it takes much longer for the two parts of the main plume to recombine when it has split. Such behaviour is consistent with the results for $\lambda = 16$, which are shown in Fig. 3(b). The box now contains two large stable plumes, which undergo dynamic splitting irregularly and independently of each other. Magnetic flux is trapped in the two regions that lie between them: in one the field is strong enough to suppress convection but the other has a weaker field that allows feeble spatially modulated oscillations to survive. Table 1 confirms that spatially averaged behaviour does not change significantly for $\lambda > 6$. We conclude therefore that an aspect ratio $\lambda = 6$ should be sufficient to deliver the true pattern of convection, i.e. one that remains unchanged as λ is indefinitely increased.

4   VARYING THE FIELD STRENGTH

The results in the previous section show that the pattern of convection is constrained in boxes with small aspect ratios. Even when the box contains several plumes (as in Fig. 2(c) for $\lambda = \frac{8}{3}$), the results may be misleading, since wider boxes allow the magnetic field to separate from the motion. We can, however, be confident that this new form of magnetoconvection is adequately represented for all $\lambda \geq 6$. In this section we therefore fix the aspect ratio by setting $\lambda = 6$ and obtain a single-parameter set of solutions by holding the Rayleigh number fixed (with $\hat{R} = 100\,000$) and varying the field strength. (Decreasing Q for fixed $\hat{R}$ is of course equivalent to increasing $\hat{R}$ for fixed Q but we are more concerned with the effects of strong fields on the pattern of convection.) The results for $5\,000 \geq Q \geq 250$ are summarized in Table 2.

4.1   The magnetically dominated regime

We begin with a magnetic field strong enough to suppress all convection and gradually decrease the field strength, until we end up in the kinematic regime. For $Q \geq 4500$, the Rayleigh number is subcritical and all perturbations to the static state, with a uniform vertical field and a uniformly stratified equilibrium atmosphere, decay to zero.

Table 2. Varying the field strength for fixed aspect ratio, $\lambda = 6$, as the Chandrasekhar number, Q, is varied. The column entries are defined in (10).

Q	pattern	$u_{\rm rms}$	$\bar{T}$	$\overline{T^4}$	$B_{\max}$
250	*flux separation*	1.554	1.114	1.560	68.0
500	*flux separation*	1.076	1.101	1.496	38.0
700	*flux separation*	0.681	1.092	1.445	24.0
750	*oscillations*	0.412	1.095	1.452	18.2
800	*oscillations*	0.374	1.094	1.448	16.0
1000	*oscillations*	0.303	1.079	1.369	11.1
1500	*oscillations*	0.147	1.049	1.213	4.4
1750	*steady convection*	0.112	1.041	1.175	3.8
2000	*steady convection*	0.092	1.036	1.151	3.6
3000	*steady convection*	0.040	1.014	1.057	2.0
4000	*steady convection*	0.010	1.001	1.004	1.3
4500	*no convection*	—	1.000	1.000	1.0
5000	*no convection*	—	1.000	1.000	1.0

Convection sets in with narrow rolls at $Q_c \approx 4\,200$. For $Q = 4\,000$, nine small, equally sized cells are rapidly formed (so $\bar{\lambda} = \frac{2}{3}$) and they remain steady throughout the ensuing motion. The pattern in each cell is similar to that in Fig. 2(a) and the magnetic field is only slightly distorted by the fluid motion. Fig. 5(a) shows the evolution of the surface temperature $T(x, z_0, t)$ when $Q = 2\,000$. Once again, the strong field enforces steady motion, somewhat more vigorous than before, but there are now only eight plumes ($\bar{\lambda} = \frac{3}{4}$).

4.2 Spatially modulated oscillations

This steady solution undergoes an oscillatory bifurcation at $Q \approx 1\,700$. For $Q = 1500$, we find oscillations that are effectively periodic. The box contains 9 plumes, all comparable in size, which are spatially modulated so that alternate plumes grow and shrink in strength. This solution closely resembles that found by Weiss *et al.* (1990) for $Q = 1\,200$ but with different boundary conditions.

The motion grows more vigorous as the field strength is further reduced. The evolution of the surface temperature for $Q = 1\,000$ is displayed in Fig. 5(b). This solution bears little resemblance to the ordered oscillations seen when $Q = 1500$; there are now only eight plumes and these oscillate aperiodically, and much more vigorously than before. With $Q = 800$, the chaotic solution has only seven plumes and convection has increased yet further in strength. Finally, at $Q = 750$, the seven plumes are no longer held separate by the field; sporadically two of them merge, only to be torn apart again soon after, as shown in Fig. 5(c). The field no longer has any persistent structure, and is tossed around by the disordered motion of the fluid.

4.3 Flux separation

As Q is reduced from 750 to 700, there is a very rapid change in the system. When the field strength passes below a certain point, global behaviour is no longer dominated by the field and we begin the approach to the kinematic regime; motion

is now so vigorous that the magnetic field is only important dynamically in the regions where it is locally strong. The evolution of the surface temperature for $Q = 700$ is shown in Fig. 3(a) and has already been discussed. We see the formation of the single convection cell that was described in §3.3 and illustrated in Fig. 4. The rms velocity has increased dramatically to nearly 70% of the surface Alfvén velocity, while the maximum velocity has more than double the value it had for $Q = 750$. The rapid oscillations that are visible in Fig. 3 are due to convectively excited magnetoacoustic modes (cf. Steffen *et al.* 1989).

Further reduction of Q produces very little qualitative change. The two large plumes form more rapidly, the motion becomes more chaotic, and the dynamic splitting and flux expulsion grow more violent. The decreased field strength allows the flow to push the flux into an ever smaller region, while the plume width increases accordingly. Fig. 5(d) shows $T(x, z_0, t)$ for $Q = 500$. In the final state, the pair of convection rolls now occupies about three-quarters of the box, with the flux sheet filling the remaining quarter. Compare this to Fig. 7(a) which shows $T(x, z_0, t)$ for $Q = 250$ in which the rolls take up over 90% of the box. The early oscillations are suppressed far more rapidly as the field distortion becomes ever more nearly kinematic. In both cases, the structure is repeatedly distorted by splitting of the plume, as in Fig. 4.

The flux sheet there has a strong, predominantly vertical field but the field lines at its edges are curved so as to be concave outwards. At the base of the layer they follow the flow but at the top the expansion is caused by magnetic pressure, which becomes increasingly important for small z in a strongly stratified atmosphere. The curvature force is almost balanced by a magnetic pressure gradient. Fig. 6 shows profiles of the vertical and horizontal components of **B** at the middle of the layer, where the field points outwards. Some of the magnetic flux is trapped in the rising plume but most of it is contained in the main flux sheet. In its centre the field is fairly uniform but B_z rises to sharp peaks at the edges of the sheet. Close inspection of the solutions shows that these peaks are maintained by slender countercells in the lower part of the box.

The abrupt transition from chaotically modulated oscillations to flux separation at $Q \approx 730$ is associated with hysteresis. In Fig. 7(b) we show the surface temperature for a run with $Q = 1\,000$. Unlike those for Fig. 5(b), which was started from small random perturbations to the static solution, the initial conditions for this run correspond to the final state for $Q = 700$, in Fig. 4. We observe that flux separation is maintained, though the flux sheet expands while the convective plume contracts and is more liable to split. As Q is further increased, flux separation eventually gives way to small scale oscillatory convection; with carefully chosen initial conditions, a transient single cell solution can nevertheless persist for a long time (over 300 dimensionless time units) even when $Q = 2\,000$.

Eventually, when Q is very small, the Lorentz force becomes negligible and the field profile is determined by diffusion. In this kinematic regime the whole box is filled by a single plume ($\bar{\lambda} = \lambda$). We have not attempted to discover whether the plume width saturates at some finite size in the absence of a magnetic field. We note, however, that N.E. Hurlburt (private communication) investigated 2D convection in a similar configuration, but with idealized bound-

ary conditions, for $\lambda = 12$ and found that although the box contained several plumes early on in the simulation, over a long period of time a single plume began to dominate whilst the others dwindled away. Of course, such a plume would be unstable to 3D perturbations in the form of cross-rolls (cf. Matthews *et al.* 1995).

The numerical results in Table 2 are summarized in Fig. 8. We note that $\overline{T^4}$ rises rapidly with decreasing Q after the onset of convection, with an increased rate of change after the onset of oscillatory convection, but that – despite a rapid increase in u_{rms} – convection becomes *less* efficient after flux separation occurs at $Q \approx 730$, owing to the appearance of stationary flux sheets.

With our scaling, the temperature fluctuations at the upper surface remain small (less than 20%) and the mean vertical temperature gradient $\bar{\beta} = 1 + z_0(1 - \overline{T})$ differs only slightly from its value in the static polytrope. To measure the efficiency of convection, we should compare the heat transport with the energy that would be conducted down the superadiabatic gradient. Thus we define a Nusselt number

$$N = \frac{\overline{T^4} - \bar{\beta}_{\mathrm{ad}}}{\bar{\beta} - \bar{\beta}_{\mathrm{ad}}}, \tag{11}$$

where the adiabatic gradient $\bar{\beta}_{\mathrm{ad}} = (m+1)(1 - 1/\gamma)$ (Hurlburt *et al.* 1989). With our choice of parameters, $N \approx 5\overline{T^4} - 4$, so the Nusselt number rises from unity (in the absence of convection) to about 4 when $Q = 250$. The difference between this and the much higher value of N for 2D convection in an incompressible (Boussinesq) fluid at the same value of $\hat{R}$ is due to the combined effects of stratification and of the stagnant flux sheet.

5   EFFECTS OF DIFFERENT BOUNDARY CONDITIONS

We have obtained solutions with 'realistic' boundary conditions (5)–(7) at $z = z_0$ for other combinations of Q and λ in addition to those already described. Fig. 9(a) shows the various types of solution found in different regions of the $Q\lambda$–parameter plane. It is clear that the sequence of transitions, from the static state to steady convection, then to spatially modulated oscillations and finally to flux separation is robust. Their order does not vary with the aspect ratio once $\lambda > 1$, although the transitions do occur at somewhat higher values of Q for $\lambda \geq 6$.

The transition from oscillations to flux separation is accompanied by a change in horizontal scale. The physical mechanism whereby narrow rolls give way to broader rolls is straightforward. Once convection is strong enough for flux expulsion to occur, motion becomes more vigorous as the width of a roll increases (up to some limit that we are unable to determine). Consider a symmetrical pair of rolls in dynamical equilibrium, with magnetic flux confined to a sheet between them. Suppose now that one roll expands slightly while the other contracts: then the expanding roll will squeeze the flux sheet to produce a stronger field, which will in turn lead to a further contraction of the smaller roll. This process can then continue until the weaker roll is eliminated. Moreover, flux transport may be facilitated if the rising plume tilts and gives rise to shearing motion (cf. Proctor *et al.* 1994). One would expect there to be parameter values

at which both the narrow-roll and the flux separated regimes are stable, as we have demonstrated.

From a mathematical point of view, one would like to associate these transitions with bifurcations from one state to another. This is feasible for solutions whose spatiotemporal structure is straightforward. Thus we can identify the initial pitchfork bifurcation from the trivial solution and the subsequent bifurcation that leads to travelling waves as Q is decreased (cf. Hurlburt *et al.* 1989). While the chaotically modulated regime present with weaker magnetic fields has complicated spatial and temporal structures, it is clear – at least qualitatively – that it retains a characteristic spatial scale that is much smaller than in the flux-separated regime. Neither of these two states has instantaneous spatial symmetries, but one quantitative way of characterising the difference between them is through their time-averaged spatial periodicity. For instance, once the slow drift is removed from the data in Figure 5(c) ($\lambda = 6, Q = 750$), the time-averaged solution has wavelength 6/7, while the wavelength of the pattern with $Q = 700$ (Figure 3(a)) is 6, the size of the box. Techniques for measuring time-averaged symmetries have been developed by Barany, Dellnitz & Golubitsky (1993), and would be useful in determining more precisely the nature of the transitions that we have observed. In particular, it should be possible to relate the transitions between the narrow-roll and flux-separated regimes to the bifurcation structure associated with subcritical behaviour in Boussinesq magnetoconvection (cf. Proctor & Weiss 1982, Fig. 31). This approach deserves further investigation.

Having ascertained the effects of varying the aspect ratio and field strength on the pattern of convection, we now go on to investigate the sensitivity of this behaviour to changes in the magnetic and thermal boundary conditions at the top of the layer.

5.1   Magnetic boundary conditions

The obvious alternative to imposing the potential field conditions (5) and (6) is to set $B_x = \partial B_z/\partial z = 0$ at $z = z_0$, as in (3), while retaining the radiative condition (7), so that **B** is vertical at the top as well as at the bottom of the layer. Results for this choice of boundary conditions are summarized in Fig. 9(b). Although the survey is less thorough than that in Fig. 9(a), it is clear that the overall pattern is similar but with transitions displaced to higher values of Q. The two sets of runs for $\lambda = 6$ are contrasted in Fig. 10. The sequences of transitions are identical, and the initial bifurcations occur very close together. However, steady solutions persist over a wider range of Q with a potential field. Moreover, flux separation appears already at $Q = 1\,500$ with a vertical field, rather than at $Q = 700$ as it does with a potential field.

The difference between the two cases is simply that, for a given value of Q, the magnetic field is less potent in Fig. 9(a) than in Fig. 9(b). The reason is clear: when the field is vertical, the curvature force vanishes at the surface – where it would be most effective. On the other hand, the potential field boundary conditions allow the Lorentz force to inhibit convection more efficiently.

It is instructive to experiment with other combinations of the magnetic boundary conditions while keeping $Q = 1000$ and $\lambda = 6$. Comparing the new results with those

in §4, we find that imposing a potential field at the top *and* the bottom has little effect. The spatially modulated oscillations are more regular and uniform, and the plumes wobble laterally as they wax and wane, reflecting the extra freedom given by a potential field. Imposing a potential field at the base, with a vertical field at the top, though unphysical, has a more marked effect: the oscillating plumes demonstrate no real regularity in either space or time, being of varying size, shape, and period. Their number varies between 7 and 8, and the pattern of behaviour is exceedingly complex.

5.2 Thermal boundary condition

In previous papers (e.g. Weiss *et al.* 1990, 1996) the temperature was held fixed at both the surface and base. In substituting the radiative boundary condition (7) at the surface, it is important to discover how those results are changed. Several sets of numerical experiments were performed in which all parameters and boundary conditions were identical, except that in one run the temperature was held constant at $z = z_0$, $z_0 + 1$, and in the other the radiative boundary condition $\frac{\partial T}{\partial z} = (\theta T)^4$ was imposed at $z = z_0$. The results of these computations demonstrate that imposing the new boundary condition has very little effect on the pattern of convection – which is not surprising, since the resulting temperature variations at the surface are comparatively small. In each case the two runs were almost identical, though the radiative boundary condition produced slightly more dynamical activity.

For comparison with earlier work, we show in Fig. 11 a time sequence from a run with $Q = 1000$ and $\lambda = 6$, with the idealized boundary conditions (2) and (3) at both boundaries. During this transient phase, there are five or six oscillating plumes of different strengths and sizes but eventually, after a very long time, the flux separates out to leave a single plume. We comment later on the implications of this run.

6 COMPARISON WITH OBSERVATIONS

The ease with which two-dimensional computations can be carried out has allowed us to make a systematic survey of the effects of varying both the aspect ratio and the boundary conditions in this problem. Not surprisingly, we find that solutions are profoundly affected if the box width λ is too small. It was already known (for idealized boundary conditions, with $Q = 1000$) that the steady solution with $\lambda = \frac{2}{3}$ gave way to spatially modulated oscillations when $\lambda = \frac{4}{3}$ (Weiss *et al.* 1990). As the aspect ratio is further increased, both spatial and temporal symmetries are broken: the solution illustrated in Fig. 11 shows aperiodic oscillations with a wide range of spatial scales for $\lambda = 6$. Within this chaotic pattern there is nevertheless a tendency for adjacent plumes to alternate in vigour. What was not expected was that this multiroll solution proved to be transient and eventually led to flux separation, with behaviour similar to that in Fig. 4. Moreover, there is a parameter range with hysteresis, where both multiroll convection and flux-separated solutions are stable; which of these is found then depends on the initial conditions. This result suggests that isolated sheets (or tubes) of magnetic flux, with fields that are sufficiently intense, may be able to maintain their integrity if they are injected into regions where small-scale convection is locally preferred.

We have established the sequence of transitions as the field strength is progressively reduced from a value that is strong enough to stop convection. These changes, from steady convection to spatially modulated oscillations (first periodic and then chaotic) and then to a regime with flux separation, are shown schematically in Fig. 10. Altering the boundary conditions simply shifts the transitions without changing the order in which they occur, so the sequence is apparently robust. We cannot, however, use these 2D results to predict the actual parameter ranges where different 3D patterns will be found on the Sun.

The clearest parallel between theory and observations is for light bridges in the umbrae of sunspots, which provide a quasi-two-dimensional configuration. Rimmele (1997) studied a light bridge with a row of three bright granules and found that the vertical velocity and continuum intensity were anticorrelated, thereby confirming that the structure was convective. He also showed that individual structures waxed and waned, with a characteristic period of about 30 min, in a manner consistent with the presence of irregular spatially modulated oscillations. The magnetic field of 1 000 G or less permits relatively large granules to develop, with a separation of $1''.0$ to $1''.5$. The light bridge therefore acts as a box with a fairly small aspect ratio (cf. Fig. 2(c)), allowing chaotic oscillations to be maintained.

High-resolution spectroscopic observations show that umbral dots are indeed a magnetoconvective phenomenon (Lites *et al.* 1991) and Rimmele (1997) has succeeded in measuring upward velocities of 50 m s^{-1} at the photosphere. Such relatively low velocities, combined with the lack of any observable variation in the strength of the magnetic field, imply that the convective plumes (which must be present to transport energy) are obscured by a radiative blanket (Lites *et al.* 1991, Weiss *et al.* 1996). Rimmele (1997) also identified periodic variations of intensity in some prominent umbral dots, with periods of about 20 min. These observations are all consistent with the behaviour of spatially modulated convective plumes, in a regime where their spatial and temporal structure is chaotic (Weiss *et al.* 1990, 1996). Recent white light observations with exceptionally high resolution have revealed a population of umbral dots with diameters ranging from $8''$ down to $0''.28$ or less, and a corresponding range of lifetimes. We have confirmed that larger aspect ratios do allow a much wider range of horizontal scales in our numerical experiments, as shown by Fig. 11.

The dark nuclei within the umbrae of large spots show no signs of convective activity. They contain magnetic fields that are stronger and more nearly vertical than elsewhere (Stanchfield, Thomas & Lites 1997). Convection must still be effective at some depth beneath these features (which are too large to be heated laterally by radiation) but is relatively weak. Our results suggest that the dark nuclei result from a form of flux separation within the umbra, leading to isolated regions with enhanced field strengths and relatively feeble motion.

The persistence of plage regions, as local flux concentrations with abnormal granulation, is another example of flux separation. Here the stronger fields allow a modified form of convection, with smaller plumes than in the ambient field-free photosphere. An idealized version of this behaviour ap-

pears in some of our runs, which show small-scale spatially modulated oscillations adjacent to broad convective plumes (see Fig. 3(b)). In our calculations, flux separation is associated with splitting of the field-free plumes, as illustrated in Fig. 4. The key feature of this process is the appearance of a downflow near the centre of the plume, as a result of buoyancy braking (Spruit *et al.* 1990). What happens next depends on the imposed geometry: with imposed axial symmetry the reversed flow remains on the axis (Steffen *et al.* 1989); in our 2D models the downflows move outwards; in 3D simulations the behaviour mimics observations of exploding granules (Nordlund 1985; Spruit *et al.* 1990; Rast *et al.* 1993; Rast 1995) and the sizes of field-free plumes are limited by this process.

These 2D investigations do improve our understanding of the interaction between magnetic fields and convection at the surfaces of stars like the Sun, but they should really be regarded as necessary preliminaries to more realistic 3D experiments. Clearly it is essential to carry out such computations in wide boxes. Preliminary calculations, in square boxes with aspect ratios $\lambda = 4$ and 8, show that flux separation still occurs and, as might be expected, leads to a richer variety of behaviour in 3D than we have found here (Tao *et al.* 1998). We have established that qualitative behaviour is relatively insensitive to the choice of boundary conditions, so pattern formation can indeed be studied in idealized models. For quantitative predictions, on the other hand, we must rely on detailed simulations (Spruit 1997). As numerical techniques and computing power improve, these two approaches are gradually converging and coming closer to the reality that is revealed by observations.

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