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Konstantopoulos, G. orcid.org/0000-0003-3339-6921, Zhong, Q., Ren, B. et al. (1 more author) (2016) Bounded Integral Control of Input-to-State Practically Stable Non-linear Systems to Guarantee Closed-loop Stability. IEEE Transactions on Automatic Control, 61 (12). pp. 4196-4202. ISSN 1558-2523

https://doi.org/10.1109/TAC.2016.2552978

Bounded Integral Control of Input-to-State Practically Stable Non-linear Systems to Guarantee Closed-loop Stability

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***Abstract*—A fundamental problem in control systems theory is that stability is not always guaranteed for a closed-loop system even if the plant is open-loop stable. With the only knowledge of the input-to-state (practical) stability (ISpS) of the plant, in this note, a bounded integral controller (BIC) is proposed which generates a bounded control output independently from the plant parameters and states and guarantees closed-loop system stability in the sense of boundedness. When a given bound is required for the control output, an analytic selection of the BIC parameters is proposed and its performance is investigated using Lyapunov methods, extending the result for locally ISpS plant systems. Additionally, it is shown that the BIC can replace the traditional integral controller (IC) and guarantee asymptotic stability of the desired equilibrium point under certain conditions, with a guaranteed bound for the solution of the closed-loop system. Simulation results of a dc/dc buck-boost power converter system are provided to compare the BIC with the IC operation.**

***Index Terms*—Integral control, non-linear systems, input-to-state stability, bounded input, small-gain theorem.**

I. INTRODUCTION

MOST engineering systems are bounded input-bounded output stable (BIBO). For this type of systems, an open-loop controller can easily bring the system in a desirable and stable operation. However, it is widely known that, when external disturbances or parameter variations occur, feedback is essential to achieve a desired performance [1], [2]. By closing the loop, stability is no longer guaranteed even for BIBO plants. Many researchers have focused on solving the stability problem of a closed-loop system, especially for the most common scenario, i.e. regulation.

During the last 40 years, integral control (IC) has been extensively used in control systems for achieving asymptotic regulation and disturbance rejection for systems with inherent parameter variations. The addition of the integrator dynamics results in an augmented system, where traditional state feedback techniques can be applied [1], [3], [4]. However, even for linear systems, closed-loop system stability with an integral control action is only guaranteed with sufficiently small integral gain and under necessary and sufficient conditions on the plant [5], [6]. Particularly, an analytic calculation of the maximum integral gain for guaranteeing closed-loop system stability of finite-dimensional linear systems can be found in [7].

The application of IC was extended to non-linear systems [1], [8], [9] with local closed-loop stability results. Semi-global results were provided in [10]–[12] for minimum-phase systems using output feedback control and high-gain observers. The idea was to transform the system into the normal form [13] and apply a saturating controller outside a compact set of interest. These results were further extended in [14] where a robust integral controller was designed according to the relative degree of the non-linear plant. Recently, conditional integrators were proposed in [15], [16], which provide the integral action inside a boundary layer and act as a stable system outside of it. In many of these works, some of the assumptions mentioned for the plant are directly related to the input-to-state stability (ISS) property [17], [18], while in [14], the generalised small-gain theorem was used [19]–[21], which represents a fundamental tool for robust stability. A different approach of IC in port-Hamiltonian systems for disturbance rejection can be also found in recent works [22], [23], where the port-Hamiltonian form is maintained and closed-loop system stability can be proven for systems with relative degree higher than one.

As demonstrated in the previous works [1], [8]–[14], the IC design for non-linear systems depends on the structure of the system (relative-degree etc.) and often results in a very complicated control scheme that requires a saturation unit to guarantee a bounded area for the controller output. The saturation unit is often applied to the traditional IC leading to a simple structure and easy implementation, but it is difficult to rigorously prove the stability, since it often leads to integrator windup and undesired oscillations. Anti-windup techniques can be used to cope with this problem, but the knowledge of the plant is often required to guarantee closed-loop system stability [24]–[28]. The proof of stability becomes even more difficult if the plant is locally ISS with the plant input considered in general as unconstrained. Therefore, the existence of a generic controller without saturation units

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The financial support from the EPSRC, UK under Grant No. EP/J01558X/1 is greatly appreciated.

that operates similar to the traditional IC independently from the non-linear plant structure and parameters, and guarantees closed-loop system stability is of significance.

In this note, a bounded integral controller (BIC) is proposed to guarantee the non-linear closed-loop stability for globally or locally input-to-state (practically) stable (ISpS) plant systems. With the only knowledge of the input-to-state practical stability (ISpS) property of the plant [17], [18], it is proven that the proposed BIC guarantees closed-loop system stability in the sense of boundedness using the generalised small-gain theorem [19], [20]. It should be noted that the system dynamics and/or parameters can be completely unknown. Although the plant input is considered unconstrained, often a given bound is introduced for stability reasons, such as for locally ISS systems. Therefore, an analytic selection of the controller parameters is presented to achieve a bounded controller output within a given range, thus extending the stability analysis to locally ISpS plant systems. Additionally, it is proven that the BIC maintains the performance of the traditional IC near the equilibrium point under some conditions. Particularly, if linearisation around an equilibrium point of an ISpS plant operating with the traditional IC results in asymptotic stability, then asymptotic stability is still maintained if the IC is replaced by the BIC. Moreover, the BIC guarantees that the solution will remain bounded, i.e. instability is avoided, even if the equilibrium point or the system parameters change. The boundedness of the solution is guaranteed in some systems even when the equilibrium point is shifted outside of the bounded range or the equilibrium point is unstable. This approach does not obsolete the IC methods proposed in the literature; in contrary, it can be easily combined with many of them to simplify the stability analysis and guarantee a given bound for the control output. Note that the proposed BIC does not use a saturation unit and as it is proven, it does not suffer from integrator windup problems. In fact, it is shown that the integration slows down near the limits without requiring any switches or knowledge of the plant parameters. Thus, the proposed BIC is expected to solve many practical and industrial problems where the traditional IC is used without any rigorous stability proof. Such an example is a dc/dc buck-boost converter system, which is simulated to verify the BIC method compared to the traditionally used IC and provide the theory that is currently missing.

II. Problem formulation

Many engineering systems are BIBO stable due to their inherent dissipative structure. In the ideal case, simple open-loop control strategies that generate a bounded control output can regulate the system output to its desired value without affecting the system stability. However, in a real environment, there are external disturbances and parameter variations that could considerably degrade the system performance. As a result, it is essential to close the loop by using feedback control, as shown in Fig. 1, in order to achieve desired performance, e.g. zero steady-state error, even when there are disturbances, parameter variations and uncertainties. The problem is that a feedback controller no longer guarantees a bounded control output, which may cause instability. In other words, the stability of the system is no longer preserved. Developing feedback control strategies that preserve the BIBO stability of the system is of significance.

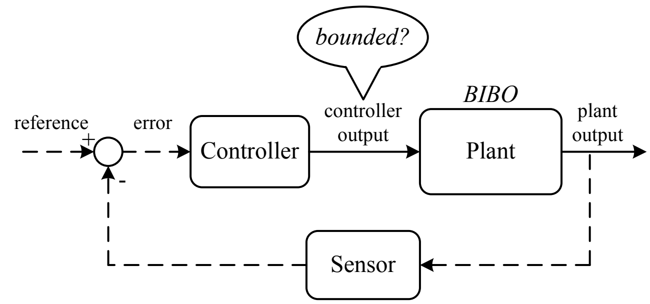


Figure 1. Closing the loop

Particularly, when a regulation problem is considered, which is the most common control objective, an IC is used to achieve zero steady-state error. Consider a general non-linear system

$$\dot{x} = f\left(x, u\right), \tag{1}$$

where $f : D \times D_u \rightarrow R^n$ is locally Lipschitz in x and u and D, D_u are open neighbourhoods of the origin for x and u, respectively. For simplicity, consider a single-input system in the form of (1) and assume that the control task is the regulation of a scalar function $g(x)$ to zero. This assumption also includes the common regulation scenario of a state variable x_i to a desired level x_i^{ref}, i.e. $g(x) = x_i^{ref} - x_i$. The traditional IC that achieves this task is given as

$$u(t) = \int_0^t k_I g\left(x(\tau)\right) d\tau, \tag{2}$$

where $k_I > 0$ represents the integral gain. Then, the IC introduces a dynamic controller that can be written as

$$\begin{aligned} u &= w && (3)\\ \dot{w} &= k_I g(x). && (4) \end{aligned}$$

However, closed-loop system stability is not always guaranteed even if the plant (1) is BIBO. Note that for a non-linear system, the BIBO or input-to-output stability is guaranteed if the plant is input-to-state stable (ISS) and the output function is $K-$bounded [17]. A generic controller that guarantees the stability of the closed-loop system will be developed in this paper.

III. Main result

In this section, the main task is to design a controller that operates similarly to the traditional IC (3)-(4) and generates a bounded output. This controller is called *Bounded Integral Controller (BIC)* and introduces a second controller state as shown below:

$$u = w \tag{5}$$

$$\begin{bmatrix} \dot{w} \\ \dot{w}_q \end{bmatrix} = \begin{bmatrix} -k\left(\frac{w^2}{u_{max}^2} + \frac{\left(w_q - b\right)^2}{\epsilon^2} - 1\right) & k_I g(x) c \\ -\frac{\epsilon^2}{u_{max}^2} k_I g(x) c & -k_q\left(\frac{w^2}{u_{max}^2} + \frac{\left(w_q - b\right)^2}{\epsilon^2} - 1\right) \end{bmatrix} \begin{bmatrix} w \\ w_q \end{bmatrix} \tag{6}$$

where w and w_q are the controller state variables, b is a non-negative constant and u_{max}, k, k_q, ϵ, c are positive constants. Consider, now, the plant system dynamics

$$\dot{x} = f(x, u, u_1) \tag{7}$$

where u describes the control input and u_1 is a vector of external uncontrolled inputs.

After applying the BIC into the general plant, the closed-loop system is described in Fig. 2, which is a composite feedback interconnection form. Here, it is assumed that the function $g(x)$ is locally Lipschitz, which is true in most control applications. Additionally, the plant system is assumed to possess the ISpS (or ISS) property which holds for most engineering systems. Then, the following theorem guarantees the ISpS property of the closed-loop system.

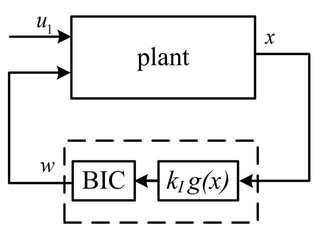


Figure 2. Closed-loop system with BIC

Theorem 1. *The feedback interconnection of plant system* (7) *with the proposed BIC* (5)-(6) *is ISpS with respect to input* u_1*, when the plant system* (7) *is ISpS with respect to both inputs* u *and* u_1*.*

Proof: For the controller dynamics (6), consider the following Lyapunov function candidate

$$V = \frac{w^2}{u_{max}^2} + \frac{w_q^2}{\epsilon^2}. \tag{8}$$

Taking the time derivative of V, it yields

$$\begin{aligned} \dot{V} &= \frac{2w\dot{w}}{u_{max}^2} + \frac{2w_q\dot{w}_q}{\epsilon^2} \\ &= -2\left(\frac{w^2}{u_{max}^2} + \frac{(w_q-b)^2}{\epsilon^2} - 1\right)\left(k\frac{w^2}{u_{max}^2} + k_q\frac{w_q^2}{\epsilon^2}\right). \end{aligned} \tag{9}$$

Its sign is related to an ellipse at the point $(0, b)$ defined by

$$C = \left\{w, w_q \in R:\ \frac{w^2}{u_{max}^2} + \frac{(w_q-b)^2}{\epsilon^2} = 1\right\}. \tag{10}$$

The derivative of the Lyapunov function $\dot{V}$ is negative outside of the ellipse C and positive inside of the ellipse except from the origin where it is zero. Note that the Lyapunov function is defined as an ellipsoid structure around the origin, while C represents a given ellipse around $(0, b)$. Defining $B_c = \left\{\frac{w^2}{u_{max}^2} + \frac{(w_q-b)^2}{\epsilon^2} \le (1+\delta)^2\right\}$, where δ is an arbitrary positive constant, from (9) it is holds that $\dot{V} < 0$ outside and on the boundary of B_c except from the origin. Consider now a closed set $\Omega_s = \{V(w, w_q) \le s\}$. One can find the value of s such that $B_c \subseteq \Omega_s$ and the boundaries of B_c and Ω_s intersect at point $(0, b+\epsilon(1+\delta))$, as shown in Fig. 3, i.e. this point should satisfy

$$\frac{w^2}{u_{max}^2} + \frac{w_q^2}{\epsilon^2} = s. \tag{11}$$

Therefore $s = \frac{(b+\epsilon(1+\delta))^2}{\epsilon^2}$ and

$$S = \left\{w, w_q \in R\colon \frac{w^2}{\left(\frac{(b+\epsilon(1+\delta))u_{max}}{\epsilon}\right)^2} + \frac{w_q^2}{(b+\epsilon(1+\delta))^2} = 1\right\} \tag{12}$$

describes the boundary of Ω_s. Hence, $\dot{V} < 0$ outside and on the boundary of Ω_s, which guarantees that the controller states w and w_q introduce an ultimate bound. As a result, it is proven that for any initial conditions $w(0)$ and $w_q(0)$, there exists a class $\mathcal{KL}$ function β and a future time instant $T \ge 0$ such that [1]

$$\left\|\begin{array}{c} w \\ w_q \end{array}\right\| \le \beta\left(\left\|\begin{array}{c} w(0) \\ w_q(0) \end{array}\right\|, t\right), \quad t \le T \tag{13}$$

and

$$\left\|\begin{array}{c} w \\ w_q \end{array}\right\| \le \frac{(u_{max}+\epsilon)(b+\epsilon(1+\delta))}{\epsilon}, \quad t \ge T. \tag{14}$$

Inequality (14) results from the norm properties $\left(\|x\|_p \le \|x\|_1, \forall p \ge 1\right)$ and taking into account from (12) that for all $t \ge T$, i.e. after the time instant that w and w_q enter ellipse S, it holds true that $|w| \le \frac{(b+\epsilon(1+\delta))u_{max}}{\epsilon}$ and $|w_q| \le b+\epsilon(1+\delta)$ which yield that $\left\|\begin{array}{c} w \\ w_q \end{array}\right\|_1 \le \frac{(u_{max}+\epsilon)(b+\epsilon(1+\delta))}{\epsilon}$.

Hence, the control states solution can be written in the form:

$$\left\|\begin{array}{c} w \\ w_q \end{array}\right\| \le \beta\left(\left\|\begin{array}{c} w(0) \\ w_q(0) \end{array}\right\|, t\right) + d \tag{15}$$

where $d = \frac{(u_{max}+\epsilon)(b+\epsilon(1+\delta))}{\epsilon}$ is a positive constant. Since inequality (15) is satisfied independently from any bounded input $g(x)$ of the controller, the controller states can be written in the general ISpS form

$$\left\|\begin{array}{c} w \\ w_q \end{array}\right\| \le \beta(\|x(0)\|, t) + \gamma_{control}\left(\sup_{0\le\tau\le t}\|k_I g(x(\tau))\|\right) + d \tag{16}$$

with zero gain, i.e. $\gamma_{control} = 0$, regardless of the selection of the initial conditions w_0, w_{q0} and the parameters k, k_q, u_{max}, b and ϵ.

Since the closed-loop system, as shown in Fig. 2, is given in the composite feedback interconnection form, the small-gain theorem given in [19], [20] can be applied. Particularly, given that the controller gain is zero, then the small-gain condition is obviously satisfied. Therefore, the closed-loop system is ISpS with respect to the external input vector u_1. ∎

The special structure of the BIC provides the opportunity of proving the ISpS property for a wide class of non-linear systems. It is obvious that if the external input u_1 of the plant is zero, the closed-loop system solution is bounded. It is also worth noting that Theorem 1 holds independently from the plant structure, the controller parameters $b \ge 0$ and u_{max}, k, k_q, $\epsilon > 0$ or the initial conditions of the BIC states.

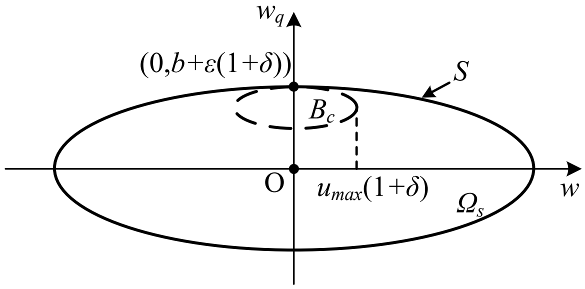


Figure 3. Boundedness of states w and w_q

Hence, BIC provides a generic controller for non-linear ISpS systems where the plant dynamics and parameters may be unknown or change during the operation.

IV. BIC WITH A GIVEN OUTPUT BOUND

A. Controller design

Although the BIC output is proven to remain bounded, a given bound is not guaranteed in general. In order for the control signal to remain inside a given bound $u \in [-u_{max}, u_{max}]$, where u_{max} denotes the maximum absolute value of the controller output, the BIC parameters can be selected as

$$b = 0,\ \epsilon = 1,\ c = \frac{w_q u_{max}^2}{u_{max}^2 - u_c^2},\ k = 0,\ k_q > 0, \tag{17}$$

where $u_c \in (-u_{max}, u_{max})$ is constant. According to this selection, the BIC dynamics (6) become

$$\begin{bmatrix} \dot{w} \\ \dot{w}_q \end{bmatrix} = \begin{bmatrix} 0 & k_I g(x) \frac{w_q u_{max}^2}{u_{max}^2 - u_c^2} \\ -k_I g(x) \frac{w_q}{u_{max}^2 - u_c^2} & -k_q \left(\frac{w^2}{u_{max}^2} + w_q^2 - 1 \right) \end{bmatrix} \begin{bmatrix} w \\ w_q \end{bmatrix} \tag{18}$$

where the initial conditions are chosen $w_0 = 0$ (initial condition of the IC, usually zero) and $w_{q0} = 1$. Now, considering the Lyapunov function candidate

$$W = \frac{w^2}{u_{max}^2} + {w_q}^2, \tag{19}$$

its derivative yields

$$\dot{W} = -2 \left(\frac{w^2}{u_{max}^2} + w_q^2 - 1 \right) k_q w_q^2 \tag{20}$$

which implies that the BIC states are on the ellipse

$$W_0 = \left\{ w, w_q \in R : \frac{w^2}{u_{max}^2} + {w_q}^2 = 1 \right\} \tag{21}$$

as shown in Fig. 4. This is due to the fact that the initial conditions are defined on W_0, where obviously

$$\dot{W} = 0 \Rightarrow W(t) = W(0) = 1,\ \forall t \geq 0 \tag{22}$$

proving that the BIC states will start and remain at all times on the ellipse W_0, i.e. the diagonal term $-k_q \left(\frac{w^2}{u_{max}^2} + w_q^2 - 1 \right)$ will be zero. This term is only used to increase the robustness with respect to external disturbances or calculation errors in the dynamics of w_q during a practical implementation. Note that the same analysis holds for any initial conditions with $w_{q0} > 0$ and w_0 defined on W_0.

By considering the following transformation

$$\begin{aligned} w &= u_{max} \sin\theta \\ w_q &= \cos\theta, \end{aligned} \tag{23}$$

it yields from the BIC dynamics (18) that

$$\dot{\theta} = \frac{k_I g(x) w_q u_{max}}{u_{max}^2 - u_c^2} \tag{24}$$

which proves that w and w_q will move on the ellipse W_0 with angular velocity $\dot{\theta}$ (Fig. 4). Therefore, it is guaranteed that $u \in [-u_{max}, u_{max}]$ for all $t \geq 0$ and as a result it extends the BIC operation to guarantee stability for locally ISpS systems. It should be noted that due to the selection of the initial conditions, the desired operation of the controller states on the ellipse is guaranteed even if $k = 0$ and c is varying such as in the present case. If it is assumed that there exists a desired equilibrium point $x = x_e$ for the plant with $u = u_e \in (-u_{max}, u_{max})$, for which $g(x_e) = 0$, this implies that w and w_q can stop at the desired equilibrium, corresponding to (u_e, w_{qe}) on $w - w_q$ plane, at which

$$\dot{\theta} = \frac{k_I g(x_e) w_{qe} u_{max}}{u_{max}^2 - u_c^2} = 0.$$

The conditions under which a possible convergence to the desired equilibrium exists are investigated in the next subsection.

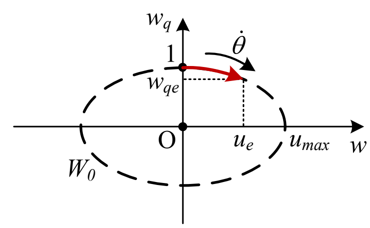


Figure 4. BIC states on $w - w_q$ plane

B. Achieving boundedness while preserving the stability of the system with the traditional IC

Consider the non-linear ISpS system of the form of (1) with the proposed BIC with the given bound (18). Since no other external inputs are present, the closed-loop system solution $x_{BIC}(t)$ will be bounded, where $x_{BIC} = \begin{bmatrix} x^T & w & w_q \end{bmatrix}^T$ is the state vector of the closed-loop system. However, since in this note the BIC is used to perform similarly to the traditional IC for achieving a desired regulation scenario, it is important to prove that the BIC does not change the behaviour of the IC near the desired equilibrium point.

Consider an ISpS plant controlled by the traditional IC (1), (3), (4). In this case assume that both f and g are continuously differentiable functions. The closed-loop system can be written in the form

$$\dot{x}_{IC} = f_{IC}(x_{IC}) \tag{25}$$

where $x_{IC} = \begin{bmatrix} x^T & w \end{bmatrix}^T$ is the state vector. Assume that $x_{ICe} = \begin{bmatrix} x_e^T & w_e \end{bmatrix}^T$ is an equilibrium point where $g(x_e) =$

0. If linearisation around the equilibrium point results in a Jacobian matrix $A_{IC} = \left.\frac{\partial f_{IC}(x_{IC})}{\partial x_{IC}}\right|_{x_{IC}=x_{ICe}}$ with $Re\lambda_i < 0$ for all eigenvalues of A_{IC}, then the equilibrium point of (25) will be asymptotically stable. However, it is not guaranteed that the solution of the closed-loop system will not escape to infinity, e.g. if initial conditions are defined away from the equilibrium point.

As it is shown in the sequel, the BIC maintains the asymptotic stability of the equilibrium point and according to the previous analysis, the proposed control method additionally guarantees a maximum bound for the closed-loop solution and a given bound for the controller output, leading to a superior performance and more rigorous theoretical analysis compared to the traditional IC.

In this framework, consider the following conditions:

1) $x_{ICe} = \left[\begin{array}{cc} x_e^T & w_e \end{array}\right]^T$ is an equilibrium point of (25) with $w_e \in (-u_{max}, u_{max})$.
2) $Re\lambda_i < 0$ for all eigenvalues of A_{IC} and for any $0 < k_I < k_{Imax}$.
3) The BIC parameter u_c satisfies

$$-u_{max}\sqrt{1-\frac{k_I}{k_{Imax}}} < u_c < u_{max}\sqrt{1-\frac{k_I}{k_{Imax}}}. \tag{26}$$

Then the following proposition can be formulated:

Proposition 2. *If Conditions 1)-3) above are satisfied, then the closed-loop system resulting from the feedback interconnection of the ISpS plant* (1) *and the BIC* (5), (18) *has an asymptotically stable equilibrium point* $\left[\begin{array}{ccc} x_e^T & w_e & w_{qe} \end{array}\right]^T$ *with* $w_{qe} = \pm\sqrt{1-\frac{w_e^2}{u_{max}^2}}$.

Proof: Based on the analysis of the previous subsection, the equilibrium point $x_{ICe} = \left[\begin{array}{cc} x_e^T & w_e \end{array}\right]^T$ of (25), where $w_e \in (-u_{max}, u_{max})$, will correspond to an equilibrium point $x_{BICe} = \left[\begin{array}{ccc} x_e^T & w_e & w_{qe} \end{array}\right]^T$ of the feedback interconnection of the ISpS plant (1) and the BIC (5), (18), where $w_{qe} = \pm\sqrt{1-\frac{w_e^2}{u_{max}^2}}$ for which $0 < w_{qe}^2 \leq 1$, since it is defined on W_0 with $w_e \in (-u_{max}, u_{max})$. According to Condition 2) all eigenvalues of

$$A_{IC} = \left[\begin{array}{cc} \left.\frac{\partial f}{\partial x}\right|_{(x_e,w_e)} & \left.\frac{\partial f}{\partial w}\right|_{(x_e,w_e)} \\ k_I \left.\frac{\partial g}{\partial x}\right|_{(x_e,w_e)} & 0 \end{array}\right]$$

have negative real parts for any $0 < k_I < k_{Imax}$. In the same framework, linearisation around $x_{BICe} = \left[\begin{array}{ccc} x_e^T & w_e & w_{qe} \end{array}\right]^T$ for the closed-loop system with the BIC results in the Jacobian

$$A_{BIC} = \left[\begin{array}{ccc} \left.\frac{\partial f}{\partial x}\right|_{(x_e,w_e)} & \left.\frac{\partial f}{\partial w}\right|_{(x_e,w_e)} & 0_{n\times 1} \\ k_I \left.\frac{\partial g}{\partial x}\right|_{(x_e,w_e)} \frac{w_{qe}^2 u_{max}^2}{u_{max}^2-u_c^2} & 0 & 0 \\ -k_I \left.\frac{\partial g}{\partial x}\right|_{(x_e,w_e)} \frac{w_e w_{qe}}{u_{max}^2-u_c^2} & -2\frac{k_q w_e w_{qe}}{u_{max}^2} & -2k_q w_{qe}^2 \end{array}\right],$$

where $w_{qe} \neq 0$ since $w_e \in (-u_{max}, u_{max})$ and w_e and w_{qe} are defined on W_0. Since $-2k_q w_{qe}^2 < 0$, then all eigenvalues of A_{BIC} will have negative real parts if matrix

$$A_{BIC1} = \left[\begin{array}{cc} \left.\frac{\partial f}{\partial x}\right|_{(x_e,w_e)} & \left.\frac{\partial f}{\partial w}\right|_{(x_e,w_e)} \\ k_I \left.\frac{\partial g}{\partial x}\right|_{(x_e,w_e)} \frac{w_{qe}^2 u_{max}^2}{u_{max}^2-u_c^2} & 0 \end{array}\right]$$

is Hurwitz. Since $0 < w_{qe}^2 \leq 1$, then

$$\begin{aligned} 0 &< k_I \frac{w_{qe}^2 u_{max}^2}{u_{max}^2 - u_c^2} \leq k_I \frac{u_{max}^2}{u_{max}^2 - u_c^2}, \\ \Rightarrow 0 &< k_I \frac{w_{qe}^2 u_{max}^2}{u_{max}^2 - u_c^2} < k_{Imax} \end{aligned}$$

taking into account (26) from Condition 3). Therefore, all eigenvalues of A_{BIC1} have negative real parts since the eigenvalues of A_{IC} are located at the left half plane for any $0 < k_I < k_{Imax}$. As a result, the equilibrium point of the closed-loop system with the BIC is asymptotically stable. It should be noted that if Condition 2) is satisfied for any $k_I > 0$, then the desired equilibrium of the closed-loop system with the BIC is asymptotically stable for any $u_c \in (-u_{max}, u_{max})$. ■

Furthermore, even if the control output tries to reach the limits during transients, i.e. $u \to \pm u_{max}$, then $w_q \to 0$ and the first equation of (18) results in $\dot{w} \to 0$ independently from the function $g(x)$. This means that the integration slows down near the limits preventing an integration windup problem. Opposed to the traditional anti-windup structures, the BIC does no stop the integration but smoothly slows it down near the limits without additional switches; hence the plant input remains a continuous-time signal, which proves the closed-loop system stability. Additionally, w and w_q stay exclusively in the first 2 quadrants in Fig. 4 for initial conditions defined on the upper semi-ellipse of W_0, and therefore they cannot move around W_0, which excludes an oscillating behaviour of the controller state dynamics around the ellipse.

The closed-loop system stability in the sense of boundedness and the given bound for the controller output have been proven in this note independently from the existence of an equilibrium point or its stability properties (stable or unstable). Therefore, if the equilibrium point changes from a stable to an unstable mode (eg. change of gain k_I) or is shifted outside the bounded range, closed-loop stability in the sense of boundedness is still maintained, opposed to the traditional IC or the IC with a saturation unit.

V. PRACTICAL EXAMPLE

In order to verify the proposed BIC in comparison to the traditional IC, the dc/dc buck-boost converter, shown in Fig. 5, is simulated. This power converter system is widely used in power applications (photovoltaic, energy storage systems, etc.) since it can regulate the dc output voltage to a higher or lower level than the dc input voltage by suitably controlling the switching element of the device.

Using average analysis [29], it has been proven that the continuous-time non-linear dynamics of the converter are

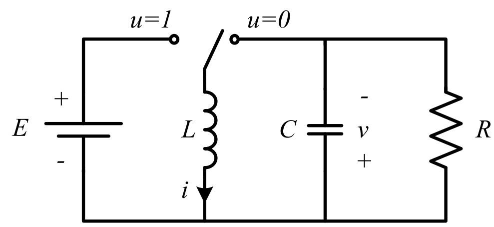


Figure 5. The dc/dc buck-boost converter

given as

$$L\frac{di}{dt} = -(1-u)v + uE \tag{27}$$
$$C\frac{dv}{dt} = (1-u)i - \frac{v}{R}, \tag{28}$$

where L and C are the converter inductance and capacitance, respectively, R is the load resistor and E is the dc input voltage. The system states are the inductor current i and the capacitor voltage v, while the control input is the duty-ratio u, which is a continuous-time signal in the range $[0,1]$. It should be noted that the system states are bounded for any $u \in [0, 1-\gamma]$, where $0 < \gamma \leq 1$, while the upper limit of the input $u = 1$ leads the inductor current to instability.

The main task is to regulate the output voltage v to a given dc reference value v_{ref}. Although several control schemes have been developed in the literature, such as traditional or cascaded PI controllers [30], passivity-based controllers [29], etc., in the industry, traditional or cascaded PI controllers are commonly used due to their simple structure and implementation. This is also due to the fact that the system dynamics can change (e.g. if a complicated load is added in the output) and the system parameters can be unknown or change during the operation. Even though, in these cases, stability may not be guaranteed, traditional controllers are still used for simplicity and are usually tuned in an empirical manner.

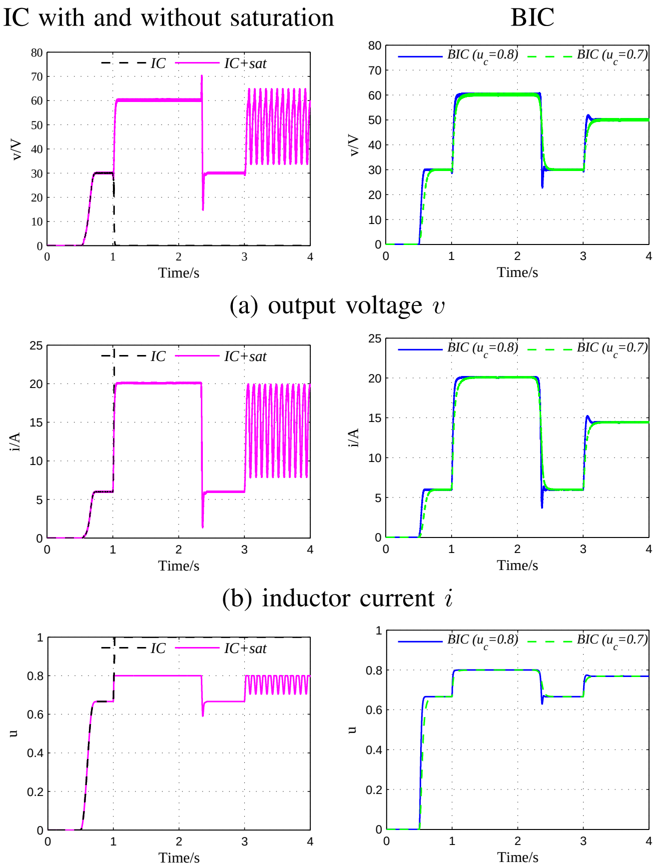
In this example, a traditional voltage IC with $g(x) = v_{ref} - v$ is investigated and compared with the BIC with a given bound. The system parameters are $L = 10mH$, $C = 30\mu F$, $R = 15\Omega$ and $E = 15V$. Initially the reference output voltage is set to $v_{ref} = 30V$. For stability reasons, in practice, it is often required the duty-ratio u to be limited below 1, usually 0.8 (i.e., $\gamma = 0.2$) to avoid a high inductor current. Since the BIC maintains the controller output in the range $[-u_{max}, u_{max}]$, one can set $u = w^2$ and $u_{max} = \sqrt{0.8}$. In this way the required range $[0, 0.8]$ for the control output can be achieved with the BIC. If the closed-loop system with the corresponding IC is linearised around the desired equilibrium point, it can be obtained (e.g. using root locus) that the equilibrium is asymptotically stable for all $0 < k_I < k_{Imax}$, where $k_{Imax} \approx 1$. Thus, the integral gain can be chosen $k_I = 0.2$ for both the IC and the BIC, where additionally two different choices of u_c are tested $u_c = \sqrt{0.65} \approx 0.8$ and $u_c = \sqrt{0.5} \approx 0.7$ that satisfy (26). Note that if the system parameters are unknown in practice, k_I and k_{Imax} are usually chosen based on experience and observation.

The converter is simulated with the traditional IC and the IC with a saturation unit in the output at $[0, 0.8]$, and is compared to the BIC with two different values of u_c. Starting with zero initial conditions for the system states and the control output, the output voltage reference is set to $v_{ref} = 30V$ at $t = 0.5s$. At time instant $t = 1s$, v_{ref} suddenly increases to $70V$ and drops back to $30V$ at $t = 2s$. Finally, at $t = 3s$, v_{ref} is set to $50V$. The time response of the system is shown in Fig. 6. Initially, both the IC with and without the saturation unit and the BIC regulate the output voltage at the desired level. However, when v_{ref} is set to $70V$, the traditional IC leads the inductor current to instability. The duty-ratio of the IC with the saturation unit saturates at the upper limit 0.8, while the BIC with either selection of u_c smoothly converges to the upper limit. In this case, the desired equilibrium is shifted outside the bounded range and the IC with the saturation suffers from integrator windup, opposed to the BIC which automatically slows down the integration. This is observed when v_{ref} changes back to $30V$ and the IC with saturation results in a larger transient. Finally, when v_{ref} is set to $50V$, the BIC converges to the desired equilibrium while the IC with saturation suffers again from integrator windup and results in an oscillatory response. Note that the different choice of u_c in the BIC design will result into slightly different transient response, since this parameter affects the angular velocity (24) of the BIC states on the desired ellipse W_0. The operation on the ellipse is illustrated in Fig. 7, where it is clear that the controller states remain on the upper semi-ellipse of W_0 as required.

It should be underlined that if the system parameters are completely unknown or change during the system operation, neither the IC or the BIC can guarantee asymptotic stability of the desired equilibrium. However, the BIC can still guarantee an ultimate bound for the closed-loop system, a given bound for the control output and the fact that it will not suffer from integrator windup issues. This is the main result of the current note which offers a replacement of the traditional IC with the BIC and can be applied in many engineering systems where the IC is used without a rigorous proof of stability.

VI. Conclusions

In this note, a bounded integral control (BIC) was proposed to guarantee closed-loop stability for a wide class of open-loop stable non-linear systems. Boundedness of the controller output signal has been achieved using the generalised small-gain theorem independently from the plant output and without external saturation units or switches, thus solving the closed-loop stability problem of many engineering systems without requiring knowledge of the plant structure or parameters. By suitably choosing the BIC parameters, a given bound for the controller output can be obtained to guarantee stability of locally ISpS plants. Therefore, for systems operating with the traditional IC, the same regulation scenario can be achieved by replacing the IC with the BIC and result in a guaranteed bounded response. The boundedness of the closed-loop system solution with the BIC is maintained even when the equilibrium point changes or becomes unstable. Simulation results of a dc/dc buck-boost converter system suitably verified the proposed BIC in comparison to the traditional IC.



(a) output voltage v

(b) inductor current i

(c) duty-ratio u

Figure 6. Simulation results of the buck-boost converter with the IC and the BIC

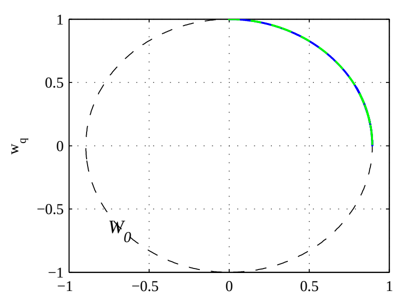


Figure 7. $w - w_q$ plane for the BIC

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