ROTATING FERMIONS

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Abstract

We investigate the rigidly rotating quantum thermal distribution of fermions in flat space-time. We find that thermal states diverge on the speed of light surface. We remove the divergences by enclosing the system inside a cylindrical boundary and investigate thermal expectation values and the Casimir effect for two sets of boundary conditions.

1 Introduction

The authors of Ref. [4] have shown that thermal states are ill-defined for quantum scalar particles on an unbounded, rigidly rotating, space-time, but are regular if the system is enclosed in a boundary inside the speed of light surface (SOL).

We perform a similar analysis for fermions. We discover that the difference between Bose-Einstein and Fermi-Dirac statistics allows fermions to exist in thermal states which diverge on the SOL, but are perfectly regular if enclosed inside a boundary located within the SOL.

2 Rotating fermions on the unbounded Minkowski space-time

We consider a Minkowski space-time rotating with a constant angular velocity $\mathbf{\Omega}$. The line element in cylindrical coordinates, with the z axis pointing along $\mathbf{\Omega}$, can be obtained from the Minkowski metric by making the substitution $\varphi \to \varphi + \Omega t$:

$$ds^2 = -(1-\rho^2\Omega^2)dt^2 + 2\rho^2\Omega\, dt\, d\varphi + d\rho^2 + \rho^2 d\varphi^2 + dz^2. \tag{1}$$

We define the surface $\rho\Omega = 1$ as the speed of light surface (SOL), since on this surface, co-rotating observers move at the speed of light.

The solutions of the Dirac equation on the metric in Eq. (1) can be written as:

$$U_k(x) = \frac{1}{2\pi} e^{-i\widetilde{E}_k t} u_k(\rho, \varphi, z), \tag{2}$$

where u_k is a four-spinor and k is a generic label distinguishing between independent modes. The eigenvalue $\widetilde{E}_k$ of the Hamiltonian is related to the Minkowski energy E_k through $\widetilde{E}_k = E_k - \Omega(m+\frac{1}{2})$, where $m+\frac{1}{2}$ is the eigenvalue of the angular momentum along the rotation axis.

If b_k is the creation operator for quanta described by the modes in Eq. (2), we can construct the following thermal expectation value (t.e.v.) at inverse temperature β [8]:

$$\left\langle b_k^\dagger b_{k'} \right\rangle_\beta = \frac{\delta_{kk'}}{e^{\beta \widetilde{E}_k} + 1}. \tag{3}$$

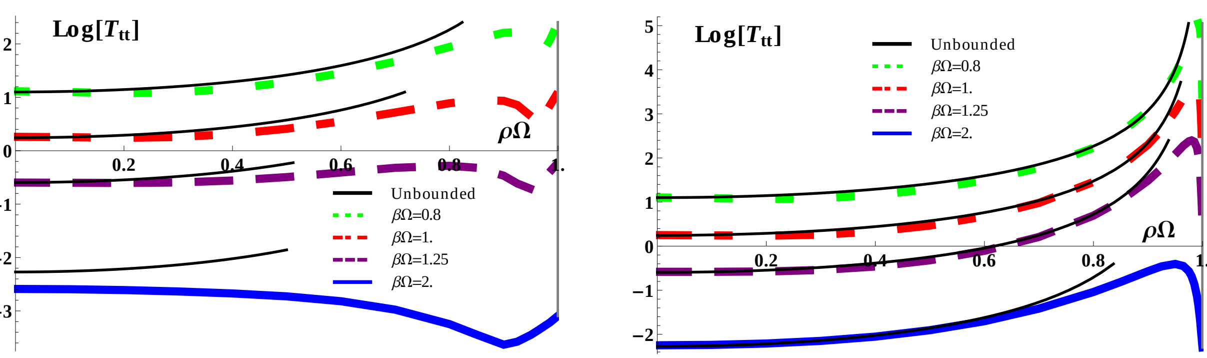
For the vacuum expectation value of products of the form $b_k^\dagger b_{k'}$ to vanish, the above expression must go to 0 as $\beta \to \infty$. Therefore, we must restrict particle modes to modes with positive frequency (i.e. positive $\widetilde{E}_k$), as discussed in Ref. [6]. Allowing particle modes to have negative frequency would give rise to temperature-independent terms in t.e.v.'s, similar to those obtained in Ref. [8].

We now present the t.e.v. of the energy density using the quantisation proposed by Ref. [6] ($\varepsilon = 1 - \rho^2\Omega^2$):

$$\langle : T_{tt} : \rangle_\beta = \frac{7\pi^2}{60\beta^4\varepsilon} + \frac{\Omega^2}{8\beta^2}\left(\frac{4}{3\varepsilon^2} - \frac{1}{3\varepsilon}\right). \tag{4}$$

Similar expressions can be obtained for the other non-vanishing components of the stress-energy tensor (i.e. for $T_{t\varphi}$, $T_{\rho\rho}$, $T_{\varphi\varphi}$ and T_{zz}), as well as for the neutrino current parallel to the rotation axis [1]. A common feature of these t.e.v.'s is their divergent behaviour as the SOL is approached, which we illustrate by the thin curves in Fig. 1.

For the scalar field, the indefinite norm requires particles to be described by modes of positive norm [7], forcing negative frequency modes into the set of particle modes. Moreover, the Bose-Einstein density of states factor diverges for frequencies approaching 0, even when the corresponding Minkowski energy is finite, requiring thermal states to have an infinite energy density at each point in the space-time.



(a) (b)

Figure 1: The t.e.v. of the energy density in the (a) spectral and (b) MIT bag models for thermal fermions inside a boundary of radius $R = \Omega^{-1}$ is compared with the corresponding energy density for the unbounded space-time (thin curves), for four values of the inverse temperature β (β is increasing from top to bottom). The two models give different results near the boundary: in the spectral case, T_{tt} decreases to a local minimum, then increases to a finite value on the SOL, while in the MIT bag model, T_{tt} increases to a local maximum, then decreases to a finite value on the SOL which is more than two times greater than in the spectral case.

3 Rotating fermions inside an infinite cylinder

The divergent nature of expectation values in the rigidly rotating space-time is related to the inclusion of a region of space which co-rotates at speeds larger than the speed of light. Following Ref. [4], we eliminate this region by enclosing the system inside a cylindrical boundary of radius R, centred on the rotation axis, such that $R\Omega \le 1$. We use two sets of boundary conditions, described below.

For consistency, a set of boundary conditions must ensure the vanishing of the time derivative of the Dirac inner product, given by:

$$i\partial_t \langle \psi, \chi \rangle = \int_{\partial V} d\Sigma\, \overline{\psi} n_\mu \gamma^\mu \chi, \tag{5}$$

where $n = n_\mu dx^\mu$ is the normal to the boundary. In the spectral model [5], Eq. (5) is satisfied mode by mode by Fourier coefficients of the wave functions ψ and χ. In order to preserve the charge conjugation invariance of the theory, the boundary conditions in the spectral model depend on the spectral index of the mode, making this formulation non-local. By contrast, the MIT bag model [2] satisfies Eq. (5) in a purely local way, by requiring that the wave functions satisfy the equation $in_\mu \gamma^\mu \psi = \psi$. Plots of the t.e.v. of the energy density in the two models are presented in Fig. 1.

4 The Casimir divergence

The presence of a boundary in a quantum system can induce a change in its vacuum state. The difference between the expectation value of the energy density for fermions of mass μ in the vacuum states of the bounded and unbounded systems is found to diverge as an inverse power of the distance to the boundary:

$$\langle T_{tt} \rangle_{\text{Casimir}} \sim \begin{cases} -\dfrac{1}{120\pi^2 R^4} \times \dfrac{1 + 10\mu R}{(1-\rho/R)^3} & \text{(MIT bag model)} \\ -\dfrac{1}{16\pi^2 R^4} \times \dfrac{1}{(1-\rho/R)^4} & \text{(spectral boundary conditions.)} \end{cases} \tag{6}$$

According to Ref. [3], the energy density should diverge as the inverse cube of the distance to the boundary for a purely local stress-energy tensor. Although it seems to violate this result, the spectral model gives a different order for the divergence because it does not satisfy the assumption of locality.

Acknowledgments

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