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Fermions on adS

Victor E. Ambruş and Elizabeth Winstanley

Abstract We construct the Feynman propagator for Dirac fermions on anti-de Sitter space-time and present an analytic expression for the bi-spinor of parallel transport. We then renormalise the vacuum expectation value of the stress-energy tensor and end by analysing its renormalised expectation value at finite temperatures.

1 Introduction

Quantum field theory (QFT) on curved spaces (CS) is a semi-classical theory for the investigation of quantum effects in gravity. Due to its simplicity, the scalar field has been the main focus of QFT on CS. However, due to the fundamental difference between the quantum behaviour of fermions and bosons, it is important to also study fermionic fields. In this paper, we consider the propagation of Dirac fermions on the anti de Sitter space-time (adS) background space-time, where the maximal symmetry can be used to obtain analytic results.

We start this paper by presenting in Sec. 2 an expression for the spinor parallel propagator [7]. Using results from geodesic theory [1, 7], an exact expression for the Feynman propagator is obtained in Sec. 3. Section 4 is devoted to Hadamard's regularisation method [8], while, in Sec. 5, the result for the renormalised vacuum expectation value (v.e.v.) of the stress-energy tensor (SET) is presented using two methods: the Schwinger-de Witt method [4] and the Hadamard method [6]. The

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exact form of the bi-spinor of parallel transport is then used in Sec. 6 to calculate the thermal expectation value (t.e.v.) of the SET for massless spinors. More details on the current work, as well as an extension to massive spinors, can be found in [2].

2 Geometric structure of adS

Anti-de Sitter space-time (adS) is a vacuum solution of the Einstein equation with a negative cosmological constant, having the following line element:

$$ds^2 = \frac{1}{\cos^2 \omega r} \left[-dt^2 + dr^2 + \frac{\sin^2 \omega r}{\omega^2} (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \quad (1)$$

The time coordinate t runs from $-\infty$ to ∞ , thereby giving the covering space of adS. The radial coordinate r runs from 0 to the space-like boundary at $\pi/2\omega$, while θ and φ are the usual elevation and azimuthal angular coordinates. In the Cartesian gauge, the line element (1) admits the following natural frame [5]:

$$\omega^{\hat{t}} = \frac{dt}{\cos \omega r}, \quad \omega^{\hat{i}} = \frac{dx^j}{\cos \omega r} \left[\frac{\sin \omega r}{\omega r} \left(\delta_{ij} - \frac{x^i x^j}{r^2} \right) + \frac{x^i x^j}{r^2} \right], \quad (2)$$

such that $\eta_{\hat{\alpha}\hat{\beta}} \omega_{\hat{\mu}}^{\hat{\alpha}} \omega_{\hat{\nu}}^{\hat{\beta}} = g_{\mu\nu}$, where $\eta_{\hat{\alpha}\hat{\beta}} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

A key role in the construction of the propagator of the Dirac field is played by the bi-spinor of parallel transport $\Lambda(x, x')$, which satisfies the parallel transport equation $n^\mu D_\mu \Lambda(x, x') = 0$ [7]. On adS, the explicit form of $\Lambda(x, x')$ is [2]:

$$\Lambda(x, x') = \frac{\cos(\omega\Delta t/2)}{\cos(\omega s/2) \sqrt{\cos \omega r \cos \omega r'}} \left\{ \cos \frac{\omega r}{2} \cos \frac{\omega r'}{2} + \frac{\mathbf{x} \cdot \hat{\gamma} \mathbf{x}' \cdot \hat{\gamma}}{r r'} \sin \frac{\omega r}{2} \sin \frac{\omega r'}{2} - \hat{\gamma} \tan \frac{\omega\Delta t}{2} \left(\frac{\mathbf{x} \cdot \hat{\gamma}}{r} \sin \frac{\omega r}{2} \cos \frac{\omega r'}{2} - \frac{\mathbf{x}' \cdot \hat{\gamma}}{r'} \cos \frac{\omega r}{2} \sin \frac{\omega r'}{2} \right) \right\}, \quad (3)$$

where $\gamma^{\hat{\alpha}} = (\hat{\gamma}^t, \hat{\gamma})$ are the gamma matrices in the Dirac representation and s is the geodesic distance between x and x' .

3 Feynman propagator on adS

The Feynman propagator $S_F(x, x')$ for a Dirac field of mass m can be defined as the solution of the inhomogeneous Dirac equation, with appropriate boundary conditions:

$$(i\not{D} - m)S_F(x, x') = (-g)^{-1/2} \delta^4(x - x'), \quad (4)$$

where D_μ denotes the spinor covariant derivative and g is the determinant of the background space-time metric. Due to the maximal symmetry of adS, the Feynman propagator can be written in the following form [7]:

$$S_F(x, x') = [\alpha_F(s) + \not{s} \beta_F(s)] \Lambda(x, x'). \quad (5)$$

The functions α_F and β_F can be determined using (4):

$$\alpha_F = \frac{\omega^3 k}{16\pi^2} \cos \frac{\omega s}{2} \left\{ -\frac{1}{\sin^2 \frac{\omega s}{2}} + 2(k^2 - 1) \ln \left| \sin \frac{\omega s}{2} \right| {}_2F_1 \left(2+k, 2-k; 2; \sin^2 \frac{\omega s}{2} \right) + (k^2 - 1) \sum_{n=0}^{\infty} \frac{(2+k)_n (2-k)_n}{(2)_n n!} \left(\sin^2 \frac{\omega s}{2} \right)^n \Psi_n \right\}, \quad (6)$$

$$\beta_F = \frac{i\omega^3}{16\pi^2} \sin \frac{\omega s}{2} \left\{ \frac{1 + k^2 \sin^2(\omega s/2)}{[\sin(\omega s/2)]^4} - k^2(k^2 - 1) \ln \left| \sin \frac{\omega s}{2} \right| {}_2F_1 \left(2+k, 2-k; 3; \sin^2 \frac{\omega s}{2} \right) - \frac{k^2(k^2 - 1)}{2} \sum_{n=0}^{\infty} \frac{(2+k)_n (2-k)_n}{(3)_n n!} \left(\sin^2 \frac{\omega s}{2} \right)^n \left(\Psi_n - \frac{1}{2+n} \right) \right\}, \quad (7)$$

where $a_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol, $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the gamma function, $k = m/\omega$,

$$\Psi_n = \psi(k+n+2) + \psi(k-n-1) - \psi(n+2) - \psi(n+1) \quad (8)$$

and $\psi(z) = d \ln \Gamma(z) / dz$ is the digamma function.

4 Hadamard renormalisation

To regularise S_F , it is convenient to use the auxiliary propagator \mathcal{G}_F , defined by analogy to flat space-time [8]:

$$S_F(x, x') = (i\not{D} + m)\mathcal{G}_F. \quad (9)$$

On adS, \mathcal{G}_F can be written using the bi-spinor of parallel transport:

$$\mathcal{G}_F(x, x') = \frac{\alpha_F}{m} \Lambda(x, x'), \quad (10)$$

where α_F is given in (6).

According to Hadamard's theorem, the divergent part \mathcal{G}_H of \mathcal{G}_F is state-independent, having the form [8]:

$$\mathcal{G}_H(x, x') = \frac{1}{8\pi^2} \left[\frac{u(x, x')}{\sigma} + v(x, x') \ln \mu^2 \sigma \right], \quad (11)$$

where $u(x, x')$ and $v(x, x')$ are finite when x' approaches x , $\sigma = -s^2/2$ is Synge's world function and μ is an arbitrary mass scale. The functions u and v can be found by solving the inhomogeneous Dirac equation (4), requiring that the regularised auxiliary propagator $\mathcal{G}_F^{\text{reg}} \equiv \mathcal{G}_F - \mathcal{G}_H$ is finite in the coincidence limit:

$$u(x, x') = \sqrt{\Delta(x, x')} \Lambda(x, x'), \quad (12)$$

$$v(x, x') = \frac{\omega^2}{2} (k^2 - 1) \cos \frac{\omega s}{2} {}_2F_1 \left(2 - k, 2 + k; 2; \sin^2 \frac{\omega s}{2} \right) \Lambda(x, x'), \quad (13)$$

where the Van Vleck-Morette determinant $\Delta(x, x') = (\omega s / \sin \omega s)^3$ on adS.

5 Renormalised vacuum stress-energy tensor

To remove the traditional divergences of quantum field theory, we employ two regularisation methods: the Schwinger–de Witt method in Sec. 5.1 and the Hadamard method in Sec. 5.2. Due to the symmetries of adS, the regularised v.e.v. of the SET takes the form $\langle T_{\mu\nu} \rangle_{\text{vac}}^{\text{reg}} = \frac{1}{4} T g_{\mu\nu}$, where $T = T^\mu{}_\mu$ is its trace. The renormalisation process has the profound consequence of shifting T for the massless (hence, conformal) Dirac field to a finite value, referred to as the conformal anomaly.

5.1 Schwinger–de Witt regularisation

By using the Schwinger–de Witt approach to investigate the singularity structure of the propagator of the Dirac field in the coincidence limit, Christensen [4] calculates a set of subtraction terms which only depend on the geometry of the background space-time, using the following formula:

$$\langle T_{\mu\nu} \rangle = \lim_{x' \rightarrow x} \text{tr} \left\{ \frac{i}{2} [\gamma_{(\mu} D_{\nu)} - \gamma_{(\mu'} D_{\nu')}] S_F(x, x') \right\}. \quad (14)$$

After subtracting Christensen's terms, we exactly recover the result obtained by Camporesi and Higuchi [3] using the Pauli-Villars regularisation method:

$$\langle T \rangle_{\text{vac}}^{\text{SdW}} = -\frac{\omega^4}{4\pi^2} \left\{ \frac{11}{60} + k - \frac{k^2}{6} - k^3 + 2k^2(k^2 - 1) \left[\ln \frac{\mu}{\omega} - \psi(k) \right] \right\}, \quad (15)$$

where μ is an arbitrary mass scale.

5.2 Hadamard regularisation

The Hadamard theorem presented in Sec. 4 allows the renormalisation to be performed at the level of the propagator. To preserve the conservation of the SET, the following definition for the SET must be used [6]:

$$\langle T_{\mu\nu} \rangle = \lim_{x' \rightarrow x} \text{tr} \left\{ \frac{i}{2} [\gamma_{(\mu} D_{\nu)} - \gamma_{(\mu'} D_{\nu')}] + \frac{1}{6} g_{\mu\nu} \left[\frac{i}{2} (\not{D} - \not{D}') - m \right] \right\} S_F^{\text{reg}}(x, x'), \quad (16)$$

where $S_F^{\text{reg}}(x, x') = (i\not{D} + m)(\mathcal{G}_F - \mathcal{G}_H)$ is the regularised propagator. The coefficient of $g_{\mu\nu}$ is proportional to the Lagrangian of the Dirac field and evaluates to zero when applied to a solution of (4). However, $S_F^{\text{reg}}(x, x')$ is not a solution of (4). The v.e.v. obtained from (16) matches perfectly the result obtained by Camporesi and Higuchi [3] using the zeta-function regularisation method (γ is Euler's constant):

$$\langle T \rangle_{\text{vac}}^{\text{Had}} = -\frac{\omega^4}{4\pi^2} \left\{ \frac{11}{60} + k - \frac{7k^2}{6} - k^3 + \frac{3k^4}{2} + 2k^2(k^2 - 1) \left[\ln \frac{\mu e^{-\gamma} \sqrt{2}}{\omega} - \psi(k) \right] \right\}. \quad (17)$$

Even though the results (15) and (17) are different for general values of the mass parameter k , they yield the same conformal anomaly. We would like to stress that the omission of the term proportional to $g_{\mu\nu}$ in (16) would increase the value of the conformal anomaly by a factor of 3.

6 Thermal stress-energy tensor

The renormalised thermal expectation value (t.e.v.) of the SET can be written as:

$$\langle T_{\mu\nu} \rangle_{\beta}^{\text{reg}} = \langle : T_{\mu\nu} : \rangle_{\beta} + \langle T_{\mu\nu} \rangle_{\text{vac}}^{\text{ren}}, \quad (18)$$

where $\beta = T^{-1}$ is the inverse temperature and the colons $::$ indicate that the operator enclosed is in normal order, i.e. with its v.e.v. subtracted. The bi-spinor of parallel transport can be used to show that

$$\langle : T_{\nu}^{\mu} : \rangle_{\beta} = \text{diag}(-\rho, p, p, p), \quad (19)$$

where ρ is the energy density and p is the pressure. If $m = 0$, we have $p = \rho/3$ and:

$$\rho]_{m=0} = -\frac{3\omega^4}{4\pi^2} (\cos \omega r)^4 \sum_{j=1}^{\infty} (-1)^j \frac{\cosh(j\omega\beta/2)}{[\sinh(j\omega\beta/2)]^4}, \quad (20)$$

with the coordinate dependence fully contained in the $(\cos \omega r)^4$ prefactor. The first term in the sum over j is within 6% of the sum, while the first two terms together are less than 1% away, for all values of $\omega\beta$. The small and large $\omega\beta$ limits can be

extracted:

$$\rho|_{m=0} = (\cos \omega r)^4 \left[\frac{7\pi^2}{60\beta^4} - \frac{\omega^2}{24\beta^2} + O(\omega^4) \right], \quad (21)$$

$$\rho|_{m=0} = \frac{6\omega^4}{\pi^2} \frac{(\cos \omega r)^4}{1 + e^{3\beta\omega/2}} \left[1 + 5e^{-\omega\beta} \frac{1 + e^{-3\omega\beta/2}}{1 + e^{-5\omega\beta/2}} + O(e^{-2\omega\beta}) \right]. \quad (22)$$

Figure 1 shows a graphical representation of the above results.

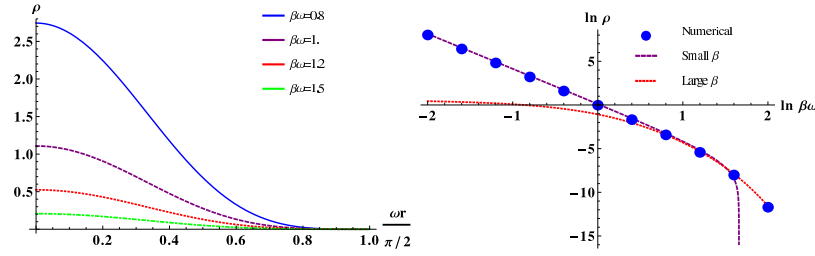


Fig. 1 a ρ between the origin ($r = 0$) and the boundary ($r\omega = \pi/2$) for $\beta\omega = 0.8, 1.0, 1.2$ and 1.4 ; b Log-log plot of ρ in terms of $\beta\omega$; comparison with the asymptotic results in (21) and (22)

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