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Some L^2 Properties of Semigroups of Measures on Lie Groups

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Abstract

We investigate the induced action of convolution semigroups of probability measures on Lie groups on the L^2-space of Haar measure. Necessary and sufficient conditions are given for the infinitesimal generator to be self-adjoint and the associated symmetric Dirichlet form is constructed. We show that the generated Markov semigroup is trace-class if and only if the measures have a square-integrable density. Two examples are studied in some depth where the spectrum can be explicitly computed, these being the n-torus and Riemannian symmetric pairs of compact type.

Key words and phrases Lie group, Lie algebra, convolution semigroup, Hunt semigroup, Hunt generator, Dirichlet form, Beurling-Deny representation, trace-class operator, Hilbert-Schmidt operator, Lévy-Khinchine formula, Riemannian symmetric pairs, spherical function, subordinator.

AMS 2000 subject classification 60B15, 60G51, 47D07, 43A05, 47N30

1 Introduction

Brownian motion on a compact Riemannian manifold is a good source of topological and geometric information (see e.g. [6], [9]). This is because it is a time homogeneous Markov process whose associated transition semigroup is precisely the heat semigroup which is generated by the Laplace-Beltrami operator acting on functions. We may enquire as to whether there are any other Markov processes on a manifold that have an intrinsic geometric character and natural candidates for investigation are Lévy processes [4]. A great deal more is known about these processes in the Lie group case than on more general manifolds (see [16] and references therein) and since a wider range of tools are available in this context it makes sense to begin our investigations here. Indeed because of the light it sheds on the properties of key operators, it seems that this can be a useful study in its own right from a purely analytic perspective.

For a Markov process to be a suitable candidate for geometric investigation it is reasonable to ask for the action of generator on the L^2-space of Haar measure to have a discrete spectrum of real-valued eigenvalues (c.f. [19], [5]). This is far from the case for all Lévy processes on Lie groups and the purpose of this paper is to identify some conditions when this holds. Throughout this paper we work with convolution semigroups of probability measures rather than Lévy processes as this seems to be a more natural framework. There is no loss of generality here. Given such a convolution semigroup we can always construct a Lévy process on the space of all real-valued mappings on the group by using Kolmogorov's construction theorem. Conversely the laws of a group-valued Lévy process form a convolution semigroup.

The organisation of this paper is as follows. We begin in section 1 by collecting together key results that we'll need about convolution semigroups and the induced "Hunt semigroup" $(T_t, t \geq 0)$ on the space of continuous functions which vanish at infinity. We describe Hunt's classification of the infinitesimal generators of these semigroups from his seminal 1956 paper [14]. We then examine the action of this semigroup on the L^2-space of a right-invariant Haar measure and give necessary and sufficient conditions for the generator to be self-adjoint extending the account that is given in Corollary 2.4 of [15]. As a consequence of these results we are able to obtain a Beurling-Deny representation for the associated symmetric Dirichlet form. Most of the results of this section can be extended to general locally compact groups by using the approach of E.Born [8].

In the second part of the paper we turn our attention to finding sufficient conditions for T_t to be trace-class for all $t > 0$. We prove that a necessary and sufficient condition is that it has a square-integrable density. Finally we focus on two cases where the spectrum can be computed explicitly. The first of these is the n-torus where we can use the classical Lévy-Khintchine formula. In the second case we employ spherical functions to investigate bi-invariant convolution semigroups on Riemannian symmetric pairs of compact type. We study the d-sphere in some detail where we can use subordination to construct specific examples.

Notation. Einstein summation convention will be used throughout this paper. If G is a locally compact group then $\mathcal{B}(G)$ is the σ-algebra of all Borel sets in G, $C_b(G)$ is the Banach space (with respect to the supremum norm) of all real-valued bounded continuous functions on G, $C_0(G)$ is the closed subspace of $C_b(G)$ which consists of functions which vanish at infinity, $C_c(G)$ is the dense linear subspace in $C_0(G)$ comprising functions of compact support and $C_c^\infty(G)$ is the subspace of $C_c(G)$ wherein all functions are required to be smooth (i.e. infinitely differentiable).

2 Hunt's Theorem and Self-Adjointness

Let G be a locally compact group with neutral element e and let $\mathcal{M}(G)$ denote the set of all Borel probability measures on G. It is a monoid with respect to the binary operation of convolution where the identity element is δ_e, the Dirac mass at e. We recall that the convolution $\mu_1 * \mu_2$ of $\mu_1, \mu_2 \in \mathcal{M}(G)$ is the unique Borel probability measure on G for which $\int_G f(\sigma)(\mu_1 * \mu_2)(d\sigma) = \int_G \int_G f(\sigma\tau)\mu_1(d\sigma)\mu_2(d\tau)$, for all real-valued bounded Borel functions f defined on G. The *reversed measure* $\widetilde{\mu}$ that is associated to each $\mu \in \mathcal{M}(G)$ is defined by $\widetilde{\mu}(A) = \mu(A^{-1})$ for each $A \in \mathcal{B}(G)$. $\mu \in \mathcal{M}(G)$ is said to be symmetric if $\widetilde{\mu} = \mu$. A family of probability measures $(\mu_t, t \geq 0)$ is a weakly continuous convolution semigroup of probability measures on G (or *convolution semigroup* for short) if $\mu_0 = \delta_e$, $\mu_{s+t} = \mu_s * \mu_t$ for all $s, t \geq 0$ and $\lim_{t\to 0} \int_G f(\sigma)\mu_t(d\sigma) = f(e)$, for all $f \in C_b(G)$. Such a semigroup is said to be *symmetric* if each μ_t is. It is easily verified that $(\mu_t, t \geq 0)$ is a convolution semigroup if and only if $(\widetilde{\mu}_t, t \geq 0)$ is a convolution semigroup. Given a convolution semigroup $(\mu_t, t \geq 0)$, we obtain a strongly continuous contraction semigroup of linear operators $(T_t, t \geq 0)$ on $C_0(G)$ via the prescription

$$(T_t f)(\sigma) = \int_G f(\sigma\tau)\mu_t(d\tau), \tag{2.1}$$

for all $t \geq 0, \sigma \in G$. It is easily verified that $L_\sigma T_t = T_t L_\sigma$ for all $\sigma \in G$ where $L_\sigma f(\tau) = f(\sigma^{-1}\tau)$ for all $f \in C_0(G), \sigma, \tau \in G$.

The infinitesimal generator of $(T_t, t \geq 0)$ will be denoted as $\mathcal{L}$. Clearly $\mathrm{Dom}(\mathcal{L})$ is L_σ-invariant and $\mathcal{L}L_\sigma f = L_\sigma \mathcal{L} f$ for all $\sigma \in G$ and all $f \in \mathrm{Dom}(\mathcal{L})$. We call $(T_t, t \geq 0)$ a *Hunt semigroup* and $\mathcal{L}$ its *Hunt generator* in view of Hunt's seminal work on classifying these operators [14] in the Lie group case which we will describe below. The corresponding Hunt semigroup associated to $(\widetilde{\mu}_t, t \geq 0)$ is denoted by $(\widetilde{T}_t, t \geq 0)$ and its generator is $\widetilde{\mathcal{L}}$, so for all

$t \geq 0, f \in C_0(G), \sigma \in G, (\widetilde{T}_t f)(\sigma) = \int_G f(\sigma\tau^{-1})\mu_t(d\tau)$. For the remainder of this section G will be an n-dimensional Lie group with Lie algebra $\mathbf{g}$ whose elements will be considered as left-invariant vector fields. We first fix a basis $(X_j, 1 \leq j \leq n)$ of $\mathbf{g}$ and define the dense linear manifold $C^2(G) := \{f \in C_0(G); X_i(f) \in C_0(G)$ and $X_iX_j(f) \in C_0(G)$ for all $1 \leq i, j \leq n\}$. There exist functions $x_i \in C_c^\infty(G), 1 \leq i \leq n$ so that $x_i(e) = 0, X_ix_j(e) = \delta_{ij}$ and $x_i(\sigma) = -x_i(\sigma^{-1})$ for all $\sigma \in G, 1 \leq i, j \leq n$. A measure ν defined on $\mathcal{B}(G - \{e\})$ is called a *Lévy measure* whenever $\int_{G-\{e\}} \{(\sum_{i=1}^n x_i(\sigma)^2) \wedge 1\} \nu(d\sigma) < \infty$.

Theorem 2.1 (Hunt's theorem) *Let $(\mu_t, t \geq 0)$ be a convolution semigroup of measures in G with Hunt generator $\mathcal{L}$ then*

1. *$C^2(G) \subseteq Dom(\mathcal{L})$.*
2. *For each $\sigma \in G, f \in C^2(G)$,*

$$\mathcal{L}f(\sigma) = b^i X_i f(\sigma) + a^{ij} X_i X_j f(\sigma) + \int_{G-\{e\}} (f(\sigma\tau) - f(\sigma) - x^i(\tau) X_i f(\sigma))\nu(d\tau), \tag{2.2}$$

where $b = (b^1, \ldots b^n) \in \mathbb{R}^n, a = (a^{ij})$ is a non-negative-definite, symmetric $n \times n$ real-valued matrix and ν is a Lévy measure on $G - \{e\}$.

Conversely, any linear operator with a representation as in (2.2) is the restriction to $C^2(G)$ of the Hunt generator corresponding to a unique convolution semigroup of probability measures.

For the proof see [14] or more recently [16]. We call (b, a, ν) the *characteristics* of $\mathcal{L}$. It is easy to see that $\mathcal{L}$ has characteristics (b, a, ν) if and only if $\widetilde{\mathcal{L}}$ has characteristics $(-b, a, \widetilde{\nu})$.

Let m be a right-invariant Haar measure on G. We will adopt the standard convention of writing $m(d\sigma)$ as $d\sigma$ within integrals. The action of each T_t when restricted to $C_c^\infty(G)$ extends to a bounded operator on $L^2(G) := L^2(G, m; \mathbb{R})$ which we continue to denote by T_t. Using standard arguments we can show that $(T_t, t \geq 0)$ is a strongly continuous contraction semigroup on $L^2(G)$ which is also Markovian in that $f \in L^2(G)$ and $0 \leq f \leq 1$ (a.e.) $\Rightarrow 0 \leq T_t f \leq 1$ (a.e.). From now on the actions of Hunt semigroups will always be considered in the L^2-setting. It is easily verified that $\widetilde{T}_t = T_t^*$ (see Proposition 2.2 in [15]) and hence $\widetilde{\mathcal{L}} = \mathcal{L}^*$.

Define $C_c^2(G) = C^2(G) \cap C_c(G)$ then since $C_c^\infty(G) \subseteq C_c^2(G)$ it follows that $C_c^2(G)$ is dense in $L^2(G)$ and by Theorem 2.1 we see that $C_c^2(G) \subseteq \text{Dom}(\mathcal{L})$

where $\mathcal{L}$ is now considered as the L^2-generator of $(T_t, t \geq 0)$. Furthermore we see that for all $f \in C_c^2(G), \sigma \in G$,

$$\mathcal{L}^* f(\sigma) = -b^i X_i f(\sigma) + a^{ij} X_i X_j f(\sigma) + \int_{G-\{e\}} (f(\sigma\tau^{-1}) - f(\sigma) + x^i(\tau) X_i f(\sigma)) \nu(d\tau), \tag{2.3}$$

(c.f. (2.7) in [15]).

Theorem 2.2 *The following are equivalent.*

(i) The convolution semigroup $(\mu_t, t \geq 0)$ is symmetric.

(ii) $T_t = T_t^*$ for each $t \geq 0$.

(iii) $\mathcal{L} = \mathcal{L}^*$.

(iv) $b = 0, \nu = \widetilde{\nu}$.

(v) For all $f \in C_c^2(G)$,

$$\mathcal{L} f(\sigma) = a^{ij} X_i X_j f(\sigma) + \frac{1}{2} \int_{G-\{e\}} (f(\sigma\tau) - 2f(\sigma) + f(\sigma\tau^{-1})) \nu(d\tau). \tag{2.4}$$

Proof. The equivalence of (i), (ii) and (iv) is stated in Corollary 2.4 of [15]. (ii) $\Leftrightarrow$ (iii) is standard semigroup theory and (iv) $\Rightarrow$ (v) is straightforward. Finally for (v) $\Rightarrow$ (i) we observe that if (v) holds then $\mathcal{L} f = \mathcal{L}^* f$ for each $f \in C_c^2(G)$. To see this it is sufficient to observe that for all $\sigma \in G$,

$$\begin{aligned} \int_{G-\{e\}} (f(\sigma\tau) - 2f(\sigma) + f(\sigma\tau^{-1})) \nu(d\tau) &= \int_{G-\{e\}} (R_\tau - 2I + R_{\tau^{-1}}) f(\sigma) \nu(d\tau) \\ &= \int_{G-\{e\}} (R_\tau - 2I + R_\tau^*) f(\sigma) \nu(d\tau), \end{aligned}$$

where $R_\tau f(\sigma) = f(\sigma\tau)$ for each $\tau \in G$. By a density argument in $C_0(G)$ we then deduce that $\mathcal{L} f = \widetilde{\mathcal{L}} f$ for each $f \in C^2(G)$. Now since by Theorem 2.1, $\mathcal{L}|_{C^2(G)}$ uniquely determines $(\mu_t, t \geq 0)$ and $\widetilde{\mathcal{L}}\Big|_{C^2(G)}$ uniquely determines $(\widetilde{\mu}_t, t \geq 0)$, we deduce that $\mu_t = \widetilde{\mu}_t$ for all $t \geq 0$, as was required. □

In Theorem 2.2 we always assumed that the operator $\mathcal{L}$ was the generator of a Hunt semigroup. We weaken this assumption in the next theorem.

Theorem 2.3 *If $\mathcal{L}$ is a linear operator defined on $C_c^2(G)$ which has the form (2.4) then it extends to the generator of a unique symmetric convolution semigroup.*

Proof. For all $f \in C_c^2(G), \sigma \in G$ we have

$$\begin{aligned} & \frac{1}{2}\int_{G-\{e\}}(f(\sigma\tau) - 2f(\sigma) + f(\sigma\tau^{-1}))\nu(d\tau) \\ = & \frac{1}{2}\left(\int_{G-\{e\}}(f(\sigma\tau) - f(\sigma) - x^i(\tau)X_i f(\sigma))\nu(d\tau)\right. \\ + & \left.\int_{G-\{e\}}(f(\sigma\tau^{-1}) - f(\sigma) + x^i(\tau)X_i f(\sigma))\nu(d\tau)\right) \\ = & \int_{G-\{e\}}(f(\sigma\tau) - f(\sigma) - x^i(\tau)X_i f(\sigma))\nu_s(d\tau). \end{aligned}$$

Since $\nu_s := \frac{1}{2}(\nu + \widetilde{\nu})$ is a Lévy measure it follows by Theorem 2.1 (after first extending the action of the operator to $C^2(G)$) that $\mathcal{L}$ extends to the generator of a unique convolution semigroup and by Theorem 2.2 that the measures are symmetric. □

Now suppose that $\mathcal{L}$ is a self-adjoint Hunt generator in $L^2(G, m)$. The next result gives a classical Beurling-Deny representation for the Dirichlet form $\mathcal{E}$ associated to $\mathcal{L}$ by the prescription

$$\mathcal{E}(f, g) := -\langle f, \mathcal{L}g\rangle,$$

for each $f, g \in \mathrm{Dom}(\mathcal{L})$.

Theorem 2.4 *For each $f, g \in C_c^2(G)$,*

$$\mathcal{E}(f,g) = a^{ij}\int_G (X_i f)(\sigma)(X_j g)(\sigma)d\sigma + \int_{(G\times G)-D}(f(\rho)-f(\sigma))(g(\rho)-g(\sigma))\mu(d\rho, d\sigma),$$

where $D := \{(\sigma,\sigma), \sigma \in G\}$ and $\mu(d\rho, d\sigma) := \nu(\sigma^{-1}d\rho)d\sigma$.

Proof. This is easily verified and is essentially contained in Proposition 2.1 of [15]. □

We remark that if the Haar measure m is both left and right invariant, which holds if e.g. G is abelian, compact, semi-simple or both connected and nilpotent (see Proposition 1.6 of [12]) then the Dirichlet form $\mathcal{E}$ is left-invariant. Indeed in this case L_σ is a unitary operator in $L^2(G)$ for each $\sigma \in G$ and so for all $f, g \in \mathrm{Dom}(\mathcal{L})$ we have

$$\mathcal{E}(L_\sigma f, L_\sigma g) = -\langle L_\sigma f, \mathcal{L}L_\sigma g\rangle = -\langle L_\sigma f, L_\sigma \mathcal{L}g\rangle = \mathcal{E}(f, g).$$

3 Trace Class Properties

In this section we seek to obtain conditions which ensure that the Hunt semigroup T_t is trace-class for all $t > 0$. First we establish a valuable fact about general semigroups of linear operators.

Proposition 3.1 *If $(R_t, t \geq 0)$ is a semigroup of bounded operators on a (real or complex) separable Hilbert space with $R_0 = I$ then R_t is trace-class for all $t > 0$ if and only if R_t is Hilbert-Schmidt for all $t > 0$.*

Proof. If R_t is trace class for some $t > 0$ it is also Hilbert-Schmidt as the trace-class operators are a subset of the Hilbert-Schmidt ones. Conversely if R_t is Hilbert-Schmidt for all $t > 0$ we write each $R_t = R_{\frac{t}{2}} R_{\frac{t}{2}}$ and use the fact that the product of two Hilbert-Schmidt operators is trace-class (see e.g. Theorem VI.22(h) in [18].) $\square$.

Note. In fact Proposition 3.1 may be generalised to the effect that R_t is trace-class for all $t > 0$ if and only if R_t is in some von-Neumann Schatten ideal for all $t > 0$. See Proposition 3.3 in [20] for details.

We now return to the study of Hunt semigroups $(T_t, t \geq 0)$ acting in $L^2(G)$ where G is now a locally compact group equipped with a left-invariant Haar measure.

Theorem 3.1 *T_t is a Hilbert-Schmidt operator for all $t > 0$ if and only if μ_t has a square-integrable density g_t for all $t > 0$. Moreover in this case*

$$T_t f(\sigma) = \int_G f(\tau) g_t(\sigma^{-1}\tau) d\tau, \tag{3.5}$$

and

$$||T_t||_2^2 = \int_G |g_t(\tau)|^2 d\tau.$$

Proof. By e.g. Theorem VI.23 in [18] we see that T_t is Hilbert-Schmidt if and only if there exists $k_t \in L^2(G \times G)$ such that $(T_t f)(\sigma) = \int_G f(\tau) k_t(\sigma, \tau) d\tau$ for all $f \in L^2(G)$. If μ_t has a density $g_t \in L^2(G)$ we simply define $k_t(\sigma, \tau) := g_t(\sigma^{-1}\tau)$ and it follows that T_t is Hilbert-Schmidt for all $t > 0$. Conversely if we assume the Hilbert-Schmidt property for the semigroup then for all $t > 0, A \in \mathcal{B}(G)$ we have

$$\mu_t(A) = T_t 1_A(e) = \int_A k_t(e, \tau) d\tau,$$

from which it follows that μ_t is absolutely continuous with respect to Haar measure with density $g_t(\tau) = k_t(e, \tau)$. Writing $\mu_t(d\tau) = g_t(\tau)d\tau$ in (2.1), we see that for $t > 0, \sigma \in G, f \in L^2(G)$,

$$T_t f(\sigma) = \int_G f(\sigma\tau) g_t(\tau) d\tau = \int_G f(\tau) g_t(\sigma^{-1}\tau) d\tau.$$

From this and the kernel representation we have $\int_G f(\tau)(k_t(\sigma, \tau) - g_t(\sigma^{-1}\tau))d\tau = 0$ for all $f \in L^2(G), \sigma \in G$ and so $k_t(\sigma, \tau) = g_t(\sigma^{-1}\tau)$ (a.e. m). The prescription for $||T_t||_2^2$ then follows from Theorem VI.23 in [18]. $\square$

By Theorem 3.1 and Proposition 3.1 we conclude that if $(T_t, t \geq 0)$ is a Hunt semigroup then T_t is trace-class for all $t > 0$ if and only if μ_t has a square-integrable density g_t for all $t > 0$.

We remark that necessary and sufficient conditions for an arbitrary probability measure on a compact Lie group to have a density have been found in [3] by using Peter-Weyl theory. Some specific classes of convolution semigroups that have a square-integrable density for all $t > 0$ have been obtained by Liao [16] (see Theorems 4.1, 4.3 and 4.4 therein.) They require G to be connected as well as compact and can be summarised as requiring that one of the following holds:

(i) $\det(a) > 0$ and ν is finite.

(ii) $\mathcal{L}_l$ is hypoelliptic and $(\mu_t, t \geq 0)$ is symmetric.

(iii) $\mathcal{L}_l$ is hypoelliptic and $(\mu_t, t \geq 0)$ is conjugate-invariant.

We now give an example of a convolution semigroup that cannot have a density. Fix $\lambda > 0$ and let ρ be a given probability measure on G. We take $(\mu_t, t \geq 0)$ to be the associated *compound Poisson semigroup* which is defined by $\mu_t := e^{-\lambda t}\delta_e + e^{-\lambda t}\sum_{n=1}^{\infty} \dfrac{(\lambda t)^n}{n!}\rho^{*^n}$, for $t > 0$ where $\rho^{*^0} := \delta_e$. Clearly the presence of the Dirac mass at e is an obstacle to absolute continuity. To explore this a little further, we define $\lambda_t := \mu_t - e^{-\lambda t}\delta_e$, for each $t > 0$. Now suppose that ρ has a density h, In this case it is easily verified that for each $t > 0, \lambda_t$ has a density $g_t := e^{-\lambda t}\sum_{n=1}^{\infty} \dfrac{(\lambda t)^n}{n!} h^{*^n}$. We then have the Lebesgue decomposition $\mu_t = e^{-\lambda t}\delta_e + \lambda_t$ for each $t > 0$.

We now return to the consideration of general convolution semigroups on Lie groups and focus on some specific cases where the spectrum of $\mathcal{L}$ can be explicitly computed.

Case 1: The n-dimensional torus

Let $(\mu_t, t \geq 0)$ be a convolution semigroup on $\mathbb{R}$. It is well known that there exists a continuous, hermitian, negative definite function $\eta : \mathbb{R}^n \to \mathbb{C}$ for which $\eta(0) = 0$ such that

$$\int_{\mathbb{R}^n} e^{iu\cdot x}\mu_t(dx) = e^{-t\eta(u)},$$

for each $u \in \mathbb{R}^n, t \geq 0$. The generic form of η is given by the Lévy-Khinchine formula

$$\eta(u) = ib\cdot u + u\cdot au + \int_{\mathbb{R}^n-\{0\}}(1 - e^{iu\cdot y} + iu\cdot y 1_{B_1}(y))\nu(dy), \tag{3.6}$$

where (b, a, ν) are the characteristics of the semigroup $(\mu_t, t \geq 0)$ and B_1 is the unit ball in $\mathbb{R}^n$ that is centered on the origin. In this case we may write the Hunt generator with respect to the basis $\{\partial_1, \ldots, \partial_n\}$ of first order partial differential operators corresponding to the natural basis in $\mathbb{R}^n$ and so for each $f \in C^2(\mathbb{R}^n), x \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{L}f(x) &= b^i\partial_i f(x) + a^{ij}\partial_i\partial_j f(x) + \\ &+ \int_{\mathbb{R}^n-\{0\}}[f(x+y) - f(x) - y^i\partial_i f(x)1_{B_1}(y)]\nu(dy). \end{aligned} \tag{3.7}$$

Now consider the n-dimensional torus $\Pi^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ and let $\gamma : \mathbb{R}^n \to \Pi^n$ be the canonical surjection. We obtain a convolution semigroup of measures on Π^n by the prescription $\mu_t^\gamma = \mu_t \circ \gamma^{-1}$ and the corresponding Hunt semigroup is given by $T_t^\gamma f = T_t(f \circ \gamma^{-1})$ for $t \geq 0, f \in C(\Pi^n)$. The generator of $(T_t^\gamma, t \geq 0)$ is denoted $\mathcal{L}^\gamma$ and $\mathcal{L}^\gamma f = \mathcal{L}(f \circ \gamma^{-1})$ for all $f \in C^2(\Pi^n)$. We identify functions on Π^n with periodic functions on $\mathbb{R}^n$ in the usual way. The characters $\{e_m, m \in \mathbb{Z}^n\}$ where $e_m(y) = e^{im\cdot y}$ for each $y \in \Pi^n$ form a complete orthonormal basis of eigenfunctions for the action of $\mathcal{L}^\gamma$ in $L^2(\Pi^n, \mathbb{C})$. Indeed a straightforward computation using (3.7) yields

$$\mathcal{L}^\gamma e_m = -\eta(m)e_m,$$

for each $m \in \mathbb{Z}^n$. It follows that T_t^γ is trace-class for $t > 0$ if and only if

$$\sum_{m\in\mathbb{Z}^n}|e^{-t\eta(m))}| = \sum_{m\in\mathbb{Z}^n} e^{-t\Re(\eta(m))} < \infty.$$

We have shown that T_t is trace-class if and only if μ_t has a square-integrable density. Some examples where the latter holds have been explicitly calculated in section 4 of [3] and we refer the reader to that paper for the proof of the assertion given in the following proposition as well as precise definitions of all terms that are employed.

Proposition 3.2 *T_t is trace-class for $t > 0$ if any of the following conditions hold:*

(i) The matrix a is non-singular.

(ii) $n = 1$ and $(\mu_t, t \geq 0)$ is α-stable with $0 < \alpha < 2$.

(iii) $n > 1$ and $(\mu_t, t \geq 0)$ is rotationally invariant and α-stable with $0 < \alpha < 2$.

(iv) The Lévy measure ν is asymptotic at the origin to that of an α-stable Lévy measure.

We now make some comments on the structure of the spectrum in the important case where we have self-adjointness. If $(\mu_t, t \geq 0)$ is symmetric then so is $(\mu_t^\gamma, t \geq 0)$ and hence $\mathcal{L}^\gamma$ is self-adjoint by Theorem 2.24. In this case

$$\eta(m) = m \cdot am + \int_{\mathbb{R}^n - \{0\}} (1 - \cos(m \cdot y))\nu(dy), \tag{3.8}$$

for all $m \in \mathbb{Z}^n$ and so $\mathcal{L}^\gamma$ has a discrete non-negative spectrum. Clearly if $m \neq 0, \eta(m) = \eta(-m)$ and it is easily deduced that if

$$(m_1+m_2) \cdot a(m_1-m_2) + 2\int_{\mathbb{R}^n - \{0\}} \sin\left(\frac{(m_1 - m_2) \cdot y}{2}\right) \sin\left(\frac{(m_1 + m_2) \cdot y}{2}\right) \nu(dy) \neq 0$$

whenever $m_1 \neq m_2$ then $\mathcal{L}^\gamma$ has the *Hodge property* (see Theorem 1.2.9 in [19]) that all its eigenvalues have multiplicity two with the exception of zero.

Taking $n = 1$ and $a \neq 0$ in (3.8) it may be of interest to study the function of a complex variable given by $z \mapsto \displaystyle\sum_{m=1}^{\infty} \frac{1}{\eta(m)^z}$. It is not difficult to verify that this series is absolutely convergent when $\Re(z) > \frac{1}{2}$ and that the corresponding function is holomorphic in this region. In the case where $\nu \equiv 0$ and $a = 1$ we obtain the Riemann zeta function evaluated at $\frac{z}{2}$.

Case 2: Compact Symmetric Pairs

Let G be a compact, connected semisimple Lie group and K be a closed subgroup of G. We assume that the pair (G, K) form a Riemannian symmetric pair of compact type. This means (in particular) that G/K is a compact globally Riemannian symmetric space. Full details can be found in [11]. Let π be an irreducible unitary representation of G so that for each $g \in G, \pi(g)$ is a unitary matrix acting in a complex finite-dimensional Hilbert space H_π with dimension d_π. $\text{Irr}_K(G)$ will denote the set of all (equivalence classes) of irreducible representations of G which are *spherical* in the sense that if $\pi \in \text{Irr}_K(G)$ then there exists $\psi_\pi \in H_\pi$ such that $\pi_k\psi_\pi = \psi_\pi$ for all $k \in K$. We note that $\text{Irr}_K(G)$ is a countable set. For each $\pi \in \text{Irr}_K(G)$ define $\phi_\pi \in C(G)$ by

$$\phi_\pi(\sigma) = \int_K \text{tr}(\pi(\sigma^{-1}k))dk,$$

for all $\sigma \in G$. These are precisely the *spherical functions* on (G, K) i.e. the unique mappings $f \in C(G, \mathbb{C})$ for which

$$\int_K f(\sigma k \tau)dk = f(\sigma)f(\tau), \tag{3.9}$$

for all $\sigma, \tau \in G$. We refer the reader to [12] for general facts about spherical functions. We shall make use of the well-known fact that $\{\sqrt{d_\pi}\phi_\pi, \pi \in \text{Irr}_K(G)\}$ is a complete orthonormal basis for the Hilbert space $L^2_K(G) := \{f \in L^2(G), f(k_1\sigma k_2) = f(\sigma), \text{for all } k_1, k_2 \in K \text{ and almost all } \sigma \in G\}$.

Now let $(\mu_t, t \geq 0)$ be a spherical convolution semigroup on G i.e. $\mu_t(k_1Ak_2) = \mu_t(A)$ for all $t > 0, k_1, k_2 \in K$ and all $A \in \mathcal{B}(G)$. We choose our basis $\{X_1, \ldots, X_n\}$ of $\mathbf{g}$ to be orthonormal with respect to the inner product given by minus the Killing form on $\mathbf{g}$. Following Liao [16] (equation 2.16, p.40) we may also choose the functions $x_i (1 \leq i \leq n)$ to be such that

$$x^i(k\sigma)X_i = x^i(\sigma)\text{Ad}(k)X_i, \tag{3.10}$$

for each $\sigma \in G, k \in K$ where Ad is the adjoint representation of G. It follows from work of Gangolli [10] (see also [17] and [1]) that $b = 0$, the matrix $a = cI$ where $c \geq 0$ and ν is a spherical measure i.e. that $\nu(k_1Ak_2) = \nu(A)$ for all $k_1, k_2 \in K$ and all $A \in \mathcal{B}(G)$. Now if we write $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ where $\mathcal{L}_1$ is the second order differential operator part of $\mathcal{L}$ then $\mathcal{L}_1$ is right-invariant for the action of K as well as being left-invariant under that of G and hence, by properties of spherical functions [12], for each $\pi \in \text{Irr}_K(G)$

$$\mathcal{L}_1\phi_\pi = -\beta_\pi\phi_\pi, \tag{3.11}$$

for some $\beta_\pi \geq 0$. We also have Gangolli's Lévy-Khinchine formula ([10], [17]):

$$\int_G \phi_\pi(\sigma)\mu_t(d\sigma) = \exp\{-t\eta_\pi\},$$

$$\text{where } \eta_\pi = \beta_\pi + \int_G (1 - \phi_\pi(\tau))\nu(d\tau). \tag{3.12}$$

Since $\phi_\pi \in C^\infty(G)$ we may compute the action of the Hunt generator $\mathcal{L}$ on spherical functions. This is carried out in the following theorem

Theorem 3.2 *For each* $\pi \in Irr_K(G)$,

$$\mathcal{L}\phi_\pi = -\eta_\pi \phi_\pi. \tag{3.13}$$

Proof. We fix a Cartan involution Θ of $\mathbf{g}$ and we let $\mathbf{k}$ and $\mathbf{p}$ be the eigenspaces of Θ corresponding to eigenvalues 1 and -1 (respectively). Then $\mathbf{k}$ is the Lie algebra of K and we have the Cartan decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$. Suppose that $\dim(\mathbf{k}) = m$ and choose the basis $\{X_1, \ldots, X_n\}$ so that $X_1, \ldots, X_m \in \mathbf{k}$ and $X_{m+1}, \ldots, X_n \in \mathbf{p}$. For each $\pi \in \mathrm{Irr}_K(G)$, ϕ_π is K-bi-invariant and hence for each $X \in \mathbf{k}, \sigma \in G$,

$$X\phi_\pi(\sigma) = \frac{d}{da}\phi_\pi(\sigma \exp aX)\bigg|_{a=0} = \frac{d}{da}\phi_\pi(\sigma)\bigg|_{a=0} = 0.$$

Consequently using the fact that ν is spherical and Fubini's theorem we have

$$\begin{aligned}
\mathcal{L}_2\phi_\pi(\sigma) &= \int_{G-\{e\}} [\phi_\pi(\sigma\tau) - \phi_\pi(e) - \sum_{j=m+1}^{n} x^j(\tau)X_j\phi_\pi(\sigma)]\nu(d\tau) \\
&= \int_{G-\{e\}} \left[\int_K \phi_\pi(\sigma k\tau)dk - 1 - \sum_{j=m+1}^{n}\int_K x^j(k\tau)X_j\phi_\pi(\sigma)dk\right]\nu(d\tau).
\end{aligned}$$

Now for each $\tau \in G$, let $Y_\tau = \sum_{j=m+1}^{n}\int_K x^j(k\tau)X_j dk$. Then $Y_\tau \in \mathbf{p}$. Furthermore for each $l \in K$, we have by (3.10)

$$\begin{aligned}
\mathrm{Ad}(l)Y_\tau &= \sum_{j=m+1}^{n}\int_K x^j(\tau)(\mathrm{Ad}(l) \circ \mathrm{Ad}(k))X_j dk \\
&= \sum_{j=m+1}^{n}\int_K x^j(\tau)(\mathrm{Ad}(lk))X_j dk \\
&= \sum_{j=m+1}^{n}\int_K x^j(lk\tau)X_j dk = Y_\tau,
\end{aligned}$$

and so $Y_\tau = 0$ by Theorem 2 of [17]. Using (3.9) we hence deduce that

$$\mathcal{L}_2\phi_\pi(\sigma) = \left(\int_{G-\{e\}}(\phi_\pi(\tau) - 1)\nu(d\tau)\right)\phi_\pi(\sigma),$$

and the required result follows from this result and (3.11) via (3.12). □

Note It is conjectured that a similar result to (3.13) holds in the more general context of convolution semigroups in Gelfand pairs (G, K) where G

is compact (see [13]) and in an appropriate class of commutative hypergroups (see section 4.5 in [7]).

We immediately deduce from Theorem 3.2 that T_t is trace class for all $t > 0$ (in the Hilbert space $L^2_K(G)$) if and only if

$$\sum_{\pi \in \mathrm{Irr}_K(G)} |e^{-t\eta_\pi}| < \infty. \tag{3.14}$$

Let π be the canonical surjection from G to G/K. There is a one to one correspondence between convolution semigroups $(\mu_t, t \geq 0)$ of spherical measures on G and convolution semigroups $(\kappa_t, t \geq 0)$ of K-invariant measures on G/K (in the sense of [17]) given by $\kappa_t = \mu_t \circ \pi^{-1}$ for $t > 0$. The corresponding C_0-semigroup $(Q_t, t \geq 0)$ on $C(G/K)$ is given by $Q_t f \circ \pi = T_t(f \circ \pi)$ for each $t \geq 0, f \in C(G/K)$. Every spherical function ψ on G/K is induced by a spherical function ϕ on G so that $\phi = \psi \circ \pi$. The normalised Haar measure on G projects to a G-invariant measure on G/K and there is a canonical isomorphism between the Hilbert spaces $L^2_K(G)$ and $L^2_K(G/K) := \{f \in L^2(G/K); f(k\sigma K) = f(\sigma K)$ for all $k \in K$ and almost all $\sigma \in G\}$. The upshot of these facts is that

$$\int_G \phi_\pi(\sigma)\mu_t(d\sigma) = \int_{G/K} \psi_\pi(y)\kappa_t(dy),$$

for each $\pi \in \mathrm{Irr}_K(G)$ and that Q_t is trace-class in $L^2_K(G/K)$ if and only if T_t is trace class in $L^2_K(G)$ for each $t > 0$.

Example *Spherical convolution semigroups on $SO(d+1)$*

Here we take $G = SO(d+1)$ and $H = SO(d)$ so that G/K is the d-sphere S^d. The spherical functions are of the form $r \to P_{n,d}(\cos(r))$ for $n \in \mathbb{Z}_+$ where $P_{n,d}$ is the nth degree ultraspherical (Gegenbauer) polynomial and r is the geodesic distance from the north pole $o = \pi(e)$. In this case we have for each $n \in \mathbb{Z}_+$

$$\eta(n) = cn(n+d-1) + \int_0^\infty (1 - P_{n,d}(\cos(r)))\omega(dr), \tag{3.15}$$

where $\int_0^\infty r^2\omega(dr) < \infty$ (see [13]). Since $|P_{n,d}(\cos(r))| \leq 1$ for all $0 \leq r < \infty$, we see that a sufficient condition for T_t to be trace-class for $t > 0$ is that $c > 0$ since in this case

$$\sum_{n=0}^\infty |e^{-t\eta(n)}| \leq \sum_{n=0}^\infty e^{-cn(n+d-1)t} < \infty.$$

We now consider a class of "non-Gaussian" Hunt semigroups on $SO(d+1)$. Let $(\mu_t, t \geq 0)$ correspond to " spherical c-Brownian motion" so that $\eta(n) = cn(n+d-1)$ for all $n \in \mathbb{Z}_+$ where $c > 0$. Now let $(\rho_t^\alpha, t \geq 0)$ be an α-stable subordinator on $\mathbb{R}^+$ where $0 < \alpha < 1$, so that its characteristics are $(0, 0, \lambda^\alpha)$ where $\lambda^\alpha(dx) = \dfrac{\alpha}{\Gamma(1-\alpha)} \dfrac{dx}{x^{1+\alpha}}$. We form a new convolution semigroup $(\mu_t^\alpha, t \geq 0)$ of spherical measures on $SO(d+1)$ by the prescription $\mu_t^\alpha(A) = \int_{(0,\infty)} \mu_s(A) \rho_t^\alpha(ds)$ for each $A \in \mathcal{B}(SO(d+1)), t \geq 0$. Furthermore the characteristics of $(\mu_t^\alpha, t \geq 0)$ are $(0, 0, m_{\lambda,\mu}^\alpha)$ where $m_{\lambda,\mu}^\alpha(B) = \int_{(0,\infty)} \mu_s(B) \lambda^\alpha(ds)$ for $B \in \mathcal{B}(SO(d+1) - \{0\})$ (see [2] for details). If $\{X_1, \ldots, X_{d+1}\}$ is the given basis for the Lie algebra $\mathbf{so(d+1)}$ then the Hunt generator of $(\mu_t^\alpha, t \geq 0)$ may be written as $-c^\alpha(-X_1^2 - \cdots - X_{d+1}^2)^\alpha$ and (3.15) takes the form $\eta^\alpha(n) = -c^\alpha n^\alpha (n+d-1)^\alpha$. By similar arguments to those used to establish Proposition 3.2 (ii) we can easily verify that for all $0 < \alpha < 1$, the Hunt semigroup is trace-class for all $t > 0$.

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