The Explicit Formula for the Hodrick-Prescott Filter in Finite Sample

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Abstract

We derive the exact expression for the weights of the Hodrick-Prescott (HP) filter in finite sample without making any assumptions about the statistical properties of the time series. We use the results to give insights about the properties of the HP filter and to build a fast algorithm with computational improvements by a factor of up to three times in samples typical in economics.

JEL codes: C1, C6, E3. *Keywords:* trend component; cyclical component; smoothing parameter; Sherman-Morrison.

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1 Introduction

In the past few decades, there has been an increasing interest among economists in techniques for detrending data and for representing their underlying trends. Without any consensus about which model represents the trend best, a popular alternative to model-based detrending is to use smoothing filters. Probably the filter that raised the most interest in economics is the Hodrick-Prescott (HP) filter (Hodrick and Prescott (1997)). The HP filter has, for a long period, been central for business cycle research (see King and Rebelo (1999); Stock and Watson (1999)) and is widely used.[1]

In this paper we derive the explicit formulae for the weights of the HP filter in finite sample, without making assumptions about the statistical properties of the data. We then develop an algorithm for implementing the filter on computers, which is up to three times faster with sample sizes typical in economics.

Given a sample of size n from a time series $\{y_i\}_{i=1}^n$, written as a column vector $\boldsymbol{y} = (y_1, \ldots, y_n)'$, the HP filter, as defined in Hodrick and Prescott (1997), decomposes each y_i into a trend component τ_i (the long-term growth of the time series) and a cyclical component c_i (the deviation from the long-term growth), i.e. $y_i = \tau_i + c_i, \quad i = 1, \ldots, n$. The trend component estimates $\{\hat{\tau}_i\}_{i=1}^n$, written as a column vector $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_n)'$, are obtained as the solution to the constrained minimization problem

$$\min_{\tau_1, \cdots, \tau_n} \sum_{i=1}^{n} (y_i - \tau_i)^2 + \alpha \sum_{i=2}^{n-1} \left(\tau_{i+1} - 2\tau_i + \tau_{i-1}\right)^2, \tag{1}$$

where α is a positive (smoothing) parameter that penalizes the variability in the trend component. For finite sample size n, the unique solution to the minimization problem

[1] A few examples of many recent articles that apply the HP filter are: Bai and Zhang (2010); Coibion and Gorodnichenko (2011); Madeira (2014); Ramadorai (2012).

in (1) is

$$\hat{\boldsymbol{\tau}} = (\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1} \boldsymbol{y}, \tag{2}$$

see Danthine and Girardin (1989) who obtains the solution in (2) without any assumptions about the data generating process of $\boldsymbol{y}$, where $\boldsymbol{I}_n$ is an $n \times n$ identity matrix and $\boldsymbol{F}$ is the pentadiagonal $n \times n$ matrix

$$\boldsymbol{F} = \begin{pmatrix} 1 & -2 & 1 & & & & \\ -2 & 5 & -4 & \ddots & & & \\ 1 & -4 & 6 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 6 & -4 & 1 \\ & & & \ddots & -4 & 5 & -2 \\ & & & & 1 & -2 & 1 \end{pmatrix},$$

for $n \geq 5$, where undisplayed elements are zero. Both the trend component and the cyclical component estimates are weighted averages of the y_i's, i.e. $\hat{\tau}_i = \sum_{j=1}^{n} p_{i,j} y_j$, $\hat{c}_i = y_i - \hat{\tau}_i$, $i = 1, \ldots, n$, with $p_{i,j}$ being the $(i,j)^{th}$ element of $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$, the weights of the HP filter in finite sample.

McElroy (2008) derived the exact formula for the weights of the bi-infinite length HP filter obtained from the first order conditions for $\hat{\tau}_i$ from (1) by letting $i = -\infty, \ldots, +\infty$. [2] However, McElroy's formulae can be seen as approximations to the finite-sample weights $p_{i,j}$ only when near the middle of the sample. More recently, De

[2] See King and Rebelo (1993). Denoting by L and L^{-1} the backward and the forward operator respectively, the low-pass bi-infinite HP filter is: $\theta(L) = (\alpha(1-L)^2(1-L^{-1})^2 + 1)^{-1}$, while $1 - \theta(L) = ((1-L)^2(1-L^{-1})^2)\left((1-L)^2(1-L^{-1})^2 + \alpha^{-1}\right)^{-1}$ is the high-pass bi-infinite HP filter.

Jong and Sakarya (2015) have derived a new representation for the finite-sample $p_{i,j}$. In this paper we obtain the *exact* finite-sample formulae for $p_{i,j}$. Our formulae are exact and simpler than those of De Jong and Sakarya. This is the first contribution of our paper.

The second contribution of our paper is to show that $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ can be computed using only a few matrices of size $m \times m$, where $m = \lfloor n/2 \rfloor$ is the least integer of $n/2$. This can reduce the computational time by a factor of up to three times.

This paper is organized as follows. In Section 2, we derive the formula for the exact weights of the HP filter for finite sample without making assumptions about the data generating process. In Section 3 we introduce the results regarding the reduction in the computation time of the HP filter. The proofs are relegated to the Supplemental Appendix where a simulation study is also conducted in order to show the reduction in the computational time using the results from Section 3. We use the notation in Abadir and Magnus (2002). The Supplemental Appendix and the Matlab programs containing our results (for Theorem 1, Corollaries 6 and 7) are provided on the website of *The Review of Economics and Statistics*.

2 The exact weights of the HP filter

In order to obtain the exact inverse of $\boldsymbol{I}_n + \alpha \boldsymbol{F}$ from (2) note that $\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{Q} - \boldsymbol{g}\boldsymbol{g}' - \boldsymbol{P}_n \boldsymbol{g}\boldsymbol{g}' \boldsymbol{P}_n$, where $\boldsymbol{P}_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ is a permutation matrix of size $n \times n$, $\boldsymbol{Q}$ is a tridiagonal matrix of size $n \times n$,

$$\boldsymbol{Q} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \tag{3}$$

and $\boldsymbol{g} = (-2, 1, \boldsymbol{0}')'$ is a $n \times 1$ column vector. The pentadiagonal matrix $\boldsymbol{QQ}$ has full rank, while $\boldsymbol{gg}'$ and $\boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n$ have rank 1 which allows us to obtain a simple expression for $(\boldsymbol{I}_n + \alpha\boldsymbol{F})^{-1}$ by applying the Sherman-Morrison formula (Abadir and Magnus (2005), p. 248) twice. Note that $\boldsymbol{Q}$ has distinct eigenvalues, $\gamma_j = 2 - 2\cos\left(\frac{\pi j}{n+1}\right)$, $j = 1, \ldots, n$, and corresponding (column) eigenvector $\boldsymbol{x}_j = (x_{1,j}, \ldots, x_{n,j})'$ with

$$x_{i,j} = \left(\frac{2}{n+1}\right)^{1/2} \sin\left(\frac{\pi i j}{n+1}\right), \tag{4}$$

where $i, j = 1, \ldots, n$.

Theorem 1 below gives the exact inverse of $\boldsymbol{I}_n + \alpha\boldsymbol{F}$ in terms of only α, n and the eigenvalues/eigenvectors of $\boldsymbol{Q}$. We denote by $\boldsymbol{T}$ the $n \times n$ matrix of eigenvectors of $\boldsymbol{Q}$ with typical element $x_{i,j}$. Also denote by $\boldsymbol{\Lambda}$ the $n \times n$ diagonal matrix:

$$\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \tag{5}$$

with $\lambda_j = 1 + \alpha\gamma_j^2$, $j = 1, \ldots, n$, the eigenvalues of $\boldsymbol{I}_n + \alpha\boldsymbol{QQ}$. Denote by k_i, $i = 1, 2$, two scalars defined as

$$k_i = \frac{2\alpha}{1 - 2\alpha \sum\limits_{j \in \boldsymbol{n}_i} (2x_{1,j} - x_{2,j})^2 \lambda_j^{-1}}, \tag{6}$$

where $\boldsymbol{n}_1 = (1, 3, 5, \ldots, n)'$ if n is odd, else $\boldsymbol{n}_1 = (1, 3, 5, \ldots, n-1)'$ if n is even, and $\boldsymbol{n}_2 = (2, 4, 6, \ldots, n)'$ if n is even, else $\boldsymbol{n}_1 = (2, 4, 6, \ldots, n-1)'$ if n is odd; $\sum_{j \in \boldsymbol{n}_i}$ denotes the summation over $\boldsymbol{n}_i$, $i = 1, 2$. Finally let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ denote two $n \times n$ matrices with typical element for row i and column j,

$$\frac{(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})}{\lambda_i \lambda_j}, \tag{7}$$

where $i, j = 1, \ldots, n$, for $i+j$ even and j odd in $\boldsymbol{K}_1$, and $i+j$ even and j even in $\boldsymbol{K}_2$, the rest of the elements being zero. We are now in the position to give the following theorem.

Theorem 1. *Given $\alpha > 0$, $5 \leq n < \infty$, the inverse of the matrix in (2) is:*

$$(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1} = \boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T} + k_1 \boldsymbol{T}\boldsymbol{K}_1\boldsymbol{T} + k_2 \boldsymbol{T}\boldsymbol{K}_2\boldsymbol{T}, \tag{8}$$

where $\boldsymbol{\Lambda}^{-1} = diag\left(\lambda_1^{-1}, \ldots, \lambda_n^{-1}\right)$ with λ_j given after (5); $\boldsymbol{T}$ is the matrix of eigenvectors of the matrix $\boldsymbol{Q}$ (from (3)) with typical element (4); $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ have typical element as in (7); the scalars k_1 and k_2 are given in (6).

The result in (8) is valid for any finite $n \geq 5$ without making assumptions about the data generating process of $\boldsymbol{y}$.[3] The proof is relegated to the Supplemental Appendix.

[3] The HP filter can be given a model-based interpretation if one assumes that $y_i = \tau_i + c_i$, $i = 1, \ldots, n$, is the smooth trend model; see Harvey (1989) Section 2.3.6. Then, the estimates of τ_i and c_i from the smooth trend model obtained using the Kalman filter plus smoothing are identical to $\hat{\tau}_i = \sum_{j=1}^{n} p_{i,j} y_j$, $\hat{c}_i = 1 - \hat{\tau}_i$ for $i = d+1, \ldots, n$, where d is a positive integer which depends upon the initialisation of the Kalman filter. Pollock (2007) also proposed an alternative solution to the minimization problem in (2) under different model-based assumptions than those for the smooth trend model. See Section A.6 from the Supplemental Appendix for more details.

Corollary 1. *Let* $\hat{\tau}_i = \sum_{j=1}^n p_{i,j} y_j$ *be the trend component estimate for observation* y_i, $i = 1, \ldots, n$, $5 \le n < \infty$, *and* $\alpha > 0$. *The weights* $p_{i,j}$ *(the elements of the matrix in* (8)*) are given by:*

$$p_{i,j} = \sum_{s=1}^{n} \frac{x_{i,s} x_{s,j}}{\lambda_s} \tag{9}$$

$$+ k_1 \sum_{t \in \boldsymbol{n}_1} \sum_{s \in \boldsymbol{n}_1} x_{i,s} \frac{(2x_{1,s} - x_{2,s})(2x_{1,t} - x_{2,t})}{\lambda_s \lambda_t} x_{t,j} \tag{10}$$

$$+ k_2 \sum_{t \in \boldsymbol{n}_2} \sum_{s \in \boldsymbol{n}_2} x_{i,s} \frac{(2x_{1,s} - x_{2,s})(2x_{1,t} - x_{2,t})}{\lambda_s \lambda_t} x_{t,j} \tag{11}$$

where k_i, $i = 1, 2$, *is given in* (6), $\sum_{t \in \boldsymbol{n}_i}$ *and* $\sum_{s \in \boldsymbol{n}_i}$ *denote summation over* $\boldsymbol{n}_i$ *which is defined after* (6), $i = 1, 2$.

The proof follows by simply computing the matrix multiplications in Theorem 1. Note that $x_{i,j} = x_{j,i}$, hence the matrices $\boldsymbol{T}$, $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ and $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ are symmetric. Also, note that $x_{i,j} = (-1)^{j-1} x_{n+1-i,j}$ and $x_{i,j} = (-1)^{i-1} x_{i,n+1-j}$, $i, j = 1, \ldots, n$. These imply that $x_{i,s} x_{s,j} = x_{n+1-i,s} x_{s,n+1-j}$, $s = 1, \ldots, n$. Hence we have the following property for the weights: $p_{i,j} = p_{n+1-i,n+1-j}$ which indicates that $\boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T}$, $\boldsymbol{T}\boldsymbol{K}_1\boldsymbol{T}$, $\boldsymbol{T}\boldsymbol{K}_2\boldsymbol{T}$ and $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ are centrosymmetric (symmetric about their center) and bisymmetric (symmetric about the main diagonals).

For large n and away from the end points of the sample, we have the following corollary for the terms in $p_{i,j}$.

Corollary 2. *Pointwise in* $i > 0$, $j > 0$ *and* $\alpha > 0$, *as* $n \to \infty$,

(a) the limit of the constants in (6) is $\lim_{n\to\infty} k_1 = \lim_{n\to\infty} k_2 = k$, *where*

$$k = 2\alpha \left(1 - 4\alpha \int_0^1 \frac{16 \left(\sin(r\pi)\right)^4 \left(\sin(2r\pi)\right)^2}{1 + 16\alpha \left(\sin(r\pi)\right)^4} \mathrm{d}r \right)^{-1}; \tag{12}$$

(b) the limit of the term in (9) is

$$\lim_{n\to\infty}\sum_{s=1}^{n}\frac{x_{i,s}x_{s,j}}{\lambda_s}=2\int_0^1\frac{\sin(ir\pi)\sin(jr\pi)}{1+16\alpha\left(\sin(r\pi)\right)^4}\,\mathrm{d}r; \tag{13}$$

(c) the limit of terms in (10) and (11) is

$$\begin{aligned}
& \lim_{n\to\infty}\sum_{t\in\boldsymbol{n}_1}\sum_{s\in\boldsymbol{n}_1}x_{i,s}\frac{(2x_{1,s}-x_{2,s})(2x_{1,t}-x_{2,t})}{\lambda_s\lambda_t}x_{t,j}\\
= & \lim_{n\to\infty}\sum_{t\in\boldsymbol{n}_2}\sum_{s\in\boldsymbol{n}_2}x_{i,s}\frac{(2x_{1,s}-x_{2,s})(2x_{1,t}-x_{2,t})}{\lambda_s\lambda_t}x_{t,j}\\
= & 1024\int_0^1\int_0^1\frac{\sin(2ir\pi)\left(\sin(r\pi)\right)^4}{\left(1+16\alpha\left(\sin(r\pi)\right)^4\right)}\\
\times & \frac{\left(\sin(u\pi)\right)^4\sin(2ju\pi)}{\left(1+16\alpha\left(\sin(u\pi)\right)^4\right)}\,\mathrm{d}r\,\mathrm{d}u.
\end{aligned} \tag{14}$$

See the Supplemental Appendix for the proof. The matrix $\boldsymbol{K}_1$ ($\boldsymbol{K}_2$) has the odd (even) rows and columns equal to zero. The nonzero elements of these matrices are weighted by k_1 and k_2 which are identical only for $n\to\infty$, as it can be seen from Corollary 2(a). Furthermore, the second term in (12) is, as $\alpha\to\infty$,

$$\begin{aligned}
& \lim_{\alpha\to\infty}\alpha\int_0^1\frac{16\left(\sin(r\pi)\right)^4\left(\sin(2r\pi)\right)^2}{1+16\alpha\left(\sin(r\pi)\right)^4}\,\mathrm{d}r\\
&=\int_0^1\left(\sin(2r\pi)\right)^2\mathrm{d}r=\frac{1}{2}.
\end{aligned} \tag{15}$$

Hence, from (12) and (15) it follows that $\lim_{\alpha\to\infty}\lim_{n\to\infty}k_1=\lim_{\alpha\to\infty}\lim_{n\to\infty}k_2=\infty$. Also, as $\alpha\to\infty$, the limit of (13) is

$$\lim_{\alpha\to\infty}\int_0^1\frac{\sin(ir\pi)\sin(jr\pi)}{1+16\alpha\left(\sin(r\pi)\right)^4}\,\mathrm{d}r=0 \tag{16}$$

pointwise in $i>0$ and $j>0$, away from the end-points of the sample. Moreover, away from the end-points of the sample, by l'Hôpital's rule, (14) converges to zero as

α and $n \to \infty$. Thus, as n and α become larger, the weights become smaller.

3 Reducing the computation time in the HP filter

Theorem 1 and Corollary 1 allow us to greatly reduce the computation time of the weights in the HP filter by working with matrices of size $m \times m$, where $m = \lfloor n/2 \rfloor$ is the least integer of $n/2$, instead of matrices of size $n \times n$. To illustrate this we denote by $\boldsymbol{P}_m$ a similar permutation matrix to $\boldsymbol{P}_n$ given before (3), but of size $m \times m$, and give the following corollaries.

Corollary 3. *The matrix $\boldsymbol{T\Lambda}^{-1}\boldsymbol{T}$ from (8) can be written for n even as:*

$$\boldsymbol{T\Lambda}^{-1}\boldsymbol{T} = \begin{pmatrix} \boldsymbol{V}_1 & \boldsymbol{V}_2 \\ \boldsymbol{P}_m\boldsymbol{V}_2\boldsymbol{P}_m & \boldsymbol{P}_m\boldsymbol{V}_1\boldsymbol{P}_m \end{pmatrix}, \tag{17}$$

and for n odd as:

$$\boldsymbol{T\Lambda}^{-1}\boldsymbol{T} = \begin{pmatrix} \boldsymbol{V}_1 & \boldsymbol{v} & \boldsymbol{V}_2 \\ \boldsymbol{v}' & v_{m+1,m+1} & \boldsymbol{v}'\boldsymbol{P}_m \\ \boldsymbol{P}_m\boldsymbol{V}_2\boldsymbol{P}_m & \boldsymbol{P}_m\boldsymbol{v} & \boldsymbol{P}_m\boldsymbol{V}_1\boldsymbol{P}_m \end{pmatrix}, \tag{18}$$

where $\boldsymbol{V}_1$ is a $m \times m$ matrix with typical element given by (9), i and $j = 1, \dots, m$; $\boldsymbol{V}_2$ is a $m \times m$ matrix with typical element given by (9), $i = 1, \dots, m$, and $j = m+1, \dots, n$, if n is even, or $j = m+2, \dots, n$, if n is odd; $\boldsymbol{v}$ is a column vector of length m with typical element as in (9) with $i = 1, \dots, m$, and $j = m+1$; $v_{m+1,m+1}$ is given by (9) where i and $j = m+1$.

Corollary 4. *The matrix* $\boldsymbol{TK}_1\boldsymbol{T}$ *from (8) can be written for* n *even as:*

$$\boldsymbol{TK}_1\boldsymbol{T} = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{DP}_m \\ \boldsymbol{P}_m\boldsymbol{D}' & \boldsymbol{P}_m\boldsymbol{DP}_m \end{pmatrix}, \tag{19}$$

and for n *odd as:*

$$\boldsymbol{TK}_1\boldsymbol{T} = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{d} & \boldsymbol{DP}_m \\ \boldsymbol{d}' & d_{m+1,m+1} & \boldsymbol{d}'\boldsymbol{P}_m \\ \boldsymbol{P}_m\boldsymbol{D}' & \boldsymbol{P}_m\boldsymbol{d} & \boldsymbol{P}_m\boldsymbol{DP}_m \end{pmatrix}, \tag{20}$$

where $\boldsymbol{D}$ *is a* $m \times m$ *matrix with typical element given by (10),* i *and* $j = 1, \ldots, m$*;* $\boldsymbol{d}$ *is a column vector of length* m *with typical element as in (10), where* $i = 1, \ldots, m$*, and* $j = m + 1$*;* $d_{m+1,m+1}$ *is the term in (10) with* i *and* $j = m + 1$*.*

Corollary 5. *The matrix* $\boldsymbol{TK}_2\boldsymbol{T}$ *from (8) can be written for* n *even as:*

$$\boldsymbol{TK}_2\boldsymbol{T} = \begin{pmatrix} \boldsymbol{E} & -\boldsymbol{EP}_m \\ -\boldsymbol{P}_m\boldsymbol{E}' & \boldsymbol{P}_m\boldsymbol{EP}_m \end{pmatrix}, \tag{21}$$

and n *odd as:*

$$\boldsymbol{TK}_2\boldsymbol{T} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{e} & -\boldsymbol{EP}_m \\ \boldsymbol{e}' & e_{m+1,m+1} & -\boldsymbol{e}'\boldsymbol{P}_m \\ -\boldsymbol{P}_m\boldsymbol{E}' & -\boldsymbol{P}_m\boldsymbol{e} & \boldsymbol{P}_m\boldsymbol{EP}_m \end{pmatrix}, \tag{22}$$

where $\boldsymbol{E}$ *is a* $m \times m$ *matrix with typical element given by (11),* i *and* $j = 1, \ldots, m$*;* $\boldsymbol{e}$ *is a column vector of length* m *with typical element as in (11), where* $i = 1, \ldots, m$*, and* $j = m + 1$*;* $e_{m+1,m+1}$ *is the term in (11) with* i *and* $j = m + 1$*.*

The proofs of Corollaries 3-5 follow from Weaver (1985), the corollaries being a simple consequence of the fact that $\boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T}$, $\boldsymbol{T}\boldsymbol{K}_1\boldsymbol{T}$ and $\boldsymbol{T}\boldsymbol{K}_2\boldsymbol{T}$ are centrosymmetric.[4] An important consequence of Corollaries 3, 4 and 5 is the following simplification of Theorem 1.

Corollary 6. *Denote* $\tilde{\boldsymbol{V}}_1 = \boldsymbol{V}_1 + \boldsymbol{D} + \boldsymbol{E}$ *and* $\tilde{\boldsymbol{V}}_2 = \boldsymbol{V}_2\boldsymbol{P}_m + \boldsymbol{D}' - \boldsymbol{E}'$*. For n even,*

$$(\boldsymbol{I}_n + \alpha\boldsymbol{F})^{-1} = \begin{pmatrix} \tilde{\boldsymbol{V}}_1 & \boldsymbol{V}_2 + (\boldsymbol{D} - \boldsymbol{E})\,\boldsymbol{P}_m \\ \boldsymbol{P}_m\tilde{\boldsymbol{V}}_2 & \boldsymbol{P}_m\tilde{\boldsymbol{V}}_1\boldsymbol{P}_m \end{pmatrix}, \tag{23}$$

and for n odd,

$$(\boldsymbol{I}_n + \alpha\boldsymbol{F})^{-1} = \begin{pmatrix} \tilde{\boldsymbol{V}}_1 & \boldsymbol{a} & \boldsymbol{V}_2 + (\boldsymbol{D} - \boldsymbol{E})\boldsymbol{P}_m \\ \boldsymbol{a}' & a & \boldsymbol{z}'\boldsymbol{P}_m \\ \boldsymbol{P}_m\tilde{\boldsymbol{V}}_2 & \boldsymbol{P}_m\boldsymbol{z} & \boldsymbol{P}_m\tilde{\boldsymbol{V}}_1\boldsymbol{P}_m \end{pmatrix}, \tag{24}$$

where $\boldsymbol{a} = \boldsymbol{v} + \boldsymbol{d} + \boldsymbol{e}$, $\boldsymbol{z} = \boldsymbol{v} + \boldsymbol{d} - \boldsymbol{e}$, $a = v_{m+1,m+1} + d_{m+1,m+1} + e_{m+1,m+1}$.

Corollary 6 suggests that $(\boldsymbol{I}_n + \alpha\boldsymbol{F})^{-1}$ which is of size $n \times n$, can be computed using only the matrices $\boldsymbol{P}_m, \boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{D}, \boldsymbol{E}$ which are of (smaller) size $m \times m$. The formulae for computing these matrices are given in the next corollary where we use the following notation. We denote by $\odot$ the Hadamard product. Let $\boldsymbol{T}_1$ be a $m \times m$ matrix with typical element given in (4), but with i and $j = 1, \ldots, m$. Let $\boldsymbol{J}$ denote a $m \times m$ matrix given by $\boldsymbol{J} = (\boldsymbol{\imath}, -\boldsymbol{\imath}, \boldsymbol{\imath}, \ldots, \boldsymbol{\imath}, -\boldsymbol{\imath})$, where $\boldsymbol{\imath}$ is a column vector of ones of size $m \times 1$. Denote $\widetilde{\boldsymbol{T}} = \boldsymbol{T}_1 \odot \boldsymbol{J}$. Using the properties of $x_{i,j}$ mentioned before

[4] In the upper-right corners of (19) and (20) we have a permutation of $\boldsymbol{D}$. This follows by noticing that for s odd, $x_{j,s} = x_{s,n+1-j}$. As a consequence, when $\hat{\tau}_i$ is computed, y_j and y_{n+j-1} receive the same weight, i and $j = 1, \ldots, n$. In the upper-right corners of (21) and (22) we have a permutation of $-\boldsymbol{E}$. This follows by noticing that for s even, $x_{j,s} = -x_{s,n+1-j}$. As a consequence, when $\hat{\tau}_i$ is computed, y_j and y_{n+j-1} receive the same weight, but of opposite sign, i and $j = 1, \ldots, n$.

Corollary 2, we have an alternative representation of the matrix $\boldsymbol{T}$ in terms of a 2×2 block matrix for n even,

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_1 & \widetilde{\boldsymbol{T}}'\boldsymbol{P}_m \\ \boldsymbol{P}_m\widetilde{\boldsymbol{T}}_1 & (-1)^l \boldsymbol{P}_m\widetilde{\boldsymbol{T}}'\boldsymbol{P}_m \odot \boldsymbol{J} \end{pmatrix}, \tag{25}$$

and in terms of a 3×3 matrix for n odd,

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_1 & \boldsymbol{x}_1 & \widetilde{\boldsymbol{T}}'\boldsymbol{P}_m \\ \boldsymbol{x}_1' & x_{m+1,m+1} & \boldsymbol{x}_2' \\ \boldsymbol{P}_m\widetilde{\boldsymbol{T}} & \boldsymbol{x}_2 & (-1)^l \boldsymbol{P}_m\widetilde{\boldsymbol{T}}'\boldsymbol{P}_m \odot \boldsymbol{J} \end{pmatrix}, \tag{26}$$

where $\boldsymbol{x}_1$ is a $m \times 1$ column vector with typical element given in (4) with $i = 1, \dots, m$, and $j = m+1$; $\boldsymbol{x}_2$ is a $m \times 1$ column vector with typical element given in (4) with $i = m+2, m+3 \dots, 2m$, and $j = m+1$; the scalar $x_{m+1,m+1}$ is computed as in (4) with i and $j = m+1$, and

$$l = \begin{cases} 2, & \text{if } n = 4j \text{ or } n = 4j - 1, \text{ with } j \in \mathbb{N}, \\ 1, & \text{for the other values of } n. \end{cases} \tag{27}$$

Note that $\boldsymbol{T}$ from (25)-(26) is not centrosymmetric.

Let $\boldsymbol{b}$ denote the $m \times 1$ vector with typical element given by $\cos(\pi j/(n+1))$, $j = 1, \dots, m$. Since $\cos(\pi j/(n+1)) = -\cos(\pi(n+1-j)/(n+1))$, then the eigenvalues of $\boldsymbol{I}_n + \alpha\boldsymbol{Q}\boldsymbol{Q}$ are given by the elements of the $n \times 1$ vector, for n even,

$$\begin{aligned} \boldsymbol{\lambda} &= \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 + 4\alpha(1 - \boldsymbol{b}) \odot (1 - \boldsymbol{b}) \\ 1 + 4\alpha(1 + \boldsymbol{P}_m\boldsymbol{b}) \odot (1 + \boldsymbol{P}_m\boldsymbol{b}) \end{pmatrix}. \end{aligned} \tag{28}$$

The matrix $\mathbf{\Lambda}$ from (5) can also be written in partitioned form, for n even,

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_1 & \mathbf{O}_{m,m} \\ \mathbf{O}_{m,m} & \mathbf{\Lambda}_2 \end{pmatrix}, \tag{29}$$

and for n odd,

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_1 & \mathbf{0}_{m,1} & \mathbf{O}_{m,m} \\ \mathbf{0}_{1,m} & \lambda_{m+1} & \mathbf{0}_{1,m} \\ \mathbf{O}_{m,m} & \mathbf{0}_{m,1} & \mathbf{\Lambda}_2 \end{pmatrix}, \tag{30}$$

where λ_{m+1} is computed as mentioned after (5) with $j = m + 1$, $\mathbf{\Lambda}_1 = \text{diag}(\boldsymbol{\lambda}_1)$ and $\mathbf{\Lambda}_2 = \text{diag}(\boldsymbol{\lambda}_2)$.

Let $\boldsymbol{G}_1$ be the $m \times m$ matrix with typical element for row s column t given by

$$\frac{(2x_{1,2s+1} - x_{2,2s+1})(2x_{1,2t+1} - x_{2,2t+1})}{\lambda_{2s+1}\lambda_{2t+1}}, \tag{31}$$

where s and $t = 0, \ldots, m - 1_{n \text{ even}}$, with $1_{n \text{ even}}$ being the indicator function which equals 1 if n is even and 0 if n odd. Let $\boldsymbol{G}_2$ be the $m \times m$ matrix with typical element for row s column t given by

$$\frac{(2x_{1,2s} - x_{2,2s})(2x_{1,2t} - x_{2,2t})}{\lambda_{2s}\lambda_{2t}}, \tag{32}$$

with $s, t = 1, \ldots, m$. Finally, let $\boldsymbol{M}_1$ be the $m \times m$ matrix with typical element $x_{i,2j+1}$, $i = 1, \ldots, m$, and $j \in \boldsymbol{m}_1$, $\boldsymbol{m}_1 = (0, \ldots, m - 1)'$ if n is even, or $j \in \boldsymbol{m}_2$, $\boldsymbol{m}_2 = (0, \ldots, m)'$ if n is odd. Let $\boldsymbol{M}_2$ be the $m \times m$ matrix with typical element $x_{i,2j}$, i and $j = 1, \ldots, m$. We are now in the position to give the following corollary.

Corollary 7. *(a) The matrices* $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{D}$, $\boldsymbol{E}$ *from (17), (19), (21) are given by*

$$\boldsymbol{V}_i = \begin{cases} \boldsymbol{W}_i, & n \text{ even}, \\ \boldsymbol{W}_i + \boldsymbol{x}_1 \lambda_{m+1}^{-1} \boldsymbol{x}_i', & n \text{ odd}, \end{cases}$$

where $i = 1, 2$, $\boldsymbol{D} = k_1 \boldsymbol{M}_1 \boldsymbol{G}_1 \boldsymbol{M}_1'$, $\boldsymbol{E} = k_2 \boldsymbol{M}_2 \boldsymbol{G}_2 \boldsymbol{M}_2'$, *with* $\boldsymbol{W}_1 = \boldsymbol{T}_1 \boldsymbol{\Lambda}_1^{-1} \boldsymbol{T}_1 + \widetilde{\boldsymbol{T}}' \boldsymbol{P}_m \boldsymbol{\Lambda}_2^{-1} \boldsymbol{P}_m \boldsymbol{T}_1 \odot \boldsymbol{J}$, *and* $\boldsymbol{W}_2 = \boldsymbol{T}_1 \boldsymbol{\Lambda}_1^{-1} \widetilde{\boldsymbol{T}}' \boldsymbol{P}_m + (-1)^l \, \widetilde{\boldsymbol{T}}' \boldsymbol{P}_m \boldsymbol{\Lambda}_2^{-1} \boldsymbol{P}_m \widetilde{\boldsymbol{T}}' \boldsymbol{P}_m \odot \boldsymbol{J}$, *where* l *is defined in (27).*

(b) For n *odd,* $\boldsymbol{v}$, $\boldsymbol{d}$, $\boldsymbol{e}$ *from (18), (20) and (22) are given by:* $\boldsymbol{v} = \boldsymbol{T}_1 \boldsymbol{\Lambda}_1^{-1} \boldsymbol{x}_1 + \boldsymbol{x}_1 \lambda_{m+1}^{-1} x_{m+1,m+1} + \widetilde{\boldsymbol{T}}' \boldsymbol{P}_m \boldsymbol{\Lambda}_2^{-1} \boldsymbol{x}_2$*; let* $i = 1, \ldots, m$, *and* $j = m+1$, *then* $\boldsymbol{d}$ *has typical element*

$$\sum_{t=0}^{m} \sum_{s=0}^{m} \left(x_{i,2s+1} \frac{2x_{1,2s+1} - x_{2,2s+1}}{\lambda_{2s+1}} \times \frac{2x_{1,2t+1} - x_{2,2t+1}}{\lambda_{2t+1}} x_{2t+1,j} \right), \tag{33}$$

and $\boldsymbol{e}$ *has typical element*

$$\sum_{t=1}^{m} \sum_{s=1}^{m} x_{i,2s} \frac{(2x_{1,2s} - x_{2,2s})(2x_{1,2t} - x_{2,2t})}{\lambda_{2s} \lambda_{2t}} x_{2t,j}. \tag{34}$$

(c) The constants k_1 *and* k_2 *from (6) are given by*

$$k_1 = \frac{2\alpha}{1 - 2\alpha \sum_{j \in \boldsymbol{m}_1} (2x_{1,2j+1} - x_{2,2j+1})^2 \lambda_{2j+1}^{-1}}, \tag{35}$$

$$k_2 = \frac{2\alpha}{1 - 2\alpha \sum_{j \in \boldsymbol{m}_2} (2x_{1,2j} - x_{2,2j})^2 \lambda_{2j}^{-1}}. \tag{36}$$

where $\boldsymbol{m}_i$, $i = 1, 2$, *was defined after* (32), *and* $\sum_{j \in \boldsymbol{m}_i}$ *denotes summation over* $\boldsymbol{m}_i$, $i = 1, 2$.

The proof is in the Supplemental Appendix. When n is odd, k_1, $\boldsymbol{M}_1$ and $\boldsymbol{G}_1$ have to be computed accordingly, as indicated in (31) and (35). A simulation study in the Supplemental Appendix (Section D, Figure 1) shows that the results in this section can reduce the computation time of the HP filter by a factor of three for sample sizes typical in macroeconomics and finance.

4 Conclusion

In this paper we obtain the exact analytical expression for the finite-sample weights of the HP filter without making assumptions about the data generating process, a result that has not been previously derived in the literature. We use the expression for the weights to build a fast algorithm that can be implemented in software. Our algorithm is up to three times faster for sample sizes typical in economics. Our results may also be used to derive analytically the moments needed in the estimation of DSGE models; to propose a solution for reducing spurious correlations/cycles and the problems these induce for inference, and to propose a data-dependent method for the choice of the smoothing parameter.

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