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3+1 magnetodynamics

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ABSTRACT

The magnetodynamics, or force-free degenerate electrodynamics, is recognized as a very useful approximation in studies of magnetospheres of relativistic stars. In this Letter, we discuss various forms of the magnetodynamic equations which can be used to study magnetospheres of black holes. In particular, we focus on the 3+1 equations which allow for curved and dynamic space–time.

Key words: black hole physics – magnetic fields – relativistic processes.

1 INTRODUCTION

In magnetospheres of pulsars and black holes, the electromagnetic field is so strong that inertia and pressure of plasma can be ignored. As a result, the Lorentz force almost vanishes, and the transport of energy and momentum is almost entirely electromagnetic (Goldreich & Julian 1969; Blandford & Znajek 1977). This justifies the name 'force-free' to describe the electrodynamics of pulsars and black holes. However, the electrodynamics of the magnetospheres is rather different from the electrodynamics in vacuum which, obviously, is also force-free. Indeed, the magnetospheric plasma is plentiful enough to support strong electric currents and screen the component of electric field parallel to the magnetic field. Electromagnetic field satisfying this condition is called 'degenerate', and for this reason Macdonald & Thorne (1982) called the electrodynamics of pulsar and black hole magnetospheres 'force-free degenerate electrodynamics' (FFDE).

For a long time, theorists were preoccupied with steady-state solutions of FFDE. Even the basic properties of FFDE as a system of time-dependent equation were not studied systematically. The first step in this direction was made only quite recently, when Uchida (1997) developed a theory of FFDE in which the electromagnetic field is described in terms of two scalar functions, called 'Euler potentials'. However, this formulation has not been very popular. In particular, it is not very convenient for numerical analysis because its basic equations, when written in components, involve mixed space and time second-order derivatives. Another approach is to use the actual Maxwell equations supplemented with a particular prescription for the electric current. This was done by Gruzinov (1999), who used the force-free condition to derive the Ohm's law. Komissarov (2002) showed that FFDE can be considered as relativistic magnetohydrodynamics (RMHD) in the limit of vanishing inertia of plasma particles. This allowed us to rewrite FFDE as a system of conservation laws similar to RMHD, including the energy–momentum conservation law. Komissarov, Barkov & Lyutikov (2007) argued that the dynamics of electromagnetic field in FFDE can be interpreted as a motion of magnetic mass–energy under the action of Maxwell stresses and proposed another name, 'magnetodynamics' (MD), for FFDE. We will be using this name in the rest of the Letter.

The formulation by Komissarov (2002) is in a covariant form and can be used to study the magnetospheres of black holes (Komissarov 2001; McKinney 2006). However, the wealth of experience accumulated in solving Maxwell equations has ensured that the formulation by Gruzinov (1999) was often found preferable (Spitkovsky 2006; Kalapotharakos & Contopoulos 2009). This prompted recent efforts to generalize Gruzinov's formulation so that it could also be used to study the magnetospheres of black holes. The starting point was the work by Thorne & Macdonald (1982), who first obtained general 3+1 equations of electrodynamics (equations 3.4 of Thorne & Macdonald 1982) and then a simplified version (equations 5.8 of the same paper) which was adapted to the case of stationary black holes. Moreover, they restricted their attention to the Boyer–Lindquist foliation of space–time. This simplified version has become most known to astrophysicists, via the follow-up paper by Macdonald & Thorne (1982) and Thorne, Price & Macdonald (1986), and widely used. Komissarov (2004) developed a different formulation, which has its roots in the works of Tamm (1924) and Plebanski (1959; see also Landau & Lifshitz 1971). In this formulations, the 3+1 equations of electrodynamics also have a very simple and familiar form. In fact, they look exactly the same as the Maxwell equations in matter. The only assumption on the space–time metric made in this formulation is that the determinant of the metric tensor of space does not depend on time. Palenzuela, Lehner & Yoshida (2010) presented, without derivation, the 3+1 equations which are free even from this constraint. They seem to have used the approach by Komissarov (2004) but reverted to the original representation of Thorne & Macdonald (1982), where only the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ are present. Their equations also include extra scalar fields, which have been introduced for purely computational reasons. The force-free Ohm's law of general relativistic MD (GRMD) was first derived in the space–time form by McKinney (2006) and then in the 3+1 form by Palenzuela

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et al. (2011). Lyutikov (2011) independently derived equations of GRMD using the simplified version of 3+1 electrodynamics by Thorne & Macdonald (1982). Thus, his equations have inherited the limitations of those by Thorne & Macdonald (1982).

In this Letter, we revert back to the 3+1 formulation of Komissarov (2004), modify it in order to allow non-stationary metric and derive the corresponding form of the force-free Ohm's law. We also present various relevant derivations and explore the connections between the different forms of the 3+1 equations.

2 3+1 ELECTRODYNAMICS

Following Thorne & Macdonald (1982), we adopt the foliation approach to the 3+1 splitting of space–time in which the time coordinate t parametrizes a suitable filling of space–time with space-like hypersurfaces described by the 3D metric tensor γ_{ij}. These hypersurfaces may be regarded as the 'absolute space' at different instances of time t. Below, we describe a number of useful results for further references. If $\{x^i\}$ are the spatial coordinates of the absolute space, then

$$\mathrm{d}s^2 = (\beta^2 - \alpha^2)\,\mathrm{d}t^2 + 2\beta_i\,\mathrm{d}x^i\,\mathrm{d}t + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \tag{1}$$

where α is called the 'lapse function' and $\boldsymbol{\beta}$ is the 'shift vector'. The 4-velocity of the local fiducial observer ('FIDO'), which can be described as being at rest in the absolute space, is

$$n_\mu = (-\alpha, 0, 0, 0)\,. \tag{2}$$

The spatial components of the projection tensor, which is used to construct pure spatial tensors,

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta\,, \tag{3}$$

coincide with the components of the spatial metric γ_{ij}. Other useful results are

$$n^\mu = \frac{1}{\alpha}(1, -\beta^i)\,, \tag{4}$$

$$g^{t\mu} = -\frac{1}{\alpha}n^\mu\,, \tag{5}$$

$$g = -\alpha^2\gamma\,, \tag{6}$$

where

$$\beta^i = \gamma^{ij}\beta_j, \quad g = \det g_{\mu\nu}, \quad \gamma = \det\gamma_{ij}\,.$$

β^i are the components of the velocity of the spatial grid relative to the local FIDO as measured using the coordinate time t and the spatial basis $\{\partial_i\}$ (Macdonald & Thorne 1982).

The covariant Maxwell equations are (e.g. Jackson 1979)

$$\nabla_\beta {}^*\!F^{\alpha\beta} = 0\,, \tag{7}$$

and

$$\nabla_\beta F^{\alpha\beta} = I^\alpha\,, \tag{8}$$

where $F^{\alpha\beta}$ is the Maxwell tensor of the electromagnetic field, ${}^*F^{\alpha\beta}$ is the Faraday tensor and I^α is the 4-vector of the electric current. The most direct way of 3+1 splitting of the covariant Maxwell equations is to write them down in components and then to introduce such spatial vectors that these equations have a particularly simple and familiar form. For example, when equation (7) is written in components, it splits into two parts:

(i) the time part,

$$\frac{1}{\sqrt{\gamma}}\partial_i\left(\alpha\sqrt{\gamma}\,{}^*F^{ti}\right) = 0\,, \tag{9}$$

(ii) the spatial part,

$$\frac{1}{\sqrt{\gamma}}\partial_t\left(\alpha\sqrt{\gamma}\,{}^*F^{jt}\right) + \frac{1}{\sqrt{\gamma}}\partial_i\left(\alpha\sqrt{\gamma}\,{}^*F^{ji}\right) = 0\,. \tag{10}$$

If we now introduce the spatial vectors $\boldsymbol{B}$ and $\boldsymbol{E}$ via

$$B^i = \alpha\,{}^*F^{it} \tag{11}$$

and

$$E_i = \frac{\alpha}{2}e_{ijk}\,{}^*F^{jk}\,, \tag{12}$$

where

$$e_{ijk} = \sqrt{\gamma}\epsilon_{ijk}\,, \qquad e^{ijk} = \frac{1}{\sqrt{\gamma}}\epsilon^{ijk}, \tag{13}$$

is the Levi-Civita tensor of the absolute space and $\epsilon_{ijk} = \epsilon^{ijk}$ is the 3D Levi-Civita symbol, then equations (9) and (10) read

$$\nabla\cdot\boldsymbol{B} = 0\,, \tag{14}$$

$$\frac{1}{\sqrt{\gamma}}\partial_t(\sqrt{\gamma}\boldsymbol{B}) + \nabla\times\boldsymbol{E} = 0\,, \tag{15}$$

where ∇ is the covariant derivative of the absolute space. Similarly, equation (8) splits into

$$\nabla\cdot\boldsymbol{D} = \rho\,, \tag{16}$$

$$\frac{1}{\sqrt{\gamma}}\partial_t(\sqrt{\gamma}\boldsymbol{D}) - \nabla\times\boldsymbol{H} = -\boldsymbol{J}\,, \tag{17}$$

where

$$D^i = \alpha F^{ti}\,, \tag{18}$$

$$H_i = \frac{\alpha}{2}e_{ijk}F^{jk} \tag{19}$$

and

$$\rho = \alpha I^t, \quad J^k = \alpha I^k\,. \tag{20}$$

Similar to any highly ionized plasma, the pair plasma of black hole magnetospheres has essentially zero electric and magnetic susceptibilities. In such a case, the Faraday tensor is simply dual to the Maxwell tensor:

$${}^*F^{\alpha\beta} = \frac{1}{2}e^{\alpha\beta\mu\nu}F_{\mu\nu}, \tag{21}$$

$$F^{\alpha\beta} = -\frac{1}{2}e^{\alpha\beta\mu\nu}\,{}^*F_{\mu\nu}\,. \tag{22}$$

Here,

$$e_{\alpha\beta\mu\nu} = \sqrt{-g}\,\epsilon_{\alpha\beta\mu\nu}\,, \qquad e^{\alpha\beta\mu\nu} = -\frac{1}{\sqrt{-g}}\,\epsilon^{\alpha\beta\mu\nu} \tag{23}$$

is the Levi-Civita alternating tensor of space–time, and $\epsilon_{\alpha\beta\mu\nu} = \epsilon^{\alpha\beta\mu\nu}$ is the 4D Levi-Civita symbol. This allows us to obtain the following alternative expressions for $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{D}$ and $\boldsymbol{H}$:

$$B^i = \frac{1}{2}e^{ijk}F_{jk}\,, \tag{24}$$

$$E_i = F_{it}\,, \tag{25}$$

$$D^i = \frac{1}{2}e^{ijk}\,{}^*F_{jk}\,, \tag{26}$$

$$H_i = {}^*F_{ti}\,. \tag{27}$$

Moreover, from the above definitions, one immediately finds the following vacuum constitutive equations:

$$\boldsymbol{E} = \alpha \boldsymbol{D} + \boldsymbol{\beta} \times \boldsymbol{B}\,, \quad (28)$$

$$\boldsymbol{H} = \alpha \boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{D}\,. \quad (29)$$

In flat space–time with Lorentzian (pseudo-Cartesian) coordinates, one has $\alpha = 1$, $\beta = 0$ and, hence, $\boldsymbol{B} = \boldsymbol{H}$ and $\boldsymbol{E} = \boldsymbol{D}$.

Each of the introduced spacial vectors can be represented by a space–time vector whose spacial part is the spacial vector in question and whose time part vanishes. As one can easily verify, these space–time vectors are given by the following covariant expressions:

$$B^\mu = -{}^*F^{\mu\nu} n_\nu\,, \quad (30)$$

$$E^\mu = \frac{1}{2}\gamma^{\mu\nu} e_{\nu\alpha\beta\gamma} k^\alpha\, {}^*F^{\beta\gamma}\,, \quad (31)$$

$$D^\mu = F^{\mu\nu} n_\nu\,, \quad (32)$$

$$H^\mu = -\frac{1}{2}\gamma^{\mu\nu} e_{\nu\alpha\beta\gamma} k^\alpha F^{\beta\gamma}\,, \quad (33)$$

$$J^\mu = 2I^{[\nu}k^{\mu]} n_\nu\,, \quad (34)$$

where $k^\alpha = \partial_t$. From these, one can see that $\boldsymbol{B}$ and $\boldsymbol{D}$ are the magnetic and electric fields as measured by FIDOs, whereas $\boldsymbol{H}$ and $\boldsymbol{E}$ are auxiliary vector fields.

It is also easy to verify that

$$\rho = -I^\nu n_\nu, \quad (35)$$

and thus ρ is the electric charge density as measured by FIDOs. However, $\boldsymbol{J}$ is not the electric current density as measured by FIDO, which we will denote as $\boldsymbol{j}$. Geometrically, $\boldsymbol{j}$ is the component of I^ν normal to n^ν. Using the projection tensor $\gamma_{\nu\mu} = g_{\nu\mu} + n_\nu n_\mu$, we find

$$\boldsymbol{J} = \alpha \boldsymbol{j} - \rho \boldsymbol{\beta}\,. \quad (36)$$

The second term in this equation accounts for the motion of spatial grid relative to FIDO, or in other words for the fact that the coordinate time direction, the basis vector ∂_t, is generally not parallel to n^ν.

When $\partial_t \gamma = 0$, these 3+1 equations have exactly the same form as the classical Maxwell equations for the electromagnetic field in matter:

$$\nabla \cdot \boldsymbol{B} = 0\,, \quad (37)$$

$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0\,, \quad (38)$$

$$\nabla \cdot \boldsymbol{D} = \rho\,, \quad (39)$$

$$-\partial_t \boldsymbol{D} + \nabla \times \boldsymbol{H} = \boldsymbol{J}\,. \quad (40)$$

This similarity explains why we prefer to denote the electric field measured by FIDO as $\boldsymbol{D}$, whereas in most papers by other researches it is denoted as $\boldsymbol{E}$.

Applying the divergence operator to equation (17), one finds the electric charge conservation law:

$$\frac{1}{\sqrt{\gamma}}\partial_t(\sqrt{\gamma}\rho) + \nabla \cdot \boldsymbol{J} = 0. \quad (41)$$

Although this is slightly different from the usual differential form of this law, its integral form is exactly the same:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\, \mathrm{d}V + \int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = 0, \quad (42)$$

where $\mathrm{d}V$ is the metric volume and $\mathrm{d}\boldsymbol{S}$ is the metric surface elements.

The limit of MD is defined by vanishing of the Lorentz force. In the covariant form, this condition reads as

$$F_{\mu\nu} I^\mu = 0\,. \quad (43)$$

In our 3+1 formulation, this equation splits into

$$\boldsymbol{E} \cdot \boldsymbol{J} = 0 \quad (44)$$

and

$$\rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} = 0\,. \quad (45)$$

From the last equation, it follows that

$$\boldsymbol{E} \cdot \boldsymbol{B} = 0\,. \quad (46)$$

When combined with the constitutive equation (28), the last equation also implies

$$\boldsymbol{D} \cdot \boldsymbol{B} = 0\,. \quad (47)$$

As first noted by Gruzinov (1999), the force-free condition allows one to express the electric current in terms of the electromagnetic field and its spacial derivatives, thus providing us with a particular form of Ohm's law. Here we repeat Gruzinov's derivation taking into account the effects of general relativity. The component of electric current normal to the magnetic field can be found directly from equation (45) via cross-multiplying its sides by $\boldsymbol{B}$. This yields

$$\boldsymbol{J}_\perp = \rho \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}\,. \quad (48)$$

In order to find the parallel component, we first note that equation (47) implies

$$\partial_t(\sqrt{\gamma}\boldsymbol{D} \cdot \boldsymbol{B}) = 0\,. \quad (49)$$

When combined with equations (15), (17) and (47), this yields

$$(\nabla \times \boldsymbol{H} - \boldsymbol{J}) \cdot \boldsymbol{B} - (\nabla \times \boldsymbol{E}) \cdot \boldsymbol{D} = 0\,, \quad (50)$$

which does not involve the time derivative of γ. From the last result, we find that

$$\boldsymbol{J}_\parallel = \frac{\boldsymbol{B} \cdot (\nabla \times \boldsymbol{H}) - \boldsymbol{D} \cdot (\nabla \times \boldsymbol{E})}{B^2}\boldsymbol{B}\,. \quad (51)$$

Collecting all these results, we can write the most general 3+1 system of GRMD as

$$\frac{1}{\sqrt{\gamma}}\partial_t(\sqrt{\gamma}\boldsymbol{B}) + \nabla \times \boldsymbol{E} = 0\,, \quad (52)$$

$$\frac{1}{\sqrt{\gamma}}\partial_t(\sqrt{\gamma}\boldsymbol{D}) - \nabla \times \boldsymbol{H} = -\boldsymbol{J}\,, \quad (53)$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \quad (54)$$

where

$$\boldsymbol{E} = \alpha \boldsymbol{D} + \boldsymbol{\beta} \times \boldsymbol{B}\,, \quad (55)$$

$$\boldsymbol{H} = \alpha \boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{D}\,, \quad (56)$$

$$\boldsymbol{J} = \rho \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} + \frac{\boldsymbol{B} \cdot (\nabla \times \boldsymbol{H}) - \boldsymbol{D} \cdot (\nabla \times \boldsymbol{E})}{B^2}\boldsymbol{B} \quad (57)$$

and

$$\rho = \nabla \cdot \boldsymbol{D}\,. \tag{58}$$

It is easy to see that in flat space–time with Lorentzian coordinates, where $\alpha = 1$, $\beta = 0$ and $\partial_t \gamma = 0$, this system is reduced to that of Gruzinov (1999). Under the conditions $\partial_t \gamma = 0$ and $\nabla \cdot \boldsymbol{\beta} = 0$, it is reduced to that of Lyutikov (2011).

MD can be considered as relativistic MHD in the limit of vanishing particle inertia (Komissarov 2002). The explicit condition of MHD approximation is vanishing of the electric field in the fluid frame. This implies that in any other frame, the component of electric field parallel to the magnetic one always vanishes, and the magnetic field is stronger than the electric one. These conditions can be written in the covariant form as

$${}^*F_{\mu\nu} F^{\mu\nu} = 0 \quad \text{and} \quad F_{\mu\nu} F^{\mu\nu} > 0\,. \tag{59}$$

In our 3+1 notation, these yield

$$\boldsymbol{B} \cdot \boldsymbol{D} = 0 \quad \text{and} \quad B^2 - D^2 > 0\,. \tag{60}$$

In computer simulation, one has to make sure that the initial solution satisfies both these conditions. The first constraint is preserved exactly by the differential equations of MD. However, the second constraint can be violated (Komissarov 2002). Slow shocks of RMHD can transform plasma from magnetically dominated to particle-dominated state (Lyubarsky 2005). However, slow waves are not allowed in the MD approximation (Komissarov 2002). This limitation can be behind many violations of the second condition (60) in MD.

Substituting the expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the constitutive equations into equations (15) and (17) and expanding the double cross-product terms, one finds

$$\partial_t \boldsymbol{B} - \mathcal{L}_{\boldsymbol{\beta}} \boldsymbol{B} + \nabla \times \alpha \boldsymbol{D} = \eta \boldsymbol{B} \tag{61}$$

and

$$\partial_t \boldsymbol{D} - \mathcal{L}_{\boldsymbol{\beta}} \boldsymbol{D} - \nabla \times \alpha \boldsymbol{B} = \eta \boldsymbol{D} - \alpha \boldsymbol{j}\,, \tag{62}$$

where $\eta = \nabla \cdot \boldsymbol{\beta} - \partial_t (\ln \sqrt{\gamma})$ and $\mathcal{L}_{\boldsymbol{\beta}}$ is the Lie derivative along the shift vector (e.g. $\mathcal{L}_{\boldsymbol{\beta}} \boldsymbol{B} = (\boldsymbol{\beta} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{\beta}$). This is another useful form of the most general Faraday and Ampére equations of 3+1 GR electrodynamics (Palenzuela et al. 2010).[1] One can show that

$$\eta = \alpha\, \mathrm{Tr}(K), \tag{63}$$

where $\mathrm{Tr}(K) = \gamma^{ik} K_{ik}$ is the trace of the external curvature tensor of the absolute space

$$K_{ik} = \frac{1}{2\alpha} \left(\beta_{i;k} + \beta_{k;i} - \partial_t \gamma_{ik} \right) \tag{64}$$

(Misner, Thorne & Wheeler 1973). When both $\partial_t \gamma = 0$ and $\nabla \cdot \boldsymbol{\beta} = 0$, these equations reduce to

$$\partial_t \boldsymbol{B} - \mathcal{L}_{\boldsymbol{\beta}} \boldsymbol{B} + \nabla \times \alpha \boldsymbol{D} = 0 \tag{65}$$

and

$$-\partial_t \boldsymbol{D} + \mathcal{L}_{\boldsymbol{\beta}} \boldsymbol{D} + \nabla \times \alpha \boldsymbol{B} = \alpha \boldsymbol{j}\,. \tag{66}$$

These are the 3+1 equations of black hole electrodynamics by Macdonald & Thorne (1982). We note here that although the condition $\nabla \cdot \boldsymbol{\beta} = 0$ is satisfied by the Boyer–Lindquist metric of Kerr black holes, it is not satisfied by the Kerr–Schild metric, which is also widely used in black hole studies.

In terms of the physical quantities measured by FIDOs, vanishing of the Lorentz force has the familiar form[2]

$$\rho \boldsymbol{D} + \boldsymbol{j} \times \boldsymbol{B} = 0\,. \tag{67}$$

The force-free electric current $\boldsymbol{j}$ can now be obtained in exactly the same fashion as we did earlier for $\boldsymbol{J}$. The normal component of $\boldsymbol{j}$ is obviously

$$\boldsymbol{j}_\perp = \rho \frac{\boldsymbol{D} \times \boldsymbol{B}}{B^2}\,. \tag{68}$$

In order to find the parallel component, we apply the operator $\partial_t - \mathcal{L}_{\boldsymbol{\beta}}$ to $\boldsymbol{B} \cdot \boldsymbol{D} = 0$. This yields

$$\boldsymbol{j}_\parallel = \frac{\boldsymbol{B} \cdot (\nabla \times \alpha \boldsymbol{B}) - \boldsymbol{D} \cdot (\alpha \nabla \times \boldsymbol{D})}{\alpha B^2} \boldsymbol{B}\,. \tag{69}$$

Given the identity $\boldsymbol{A} \cdot (\nabla \times \alpha \boldsymbol{A}) \equiv \alpha \boldsymbol{A} \cdot (\nabla \times \boldsymbol{A})$, the final expression for the force-free current does not actually involve either the shift vector or the lapse function and has exactly the same form as in special relativity,

$$\boldsymbol{j} = \rho \frac{\boldsymbol{D} \times \boldsymbol{B}}{B^2} + \frac{\boldsymbol{B} \cdot (\nabla \times \boldsymbol{B}) - \boldsymbol{D} \cdot (\nabla \times \boldsymbol{D})}{B^2} \boldsymbol{B} \tag{70}$$

(Palenzuela et al. 2011).

3 THE 4-VECTOR OF FORCE-FREE CURRENT

Finally, we briefly discuss the space–time formulation of MD. If one prefers to deal with the 4-tensor Maxwell–Ampére equation (8) instead of the energy–momentum equation (as in Komissarov 2002), then the key issue is the expression for the 4-vector of force-free current. This expression was found by McKinney (2006). However, it can be simplified a little bit further. Here we explain this and give a slightly different derivation.

From the definitions (30 and 32), it follows that

$$F^{\alpha\beta} = n^\alpha D^\beta - D^\alpha n^\beta - e^{\alpha\beta\nu\xi} B_\nu n_\xi \tag{71}$$

and

$${}^*F^{\alpha\beta} = -n^\alpha B^\beta + B^\alpha n^\beta - e^{\alpha\beta\nu\xi} D_\nu n_\xi\,. \tag{72}$$

Then the force-free condition (43) reads

$$\rho D^\beta + e^{\xi\beta\alpha\nu} n_\xi I_\alpha B_\nu = 0. \tag{73}$$

From this, we find that

$$D^\beta B_\beta = 0 \tag{74}$$

and

$$I^\mu = \frac{\rho}{B^2} e^{\gamma\mu\beta\delta} n_\gamma D_\beta B_\delta + \frac{(I^\nu B_\nu)}{B^2} B^\mu + \rho n^\mu\,. \tag{75}$$

One can see that the spacial part of I^μ, which we will denote as $\mathcal{J}^\mu = \mathcal{J}^\mu_\parallel + \mathcal{J}^\mu_\perp$, has the following components parallel and perpendicular to B^μ:

$$\mathcal{J}^\mu_\parallel = \frac{(I^\nu B_\nu)}{B^2} B^\mu\,, \tag{76}$$

$$\mathcal{J}^\mu_\perp = \frac{\rho}{B^2} e^{\gamma\mu\beta\delta} n_\gamma D_\beta B_\delta\,. \tag{77}$$

The coefficient $I^\nu B_\nu$ in equation (76) can be expressed in terms of the electric and magnetic fields and their derivatives, making

[1] In Palenzuela et al. (2010), as well as in Thorne & Macdonald (1982) and many other papers, the variable $\boldsymbol{D}$ is denoted as $\boldsymbol{E}$, following its interpretation as the electric field measured by the local FIDO of the space–time foliation.

[2] This equation can also be obtained via substituting expressions (28) and (36) into equation (45).

this equation an explicit expression for $\mathcal{J}_{\parallel}^{\mu}$. Following McKinney (2006), we first contract the Maxwell–Ampére law (8) with B^{μ} to find that

$$I^{\alpha} B_{\alpha} = -B^{\alpha} D_{\alpha,\beta} n^{\beta} - e^{\alpha\beta\nu\xi} B_{\alpha} B_{\nu,\beta} n_{\xi} \,, \tag{78}$$

where the comma indicates partial derivative. Then we contract the Maxwell–Faraday equation (7) with D_{ν} to find that

$$B^{\alpha} D_{\alpha,\beta} n^{\beta} = -e^{\alpha\beta\nu\xi} D_{\alpha} D_{\nu,\beta} n_{\xi} \,. \tag{79}$$

Thus,

$$\begin{aligned} I^{\alpha} B_{\alpha} &= e^{\xi\alpha\beta\nu} n_{\xi} (B_{\alpha} B_{\nu,\beta} - D_{\alpha} D_{\nu,\beta}) \\ &= e^{\xi\alpha\beta\nu} n_{\xi} (B_{\alpha} B_{\nu;\beta} - D_{\alpha} D_{\nu;\beta}) \,, \end{aligned} \tag{80}$$

where the semicolon stands for covariant differentiation. The corresponding expression in McKinney (2006) is a little bit different because it includes the term $B^{\alpha} D^{\beta} (n_{\beta;\alpha} + n_{\alpha;\beta})$, which equals to zero. Collecting all the results, we obtain

$$\begin{aligned} \mathcal{J}^{\mu} = &\frac{\rho}{B^2} e^{\gamma\mu\beta\delta} n_{\gamma} D_{\beta} B_{\delta} \\ &+ B^{\mu} \frac{e^{\xi\alpha\beta\nu} n_{\xi} (B_{\alpha} B_{\nu;\beta} - D_{\alpha} D_{\nu;\beta})}{B^2} \,, \end{aligned} \tag{81}$$

$$I^{\mu} = \rho n^{\mu} + \mathcal{J}^{\mu} \,. \tag{82}$$

It is easy to verify that in the 3+1 notation, equation (81) is identical to equation (70), which does not include either the lapse function or the shift vector, or the time derivatives of $\boldsymbol{B}$ and $\boldsymbol{D}$.

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