On Stability of a Distributed Averaging PI Frequency and Active Power Controlled Differential-Algebraic Power System Model

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Abstract— **We consider the problems of stability, frequency restoration and optimal steady-state resource allocation in a heterogeneous and structure-preserving differential-algebraic equation (DAE) power system model. Thereby, we include constant-power-controlled loads (CPCLs) and constant-power-controlled sources (CPCSs) explicitly in the analysis and network control design. This results in a power system model with mixed algebraic as well as first- and second-order differential dynamics. We show that the abovementioned control objectives can be achieved via a distributed averaging proportional integral (DAPI) control and, in particular, extend the stability proof in [1] to the resulting closed-loop DAE system.**

I. Introduction

Motivated by environmental, economic and societal aspects, countries worldwide are seeking to build reliable, efficient and sustainable future energy systems [2], [3]. Such systems combine distributed power generation based on renewables and demand response with advanced measurement, communication and control techniques [4]. Therefore, they are termed Smart Grids. Key to the efficient implementation of Smart Grids is the development of advanced modular control schemes guaranteeing a reliable and efficient system operation [4]. When addressing this aspect, it is important to realize that most renewable generation sources as well as storage units are either DC sources (photovoltaic plants, fuel cells, batteries) or operated at variable or high-speed frequency (wind turbines, microturbines). This implies that they have to be connected to an AC network via AC inverters [3]. Such inverters are power electronic devices, which possess significantly different dynamic and physical characteristics from synchronous generators (SGs)—the electro-mechanical network interface employed in conventional power plants [5].

Independently of their particular network interface, generation units are usually operated in either of the following two operation modes: grid-forming or grid-feeding mode [6], [7]. Thereby, grid-forming units are mainly responsible for frequency and voltage control, while grid-feeding units are controlled such that they provide a pre-specified amount of active and reactive power to the grid, i.e., they are constant-power-controlled sources (CPCSs) [6], [7]. Consequently, most previous work on power system stability in the presence of renewable generation units, e.g., [1], [8], [9], has mainly focused on units in grid-forming mode. Yet, it is foreseen essential in Smart Grids to actively incorporate renewable grid-feeding units and flexible loads in network control and ancillary service tasks [4]. Furthermore, also more and more loads are interfaced to the network via inverters and operated as constant-power-controlled loads (CPCLs).

Motivated by the above considerations, in this paper we include CPCLs and CPCSs explicitly in the modeling, analysis, and network control. This implies that—unlike most other work on stability analysis of power systems [10], [11] and recent articles on microgrid studies [7], [8]—we don't work with the Kron-reduced network model [5], [12], but instead resort to structure-preserving models [13]–[15]. Thereby, we follow the standard praxis to represent CPCLs and CPCSs by algebraic power balance equations [6]. In addition, we consider a diverse generation pool composed of inverter-interfaced units, SG-interfaced units, as well as frequency-responsive loads [5], [13]. Consequently, the derived power system model is a heterogeneous and structure-preserving *differential-algebraic equation* (DAE) system. We then focus on the problems of stability, secondary frequency control and optimal active power dispatch for this DAE power system model. To that end and following [1], [9], we employ a distributed averaging proportional integral (DAPI) frequency and active power control. We show that the DAPI control is well-suited to achieve the abovementioned control objectives.

To establish our stability result, we build upon previous work on stability analysis of semi-explicit index-one DAE models [15], [16], which we briefly review for self-consistency and to adapt the notation and tools to our needs. A similar analysis has been carried out in [17] for a related Hamiltonian DAE power system model without CPCLs and CPCSs, while an explicit reduced ODE model for a DAE model with SGs and CPCLs has been derived in [18].

Notation. We define the sets $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} | x \geq 0\}$, $\mathbb{R}_{>0} := \{x \in \mathbb{R} | x > 0\}$ and $\mathbb{R}_{<0} := \{x \in \mathbb{R} | x < 0\}$. Let $x := \mathrm{col}(x_i) \in \mathbb{R}^n$ denote a vector with entries x_i, $\mathbb{0}_n$ the zero vector, $\mathbb{1}_n$ the one vector, I_n the $n \times n$ identity matrix, $\mathbb{0}_{n\times n}$ the $n \times n$ matrix with all entries equal to zero and $\mathrm{diag}(a_i)$ an $n \times n$ diagonal matrix with diagonal entries $a_i \in \mathbb{R}$. Likewise, $A = \mathrm{blkdiag}(A_i)$ denotes a block-diagonal matrix with block-diagonal matrix entries A_i. For $A \in \mathbb{R}^{n\times n}$, $A < (>) 0$ means that A is symmetric negative (positive) definite. For $z = \mathrm{col}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ and sets $\mathbb{X} = \{1, \ldots, n\}$, $\mathbb{Y} = \{n+1, \ldots, n+m\}$, we let $z_{\mathbb{X}} = \mathrm{col}(x_1, \ldots, x_n)$ and $z_{\mathbb{Y}} = \mathrm{col}(x_{n+1}, \ldots, x_{n+m}\}$. Also, ∇f denotes the transpose of the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$. For a function $f : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$, $(x, y) \to f(x, y)$, we employ the notation $\nabla_{\mathbb{X}} f = ((\partial f)/(\partial x))^T$.

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II. Stability theory for DAE systems

We briefly state the main theoretical results used to establish the stability claims in the present paper. The theory is mainly taken from [16] with minor modifications in notation.

Following [16], we consider the autonomous DAE system

$$\dot{x} = f(x, y), \tag{1a}$$
$$0 = g(x, y), \tag{1b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, with admissible initial conditions $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying the algebraic constraint

$$0 = g(x_0, y_0), \tag{2}$$

and where the vector fields are $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. The solutions of (1) starting at (x_0, y_0) are denoted by $\big(x(x_0, y_0, t),\ y(x_0, y_0, t)\big)$ for $t \geq 0$ in the domain of the solution. At times, it will be convenient to use the notation $z = \mathrm{col}(x, y) \in \mathbb{R}^{n+m}$. We denote the maximal domain of a solution of (1) by $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$. We omit the explicit parametrization (x_0, y_0, t) whenever it is clear from the context. We make the following assumptions.

*Assumption 2.1 (***Equilibria***):* The system (1) possesses an equilibrium point $z^* = \mathrm{col}(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^m$.

*Assumption 2.2 (***Regularity***):* Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open connected set containing (x^*, y^*). The functions f and g are twice continuously differentiable in Ω, and the Jacobian of g with respect to y has constant full rank on Ω

$$\mathrm{rank}\,(\nabla_y g(x, y)) = m \quad \forall (x, y) \in \Omega.$$

Assumption 2.2 ensures existence and uniqueness of solutions of (1) in Ω over the interval $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$ for any $\big(x(x_0, y_0, t),\ y(x_0, y_0, t)\big) \in \Omega$ satisfying (2) [16, Theorem 1]. In addition, Assumption 2.2 together with Assumption 2.1 has the following important implication - the proof of which follows directly from the implicit function theorem [19].

*Lemma 2.3 (***Correspondence of solutions***):* Consider the system (1) with Assumptions 2.2 and 2.1. Then there exists an open set $\Omega^* = (\mathcal{U}(x^*) \times \mathcal{U}(y^*)) \subset \Omega$ containing (x^*, y^*), and a unique twice continuously differentiable function $u : \mathcal{U}(x^*) \to \mathcal{U}(y^*)$, such that for all $(x_0, y_0) \in \Omega^*$ and for all $t \in \mathcal{I}$ the solution $\big(x(x_0, y_0, t),\ y(x_0, y_0, t)\big) \subseteq \Omega^*$ of the DAE system (1) (remaining in Ω^*) is identical to the solution $\big(x(x'_0, y'_0, t),\ y(x'_0, y'_0, t)\big)$ of the associated ODE system

$$\dot{x}' = f(x', u(x')), \tag{3a}$$
$$y' = u(x'), \tag{3b}$$

where $(x'_0, y'_0) = (x_0, y_0 = u(x_0))$.

DAEs of the form (1) satisfying the regularity property in Assumption 2.2 are referred to as *semi-explicit index-one DAEs*. We employ the following definition of stability.

*Definition 2.4 (***Stability***):* Let $z^* = \mathrm{col}(x^*, y^*) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an interior point of Ω and an equilibrium point of (1). Let $z_0 = \mathrm{col}(x_0, y_0) \in \Omega$ satisfy (2). Then, z^* is said to be

- stable, if for each positive real ε there is a real constant $\delta = \delta(\varepsilon) > 0$, such that
$$\|z - z_0\| < \delta \quad \Rightarrow \quad \|z - z^*\| < \varepsilon, \quad \forall t \geq 0,$$
- unstable if it is not stable,
- asymptotically stable (AS) if it is stable and there exists a real constant $r > 0$, such that
$$\|z - z_0\| < r \quad \Rightarrow \quad \lim_{t \to \infty} z(t, z_0) = z^*.$$

The following theorem gives a sufficient stability criterion for the DAE (1). An equivalent claim is made in [16], yet without providing an explicit proof of 2) below[1].

*Theorem 2.5 (***Lyapunov/LaSalle stability criterion***):* Consider the system (1) with Assumptions 2.1 and 2.2. Let $\Omega_S \subset \Omega$ containing (x^*, y^*). Suppose that there exists a continuously differentiable function $S : \Omega_S \to \mathbb{R}$, such that (x^*, y^*) is a strict minimum of S. Furthermore, suppose that $\dot{S}(x, y) \leq 0$ for all $(x, y) \in \Omega_S$. Then, the following statements hold:

1) (x^*, y^*) is a stable equilibrium point with a local Lyapunov function $\mathcal{V}(x, y) = S(x, y) - S(x^*, y^*) \geq 0$ for (x, y) near (x^*, y^*).
2) Suppose, in addition, that no solution of (1) other than $(x(t), y(t)) \equiv (x^*, y^*)$ remains in $\{(x, y) \in \Omega_c \,|\, \dot{S}(x, y) = 0\}$ for all $t \geq 0$, where $\Omega_c = \{(x, y) \in \Omega_S \,|\, S(x, y) \leq c\}$ is a compact sublevel set for some $c > S(x^*, y^*)$. Then (x^*, y^*) is an AS equilibrium point.

Proof: The claim is established by following [16, Theorem 3] and [20, Lemma 3.2.4]. Lemma 2.3 implies that there exists a neighborhood $\Omega^* = (\mathcal{U}(x^*) \times \mathcal{U}(y^*)) \subset \Omega$ of (x^*, y^*), in which the DAE (1) is equivalent to the ODE (3) on the domain $\mathcal{I} \times \Omega^*$. Furthermore, with the standing assumptions, we have that for all $(x, y) \in \Omega^* \cap \Omega_S \setminus \{(x^*, y^*)\}$

$$S(x^*, y^*) = S(x^*, u(x^*)) < S(x, y) = S(x, u(x)) = S(x),$$
$$\dot{S}(x, y) = \dot{S}(x, u(x)) = \dot{S}(x) \leq 0.$$

Hence, by standard Lyapunov theory for ODEs [21], x^* is a stable equilibrium point of the ODE (3). Furthermore, the set $\mathbb{X} := \{x \in \mathbb{R}^n \,|\, S(x) \leq \gamma\} \subset \mathcal{U}(x^*)$ with $\mathcal{U}(x^*)$ given in Lemma 2.3 is compact and forward invariant for some $\gamma > S(x^*)$ close enough to $S(x^*)$. Consequently, existence and uniqueness is guaranteed for $\mathcal{I} = \mathbb{R}_{\geq 0}$ and $\forall x_0 \in \mathbb{X}$ [21, Theorem 3.3]. In addition, by Lemma 2.3 we have that $u : \mathcal{U}(x^*) \to \mathcal{U}(y^*)$ is a continuous mapping. Hence, $y = u(x)$ is bounded on the compact domain $\mathbb{X} \subset \mathcal{U}(x^*)$. Because of this and Lemma 2.3, existence, uniqueness, and stability of a solution of the DAE system (1) on $\mathbb{X} \times u(\mathbb{X}) \subseteq \Omega_S$ is implied by the same properties of the associated ODE system (3) on $\mathbb{X}$. Furthermore, injectivity of the map u [19] implies that $u(x) = y^* \Leftrightarrow x = x^*$. Hence, if, in addition, the assumption in 2) is satisfied, invoking LaSalle's invariance principle [21] on $\Omega_c \cap \mathbb{X}$ yields that x^* is an AS equilibrium point of the ODE (3). By analogous arguments as above, we conclude that (x^*, y^*) is an AS equilibrium of the DAE (1). ∎

[1] The claims of Theorem 2.5 are based on the standard implicit function theorem [19] and are thus only valid in an open neighborhood of (x^*, y^*).

III. Differential-Algebraic Power System Model, Problem Statement and DAPI Control

A. Differential-Algebraic Power System Model

We consider a structure-preserving power system model composed by $n \geq 1$ nodes and denote the set of network nodes by $\mathcal{N} := \{1, \ldots, n\}$. We make the standard assumptions that the line admittances are purely inductive and that the voltage amplitudes $V_i \in \mathbb{R}_{>0}$ at all nodes $i \in \mathcal{N}$ are constant [5]. Then, two nodes i and j in the network are connected by a nonzero susceptance $B_{ij} \in \mathbb{R}_{<0}$. The set of neighbors of the i-th node is denoted by $\mathcal{N}_i = \{j \in \mathcal{N} \mid B_{ij} \neq 0\}$. We associate a phase angle $\theta_i \in \mathbb{R}$ to each node $i \in \mathcal{N}$, and use the common short-hand $\theta_{ij} := \theta_i - \theta_j$, $i \in \mathcal{N}$, $j \in \mathcal{N}$. The electrical frequency at the i-th node is given by $\dot{\theta}_i = \omega_i \in \mathbb{R}$. In addition, we assume that the power system is connected, that is, that for all pairs $(i,j) \in \mathcal{N} \times \mathcal{N}$, $i \neq j$, there exists an ordered sequence of nodes from i to j such that any pair of consecutive nodes in the sequence is connected by a power line represented by an admittance.

We consider a heterogeneous network with three distinct sets of nodes $\mathcal{N} = \mathcal{P} \cup \mathcal{F} \cup \mathcal{G}$, corresponding to passive buses, buses equipped with frequency-responsive loads or inverters, and buses connecting SGs and inverters with power measurement filters. Passive buses represent buses at which either CPCSs or CPCLs are connected at. Here, the qualifier *passive* means that these units do not contribute to primary frequency control and is not related to the control-theoretic notion of passivity [20]. Following standard praxis [6], we model each CPCL and each CPCS by an algebraic equation. Hence, the set of *passive* network nodes is given by $\mathcal{P} := \{1, \ldots, p\}$, where $n > p \geq 0$ is the number of CPCLs and CPCSs in the network. Furthermore, we assume that first-order frequency-responsive loads [5], [13] and inverter-interfaced grid-forming units with instantaneous power measurements and primary droop control [3] are connected at $n > f \geq 0$ nodes $\mathcal{F} := \{p+1, \ldots, p+f\}$. Finally, SG-interfaced units, synchronous motor loads, as well as droop-controlled inverter-interfaced units with filtered power measurements (that admit a mathematically equivalent representation to SGs [7]) are connected at $n > g \geq 1$ nodes $\mathcal{G} := \{p+f+1, \ldots, n\}$. With these considerations, the DAE power system model considered in this paper is

$$
\begin{aligned}
i \in \mathcal{G}: \quad & \dot{\theta}_i = \omega_i, && \text{(4a)}\\
& M_i \dot{\omega}_i = -D_i(\omega_i - \omega^d) + P_i^d + u_i - P_i, && \text{(4b)}\\
i \in \mathcal{F}: \quad & D_i \dot{\theta}_i = D_i \omega^d + P_i^d + u_i - P_i, && \text{(4c)}\\
i \in \mathcal{P}: \quad & 0 = P_i^d + u_i - P_i, && \text{(4d)}
\end{aligned}
$$

where the active power flow at the i-th node is given by

$$P_i = \sum\nolimits_{j \in \mathcal{N}} |B_{ij}| V_i V_j \sin(\theta_{ij}).$$

Here, $P_i^d \in \mathbb{R}$ are the active power setpoints of the network components (positive for sources and negative for loads), $M_i \in \mathbb{R}_{>0}$ is the (virtual) inertia, $D_i \in \mathbb{R}_{>0}$ the droop, damping, or frequency-sensitivity coefficient and ω^d the nominal frequency. In addition, we assume that the active power demand of each network component can be influenced by its respective control input $u_i \in \mathbb{R}^2$. We refer the reader to [6], [7] for a detailed modeling of the system components.

B. Problem statement

The overarching objective in power system operation is to balance load and generation. If this power balance is not met, then the synchronous frequency in the network deviates from its desired nominal value $\omega^d = 2\pi \cdot 50\,\mathrm{Hz}$ (respectively, $2\pi \cdot 60\,\mathrm{Hz}$). Indeed, assume that (4) possesses a synchronized solution with constant frequencies $\dot{\theta}_i^*(t) = \omega^*$ and constant $u_i^*(t)$ for all $i \in \mathcal{N}$ and for all $t \geq 0$. Then by summing over all equations (4b), (4c), (4d) and noting that $\sum_{i \in \mathcal{N}} P_i^d = 0$, we obtain the *net power balance*

$$\underbrace{\sum\nolimits_{i \in \mathcal{F} \cup \mathcal{G}} D_i(\omega^* - \omega^d)}_{\text{primary control}} = \underbrace{\sum\nolimits_{i \in \mathcal{N}} P_i^d}_{\text{injection setpoints}} + \underbrace{\sum\nolimits_{i \in \mathcal{N}} u_i^*}_{\text{secondary control}},$$

where the left-hand side is due to *primary frequency droop control* and frequency damping, the first term on the right-hand side is the nominal power balance (due to controllable generation scheduled according to a load and renewable forecast), and the second term on the right-hand side is due to the action of *secondary frequency control* [22]. The power setpoints P_i^d of the CPCLs and CPCSs are usually uncertain and not known exactly. Hence, $\omega^* = \omega^d$ only if the secondary control inputs u_i^* compensate for this uncertainty.

Aside from merely balancing load and generation via secondary control, a *tertiary* control objective is to allocate the additional injections u_i in an optimal fashion accounting for generation costs and capacity via an economic generation dispatch [23]. We summarize this discussion as follows.

*Problem 3.1 (***Optimal secondary control***):* Consider the system (4). Design a control law for the control inputs u_i such that the following performance objectives are satisfied.

1) Zero steady-state frequency deviation, that is,

$$\lim_{t \to \infty} \|\omega_i - \omega^d\| = 0, \quad \forall i \in \mathcal{N}, \quad \omega^d \in \mathbb{R}_{>0}.$$

2) Optimal steady-state resource allocation, that is,

$$
\begin{aligned}
\text{minimize}_{u^*} & \sum\nolimits_{i \in \mathcal{N}} A_i (u_i^*)^2, \quad A_i \in \mathbb{R}_{>0},\\
\text{subject to} & \sum\nolimits_{i \in \mathcal{N}} u_i^* + \sum\nolimits_{i \in \mathcal{N}} P_i^d = 0,
\end{aligned}
\tag{5}
$$

where $A_i > 0$ is the cost coefficient for source $i \in \mathcal{N}$, and u_i^* in (5) is understood as the steady-state of $u_i(t)$.

The optimization problem (5) is (strictly) convex, and the essential insight from the optimality conditions [24] is that all units should produce at *identical marginal costs*

$$A_i u_i^* = A_j u_j^* \text{ for all } i \in \mathcal{N},\, j \in \mathcal{N}. \tag{6}$$

A special case of the identical marginal cost requirement is the *proportional power sharing* objective $u_i^*/\overline{P_i} = u_j^*/\overline{P_j}$, where $\overline{P_i} \in \mathbb{R}_{>0}$ is the rating of source i [3]. Thus, power sharing is a special case of the optimal allocation problem (5).

[2]We remark that all our results also hold in presence of uncontrolled nodes with $u_i = 0$ which do not contribute to ancillary system services.

C. Distributed Averaging PI (DAPI) Control

Inspired by [1], we consider the following control law to address Problem 3.1

$$u_i = -K_i s_i - R_i q_i, \quad \dot{s}_i = \omega_i - \omega^d\,, \tag{7a}$$

$$\dot{q}_i = \sum\nolimits_{j\in\mathcal{N}} a_{ij}\left(A_i u_i - A_j u_j\right)\,, \quad i \in \mathcal{N}\,, \tag{7b}$$

where $K_i > 0$, $R_i > 0$ for $i \in \mathcal{N}$ are control gains, and the weights $a_{ij} \geq 0$ for $i, j \in \mathcal{N}$ induce an undirected and connected communication graph, i.e., $a_{ij} = a_{ji} > 0$ when the local controllers at buses i and j can communicate, otherwise $a_{ij} = a_{ji} = 0$. Observe that (7b) enforces control signals that in steady-state achieve identical marginal costs as in (6). Let $p_i := K_i s_i + R_i q_i$, then $u_i = -p_i$ and (7) reads as the distributed averaging-based PI (DAPI) controller [9]

$$\dot{p}_i = K_i(\omega_i - \omega^d) - R_i \sum\nolimits_{j\in\mathcal{N}} a_{ij}\left(A_i p_i - A_j p_j\right), i \in \mathcal{N}. \tag{8}$$

In order to obtain a compact closed-loop model representation, it is convenient to introduce the matrices

$$D_\mathcal{G} = \mathrm{diag}(D_i) \in \mathbb{R}^{g\times g},\ D_\mathcal{F} = \mathrm{diag}(D_i) \in \mathbb{R}^{f\times f},$$
$$M = \mathrm{diag}(M_i) \in \mathbb{R}^{g\times g},\ K = \mathrm{diag}(k_i) \in \mathbb{R}^{n\times n},$$
$$A = \mathrm{diag}(A_i) \in \mathbb{R}^{n\times n}$$

and the vectors

$$P^d_\mathcal{G} = \mathrm{col}(P^d_i) \in \mathbb{R}^g,\ P^d_\mathcal{F} = \mathrm{col}(P^d_i) \in \mathbb{R}^f,$$
$$p^d_\mathcal{P} = \mathrm{col}(P^d_i) \in \mathbb{R}^l,\ p = \mathrm{col}(p_i) \in \mathbb{R}^n.$$

Also, we introduce the *potential function* $U : \mathbb{R}^n \to \mathbb{R}$,

$$U(\theta) = -\frac{1}{2}\sum\nolimits_{\{i,j\}\in\mathcal{N}\times\mathcal{N}} |B_{ij}|V_i V_j \cos(\theta_{ij}).$$

Observe that due to symmetry of the power flows P_i,

$$\mathbb{1}_n^T \nabla_\theta U(\theta) = \sum\nolimits_{i=1}^n P_i = 0. \tag{9}$$

Combining (4) with (8), yields the overall closed-loop system

$$\dot{\theta} = \omega, \tag{10a}$$
$$M\dot{\omega}_\mathcal{G} = -D_\mathcal{G}(\omega_\mathcal{G} - \omega^d \mathbb{1}_g) + P^d_\mathcal{G} - \nabla_{\theta_\mathcal{G}} U(\theta) - p_\mathcal{G}, \tag{10b}$$
$$D_\mathcal{F}\dot{\theta}_\mathcal{F} = D_\mathcal{F}\omega^d \mathbb{1}_f + P^d_\mathcal{F} - \nabla_{\theta_\mathcal{F}} U(\theta) - p_\mathcal{F}, \tag{10c}$$
$$\mathbb{0}_p = P^d_\mathcal{P} - \nabla_{\theta_\mathcal{P}} U(\theta) - p_\mathcal{P}, \tag{10d}$$
$$\dot{p} = K(\omega - \mathbb{1}_n\omega^d) - R\mathcal{L}Ap, \tag{10e}$$

where $\mathcal{L} = \mathcal{L}^T \in \mathbb{R}^{n\times n}$ is the Laplacian matrix induced by the communication network with weights a_{ij}.

Remark 3.2: Many renewable CPCSs are fluctuating. Implementing the control law (8) on such a plant requires a certain margin in which this unit can adjust its active power injection. One way of doing this is to reserve a certain power margin (i.e., *derating*), when determining the setpoint P^d_i.□

Remark 3.3: The control law (8) requires knowledge of the frequencies $\omega_\mathcal{P}$, i.e., the time derivatives of $\theta_\mathcal{P}$. In practice, this information is typically available, as any CPCL or CPCS synchronizes its current to the network frequency, e.g., through a phase-locked loop device. For SGs or grid-forming inverters $\omega_{\mathcal{F}\cup\mathcal{G}}$ is directly measurable, respectively, an internal controller variable. □

Remark 3.4: In the present case, the variables $\theta_\mathcal{P}$ represent algebraic states in the model (10). We remark that, if Assumption 2.2 holds, it is possible to express the derivative of $\theta_\mathcal{P}$ via the implicit function theorem, see [16]. □

IV. Stability of the Closed-Loop System

In this section, we analyze stability of the closed loop (10).

A. Synchronized motion

For the analysis of the system (10), it is convenient to introduce the notion of a synchronized motion.

*Definition 4.1 (***Synchronized motion):** A solution $\mathrm{col}(\theta^*, \omega^*_\mathcal{G}, p^*) \in \mathbb{R}^n \times \mathbb{R}^g \times \mathbb{R}^n$ of the system (10) is a synchronized motion if $\omega^*_\mathcal{G}$ and p^* are constant vectors and

$$\theta^*(t) \in \Theta := \left\{\theta(t) \in \mathbb{R}^n \;\middle|\; |\theta_{ij}| < \frac{\pi}{2}, i \in \mathcal{N},\ j \in \mathcal{N}_i\right\},$$

for all $t \geq 0$ such that $\theta^*_{ij}(t)$ are constant for all $i \in \mathcal{N}, j \in \mathcal{N}$.

The name *synchronized motion* stems from the fact that constant phase differences $\theta^*_{ik}(t)$, for all $t \geq 0$ and $i, k \in \mathcal{N}$, in the power system model (10) readily imply synchronized frequencies, that is, $\dot{\theta}^*_i = \omega^*$, $\forall i \in \mathcal{N}$, for some $\omega^* \in \mathbb{R}$.

*Lemma 4.2 (***Synchronized motion):** The system (10) possesses at most one synchronized motion. Moreover, this synchronized motion satisfies

$$\omega^*_\mathcal{G} = \mathbb{1}_g\omega^d,\ \ p^* = cA^{-1}\mathbb{1}_n,\ \ c = \frac{1}{\mathbb{1}_n^T A^{-1}\mathbb{1}_n}\sum_{i\in\mathcal{N}} P^d_i, \tag{11}$$

where p^* is the unique minimizer of (5) in Problem 3.1.

Proof: From the fact that $\dot{\theta}^*_i = \omega^*$ for all $t \geq 0$, for all $i \in \mathcal{N}$, and for some $\omega^* \in \mathbb{R}$ together with (10e), we have

$$(\omega^* - \omega^d)K\mathbb{1}_n = R\mathcal{L}Ap^*. \tag{12}$$

Recall that $\mathbb{1}_n^T\mathcal{L} = 0$ and that K and R are diagonal matrices with positive diagonal entries. Hence, premultiplying both sides in (12) with $\mathbb{1}_n^T R^{-1}$ yields $(\omega^* - \omega^d)\mathbb{1}_n^T R^{-1}K\mathbb{1}_n = 0$ which implies $\omega^* = \omega^d$. Consequently, $p^* = \alpha A^{-1}\mathbb{1}_n$ for some $\alpha \in \mathbb{R}$. Thus, $u^* = -p^*$ achieves identical marginal costs (6) and is the unique minimizer (due to strict convexity) of (5); see [25]. Furthermore, we have from (10b)-(10d) that

$$-p^* + P^d = \nabla_{\theta^*} U^*, \tag{13}$$

which with (9) yields $\mathbb{1}_n^T p^* = \sum_{i\in\mathcal{N}} P^d_i$. From the fact that $p^* = \alpha A^{-1}\mathbb{1}_n$, we obtain $\alpha = c$ with c from (11). It follows from [26] that (13) has at most one solution $\theta^* \in \Theta$. ■

B. Stability

We analyze the stability of a synchronized motion of the closed-loop system (10) under the following parametric assumption on the DAPI controller gains.

Assumption 4.3: **(Controller gains)** The gains of the DAPI controller (8) satisfy $AK = R = T^{-1}$ with T being a diagonal matrix with positive diagonal elements. □

Observe that Assumption 4.3 couples the *frequency bias* and *averaging* gains in the controller (8). While this assumption removes a degree of freedom in tuning the controller (8), we feel that it is not particularly restrictive for the closed-loop performance. Simulations show that all of the following results also hold true without Assumption 4.3.

Error states & incremental variables: The left-hand side of the defining equation (13) is a vector with zero average but of arbitrary magnitude, while the right-hand side (the power flows) is bounded. Hence, a synchronized motion as in Lemma 4.2 may not exist. Therefore, we make the following natural power-balance assumption, see [8].

*Assumption 4.4 (***Existence of synchronized motion***):* The closed-loop system (10) possesses a synchronized motion $(\theta^*, \omega_{\mathcal{G}}^*, p^*) \in \mathbb{R}^n \times \mathbb{R}^g \times \mathbb{R}^n$. □

Under Assumption 4.4, we introduce the error states

$$\tilde{\omega}(t) := \dot{\theta}(t) - \omega^* = \omega(t) - \omega^* \in \mathbb{R}^n,$$
$$\tilde{\theta}(t) := \theta(0) + \int_0^t \tilde{\omega}(\tau)d\tau \in \mathbb{R}^n,\ \tilde{p}(t) = p(t) - p^* \in \mathbb{R}^n.$$

Furthermore, by noting that the power flows $\nabla_\theta U(\theta)$ only depend upon angle differences, we express all angles relative to a reference node. For the later analysis it is convenient to choose a reference node in $\mathcal{G}$, say node $n \in \mathcal{G}$, that is,

$$\phi := \mathcal{R}\tilde{\theta}, \quad \mathcal{R} := \begin{bmatrix} I_{(n-1)} & -\mathbb{1}_{(n-1)} \end{bmatrix}. \tag{14}$$

For ease of notation, define the constant $\phi_n := 0$, which is not part of the vector $\phi \in \mathbb{R}^{n-1}$. Then, equations (10) become

$$\dot{\phi} = \mathcal{R}\tilde{\omega}, \tag{15a}$$
$$\begin{bmatrix} \mathbb{0}_p \\ \mathbb{0}_f \\ M\dot{\tilde{\omega}}_{\mathcal{G}}, \end{bmatrix} = \begin{bmatrix} \mathbb{0}_p \\ -D_{\mathcal{F}}\tilde{\omega}_{\mathcal{F}} \\ -D_{\mathcal{G}}\tilde{\omega}_{\mathcal{G}} \end{bmatrix} - \mathcal{R}^T(\nabla_\phi U - \nabla_\phi U^*)) - \tilde{p}, \tag{15b}$$
$$T\dot{\tilde{p}} = A^{-1}\tilde{\omega} - \mathcal{L}A\tilde{p}, \tag{15c}$$

which is a DAE system of the form (1) with $x = \mathrm{col}(\phi_{\mathcal{F}}, \phi_{\mathcal{G}}, \tilde{\omega}_{\mathcal{G}}, \tilde{p})$, $y = \phi_{\mathcal{P}}$ and $z = \mathrm{col}(y, x) \in \mathbb{R}^{2n-1+g}$. Here, we have used the fact that with (14) it follows that

$$\nabla_\phi U(\tilde{\theta}(\phi)) = \left(\frac{\partial U(\tilde{\theta}(\phi))}{\partial \tilde{\theta}} \frac{\partial \tilde{\theta}(\phi)}{\partial \phi}\right)^T = \left(\frac{\partial U(\tilde{\theta}(\phi))}{\partial \tilde{\theta}} \begin{bmatrix} I_{(n-1)} \\ \mathbb{0}_{(n-1)}^T \end{bmatrix}\right)^T \tag{16}$$

and, hence,

$$\mathcal{R}^T \nabla_\phi U(\tilde{\theta}(\phi)) = \mathcal{R}^T \begin{bmatrix} I_{(n-1)} & \mathbb{0}_{(n-1)} \end{bmatrix} \nabla_{\tilde{\theta}} U(\tilde{\theta}) = \nabla_{\tilde{\theta}} U(\tilde{\theta}),$$

where the last equality is obtained from the fact that $\mathbb{1}_n^T \nabla_{\tilde{\theta}} U(\tilde{\theta}) = 0$. Finally, we have used the shorthand $\nabla_\phi U^*(\tilde{\theta}^*(\phi^*))$ as in (13). Observe that the system (15) possesses a unique equilibrium $z^* = (\phi^*, \tilde{\omega}_{\mathcal{G}}^*, \tilde{p}^*) = (\phi^*, \mathbb{0}_g, \mathbb{0}_n)$ with $\phi^* \in \mathcal{R}\Theta$ if and only if the system (10) possesses a synchronized motion. The latter claim follows since, given $\phi^* = \mathcal{R}\theta^* \in \mathcal{R}\Theta$, the corresponding value of $\theta^* \in \Theta$ can be uniquely recovered up to a uniform shift of all angles (modulo 2π). Furthermore, $\theta^* \in \Theta$ is unique by Lemma 4.2. Thus, Assumption 4.4 implies existence and uniqueness of the associated z^*. Likewise, AS of this z^* implies asymptotic convergence of trajectories of the system (10) to the unique synchronized motion up to a uniform shift of all angles.

Main result: The lemma below establishes regularity of the DAE (15) and is fundamental for our stability claim.

*Lemma 4.5 (***Regularity***):* Consider the system (15) with Assumption 4.4. Then Assumption 2.2 is satisfied locally near the equilibrium $z^* = (\phi^*, \tilde{\omega}_{\mathcal{G}}^*, \tilde{p}^*) = (\phi^*, \mathbb{0}_g, \mathbb{0}_n)$, with $\phi^* \in \mathcal{R}\Theta$, corresponding to a synchronized motion.

Proof: It is well known that Assumption 4.4 together with the assumed connectedness of the electrical network imply that the partial derivative [7]–[9], [25]

$$L(\theta(\phi))\big|_{\phi^*} = \frac{\partial(\nabla_\theta U(\theta))}{\partial \theta}\bigg|_{\phi^*}$$

is the Laplacian of an undirected and connected graph with weights $|B_{ij}|V_iV_j\cos(\theta_{ij}(\phi^*)) \geq 0$. Thus, the Jacobian

$$L_{\mathcal{P}}(\phi)\big|_{\phi^*} = \frac{\partial(\nabla_{\phi_{\mathcal{P}}} U(\phi))}{\partial \phi_{\mathcal{P}}}\bigg|_{\phi^*} = \frac{\partial(\nabla_{\theta_{\mathcal{P}}} U(\theta))}{\partial \theta_{\mathcal{P}}}\bigg|_{\phi^*}$$

is a principal minor of a Laplacian matrix of an undirected connected graph and, hence, nonsingular. By continuity of $L_{\mathcal{P}}$ in its argument ϕ we conclude that there exists an open connected set Ω on which $L_{\mathcal{P}}$ has constant full rank. Hence, Assumption 2.2 is satisfied, completing the proof. ■

Our main result of this section is as follows.

*Proposition 4.6 (***Stability of equilibria***):* Consider the system (15) with Assumption 4.4. Then the equilibrium point $z^* = (\phi^*, \tilde{\omega}_{\mathcal{G}}^*, \tilde{p}^*) = (\phi^*, \mathbb{0}_g, \mathbb{0}_n)$, with $\phi^* \in \mathcal{R}\Theta$, corresponding to a synchronized motion is locally AS.

Proof: The stability claim is established by invoking Theorem 2.5. To this end, recall that Lemma 4.5 implies that with Assumption 4.4 both Assumptions 2.1 and 2.2 are satisfied for the system (15). Consider an incremental Lyapunov function candidate inspired by [1], [15]–[17]

$$\mathcal{V}(\tilde{\omega}_{\mathcal{G}}, \phi, \tilde{p}) = \frac{1}{2}\tilde{\omega}_{\mathcal{G}}^T M \tilde{\omega}_{\mathcal{G}} + U(\tilde{\theta}(\phi)) - U(\tilde{\theta}^*(\phi^*)) - \nabla_\phi U(\tilde{\theta}^*(\phi^*))^T(\phi - \phi^*) + \frac{1}{2}\tilde{p}^T AT\tilde{p}. \tag{17}$$

Following Theorem 2.5, we start by showing that $\mathcal{V}$ is locally positive definite around z^*. It is easily verified that

$$\nabla \mathcal{V}|_{z^*} = \mathrm{col}\left(\nabla_\phi U - \nabla_\phi U^*,\ M\tilde{\omega}_{\mathcal{G}},\ AT\tilde{p}\right)\big|_{z^*} = \mathbb{0}_{(2n-1+g)}.$$

Hence, z^* is a critical point of $\mathcal{V}$. Furthermore, the Hessian of $\mathcal{V}$ evaluated at z^* is given by

$$\nabla^2 \mathcal{V}|_{z^*} = \mathrm{blkdiag}\left(\tilde{\mathcal{L}}, M, AT\right) \in \mathbb{R}^{(2n-1+g)\times(2n-1+g)},$$

where the matrix $\tilde{\mathcal{L}} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a principal minor of a Laplacian matrix (and thus positive definite [8, Lemma 5.8]) with elements $\tilde{l}_{ii} := \sum_{q=1}^n |B_{iq}|V_iV_q\cos(\theta_{iq}^*)$, $\tilde{l}_{ik} := -|B_{ik}|V_iV_k\cos(\theta_{ik}^*)$. Since AT and $\tilde{\mathcal{L}}$ are positive definite, the Hessian $\nabla^2\mathcal{V}|_{z^*}$ is positive definite. Consequently, z^* is a strict minimum of $\mathcal{V}$.

Next, we evaluate the derivative of the function $\mathcal{V}$ defined in (17) along trajectories of the system (15). This gives

$$\dot{\mathcal{V}} = \tilde{\omega}_{\mathcal{G}}^T M \dot{\tilde{\omega}}_{\mathcal{G}} + \left(\nabla_\phi U(\phi) - \nabla_\phi U(\phi^*)\right)^T \dot{\phi} + \tilde{p}^T AT\dot{\tilde{p}}. \tag{18}$$

Furthermore, an inspection of (15b) yields that

$$\mathcal{R}^T\left(\nabla_\phi U-\nabla_\phi U^*\right)=\begin{bmatrix}-\tilde{p}_{\mathcal{P}}\\ -D_{\mathcal{F}}\tilde{\omega}_{\mathcal{F}}-\tilde{p}_{\mathcal{F}}\\ -M\dot{\tilde{\omega}}_{\mathcal{G}}-D_{\mathcal{G}}\tilde{\omega}_{\mathcal{G}}-\tilde{p}_{\mathcal{G}}\end{bmatrix}. \quad (19)$$

Defining $\zeta := \operatorname{col}\left(\tilde{\omega}_{\mathcal{G}}, \nabla_{\phi_{\mathcal{F}}}U-\nabla_{\phi_{\mathcal{F}}}U^*, \tilde{p}\right) \in \mathbb{R}^{g+f+n}$, inserting (19) in (18) and using (15c), gives $\dot{\mathcal{V}}=\zeta^T\mathcal{Q}\zeta$, where the block entries of $\mathcal{Q}=\mathcal{Q}^T$ are given by

$$\begin{aligned}&\mathcal{Q}_{11}=-D_{\mathcal{G}},\quad \mathcal{Q}_{12}=\mathbb{0}_{g\times f},\quad \mathcal{Q}_{13}=\mathbb{0}_{g\times n},\\ &\mathcal{Q}_{22}=-D_{\mathcal{F}}^{-1},\quad \mathcal{Q}_{23}=-\begin{bmatrix}\mathbb{0}_{f\times p} & D_{\mathcal{F}}^{-1} & \mathbb{0}_{f\times g}\end{bmatrix},\\ &\mathcal{Q}_{33}=-A\mathcal{L}A-\operatorname{blkdiag}(\mathbb{0}_{p\times p}, D_{\mathcal{F}}^{-1}, \mathbb{0}_{g\times g}).\end{aligned}$$

To prove that $\dot{\mathcal{V}}\leq 0$, note that, as $D_{\mathcal{G}}>0$ and $D_{\mathcal{F}}>0$, $\mathcal{Q}_{11}<0$ and $\mathcal{Q}_{22}<0$. In addition, from the property that $v^T\mathcal{L}v>0$ for any nonzero $v\in\mathbb{R}^n\setminus\{\mathbb{1}_n\}$, it follows that $\mathcal{Q}_{33}<0$. Furthermore, we see that the quadratic submatrix of $\mathcal{Q}$ formed by $\mathcal{Q}_{22}$, $\mathcal{Q}_{23}$ and $\mathcal{Q}_{33}$ is negative semidefinite with a zero eigenvalue with geometric multiplicity one and a corresponding right-eigenvector $v_0 := \beta\operatorname{col}(-A_{\mathcal{F}}^{-1}\mathbb{1}_f, A^{-1}\mathbb{1}_n)$, $\beta\in\mathbb{R}$, where $A_{\mathcal{F}}$ denotes the (diagonal) submatrix of A corresponding to the nodes in the set $\mathcal{F}$. Hence, $\mathcal{Q}\leq 0$, which implies that $\dot{\mathcal{V}}\leq 0\quad\forall z\in\mathbb{R}^{2n-1+g}$ and by Theorem 2.5 z^* is a stable equilibrium point. In order to establish asymptotic stability, we observe that the above arguments also have the following implication

$$\dot{\mathcal{V}}\equiv 0\quad\Leftrightarrow\quad\zeta\equiv\operatorname{col}(\mathbb{0}_g, -\beta A_{\mathcal{F}}\mathbb{1}_f, \beta A^{-1}\mathbb{1}_n).$$

From (15c) we have that $\tilde{p}\equiv\beta A^{-1}\mathbb{1}_n$ implies that $\tilde{\omega}\equiv\mathbb{0}_n$. Hence, ϕ is constant. Thus, the invariant set $\dot{\mathcal{V}}(z(t))\equiv 0$ is an equilibrium set. Lemma 4.2 implies that the system (15) possesses at most one equilibrium with $\phi^*\in\mathcal{R}\Theta$, i.e., z^*, and z^* is an isolated minimum of $\mathcal{V}$, as shown before. Hence, there is a compact neighborhood of z^* where no other equilibrium exists and, by Theorem 2.5, z^* is AS. ■

The corollary below follows immediately by combining Lemma 4.2 with Proposition 4.6.

Corollary 4.7: Consider the closed-loop system (10) with Assumptions 4.3, 4.4. The controller (8) solves Problem 3.1.

Proof: Recall that a synchronized motion of (10) corresponds to the equilibrium $z^*=(\phi^*, \tilde{\omega}_{\mathcal{G}}^*, \tilde{p}^*)=(\phi^*, \mathbb{0}_g, \mathbb{0}_n)$ of (15). By Lemma 4.2, the solution z^* of the system (15) satisfies the optimality criteria in item 2) of Problem 3.1. By Lemma 4.2, $\omega^*=\omega^d\mathbb{1}_n$ and Proposition 4.6 guarantees that there exists an open neighborhood of z^*, such that all trajectories of the system (15) starting in this neighborhood converge asymptotically to z^*, which implies that $\lim_{t\to\infty}\|\omega^*-\omega^d\|=0$, i.e., item 1) of Problem 3.1. ■

V. Conclusions

We have shown that the DAPI control solves the problems of stability, frequency restoration and optimal dispatch in a structure-preserving DAE power system model and, at the same time, allows to actively integrate CPCSs and CPCLs in the network control tasks. The latter item is considered to be a fundamental cornerstone in Smart Grids. The stability claim has been derived based on previous results on stability of index-one DAEs as in power system models [16].

Future work will extend the presented analysis to power system models with variable voltage magnitudes and, possibly, distributed voltage/reactive power controls [27], [28]. Another interesting aspect is the effect of clock drifts on performance of the DAPI control, see [29]. In addition, we plan to improve the theoretical results by explicitly indicating a region in which the asymptotic stability and the equivalence of the DAE system (1) to an ODE system can be guaranteed.

References

[1] C. Zhao, E. Mallada, and F. Dörfler, "Distributed frequency control for stability and economic dispatch in power networks," in *American Control Conference*, 2015, pp. 2359–2364.

[2] J. A. Taylor, S. V. Dhople, and D. S. Callaway, "Power systems without fuel," *Renewable and Sustainable Energy Reviews*, vol. 57, pp. 1322–1336, 2016.

[3] Q.-C. Zhong and T. Hornik, *Control of Power Inverters in Renewable Energy and Smart Grid Integration*. Wiley-IEEE Press, 2013.

[4] H. Farhangi, "The path of the smart grid," *IEEE Power and Energy Magazine*, vol. 8, no. 1, pp. 18 –28, 2010.

[5] P. Kundur, *Power System Stability and Control*. McGraw, 1994.

[6] J. Schiffer, D. Zonetti, R. Ortega, A. Stankovic, T. Sezi, and J. Raisch, "Modeling of microgrids-from fundamental physics to phasors and voltage sources," *arXiv preprint arXiv:1505.00136*, 2015.

[7] J. Schiffer, "Stability and power sharing in microgrids," Ph.D. dissertation, Technische Universität Berlin, 2015.

[8] J. Schiffer, R. Ortega, A. Astolfi, J. Raisch, and T. Sezi, "Conditions for stability of droop-controlled inverter-based microgrids," *Automatica*, vol. 50, no. 10, pp. 2457–2469, 2014.

[9] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Synchronization and power sharing for droop-controlled inverters in islanded microgrids," *Automatica*, vol. 49, no. 9, pp. 2603–2611, 2013.

[10] H.-D. Chiang, C. C. Chu, and G. Cauley, "Direct stability analysis of electric power systems using energy functions: Theory, applications, and perspective," *Proceedings of the IEEE*, vol. 83, no. 11, pp. 1497–1529, 1995.

[11] R. Ortega, M. Galaz, A. Astolfi, Y. Sun, and T. Shen, "Transient stabilization of multimachine power systems with nontrivial transfer conductances," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 60–75, 2005.

[12] F. Dörfler and F. Bullo, "Kron reduction of graphs with applications to electrical networks," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 60, no. 1, pp. 150–163, 2013.

[13] A. R. Bergen and D. J. Hill, "A structure preserving model for power system stability analysis," *IEEE Transactions on Power Apparatus and Systems*, vol. 100, no. 1, pp. 25–35, 1981.

[14] R. J. Davy and I. A. Hiskens, "Lyapunov functions for multimachine power systems with dynamic loads," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 44, no. 9, pp. 796–812, 2002.

[15] D. J. Hill, H. I. A, and I. M. Mareels, "Stability theory of differential/algebraic models of power systems," *Sadhana*, vol. 18, no. 5, pp. 731–747, 1993.

[16] D. J. Hill and I. M. Mareels, "Stability theory for differential/algebraic systems with application to power systems," *IEEE Transactions on Circuits and Systems*, vol. 37, no. 11, pp. 1416–1423, 1990.

[17] N. Monshizadeh and C. De Persis, "Agreeing in networks: unmatched disturbances and algebraic constraints," *submitted*, 2015.

[18] N. Monshizadeh, C. De Persis, A. van der Schaft, and J. Scherpen, "A networked reduced model for electrical networks with constant power loads," in *American Control Conference*, 2016, submitted.

[19] R. Abraham, J. E. Marsden, and T. S. Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2nd ed., ser. Appl. Math. Sci. Springer, 1988, vol. 75.

[20] A. J. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*, 2nd ed. Springer, 1999.

[21] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[22] J. Machowski, J. W. Bialek, and J. R. Bumby, *Power System Dynamics*, 2nd ed. Wiley, 2008.

[23] A. J. Wood and B. F. Wollenberg, *Power Generation, Operation, and Control*, 2nd ed. John Wiley & Sons, 1996.

[24] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambr. Uni. Press, 2004.

[25] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, "Breaking the hierarchy: Distributed control & economic optimality in microgrids," *IEEE Trans. on Control of Network Systems*, vol. PP, no. 99, pp. 1–1, 2015.

[26] A. Araposthatis, S. Sastry, and P. Varaiya, "Analysis of power-flow equation," *Int. Jour. of Electr. Power & Energy Systems*, vol. 3, no. 3, pp. 115–126, 1981.

[27] J. Schiffer, T. Seel, J. Raisch, and T. Sezi, "A consensus-based distributed voltage control for reactive power sharing in microgrids," in *European Control Conference*, 2014, pp. 1299–1305.

[28] J. Simpson-Porco, Q. Shafiee, F. Dörfler, J. Vasquez, J. Guerrero, and F. Bullo, "Secondary frequency and voltage control of islanded microgrids via distributed averaging," *IEEE Trans. on Ind. Electr.*, vol. 62, no. 11, pp. 7025–7038, 2015.

[29] J. Schiffer, R. Ortega, C. A. Hans, and J. Raisch, "Droop-controlled inverter-based microgrids are robust to clock drifts," in *American Control Conference*, 2015, pp. 2341–2346.