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Variational Coupling of Wave Slamming against Elastic Masts

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1 Introduction

We present a novel approach to fluid-structure interactions (FSI) that preserves energy and phase-space structure owing to the variational and Hamiltonian techniques used. We posit a variational principle (VP), for nonlinear potential-flow wave dynamics coupled to a nonlinear hyperelastic mast, and derive its linearization. Both linear and nonlinear formulations can then be discretized in a classical-mechanical VP, using finite element expansions.

2 Nonlinear Variational Formulation

Potential flow water waves: We consider water as an incompressible fluid with density ρ . The vector velocity field $\mathbf{u} = \mathbf{u}(x, y, z, t)$ has zero divergence, $\nabla \cdot \mathbf{u} = 0$, with spatial coordinates $\mathbf{x} = (x, y, z)^T$, and time coordinate t . Gravity acts in the negative z -direction and the associated acceleration of gravity is g . The velocity is expressed in terms of a scalar velocity potential $\phi = \phi(x, y, z, t)$ such that $\mathbf{u} = \nabla \phi$. In a 3D domain $[0, L_x] \times [0, L_y] \times [0, h(x, y, t)]$ with solid walls at $x = 0$ and $x = L_x$, $y = 0$ and $y = L_y$, and the flat bottom $z = 0$, Luke's [4] VP for potential-flow water waves reads

$$\begin{aligned} 0 &= \delta \int_0^T \int_0^{L_x} \int_0^{L_y} \int_0^{h(x,y,t)} -\rho \partial_t \phi \, dz \, dx \, dy - \mathcal{H} \, dt \\ &\equiv \delta \int_0^T \int_0^{L_x} \int_0^{L_y} \int_0^{h(x,y,t)} -\rho \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(z - H_0) \right) \, dz \, dx \, dy \, dt \end{aligned} \quad (1)$$

for a fluid of constant depth and the single-valued free surface at $z = h(x, y, t)$. Here $h = h(x, y, t)$ is the water depth and H_0 the rest-state water level. The energy or Hamiltonian \mathcal{H} consists of the sum of kinetic and potential energies. We use integration by parts in time together with Gauss' law with outward normal $\hat{\mathbf{n}} = (-\nabla h, 1)^T / \sqrt{1 + |\nabla h|^2}$ at the free surface. The passive and constant air pressure is denoted by p_a . Then, variation of (1) yields

$$\begin{aligned} 0 &= \int_0^T \int_0^{L_x} \int_0^{L_y} \int_0^{h(x,y,t)} \rho \nabla^2 \phi \, \delta \phi \, dz \, dy \, dx - \int_{\partial \Omega_w} \rho \nabla \phi \cdot \hat{\mathbf{n}} \, \delta \phi \, dS \\ &\quad + \int_0^{L_x} \int_0^{L_y} \rho \left(-\partial_z \phi + \partial_x \phi \partial_x h + \partial_y \phi \partial_y h + \partial_t h \right) \Big|_{z=h} \delta \phi \Big|_{z=h} + (p - p_a) \Big|_{z=h} \delta h \, dy \, dx \, dt, \end{aligned} \quad (2)$$

in which the pressure $p - p_a$ here acts as a shorthand placeholder for the Bernoulli expression $-\rho \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(z - H_0) \right)$. The equations of motion emerge from the above relation (2), together with normal-flow boundary conditions $\nabla \phi \cdot \hat{\mathbf{n}} = 0$ with outward normal $\hat{\mathbf{n}}$ at solid walls $\partial \Omega_w$, see [4].

Geometrically nonlinear elastic mast. We consider a nonlinear hyperelastic model for an elastic material in which the geometric nonlinearity of the displacements is also taken into account. The constitutive law is such that, after linearization, it satisfies a linear Hooke's law. The choice of this model is guided by our goal to couple the potential-flow water-wave model to a weakly nonlinear elastic model.

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We first model the positions $\mathbf{X} = \mathbf{X}(a, b, c, t) = (X, Y, Z)^T = (X_1, X_2, X_3)^T$ of an infinitesimal 3D element of solid material as a function of Lagrangian coordinates $\mathbf{a} = (a, b, c)^T = (a_1, a_2, a_3)^T$ and time t . At time $t = 0$ we take $\tilde{\mathbf{X}}(\mathbf{a}, 0) = \mathbf{a}$. The displacements $\tilde{\mathbf{X}}$ follow from the positions as $\tilde{\mathbf{X}} = \mathbf{X} - \mathbf{a}$. The velocity of the displacements is $\partial_t \tilde{\mathbf{X}} = \mathbf{U} = (U, V, W)^T = (U_1, U_2, U_3)^T$, where the displacement velocity $\mathbf{U} = \mathbf{U}(\mathbf{a}, t)$ is again a function of Lagrangian coordinates \mathbf{a} and time t . The variational formulation of the elastic material is close to the variational formulation of a linear elastic solid obeying Hooke's law, but with one difference: the material is Lagrangian with finite, rather than infinitesimal, displacements. The variational formulation then consists of the kinetic and potential energies in the Lagrangian framework. In the linear case, $\mathbf{X} = \mathbf{a} + \tilde{\mathbf{X}}(\mathbf{a}, t) \approx \mathbf{a} + \tilde{\mathbf{X}}(\mathbf{x}, t)$ since we take $\mathbf{a} = \mathbf{x}$ for small $\tilde{\mathbf{X}}$. The VP for the hyperelastic model is then as follows [2]

$$0 = \delta \int_0^T \iiint_{\Omega_0} \rho_0 \mathbf{U} \cdot \partial_t \mathbf{X} - \frac{1}{2} \rho_0 |\mathbf{U}|^2 - \rho_0 g Z - \frac{1}{2} \lambda [\text{tr}(\mathbf{E})]^2 - \mu \text{tr}(\mathbf{E}^2) da db dc dt, \quad (3)$$

with $\rho_0 = \rho_0(\mathbf{a})$ and the Green-Lagrangian strain tensor $E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) = E_{ji}$ with $F_{ij} = \partial X_i / \partial a_j$. Evaluation of the variation in (3) yields

$$\begin{aligned} 0 = & \delta \int_0^T \iiint_{\Omega_0} \rho_0 (\partial_t \mathbf{X} - \mathbf{U}) \cdot \delta \mathbf{U} - \rho_0 \partial_t \mathbf{U} \cdot \delta \mathbf{X} - \rho_0 \delta l_3 \delta X_l \\ & + \partial_{a_i} (\lambda \text{tr}(\mathbf{E}) F_{li} + 2\mu E_{ki} F_{lk}) \delta X_l da db dc \\ & - \iint_{\partial \Omega_0} n_i (\lambda \text{tr}(\mathbf{E}) F_{li} + 2\mu E_{ki} F_{lk}) \delta X_l dS dt, \end{aligned} \quad (4)$$

in which we have used the temporal end-point conditions $\delta \mathbf{X}(0) = \delta \mathbf{X}(T) = 0$.

Given the arbitrariness of the respective variations, the resulting equations of motion become

$$\delta \mathbf{U} : \quad \partial_t \mathbf{X} = \mathbf{U} \quad \text{in } \Omega_0 \quad (5a)$$

$$\delta X_l : \quad \rho_0 \partial_t U_l = -\rho_0 g \delta_{3l} + \partial_{a_i} (\lambda \text{tr}(\mathbf{E}) F_{li} + 2\mu E_{ki} F_{lk}) = -\rho_0 g \delta_{3l} + \partial_{a_i} T_{li} \quad \text{in } \Omega_0 \quad (5b)$$

$$\delta X_l : \quad 0 = n_i (\lambda \text{tr}(\mathbf{E}) F_{li} + 2\mu E_{ki} F_{lk}) = n_i T_{li} \quad \text{on } \partial \Omega_0 \quad (5c)$$

with stress tensor $T_{li} = \lambda \text{tr}(\mathbf{E}) F_{li} + 2\mu E_{ki} F_{lk}$.

Linearized elastic dynamics. Given that $\mathbf{X} = \mathbf{a} + \tilde{\mathbf{X}}$, we find that [3]

$$\mathbf{E} = \frac{1}{2} \left(\left(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{a}} \right)^T + \left(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{a}} \right) \right) + \frac{1}{2} \left(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{a}} \right)^T \cdot \left(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{a}} \right). \quad (6)$$

The linearization entails that $\mathbf{a} = \mathbf{x}$ such that $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}(\mathbf{x}, t)$. To be precise, we define $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)^T = \mathbf{a} - \mathbf{x}$, Taylor expand around $\mathbf{a} = \mathbf{x}$ and use Taylor's remainder theorem to yield

$$\mathbf{X}(\mathbf{a}, t) = \mathbf{a} + \tilde{\mathbf{X}}(\mathbf{a}, t) = \mathbf{x} + \boldsymbol{\epsilon} + \tilde{\mathbf{X}}(\mathbf{x}, t) + \boldsymbol{\epsilon}^T \frac{\partial \tilde{\mathbf{X}}}{\partial (\mathbf{x} + \boldsymbol{\epsilon})} \Big|_{\mathbf{x} + \boldsymbol{\epsilon} = \boldsymbol{\zeta}} \quad (7)$$

for $|\mathbf{a}| \leq \boldsymbol{\zeta} \leq |\mathbf{x}|$. Hence, we find that

$$\frac{\partial \mathbf{X}(\mathbf{a}, t)}{\partial \mathbf{a}} = \mathbf{I} + \frac{\partial \tilde{\mathbf{X}}(\mathbf{a}, t)}{\partial \mathbf{a}} = \mathbf{I} + \frac{\partial \tilde{\mathbf{X}}(\mathbf{x}, t)}{\partial \mathbf{x}} + \mathcal{O}(\boldsymbol{\epsilon}) \quad \text{and} \quad \frac{\partial \tilde{\mathbf{X}}(\mathbf{a}, t)}{\partial \mathbf{a}} = \frac{\partial \tilde{\mathbf{X}}(\mathbf{x}, t)}{\partial \mathbf{x}} + \mathcal{O}(\boldsymbol{\epsilon}). \quad (8)$$

Consequently, the linearized version \mathbf{e} of \mathbf{E} is [3]

$$\mathbf{e} = \frac{1}{2} \left(\left(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{x}} \right)^T + \left(\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{x}} \right) \right) \quad \text{or} \quad e_{ij} = \frac{1}{2} \left(\frac{\partial X_j}{\partial x_i} + \frac{\partial X_i}{\partial x_j} \right). \quad (9)$$

Moreover, $\text{tr}(\mathbf{E})^2 = E_{ii}E_{jj} \approx e_{ii}e_{jj}$ and $\text{tr}(\mathbf{E} \cdot \mathbf{E}) = E_{ij}^2 \approx e_{ij}^2$, whence the standard VP for linear elastodynamics emerges:

$$0 = \delta \int_0^T \iiint_{\Omega_0} \rho_0 \mathbf{U} \cdot \partial_t \tilde{\mathbf{X}} - \frac{1}{2} \rho_0 |\mathbf{U}|^2 - \rho_0 g Z - \frac{1}{2} \lambda e_{ii}e_{jj} - \mu e_{ij}^2 dx dy dz dt. \quad (10)$$

In the limit of small displacements, the following approximations hold

$$\text{tr}(\mathbf{E}) F_{li} = E_{jj} F_{li} \approx e_{jj} \delta_{li}, \quad E_{ki} F_{lk} \approx e_{ik} \delta_{lk} = e_{il}. \quad (11)$$

Either by linearizing (5) or taking the variation of (10), the linearized equations of motion emerge.

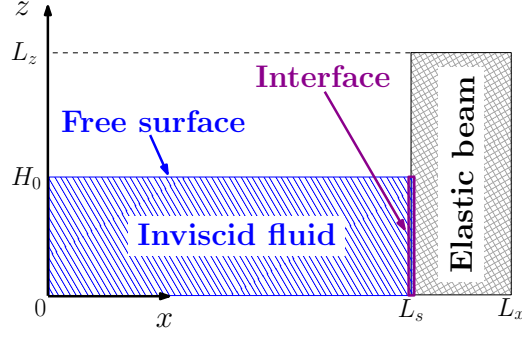


Figure 1: Geometry of the linearized or rest system: fluid (hatched) and elastic beam (cross-hatched).

2.1 Coupled Model

The domain occupied by the fluid is denoted by Ω and the domain occupied by the hyperelastic material by Ω_0 . For simplicity we consider a block of hyperelastic material. The interface between the fluid and solid domain is parameterized by $\mathbf{X}_s = \mathbf{X}(L_s, b, c, t)$ and, at rest, $\mathbf{X} = \mathbf{a}$ for Cartesian $a \in [L_s, L_x]$, $b \in [0, L_y]$, $c \in [0, L_z]$, while the fluid domain at rest is $x \in [0, L_s]$, $y \in [0, L_y]$, $z \in [0, H_0]$. The (outward-from-fluid) normal at this interface $\mathbf{X}(L_s, b, c, t)$ with $b \in [0, L_y]$, $c \in [0, L_z]$ is $\hat{\mathbf{n}} = \partial_b \mathbf{X} \times \partial_c \mathbf{X} / |\partial_b \mathbf{X} \times \partial_c \mathbf{X}|$. The nonlinear hyperelastic material is assumed to be stiff and nearly linear, such that at the interface $X \approx L_s$, $Y \approx b$ and $Z \approx c$, whence $\hat{\mathbf{n}} \approx (1, 0, 0)^T$. A sketch of this linearized domain or domain at rest is given in Fig. 1. This confirms that our expression is the outward normal to the fluid domain at the fluid-structure interface.

We assume that the elastic material is sufficiently stiff so that the fluid and elastic domains are

$$\Omega: z \in (0, h(x, y, t)), y \in (0, L_y), x \in (0, x_s(y, z, t)); \quad \Omega_0: a \in (L_s, L_x), b \in (0, L_y), c \in (0, L_z), \quad (12)$$

where we remark that this is an implicit description of the fluid domain, because the waterline height z at the fluid-beam interface is defined by $z = h(x_s(y, z, t), y, t)$. We therefore introduce a new horizontal coordinate $\chi = L_s x / x_s(y, z, t)$ such that $\Omega: \chi \in (0, L_s)$, $y \in (0, L_y)$, $z \in (0, h(\chi, y, t))$. Alternatively, we can introduce $x_s(y, z, t) \circ \mathbf{X}_s(L_s, b, c, t) = X_s(L_s, b, c, t)$ as an unknown and use a Lagrange multiplier $\gamma = \gamma(b, c, t)$ to equate $x_s(y = Y(L_s, b, c, t), z = Z(L_s, b, c, t))$ to $X(L_s, b, c, t)$.

As the coupled fluid-structure VP in $\{x, y, z, t\}$ -coordinates, we take the sum of the two VPs

$$\begin{aligned} 0 = & \delta \iiint_{\Omega} -\rho \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(z - H_0) \right) dz dx dy \\ & + \int_0^{L_y} \int_0^{L_z} \gamma \left(x_s(Y(L_s, b, c, t), Z(L_s, b, c, t), t) - X(L_s, b, c, t) \right) db dc \\ & + \iiint_{\Omega_0} \rho_0 \mathbf{U} \cdot \partial_t \mathbf{X} - \frac{1}{2} \rho_0 |\mathbf{U}|^2 - \rho_0 g Z - \frac{1}{2} \lambda [\text{tr}(\mathbf{E})]^2 - \mu \text{tr}(\mathbf{E}^2) da db dc dt. \end{aligned} \quad (13)$$

For non-breaking waves, a coordinate change then becomes suitable, from coordinates $\{x, y, z, t\}$ to $\{\chi = L_s \frac{x}{x_s(y, z, t)}, y, z, t\}$ and fluid domain Ω . In these new coordinates, using transformation formulae, (13) becomes

$$\begin{aligned} 0 = & \delta \int_0^T \int_0^{L_s} \int_0^{L_y} \int_0^{h(\chi, y, t)} -\rho \left(\frac{x_s}{L_s} \partial_t \phi - \frac{\chi}{L_s} \partial_t x_s \partial_{\chi} \phi \right. \\ & + \frac{1}{2} \frac{L_s}{x_s} (\partial_{\chi} \phi)^2 + \frac{1}{2} \frac{x_s}{L_s} (\partial_y \phi - \frac{\chi}{x_s} \partial_y x_s \partial_{\chi} \phi)^2 \\ & + \left. \frac{1}{2} \frac{x_s}{L_s} (\partial_z \phi - \frac{\chi}{x_s} \partial_z x_s \partial_{\chi} \phi)^2 + \frac{x_s}{L_s} g(z - H_0) \right) dz dy d\chi \\ & + \int_0^{L_y} \int_0^{L_z} \rho \gamma \left(x_s(Y(L_s, b, c, t), Z(L_s, b, c, t), t) - X(L_s, b, c, t) \right) db dc \\ & + \int_{L_s}^{L_x} \int_0^{L_y} \int_0^{L_z} \rho_0 \mathbf{U} \cdot \partial_t \mathbf{X} - \frac{1}{2} \rho_0 |\mathbf{U}|^2 - \rho_0 g Z - \frac{1}{2} \lambda [\text{tr}(\mathbf{E})]^2 - \mu \text{tr}(\mathbf{E}^2) da db dc dt. \end{aligned} \quad (14)$$

3 Linearized Wave-Beam Dynamics for FSI

We linearize (14) around a state of rest. Small-amplitude perturbations around this rest state are denoted by tilded variables and introduced as follows

$$x_s = L_s + \tilde{x}_s, \quad \phi = 0 + \phi, \quad h = H_0 + \eta, \quad \mathbf{X} = \mathbf{x} + \tilde{\mathbf{X}}, \quad \mathbf{U} = \mathbf{0} + \tilde{\mathbf{U}}. \quad (15)$$

After linearizing (14), we obtain the VP

$$\begin{aligned} 0 = & \delta \int_0^T \int_0^{L_y} \int_0^{H_0} \rho \phi_w \partial_t \tilde{x}_s \, dy \, dz - \int_0^{L_s} \int_0^{L_y} \int_0^{H_0} \rho \left(\frac{1}{2} (\partial_\chi \phi)^2 + \frac{1}{2} (\partial_y \phi)^2 + \frac{1}{2} (\partial_z \phi)^2 \right) \, dz \, dy \, d\chi \\ & + \int_0^{L_y} \int_0^{L_s} \rho \phi_s \partial_t \eta - \frac{1}{2} \rho g \eta^2 \, dy \, d\chi + \int_0^{L_y} \int_0^{H_0} \rho \gamma \left(\tilde{x}_s(y, z, t) - \tilde{X}(L_s, y, z, t) \right) \, dy \, dz \\ & + \int_{L_s}^{L_x} \int_0^{L_y} \int_0^{L_z} \rho_0 \tilde{\mathbf{U}} \cdot \partial_t \tilde{\mathbf{X}} - \frac{1}{2} \rho_0 |\tilde{\mathbf{U}}|^2 - \frac{1}{2} \lambda e_{ii} e_{jj} - \mu e_{ij}^2 \, dx \, dy \, dz \, dt, \end{aligned} \quad (16)$$

which, being the variation of a quadratic form, yields the dynamics linearized around a state of rest with $\phi_w = \phi(L_s, y, z, t)$ and $\phi_s = \phi(\chi, y, H_0, t)$. After using $\delta \tilde{\mathbf{X}}(\mathbf{x}, 0) = \delta \tilde{\mathbf{X}}(\mathbf{x}, T) = 0$ and $\delta \eta(\chi, y, 0) = \delta \eta(\chi, y, T) = 0$, the variation in (16) yields

$$\delta \phi_w : \quad \rho \partial_t \tilde{x}_s = \rho \partial_\chi \phi \quad \text{at } \chi = L_s, \quad \delta \tilde{x}_s : \quad \rho \partial_t \phi_w = \gamma \quad \text{at } \chi = L_s \quad (17a)$$

$$\delta \gamma : \quad \tilde{x}_s(y, z, t) = \tilde{X}(L_s, y, z, t), \quad \delta \tilde{X}_j(L_s, y, z, t) : \quad \gamma \delta_{1j} = T_{1j} \quad \text{at } x = L_s \quad (17b)$$

$$\delta \phi_s : \quad \partial_t \eta = \partial_z \phi \quad \text{at } z = H_0, \quad \delta \eta : \quad \partial_t \phi_s = -g \eta \quad \text{at } z = H_0 \quad (17c)$$

$$\delta \phi : \quad (\partial_{\chi\chi} + \partial_{yy} + \partial_{zz}) \phi = 0 \quad \text{in } \bar{\Omega} \quad (17d)$$

$$\delta \tilde{\mathbf{U}} : \quad \partial_t \tilde{\mathbf{X}} = \tilde{\mathbf{U}} \quad \text{in } \bar{\Omega}_0, \quad \delta \tilde{\mathbf{X}} : \quad \partial_t \tilde{\mathbf{U}}_j = \nabla_k T_{jk} \quad \text{in } \bar{\Omega}_0 \quad (17e)$$

with $\bar{\Omega}_0 : x \in [L_s, L_x], y \in [0, L_y], z \in [0, L_z]$ and $\bar{\Omega} : \chi \in [0, L_s], y \in [0, L_y], z \in [0, H_0]$. Note that we replaced Lagrangian coordinates (a, b, c) by (x, y, z) in the linearization but kept coordinates (χ, y, z) in the fluid domain.

4 Discussion and Conclusion

After elimination of the Lagrange multiplier γ , the system (17) of linearized water-wave dynamics coupled to an elastic beam, *i.e.*, a system of linearized fluid-structure interaction (FSI) equations, is equivalent to the FSI with *ad hoc* coupling derived in [5], in which the linear equations are discretized using dis/continuous variational finite element methods, employing techniques from [1], leading to fully coupled and stable FSI with overall energy conservation, *i.e.*, without any energy loss between the subsystems. We shall also present these results. The numerical extension of these FSI to the nonlinear realm is planned as future research.

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