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THE DECOMPOSITION MATRICES OF THE BRAUER ALGEBRA OVER THE COMPLEX FIELD

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ABSTRACT. The Brauer algebra was introduced by R. Brauer in 1937 as a tool in invariant theory. The problem of determining the Cartan decomposition matrix of the Brauer algebra over the complex field has remained open since then. Here we determine this fundamental invariant.

1. INTRODUCTION

For each field k, natural number n and parameter $\delta \in k$, the Brauer algebra $B_n(\delta)$ is a finite dimensional algebra, with a basis of pair partitions of the set $\{1, 2, ..., 2n\}$ [4]. Indeed there is a $\mathbb{Z}[\delta]$-algebra $B_n^{\mathbb{z}}$ (for δ indeterminate), free of finite rank as a $\mathbb{Z}[\delta]$-module, that passes to each Brauer algebra by the natural base change. Further, there is a collection of modules $\{\Delta^{\mathbb{z}}(\lambda)\}_{\lambda \in \Lambda^n}$ for $B_n^{\mathbb{z}}$ that are $\mathbb{Z}[\delta]$-free modules of known rank, so that

$$\Delta^k(\lambda) = k \otimes_{\mathbb{Z}[\delta]} \Delta^{\mathbb{z}}(\lambda)$$

are $B_n(\delta)$-modules; and there is a choice of field k extending $\mathbb{Z}[\delta]$ for which $\{\Delta^k(\lambda)\}_{\lambda \in \Lambda^n}$ is a complete set of simple modules. (The index set is $\Lambda^n = \Lambda_n \cup \Lambda_{n-2} \cup ... \cup \Lambda_{n_0}$ where Λ_n is the set of integer partitions of n, and $n_0 = 0$ for n even and $= 1$ for n odd [5].) Accordingly we are presented with the following tasks in studying the representation theory of $B_n(\delta)$:

(1) There are finitely many isomorphism classes of simple modules — index these.

(2) Describe the blocks (the reflexive-symmetric-transitive closure of the relation on the index set for simples given by $\lambda \sim \mu$ if simple modules $L(\lambda)$ and $L(\mu)$ are composition factors of the same indecomposable projective module).

(3) Describe the composition multiplicities of indecomposable projective modules (which follow from the composition multiplicities for the $\Delta^k(\lambda)$ (see for example [10, §16],[2, §1.9])).

Over the complex field, (1) was effectively solved in [5] (an index set is Λ^n, or $\Lambda^n \setminus \Lambda_0$ if $\delta = 0$ and n even), and (2) was solved in [7] (see references therein for other important contributions). Here we solve (3).

The layout of the paper is as follows. For each n, δ we wish to determine the Cartan decomposition matrix C given by $C_{\lambda\mu} = [P(\lambda) : L(\mu)]$, the composition multiplicity, where $\{P(\lambda)\}_{\lambda \in \Lambda^{n,\delta}}$ and $\{L(\lambda)\}_{\lambda \in \Lambda^{n,\delta}}$ are complete sets of indecomposable projective and simple modules respectively. We firstly recall some organisational results to this end. We construct the modules $\Delta(\lambda)$, such that projective modules are filtered by these, with well-defined composition multiplicities denoted

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$(P(\lambda) : \Delta(\mu))$; and that $C = DD^T$, where $D_{\lambda\mu} = (P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]$ (so D is what might be called the Δ-decomposition matrix). Then we construct an inverse limit for the sets $\{\Lambda^{n,\delta}\}_n$ and show that the Cartan decomposition matrices (and the Ds) for all n can be obtained by projection from a corresponding limit. Next we give an explicit matrix D for each δ (this construction takes up the majority of the paper, and uses the block result [8, 9]). And finally we prove, in Section 7, by an induction on n, that this D is the limit Δ-decomposition matrix.

It is probably helpful to note that the original route to the solution of the problem was slightly different. It proceeded from a conjecture, following [20, §1.2], that D would consist of evaluations of parabolic Kazhdan–Lusztig polynomials for a certain reflection group given in, and parabolic determined by, our joint work in [8]. This is essentially correct, as it turns out, and without this idea we would not have had a candidate for D, the form of which then drives the proof of the Theorem. However the proof does not, in the end, lie entirely within the realms of Kazhdan-Lusztig theory and alcove geometry. Accordingly we do not rely on this framework, but instead use a more general one within which the proof proceeds uniformly. We return to discuss our parabolic Kazhdan-Lusztig polynomial solution in a second part to the paper: section 8 and thereafter.

As the derivation of our main result is somewhat involved, we end here with a brief preview of the result itself. For each fixed $\delta \in \mathbb{Z}$ (the cases $\delta \notin \mathbb{Z}$ are semisimple [23]), the rows and columns of the limit Δ-decomposition matrix D may be indexed by Λ, the set of all integer partitions. This matrix may be decomposed, of course, as a direct sum of matrices for the limit blocks. In this sense we may describe the blocks by a partition of Λ. As we shall see, there is a map for each block to the set $P_{even}(\mathbb{N})$ of subsets of $\mathbb{N}$ of even degree. Under these maps all the block summands of D (and for all δ) are identified with the same matrix. Thus we require only to give a closed form for the entries of this matrix. The closed form is given in Section 5, but an indication of its structure is given by a truncation to a suitable finite rank. Such a truncation is given in Figure 7 (the entries in this matrix encode polynomials that will be used later, and which must be evaluated at 1 to give the decomposition numbers; the *blank* entries evaluate to zero, and all other entries evaluate to 1).

This paper is a contribution toward a larger project, with Cox and De Visscher, aiming to compute the decomposition matrices of the Brauer algebras over fields of finite characteristic. This is a very much harder problem again (it includes the representation theory of the symmetric groups over the same fields as a sub-datum — see [8]), and so it is appropriate to present the characteristic zero case separately.

2. Brauer diagrams and Brauer algebras

We mainly base our exposition on the notations and terminology of [7], as well as key results from that paper. For self-containedness, however, we review the notation here. Our hypotheses are slightly more general than in [7], however many of the proofs in [7] go through essentially unchanged (as we shall indicate, where appropriate). We shall also make use of a categorical formulation of the Brauer algebra (a subcategory of the partition algebra category of [19, §7]).

(**2.1**) For $n \in \mathbb{N}$ we write S_n for the symmetric group, and $\underline{n} := \{1, 2, .., n\}$ and $\underline{n}' := \{1', 2', .., n'\}$ (and so on). For S a set we write $P(S)$ for the power set and J_S for the set of pair-partitions of S. We define $J_{n,m} = J_{\underline{n} \cup \underline{m}'}$. For example, in

$J_{n,n}$ let us define

$$U_{ij} = \{\{1,1'\},\{2,2'\},...,\{i,j\},\{i',j'\},...,\{n,n'\}\};\tag{1}$$

$$(ij) = \{\{1,1'\},\{2,2'\},...,\{i,j'\},\{i',j\},...,\{n,n'\}\}.$$

(**2.2**) An (n,m)-Brauer diagram is a representation of a pair partition of a row of n and a row of m vertices, arranged on the top and bottom edges (respectively) of a rectangular frame. Each part is drawn as a line, joining the corresponding pair of vertices, in the rectangular interval. We identify two diagrams if they represent the same partition. Write $\mathbf{Br}(n,m)$ for the set of (n,m)-Brauer diagrams (up to this identification). It will be evident that these diagrams can be used to describe elements of $J_{n,m}$. (In what follows it is usually safe to simply identify a diagram with its partition. If we need to emphasise the formal distinction we may write $d \mapsto [d]$ for the map $\mathbf{Br}(n,m) \xrightarrow{\sim} J_{n,m}$.) For example,

 $\mapsto$ $U_{24} \in J_{6,6}$
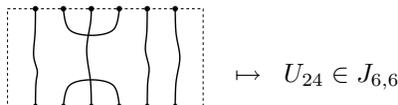
We then define a 'multiplication' $*$ as a composite map

$$J_{n,m} \times J_{m,l} \xrightarrow[\alpha]{} \mathbb{N}_0 \times J_{n,l} \xrightarrow[\beta]{} \mathbb{Z}[\delta]J_{n,l}$$

$$* = \beta \circ \alpha$$

as follows. Suppose d',d'' are diagrams representing the pair-partitions to be composed. Firstly vertically juxtapose the diagrams so that the two sets of m vertices meet (i.e. with d' over d''). This produces a diagram for an element d of $J_{n,l}$ (the pair partition of the vertices on the exterior of the combined frame); together with some number c of closed loops, which loops we discard. Thus we have a pair $(c,d) \in \mathbb{N}_0 \times J_{n,l}$. The final image is then $\delta^c d$ (i.e. we replace each closed loop formed in diagram composition by a factor δ).

For k a commutative ring and $\delta \in k$ we have a k-linear category $\mathbf{Br}_\delta$ with object set $\mathbb{N}_0$; and for each pair (n,m) of objects a hom set $kJ_{n,m}$ (or equivalently $k\mathbf{Br}(n,m)$); and composition k-linearly extending $*$. (Here we allow $k = \mathbb{Z}[\delta]$ or any suitable base change.)

(**2.3**) The Brauer algebra $B_n(\delta)$ over k is the free k-module with basis $\mathbf{Br}(n,n)$ and the category composition. We write simply B_n for $B_n(\delta)$ where no ambiguity arises.

(**2.4**) Write $\mathbf{Br}^{\leq l}(m,n)$ for the subset of $\mathbf{Br}(m,n)$ consisting of diagrams with $\leq l$ propagating lines (lines from top to bottom); $\mathbf{Br}^l(m,n)$ for the subset with l propagating lines; and $\mathbf{Br}^{\underline{l}}(m,n)$ for the subset of these in which no pair of the l propagating lines cross each other. Write 1_r for the identity diagram in $\mathbf{Br}(r,r)$. Note that $\mathbf{Br}^l(l,l)$ can be identified with the symmetric group S_l, so the category composition defines a bijection:

$$\mathbf{Br}^{\underline{l}}(m,l) \times \mathbf{Br}^l(l,l) \rightarrow \mathbf{Br}^l(m,l)\tag{2}$$

In particular $k\mathbf{Br}^l(m,l)$ is a free (right) kS_l-module of rank the number of elements in $\mathbf{Br}^{\underline{l}}(m,l)$.

(**2.5**) Define a product $\otimes : \mathbf{Br}(m,n) \times \mathbf{Br}(r,s) \rightarrow \mathbf{Br}(m+r, n+s)$ by placing diagrams side by side. For example the injection $i_{m+1,m+r} : \mathbf{Br}(m,n) \hookrightarrow \mathbf{Br}(m+r, n+r)$ defined by $d \mapsto d \otimes 1_r$ adds propagating lines $\{\{m+1, n+1'\}, ..., \{m+r, n+r'\}\}$.

(**2.6**) **Example.**   The sets $\mathbf{Br}(1,1)$, $\mathbf{Br}(2,0)$ and $\mathbf{Br}(0,2)$ each have a single element, here denoted 1_1, u and u' respectively. The map $k\mathbf{Br}(1,3) \rightarrow k\mathbf{Br}(3,3)$ defined by $d \mapsto u \otimes d$ is an injection. As a right B_3-module we have $U_{12}U_{23}B_3 \cong k\mathbf{Br}(1,3)$ (pictorially, the right action corresponds to acting with diagrams from $\mathbf{Br}(3,3)$ from *below*).

(**2.7**) *Remark*. A basic 'integral' version of the Brauer algebra is the case over the ring $k = \mathbb{Z}[\delta]$. Starting from this case, there are thus two aspects to the base change to a field: the choice of k and the choice of δ. More precisely this is the choice of k equipped with the structure of $\mathbb{Z}[\delta]$-algebra. Thus we have possible intermediate steps: base change to $k[\delta]$ (k a field); base change to $\mathbb{Z}$ (a $\mathbb{Z}[\delta]$-algebra by fixing $\delta = d \in \mathbb{Z}$). Each of these ground rings is a principal ideal domain and hence a Dedekind domain, and hence amenable to a P-modular treatment (see for example [10, §16],[2]).

3. Brauer-Specht modules

Here we construct the integral representations (in the sense of [2]) that we shall need. (These pass by base change to the *standard* modules of [7].)

(**3.1**) For any commutative ring k and $\delta \in k$, we have, as an elementary consequence of the composition rule, a sequence of $B_n(\delta)$-bimodules:

$$k\mathbf{Br}(n,n) = k\mathbf{Br}^{\leq n}(n,n) \supset k\mathbf{Br}^{\leq n-2}(n,n) \supset k\mathbf{Br}^{\leq n-4}(n,n) \supset ... \supset k\mathbf{Br}^{n_0}(n,n) \tag{3}$$

($n_0 = 0, 1$ for n even, odd respectively). Note that the i-th section of the sequence (3) has basis $\mathbf{Br}^{n-2i}(n,n)$. For $n - 2i = l$ we write $k\mathbf{Br}^l(n,n)$ for this section. We have

$$k\mathbf{Br}^l(n,n) \cong \bigoplus_{w \in \mathbf{Br}^{\underline{l}}(l,n)} k\mathbf{Br}^l(n,l) \, w \tag{4}$$

as a left B_n-module; where all the summands are isomorphic to $k\mathbf{Br}^l(n,l)$ (a left B_n-module similarly, via the category composition, quotienting $k\mathbf{Br}(n,l)$ by $k\mathbf{Br}^{\leq l-2}(n,l)$).

Fixing a ring k, it will be evident that $\mathbf{Br}^l(m,l)$ is a basis for a free right kS_l-module by (2), and hence for a left-$B_m(\delta)$ right-kS_l bimodule, so long as $m \geq l$ and $m - l$ even.

(**3.2**) **Proposition**. *Fix a commutative ring k and $\delta \in k$. The free k-module $k\mathbf{Br}^l(m,l)$ (which is a left $B_m(\delta)$-module by the action in (3.1)) is a projective right kS_l-module. Hence the functor*

$$k\mathbf{Br}^l(m,l) \otimes_{kS_l} - \; : \; kS_l - mod \; \rightarrow \; B_m(\delta) - mod$$

between the categories of left-modules is exact.

Proof. As noted, $k\mathbf{Br}^l(m,l)$ is a direct sum of copies of the regular right kS_l-module. $\square$

(**3.3**) Let $\Lambda_n = \{\lambda \vdash n\}$, the set of integer partitions of n. Let Λ be the set of all integer partitions; and define

$$\Lambda^n = \Lambda_n \cup \Lambda_{n-2} \cup \ldots \cup \Lambda_{n_0}, \qquad \text{and} \qquad \Lambda^{n,0} = \Lambda^n \setminus \Lambda_0$$

For $\lambda \vdash l$ let $\mathcal{S}(\lambda)$ denote the corresponding kS_l-Specht module (see e.g. [17]). For $m \geq l$ define

$$\Delta_m(\lambda) = k\mathbf{Br}^l(m, l) \otimes_{kS_l} \mathcal{S}(\lambda)$$

as the image of this Specht module under the functor in (3.2). Varying l, we have a set $\{\Delta_m(\lambda)\}_{\lambda \in \Lambda^m}$.

(**3.4**) **Proposition**. *Fix n and suppose k is such that the left regular module $_{kS_l}kS_l$ is filtered by $\{\mathcal{S}(\lambda)\}_{\lambda \in \Lambda_l}$ for all $l \leq n$. Then the left regular module $_{B_n}B_n$ is filtered by $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$. In particular Brauer algebra projective modules over $\mathbb{C}$ (any δ) are filtered by $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$.*

Proof. Note first that if a module M is filtered by a set $\{N_i\}_i$, and these are all filtered by a set $\{N'_j\}_j$, then M is filtered by $\{N'_j\}_j$. By (3.1) the set $\{k\mathbf{Br}^l(n, l)\}_l$ gives (via the action therein) a left-B_n filtration of B_n. By Prop. 3.2 each factor itself has a filtration by Δs under the stated condition. For the last part, simply note that $\mathbb{C}S_l$ is semisimple, the modules $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ are indecomposable over $\mathbb{C}$ (for any δ — see e.g. [7], or cf. Prop.3.10 and [15, §6.2]), and each indecomposable projective P (say) a direct summand of $_{B_n}B_n$. $\square$

(**3.5**) **Proposition**. [7, Lemma 2.4] *Let $b(\lambda)$ be a basis for $\mathcal{S}(\lambda)$. Then a basis for $\Delta_m(\lambda)$ is*

$$b_{\Delta_m(\lambda)} = \{a \otimes b \; : \; (a, b) \in \mathbf{Br}^{\underline{L}}(m, l) \times b(\lambda)\}.$$

Proof. This is a set of generators by (2); and it passes to a basis (of the image) under the surjective multiplication map (using from [17] that $\mathcal{S}(\lambda)$ is a left ideal), so it is k-free. $\square$

(**3.6**) The following low rank cases form the bases for inductions later on. We have $B_0(\delta) \cong B_1(\delta) \cong k$. For $B_2(\delta)$ we have $\Delta_2(\emptyset), \Delta_2(2), \Delta_2(1^2)$, each of rank 1. These are inequivalent over $\mathbb{C}$ except when $\delta = 0$, where we have $\Delta_2(2) \xrightarrow{\sim} \Delta_2(\emptyset)$. Thus we may regard $\Delta_2(2), \Delta_2(1^2)$ as the inequivalent simple $B_2(0)$-modules, and $P_2(2)$ is the self-extension of $\Delta_2(2)$, while $P_2(1^2) = \Delta_2(1^2)$.

3.1. **Globalisation functors.** Here we define certain functors that will allow us, in Section 3.2, to manipulate composition muliplicity data for all n simultaneously (cf. [19, §4], [15, §6]).

(**3.7**) For $n + m$ even the k-module $k\mathbf{Br}(n, m)$ is an algebra bimodule. Thus there is a functor between left-module categories

$$k\mathbf{Br}(n, m) \otimes_{B_m} - \; : \; B_m - \text{mod} \; \rightarrow \; B_n - \text{mod}$$

Let us write F for the functor $k\mathbf{Br}(n-2, n) \otimes_{B_n} -$; and G for the functor $k\mathbf{Br}(n, n-2) \otimes_{B_{n-2}} -$ for any n (if $\delta \in k$ a non-unit we shall exclude the case $n = 2$ from this notation — cf. Prop.3.8).

(**3.8**) **Proposition**. *Suppose either $n > 2$ or δ invertible in k. Then*
(I) the free k-module $k\mathbf{Br}(n - 2, n)$ is projective as a right B_n-module; and indeed

$$k\mathbf{Br}(n - 2, n) \cong \; e \, (k\mathbf{Br}(n, n))$$

as a right B_n-module, for a suitable idempotent $e \in k\mathbf{Br}(n,n)$ (for $n > 2$ we may use $e = U_{12}U_{23}$; for $n = 2$ use $e = \delta^{-1}U_{12}$).
(II) Functor $F : B_n-mod \to B_{n-2}-mod$ is exact; G is a right-exact right-inverse to F.

Proof. (I) Note that $U_{12}U_{23}$ is idempotent, so $k\mathbf{Br}(1,3)$ is projective by (2.6). This argument generalises without difficulty.   (II) follows immediately (see e.g. [1]). □

(**3.9**) The first section in (3) obeys the algebra isomorphism

$$k\mathbf{Br}^{\leq n}(n,n)/k\mathbf{Br}^{\leq n-2}(n,n) \;\cong\; kS_n$$

Thus each kS_n-module induces an identical (as k-module) B_n-module, where the action of any diagram with fewer than n propagating lines is by 0. In particular, for $\lambda \vdash n$, $\mathcal{S}(\lambda) = \Delta_n(\lambda)$.

(**3.10**) **Proposition**. *Let $\lambda \in \Lambda$. Set $l = |\lambda|$ and regard $\mathcal{S}(\lambda)$ as a B_l-module as in (3.9). For $m \in \mathbb{N}$*

$$\Delta_{2m+l}(\lambda) \cong G^{\circ m}\mathcal{S}(\lambda) \;=\; G^{\circ m}\Delta_l(\lambda)$$

unless $\delta \in k$ a non-unit, and $\lambda = \emptyset$, in which case $\Delta_{2m+4}(\emptyset) \cong G^{\circ m}\Delta_4(\emptyset)$.

Proof. Via Proposition 3.5 and the various definitions (cf. [7]). (The special case arises in the isomorphism of $k\mathbf{Br}(4,2) \otimes_{B_2} k\mathbf{Br}(2,0)$ with $k\mathbf{Br}(4,0)$, which holds only if δ has an inverse.) □

(**3.11**) *Remark*. If k is a field then in particular (unless $n = 2$ and $\delta = 0$) the category $B_{n-2}-mod$ fully embeds in B_n-mod under G, and this embedding takes $\Delta_{n-2}(\lambda)$ to $\Delta_n(\lambda)$. The embedding allows us to consider a formal limit module category (we take n odd and even together), from which all $B_n - mod$ may be studied by 'localisation' (action of the functor F).

(**3.12**) **Proposition**. *The set $\{L_n(\lambda) := head\,(\Delta_n(\lambda)) \mid \lambda \vdash n, n-2, ..., n_0\}$ is a complete set of simple modules, up to isomorphism, for $B_n(\delta)$ over $\mathbb{C}$ for any $\delta \in \mathbb{C}$. These modules are pairwise nonisomorphic; provided, in case $\delta = 0$, that $\lambda = \emptyset$ is excluded.*

Proof. By Prop. 3.4 this set includes the heads of all indecomposable projectives, and hence all simples. For $\delta = 0$ one can show directly that $G^{\circ(m-1)}\Delta_2(2)$ (with simple head) maps surjectively onto $\Delta_{2m}(\emptyset)$. For the remaining cases the arguments in [7], or [15, §6.2], may be used. □
    It follows that every composition factor below the head of $\Delta_n(\lambda)$ comes from $\Delta_n(\mu)$ with $|\mu| > |\lambda|$. Thus the set $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ (or, in case $\delta = 0$, the subset $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^{n,0}}$) is a basis for the Grothendieck group. Note also, e.g. from [5], that if k is a field extending $\mathbb{Z}[\delta]$ then $B_n(\delta)$ is semisimple, so in this case the Δ-modules are a complete set of simples.

(**3.13**) If we wish to emphasise δ notationally we may write $\Delta_m^\delta(\lambda)$ for $\Delta_m(\lambda)$. On the other hand, where unambiguous we may just write $\Delta(\lambda)$. Also define $\bar{\Delta}_m(\lambda) = \Delta_m(\lambda^T)$ (transposing λ). We shall adopt analogous conventions for the simple modules $\bar{L}_n(\lambda)$ and corresponding indecomposable projectives $\bar{P}_n(\lambda)$.

(**3.14**) **Proposition**. [7, Lemma 2.6,Prop.2.7] *Let $Ind-$ and $Res-$ denote the induction and restriction functors associated to the injection $B_n(\delta) \hookrightarrow B_{n+1}(\delta)$.*
(i) We may identify the functors $Res\ G-$ and $Ind-$ (each from $B_n - mod\ \to$

$B_{n+1} - mod$).

(ii) Over the complex field we have, for each $\lambda \in \Lambda^n$, a short exact sequence

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{n+1}(\mu) \to Ind\, \Delta_n(\lambda) \to \bigoplus_{\mu \rhd \lambda} \Delta_{n+1}(\mu) \to 0$$

(recall $\mu \lhd \lambda$ if μ is obtained from λ by removing one box from the Young diagram).

Proof. (i) Unpack the definitions. (ii) Note from (i) and Prop. 3.10 that it is enough to prove the equivalent result for restriction. Use the Brauer diagram notation. Consider B_n acting on the first n strings. We may separate the diagrams out into those for which the $n + 1$-th string is propagating (which span a submodule, since action on the first n strings cannot change this property), and those for which it is not. The result follows by comparing with diagrams from the indicated terms in the sequence, using the induction and restriction rules for Specht modules. (See also [13].) □

3.2. Characters and Δ-filtration factors.
For M a module, the shorthand $M \simeq A_1//A_2//...$ (or $M \simeq //_j A_j$) means that M has a chain of submodules with sections $A_1, A_2, ...$ (up to isomorphism).

(**3.15**) Over the complex field the modules $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ have pairwise distinct characters except precisely in the case $n = 2, \delta = 0$ in (3.6). If $\delta \neq 0$ there is a unique expression for any character in terms of Δ-characters (see (3.12)). This means that the Δ-filtration multiplicities for a Δ-filtered module P, denoted $(P : \Delta_n(\lambda))$, are also uniquely defined (and, for P projective, coincide with the appropriate 'lifted' decomposition numbers [2, §1.9]).

For the case $\delta = 0$, when $n = 2$ the noted isomorphism means that these multiplicities are not uniquely defined. For all other n, however, the multiplicities of the $\Delta_n(\lambda)$'s with $\lambda \neq \emptyset$ are defined as before (consider the quotient algebra $B_n/k\mathbf{Br}^0(n, n)$ for example) and then the distinct character property of Δs precludes any remaining ambiguity. In particular, the sectioning of projectives in the block of $\Delta_n(\emptyset)$ up to $\lambda \vdash 4$ is indicated by

$$P_4(2) \simeq \Delta_4(2)//\Delta_4(\emptyset) \qquad P_4(31) \simeq \Delta_4(31)//\Delta_4(2)$$

(this is an easy direct calculation). In this sense we may treat $\delta = 0$ as a degeneration of the more general case, and treat the multiplicities $(P : \Delta_n(\lambda))$ as uniquely defined throughout. We do this hereafter.

(**3.16**) Recall from Proposition 3.10 that $G\Delta_n(\lambda) = \Delta_{n+2}(\lambda)$. By Prop.3.10 and 3.12 the character of any B_n-module over $\mathbb{C}$ can be expressed in the form

$$\chi(M) = \sum_\lambda \alpha_\lambda(M)\, \chi(\Delta(\lambda)) \qquad (\alpha_\lambda(M) \in \mathbb{Z})$$

If in addition a module M has a Δ-filtration then this is a non-negative combination and (with the caveat mentioned in (3.15)) we have from (3.8) (cf. [12, Appendix], say) that

$$(GM : \Delta_{n+2}(\lambda)) = \begin{cases} (M : \Delta_n(\lambda)) & |\lambda| < n + 2 \\ 0 & |\lambda| = n + 2 \end{cases}$$

The functor G evidently takes projectives to projectives. It also preserves indecomposability, so

$$GP_n(\lambda) = P_{n+2}(\lambda) \tag{5}$$

Combining these results we see that the multiplicities $(P(\lambda) : \Delta(\mu))$ depend on n only through the range of possible values of λ. Thus for each δ (here with $k = \mathbb{C}$) there is a semiinfinite matrix D with rows and columns indexed by Λ such that

$$(P(\lambda) : \Delta(\mu)) = D_{\lambda,\mu}$$

for any n. Note from (3.1) that $(P(\lambda) : \Delta(\lambda)) = 1$ and otherwise

$$(P(\lambda) : \Delta(\mu)) = 0 \text{ if } |\mu| \geq |\lambda| \tag{6}$$

In our case the 'standard' decomposition matrix D also determines the Cartan decomposition matrix C (see e.g. [2, §1.9]). That is $D_{\lambda,\mu} = (P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]$, so that $C = DD^T$. In particular there is an inverse limit of blocks that is a partition of Λ.

Equation(6) says that the matrix D is lower unitriangularisable. From this we have

(**3.17**) **Proposition**. *If P is a projective module containing $\Delta(\lambda)$ with multiplicity m and no $\Delta(\mu)$ with $|\mu| > |\lambda|$, then P contains $P(\lambda)$ as a direct summand with multiplicity m.* □

The induction functor takes projective modules to projective modules, and has a behaviour with regard to standard characters determined by Prop. (3.14). From this we see that

(**3.18**) **Proposition**. *For e_i a removable box of the Young diagram of λ,*

$$Ind \, P(\lambda - e_i) \cong P(\lambda) \bigoplus Q$$

where $Q = \oplus_\mu P(\mu)$ a possibly empty sum with no $\mu \geq \lambda$.

Proof: By Prop.3.17 a projective module is a sum of indecomposable projectives including all those with labels maximal in the dominance order of its standard factors. Now use (3.14). □

(**3.19**) *Remark*. From the definitions we have

$$F\Delta_n(\lambda) = \left\{ \begin{array}{ll} \Delta_{n-2}(\lambda) & |\lambda| < n \\ 0 & |\lambda| = n. \end{array} \right.$$

4. Blocks

We now assemble the results we shall need on the blocks of the Brauer algebras. These include important results from [7], [8], [9] and extensions thereof. The Young diagram inclusion partial order $(\Lambda, \subset)$ restricts to a partial order on each block (any such construction evidently survives the inverse limit). By construction this order has a transitive reduction, that is, a directed graph that describes the limit of Hasse diagrams. This graph is key to our main result, and we describe it here. For example we endow the implicit definition of graph edges given above (and in [7]) with an explicit contruction that we shall need.

FIGURE 1. (a) Possible π-rotation points. (b) A pair of rims, γ, γ' say, such that $\pi_x(\gamma) = \gamma'$. Here x is the dot shown, on the charge-0 diagonal in case $\delta = 5$.

FIGURE 2.   Examples of δ-pairs ($\delta = 1$ in cases (i) and (ii)).

4.1. **The δ-balance condition.** Recall that the *content $c(b)$* of a box b in a Young diagram is $c(b) =$ column position - row position. Block membership depends on the relative content of the labelling Young diagrams (see [7], [13]). We shall need to cast the content condition for blocks in various forms.

(**4.1**) The δ-*charge* of a box in a Young diagram is $chg(b) := \delta - 1 - 2c(b)$ (cf. the 'conjugate' function $ch(b) = \delta - 1 + 2c(b)$ used in [7, §1]).

   Let $\bar{L}(\mu)$, $\bar{L}(\lambda)$ be simple modules of $B_n(\delta)$ over $\mathbb{C}$ for given $\delta \in \mathbb{Z}$ (N.B. (3.13)). We write $\bar{L}(\mu) \sim^\delta \bar{L}(\lambda)$ (or $\mu \sim^\delta \lambda$) if they are in the same block. A pair $\mu \subset \lambda$ gives modules *in the same block* only if the skew diagram λ/μ consists of $\pm$charge pairs of boxes [8]. (We give a precise statement shortly.) For example, with $\delta = 2$ the skew $(2^2)/(2)$ contains $3, 1$, so $\bar{L}((2^2)) \not\sim^{\delta=2} \bar{L}((2))$.

(**4.2**) A *rim* is a connected skew Young diagram with no subset of shape (2^2) [17]. Now fix δ. Two rims are δ-*opposite* if there is a point x on the δ-charge=0 diagonal and a rotation by π radians of the plane about x (hereafter called a π-*rotation* and denoted π_x) that takes one into the other. See Figure 1(b) for an example (the charge-0 diagonal here corresponds to $\delta = 5$).   Note that such a rotation exchanges boxes in $\pm$charge pairs.

(**4.3**) A δ-*pair* is a skew that is a δ-opposite pair of rims such that no row of the skew is fixed by the associated π-rotation. There are several examples of δ-pairs shown in Figure 2.

(**4.4**) Define a relation $(\Lambda, \leftarrow^\delta)$ by $\mu \leftarrow^\delta \lambda$ if λ/μ is a δ-pair. Define $(\Lambda, <^\delta)$ as the partial order that is the transitive closure of this relation.

(**4.5**) **Proposition**. *(I) If $\mu \subset \lambda$ and λ/μ a δ-pair, then there is no $\mu \subset \mu' \subset \lambda$ such that μ'/μ is a δ-pair. (II) The relation $(\Lambda, \leftarrow^\delta)$ is the cover (transitive reduction) of the partial order $(\Lambda, <^\delta)$.*

Proof. (I): Let π_0 be the rotation fixing λ/μ and suppose (for a contradiction) that some π_γ fixes $\gamma = \mu'/\mu \subset \lambda/\mu$. The positive charge part of λ/μ is connected, so there exists box $b' \in \lambda/\mu'$ adjacent to $b \in \gamma$. Thus $\pi_0(b')$ lies in λ/μ adjacent to $\pi_0(b)$. Define a 'light-cone' partial order on the set of boxes occuring in Young diagrams by box $a' > a$ if a' lies below *and* to the right of the top-left-hand corner of a (and $a' \neq a$). Since λ/μ' is a skew over μ', we have $b' \nleq b$ and hence (since adjacent) $b' > b$. Thus after rotation $\pi_0(b) > \pi_0(b')$.

Suppose for a moment that $\pi_0 = \pi_\gamma$ (i.e. they are rotations about the same point). Then $\pi_0(b') < \pi_\gamma(b)$, contradicting that γ is a skew over μ. Thus $\pi_0 \neq \pi_\gamma$. Now, since $\pi_0 \neq \pi_\gamma$, π_0 fixes no pair $b, \pi_\gamma(b)$ in γ. Thus for example no charge appears more than once in γ, while all the charges appearing in γ appear twice in λ/μ. Thus λ/μ is connected and all the lesser magnitude charges also appear twice. Note that the rotation point of π_0 is necessarily half a box down and to the right of π_γ. It then follows (consider Figure 1(a) say) that the two parts, call them γ_+ and γ_-, are disconnected from each other. Let c be the lowest charge box in γ_+. The box $\pi_0(\pi_\gamma(c))$ is below and to the right of it. Thus there is a box of λ/μ to its immediate right. There cannot be a box of λ/μ above it (since γ is a skew over μ) so there is a box of λ/μ to the right of $\pi_0(\pi_\gamma(c))$. But the π_0 image of *this* is to the left of $\pi_\gamma(c) \in \gamma$, contradicting the γ skew over μ property. Done.

Claim (II) follows from (I) since $\mu \subset \lambda$ is a necessary condition for $\mu <^\delta \lambda$ so any failure of $(\Lambda, \leftarrow^\delta)$ to be a transitive reduction implies the existence of a μ' contradicting (I). $\square$

(**4.6**) **Theorem**. *Fix δ. If λ/μ is a δ-pair then $Hom(\Delta_n^\delta(\lambda^T), \Delta_n^\delta(\mu^T)) \neq 0$.*

Proof. Noting the formulation in [7, Theorem 6.5], it is enough to show that $(\Lambda, \leftarrow^\delta)$ gives the cover of the restriction of $(\Lambda, \subset)$ to each block. This follows routinely from (4.5). $\square$

Write $\Lambda^{\sim\delta}$ for the reflexive-symmetric-transitive closure of the partial order $(\Lambda, <^\delta)$. Write $[\lambda]_\delta$ for the $\Lambda^{\sim\delta}$-class of $\lambda \in \Lambda$. A pair λ, μ are δ-*balanced* if they are in the same class.

(**4.7**) **Proposition**. [7, Corollary 6.7] *The relation $\Lambda^{\sim\delta}$ gives the (transposed) block relation for $B_n(\delta)$ over the complex field.* $\square$

(**4.8**) For any n, we write $\mathrm{Proj}_\lambda-$ for the projection functor on the category $B_n(\delta)-$mod onto the block associated to the class $[\lambda]_\delta$ (i.e. the block containing $\Delta_n^\delta(\lambda^T)$). Write $\mathrm{Ind}_\lambda-$ for $\mathrm{Proj}_\lambda\mathrm{Ind}-$.

4.2. **The block graph.** Let $G_\delta(\lambda)$ be the λ-connected component of $(\Lambda, \leftarrow^\delta)$. This may thus be thought of as a directed acyclic graph (with edge $\mu \to \lambda$ if $\mu \leftarrow^\delta \lambda$).

We call this the *block graph*. The structure of $G_\delta(\lambda)$ will be crucial for the statement and proof of the main Theorem. We can describe it as follows.

(**4.9**) Firstly we embed the blocks for $\delta \in \mathbb{R}$ (and in particular for $\delta \in \mathbb{Z}$) in $\mathbb{R}^\mathbb{N}$. For $\delta \in \mathbb{R}$ define

$$\rho_\delta \ = \ -\frac{\delta}{2}(1, 1, ...) - (0, 1, 2, ...) \ \in \mathbb{R}^\mathbb{N}$$

Let $\mathbb{Z}^f$ be the subset of finitary elements of $\mathbb{Z}^\mathbb{N}$, so that $\Lambda \hookrightarrow \mathbb{Z}^f$. Define

$$\mathsf{e}_\delta : \mathbb{Z}^f \ \hookrightarrow \ \mathbb{R}^\mathbb{N} \tag{7}$$

by $\lambda \mapsto \lambda + \rho_\delta$. In other words, since $\Lambda \hookrightarrow \mathbb{Z}^f$, we have, for each δ, embedded our index set Λ into a Euclidean space. Thus our blocks $[\lambda]_\delta$ now correspond to collections of points in this space.

Example: $\mathsf{e}_2(\emptyset) = (0, 0, 0, 0, ...) - (1, 1, 1, 1, ...) - (0, 1, 2, 3, ...) \ = \ (-1, -2, -3, -4, ...)$

(**4.10**) Now consider the following reflection group actions $(ij) : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ for $i, j \in \mathbb{N}$:

$$(ij) : (v_1, v_2, ..., v_i, ..., v_j, ...) \mapsto (v_1, v_2, ..., v_j, ..., v_i, ...)$$

$$(ij)_- : (v_1, v_2, ..., v_i, ..., v_j, ...) \mapsto (v_1, v_2, ..., -v_j, ..., -v_i, ...)$$

We notationally identify a reflection with its corresponding hyperplane in $\mathbb{R}^\mathbb{N}$ where no ambiguity arises. Write $\mathcal{D}$ for the group generated by these reflections (all $i < j$); and $\mathcal{D}_+$ for the subgroup $\langle (ij) \rangle_{ij}$. Write $\mathbb{H}_\mathcal{D}$ and $\mathbb{H}_{\mathcal{D}_+}$ for the corresponding closed sets of reflection hyperplanes. Let $\sigma_i = (i\ i+1)$, $S_{\mathcal{D}_+} := \{(i\ i+1) : i \in \mathbb{N}\}$, and $S_\mathcal{D} := S_{\mathcal{D}_+} \cup \{(12)_-\}$. Write $\mathcal{D}v$ for the orbit of a point $v \in \mathbb{R}^\mathbb{N}$ under the action of $\mathcal{D}$.

Comparing the definitions of δ-pair (4.3), e_δ and $(ij)_-$ we find:

(**4.11**) **Lemma**. *Fix δ. If λ/μ is a δ-pair then*

$$\mathsf{e}_\delta(\lambda) = \ w_{\lambda/\mu} \ \mathsf{e}_\delta(\mu) \qquad where \qquad w_{\lambda/\mu} := \prod_{ij} (ij)_-,$$

where the product is over pairs of rows in the skew, from the outer pair to the inner pair. No subset of this product, applied to $\mathsf{e}_\delta(\mu)$, results in a dominant (i.e. descending) sequence. $\square$

It follows that the $\mathcal{D}$ action on λ, via this construction, traverses the block $[\lambda]_\delta$. In [8] it is shown that it intersects no other block, that is $\mathsf{e}_\delta : [\lambda]_\delta \overset{\sim}{\to} \mathcal{D}\mathsf{e}_\delta(\lambda) \cap \mathsf{e}_\delta(\Lambda)$.

Note that $(\mathcal{D}, S_\mathcal{D})$ is a Coxeter system and $(\mathcal{D}_+, S_{\mathcal{D}_+})$ a maximal parabolic [16]. An *alcove* is a connected component of $\mathbb{R}^\mathbb{N} \setminus \cup_{H \in \mathbb{H}_\mathcal{D}} H$. A *chamber* is a connected component of $\mathbb{R}^\mathbb{N} \setminus \cup_{H \in \mathbb{H}_{\mathcal{D}_+}} H$. Note that there is a 'dominant' chamber C_0 consisting of strictly decending sequences. Chamber C_0 is bounded by the hyperplane set $S_{\mathcal{D}_+}$ (as is the region of ascending sequences). Write $\mathcal{A}$ for the set of alcoves and $\mathcal{A}^+$ for the subset in C_0. Choose the 'fundamental' alcove a_0 as the one containing $v_- := (-1, -2, -3, ...)$. Let $D(\mathbb{H}_\mathcal{D})$ denote the graph with vertices $\mathcal{A}$ and an edge (a, b) whenever a, b have a common wall-facet, in the sense of [16], with a, a_0 on the same side (that is, with the usual a_0 length function, $l(b) = l(a) + 1$). Let G_{alc} denote the full subgraph with vertices $\mathcal{A}^+$. See fig.3 for a picture of G_{alc} (together with the same construction for a simpler pairing of Coxeter groups for comparison). As we shall see, one useful characterisation of $G_\delta(\lambda)$, for all $(\delta, \lambda) \in \mathbb{Z} \times \Lambda$, is

$G_{alc} \cong G_\delta(\lambda)$ [9]. Note that $G_\delta(\lambda)$ is trivial unless $\delta \in \mathbb{Z}$ so, unless stated otherwise, we take $\delta \in \mathbb{Z}$ hereafter.

Note that $\mathsf{e}_\delta(\Lambda) \subset C_0$ for all δ. Indeed $\mathsf{e}_\delta(\Lambda)$ contains only *strongly* descending sequences, meaning that $v_i - v_{i+1} \in \mathbb{N}_{>0}$ for all i. We write A^+ for the set of strongly descending sequences.

(**4.12**) Define a partial order $(\mathbb{R}^\mathbb{N}, \leq)$ by $v \leq w$ if $v_i \leq w_i$ for all i. For $v \in \mathbb{R}^\mathbb{N}$ define
$$V(v) = \mathcal{D}v \cap A^+$$
The partial order $(\mathbb{R}^\mathbb{N}, \leq)$ restricts to a partial order $(V(v), \leq)$. The latter (unlike the former) has a unique transitive reduction. This reduction thus defines a directed acyclic graph, denoted $\mathbf{G}(v)$. Comparing with (4.5) and (4.11) we have, for $(\delta, \lambda) \in \mathbb{Z} \times \Lambda$, a graph isomorphism $\mathsf{e}_\delta : G_\delta(\lambda) \xrightarrow{\sim} \mathbf{G}(\mathsf{e}_\delta(\lambda))$.

(**4.13**) We say a sequence $v \in \mathbb{R}^\mathbb{N}$ is *regular* if $v \in \mathbb{R}^{Reg} := \mathbb{R}^\mathbb{N} \setminus \cup_{H \in \mathbb{H}_\mathcal{D}} H$. If $v \in \mathbb{R}^\mathbb{N}$ is regular then it lies within an alcove. Write $\mathfrak{a}(v)$ for the alcove in which v lies.

For δ, λ such that $\mathsf{e}_\delta(\lambda)$ lies within an alcove, the underlying set bijection between G_{alc} and $\mathbf{G}(\mathsf{e}_\delta(\lambda))$ is clear using $\mathfrak{a} : V(\mathsf{e}_\delta(\lambda)) \xrightarrow{\sim} \mathcal{A}^+$, and the graph isomorphism $G_{alc} \cong G_\delta(\lambda)$ is straightforward to verify. However we will need to describe the specific isomorphisms $G_{alc} \cong G_\delta(\lambda)$ in all cases, as in [9]. We do this next.

Firstly, since $v_- = \mathsf{e}_2(\emptyset)$ is regular, $\mathfrak{a} : V(v_-) \xrightarrow{\sim} \mathcal{A}^+$ is a bijection and we may use $V(v_-)$ to *label* alcoves in $\mathcal{A}^+$. Thus a_0 becomes $(-1, -2, -3, ...)$ in this labelling, and so on.

Note further that $V(v_-)$ is the set of the descending signed permutations of v_- that have an even number of positive terms. Such a sequence v is completely determined by the (possibly empty) list of its positive terms — that is, by an element of the power set $P(\mathbb{N})$. Write $\phi_+(v)$ for this element. Let $P_{even}(\mathbb{N}) \subset P(\mathbb{N})$ denote the subset of elements of even order; then $\phi_+ : V(v_-) \xrightarrow{\sim} P_{even}(\mathbb{N})$ is a bijection and so $P_{even}(\mathbb{N})$ is another convenient labelling set for $\mathcal{A}^+$. For example $\phi_+(5, 3, 2, 1, -4, -6, ...) = \{1, 2, 3, 5\}$ (we may even abbreviate $\{1, 2, 3, 5\}$ to 1235, and so on).

(**4.14**) Consider the magnitudes of terms in a sequence in C_0. Each magnitude occurs at most twice, i.e. in a sequence of form $(..., x, ..., -x, ...)$. We call such a $\pm x$ pairing a *doubleton*. Define
$$Reg : C_0 \to C_0$$
such that $Reg(v)$ is obtained from v by removing the doubletons [9]. For $\lambda \in \Lambda$ write $p_\delta(\lambda)$ for the set of pairs of rows $\{i, j\}$ such that $\mathsf{e}_\delta(\lambda)_j = -\mathsf{e}_\delta(\lambda)_i$. This gives the set of hyperplanes $(ij)_-$ upon which $\mathsf{e}_\delta(\lambda)$ lies. The set $p_\delta(\lambda)$ is clearly not an invariant of the block; although the *singularity*
$$s_\delta(\lambda) := |p_\delta(\lambda)|$$
is. However, given (δ, λ) and $Reg(v)$ for $v \in V(\mathsf{e}_\delta(\lambda))$ we can recover v, so Reg restricts to a bijection between $V(\mathsf{e}_\delta(\lambda))$ and $V(Reg(\mathsf{e}_\delta(\lambda)))$, the dominant part of the *regular* orbit $\mathcal{D}Reg(\mathsf{e}_\delta(\lambda))$.

(**4.15**) Define
$$\begin{aligned} o_\delta : \Lambda &\to P(\mathbb{N}) \\ \lambda &\mapsto \phi_+(\mathfrak{a}(Reg(\mathsf{e}_\delta(\lambda)))) \end{aligned} \qquad (8)$$

Examples:

$(3, 3, 3, 1) \overset{\mathsf{e}_0}{\mapsto} (3, 2, 1, -2, -4, -5, ...) \overset{Reg}{\mapsto} (3, 1, -4, -5, ...) \overset{\mathfrak{a}}{\mapsto} (2, 1, -3, -4, ...) \overset{\phi_+}{\mapsto}$
$\{1, 2\};$

$(4, 3, 3, 1) \overset{\mathsf{e}_0}{\mapsto} (4, 2, 1, -2, -4, -5, -6, ...) \overset{Reg}{\mapsto} (1, -5, -6, ...) \overset{\mathfrak{a}}{\mapsto} (-1, -2, -3, ...) \overset{\phi_+}{\mapsto} \emptyset.$
Note here that a useful way to compute $\mathfrak{a}(v)$ directly in the $V(v_-)$ labelling is
that $\mathfrak{a}(v)_i$ is (up to sign) the position of v_i in the *magnitude* ordering of the set of
numbers appearing in v.

(**4.16**) Given δ and λ we define $o_\delta^\lambda : P_{even}(\mathbb{N}) \to [\lambda]_\delta$ as follows. First construct
$\mathsf{e}_\delta(\lambda)$. This fixes the doubletons and (magnitudes of) singletons for $o_\delta([\lambda]_\delta)$. Ignore
the doubletons for a moment, and work out the magnitude order for the singletons.
Now for $a \in P_{even}(\mathbb{N})$ we give the positive sign to the corresponding singletons (in
the magnitude order). The order in which the singletons can appear in a descending
sequence is uniquely determined by their sign, so we have determined the singletons
and their order in a descending sequence. The position of the doubletons is now
forced. This gives the sequence $\mathsf{e}_\delta(o_\delta^\lambda(a))$. But e_δ is readily invertible, so finally
apply this inverse.

Example: $o_{-1}^{(2)}(\{1, 2, 4, 5\})$. The doubletons of $e_{-1}(2)$ are $\{5/2, -5/2\}$. The sin-
gletons have magnitudes $\{1/2, 3/2, 7/2, 9/2, 11/2, ...\}$, written out in the magnitude
order. For $v = \{1, 2, 4, 5\}$ we give $+$ signs to $1/2$, $3/2$, $9/2$ and $11/2$ and the re-
maining singletons are negative. Thus

$$\mathsf{e}_{-1}(o_{-1}^{(2)}(\{1, 2, 4, 5\})) = (\frac{11}{2}, \frac{9}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-5}{2}, \frac{-7}{2}, \frac{-13}{2}, \frac{-15}{2}, ...)$$

so $o_{-1}^{(2)}(\{1, 2, 4, 5\}) = 5^2 4^3 2^2$.

(**4.17**) **Lemma**. *Fix δ, λ. Then o_δ and o_δ^λ are mutual inverses on $[\lambda]_\delta \leftrightarrow P_{even}(\mathbb{N})$.*
□

(**4.18**) Define a directed graph, G_{even}, with vertex set $P_{even}(\mathbb{N})$; and labelled edges:

$$a \overset{\alpha}{\to} b \quad \text{if} \quad a \backslash b = \{\alpha\}, \; b \backslash a = \{\alpha{+}1\}, \quad (\alpha \in \mathbb{N}); \qquad\qquad a \overset{12}{\to} b \quad \text{if} \quad a \backslash b = \emptyset, \; b \backslash a = \{1, 2\}.$$

See Figure 3. There is a corresponding graph G_{odd} with vertices given by $P_{odd}(\mathbb{N})$.
The *toggle map* $\tau : P_{odd}(\mathbb{N}) \to P_{even}(\mathbb{N})$ toggles the presence of 1 so as to make an
odd set even. The map τ is readily seen to pass to a graph isomorphism (the edge
labels 1 and 12 are interchanged).

With the identification of $V(v_-)$ and $\mathcal{A}^+$ in mind, recall $\phi_+ : V(\mathsf{e}_2(\emptyset)) \to$
$P_{even}(\mathbb{N})$ as the map that discards negative entries. Considering the effect of simple
reflections on $V(v_-)$, such as $(15)_-(4, 3, -1, 2, -5, ...) = (5, 3, -1, -2, -4, ...)$ we
see $\phi_+ : G_{alc} \overset{\sim}{\to} G_{even}$. Indeed:

(**4.19**) **Theorem**. [9, Cor.7.3 *et seq*] *For all δ, λ we have isomorphisms*

$$G_\delta(\lambda) \overset{\mathsf{e}_\delta}{\to} \mathbf{G}(\mathsf{e}_\delta(\lambda)) \overset{Reg}{\to} \mathbf{G}(Reg(\mathsf{e}_\delta(\lambda))) \overset{\mathfrak{a}}{\to} G_{alc} \overset{=}{\to} \mathbf{G}(\mathsf{e}_2(\emptyset)) \overset{\phi_+}{\to} G_{even}$$

In particular $o_\delta : G_\delta(\lambda) \to G_{even}$ is an isomorphism. □

(**4.20**) Recall from [16] that $\mathcal{D}$ acts simply transitively on $\mathcal{A}$, so a bijection $\zeta : \mathcal{D} \to$
$\mathcal{A}$ is defined by $1 \mapsto a_0$. Edge $(a, b) \in D(\mathbb{H}_\mathcal{D})$ can be written (a, as) for some $s \in S_\mathcal{D}$
via the right action of $\mathcal{D}$ on itself (so $D(\mathbb{H}_\mathcal{D})$ is the Cayley graph of $(\mathcal{D}, S_\mathcal{D})$). The
corresponding *right* action of $\mathcal{D}$ on $V(v_-)$ is by signed permutation of the *entries* in

FIGURE 3.   (a) The beginning of the graph $G_{even} \cong G_{alc}$. Edge labels are as in (4.18). (b) A simpler example for comparison: case $\hat{A}_2/A_2$ [16].

the sequence. For example: $(4, 3, -1, -2, -5, ...)(45) = (5, 3, -1, -2, -4, ...)$. One readily checks:

(**4.21**) **Lemma**. *The right label $s \in S_{\mathcal{D}}$ of edge (a, as) in G_{alc} passes via ϕ_+ to the label α in G_{even} in case $s = (\alpha\ \alpha+1)$ and to the label 12 in case $s = (12)_-$. (See Fig.3(a).)* □

(**4.22**) **Theorem**. *Two edges in G_{even} pass to G_{alc} edges in the same $\mathcal{D}$-orbit (up to direction) if and only if they have the same label.* □

5. DECOMPOSITION DATA

In this section we prepare the structures needed in the statement of Theorem 7.1. The idea comes from solving for parabolic Kazhdan–Lusztig polynomials for the $\mathcal{D}/\mathcal{D}_+$ system (a non-trivial exercise, cf. [3]). However the *proof* of the main result requires a more general approach.

Let $\mathbf{b} : P(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ denote the natural bijection. For example: $\mathbf{b} : \{1,3,5,6\} \mapsto 101011$ (we omit the open string of 0s on the right). Define $\mathbf{b}_\delta : \Lambda \to \{0,1\}^{\mathbb{N}}$ by $\mathbf{b}_\delta(\lambda) = \mathbf{b}(o_\delta(\lambda))$.

Via ζ, the right cosets $\mathcal{D}_+ \backslash \mathcal{D}$ have coset representatives labelled by $\mathcal{A}^+$ (see e.g. [22]). In this way we have a right action of $w \in \mathcal{D}$ on $a \in \mathcal{A}^+$. Define a graph $\mathbf{D}_+$ with vertex set the right cosets $\mathcal{D}_+ \backslash \mathcal{D}$, and an edge (a,b) whenever $b = aw$ and $l(aw) < l(a)$ for some reflection $w \in \mathcal{D}$.

It is convenient here to write actions on the left, so we write $\langle ij \rangle a$ to denote $a(ij)$ in case $(a, a(ij))$ an edge in $\mathbf{D}_+$ (resp. $\underline{\langle ij \rangle} a$ for $a(ij)_-$). Otherwise $\langle ij \rangle a$ is *undefined*.

For S a set, the 'hypercubical' directed graph $\mathbf{h}(S)$ has vertex set $P(S)$ and an edge (s,t) whenever subset t is obtained from s by deleting one element. (Hypercube edges corresponding to deleting the same element of S are 'parallel'.) Let S be a set of commuting *reflections* in $\mathcal{D}$. Let $a \in \mathcal{A}^+$ be such that (a, ar) an edge in $\mathbf{D}_+$ for all $r \in S$. Then the map $\Psi_a : P(S) \to \mathcal{A}^+$ given by $\Psi_a(S') = a \prod_{r \in S \backslash S'} r$ defines a hypercubical graph $h(a,S) = \Psi_a(\mathbf{h}(S))$. Note that the edges of $h(a,S)$ take a particularly simple form when the vertices are written as elements of $P_{even}(\mathbb{N})$ using the ϕ_+ isomorphism: here $(a, a(ij))$ an edge, for example, means that $a \in P_{even}(\mathbb{N})$ contains j not i, and $a(ij)$ is the same set except containing i not j (cf. G_{even}). Henceforth we shall write $h(a,S)$ in this way. We shall need graphs of this form for certain special choices of S.

(**5.1**) A generalisation of Brauer diagrams is to allow singleton vertices. A vertex pairing in such a diagram *covers* a vertex if the pair lie either side of it. A *TL-diagram* (TL as in Temperley–Lieb) is here a diagram drawn in the positive quadrant of the plane, consisting of a collection of vertices drawn on the horizontal part of the boundary (countable by the natural numbering from left to right); together with a collection of *non-crossing* arcs drawn in the positive quadrant, each terminating in two of the vertices, such that no vertex terminates more than one arc, and no arc covers a singleton vertex. An example is:
It will be convenient to label each arc by the associated pair of numbered vertices.

(**5.2**) Each binary sequence b has a TL-diagram $d(b)$ constructed as follows.
1. Draw a row of vertices, one for each entry in b (up to the last non-zero entry).
2. For each binary subsequence 01 draw an arc connecting the corresponding vertices.
3. Consider the sequence obtained by ignoring the vertices paired in 2. For each subsequence 01 draw an arc connecting these vertices (it will be evident that this can be done without crossing).
4. Iterate this process until termination.
5. Note that this process terminates either in the empty sequence or in a sequence of 1s then 0s (either run possibly empty). Finally connect the run of vertices binary-labelled 1 in adjacent pairs (if any) from the left. Leave the remaining vertices as singletons.

Example: $d(10011) = $ A number of examples are shown in Figure 4.

(**5.3**) For $a \in P(\mathbb{N})$ we write Γ_a for the list of arcs (i.e. pairs) in $\mathcal{T}(a) := d(\mathbf{b}(a))$ corresponding to 01 subsequences in $\mathbf{b}(a)$, and an initial 11 subsequence (i.e. if there is one in the 12-position); and Γ^a for the list of all arcs. For example, $\Gamma_{1356} =$
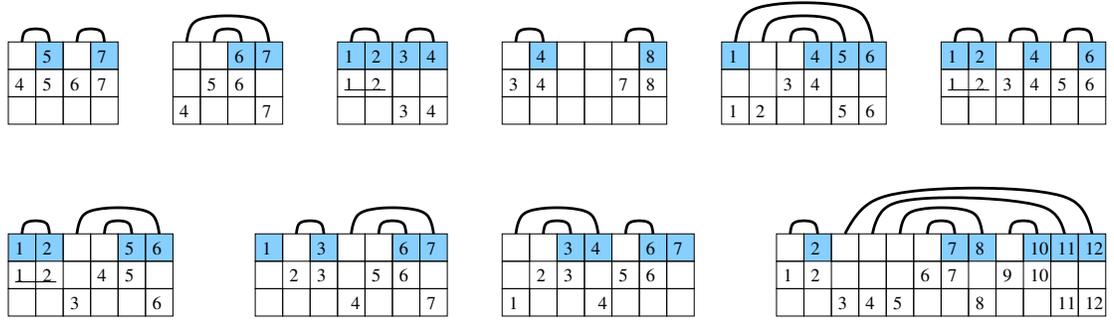
FIGURE 4. Examples for the maps from $a \in P(\mathbb{N})$ to sequences, to TL-diagrams, and then to sets of pairs Γ_a and Γ^a. In each case $\mathbf{b}(a)$ is indicated in the top row of boxes (shaded=1, unshaded=0). The second row shows the set of pairs of numbers Γ_a. The third row shows the further pairs added to obtain the set Γ^a.
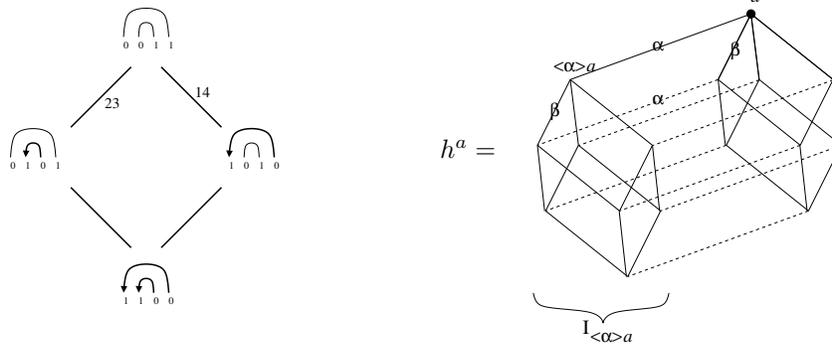

FIGURE 5. (a) Hypercube $h^{3\,4}$ (showing the TL arcs used in the construction). (b) Decomposition of h^a into sub-hypercubes (denoting $\langle \alpha\ \alpha+1 \rangle a$ by $\langle \alpha \rangle a$).

$\{\{2,3\}, \{4,5\}\}$ and $\Gamma^{1356} = \{\{2,3\}, \{4,5\}, \{1,6\}\}$. See Figure 4 for more examples. We may write $\Gamma_{\delta,\lambda}$ for $\Gamma_{o_\delta(\lambda)}$, and Γ^λ_δ for $\Gamma^{o_\delta(\lambda)}$.

(**5.4**) For $a \in P_{even}(\mathbb{N})$ define $h^a = h(a, \Gamma^a)$ (it is to be understood that $\{i,j\} \in \Gamma^a$ acts as $(ij)_-$ if $\{i,j\} \subset a$). See Figure 5 for an example showing vertices in the binary representation, so that the sequence at the other end of a given edge is obtained from the original simply by replacing $01 \to 10$ (or $11 \to 00$) at the ends of the corresponding TL arc.

  The construction for h^a also defines a hypercubical directed graph $h_\delta(\mu)$ for each pair $(\delta, \mu) \in \mathbb{Z} \times \Lambda$, obtained by applying o^μ_δ to the vertices. That is, $h_\delta(\mu) = o^\mu_\delta(h^{o_\delta(\mu)})$.

(**5.5**) We label each edge of h^a by the corresponding arc $\{\alpha, \alpha'\}$ (or simply by $\alpha\alpha'$). If label $\alpha' = \alpha+1$ for an 01-arc, we may just label the edge by α. If $\{\alpha, \alpha'\} = \{1,2\}$ for a 11-arc we may just label the edge by $\underline{12}$. (These α-edges and $\underline{12}$-edges then coincide with edges of G_{even}, although other edges do not.)

(a)

776532

$(35)_-$ $(26)_-$ $(16)-(25)-(34)-(26)-(35)-$

775522   766531   652210

765521   642110   552200

542100

(b)

1356

23 45 $\underline{16}$

1256   1346   35

1246   25   34

24

(c)

$(6,5,3,1,-2,-4,...)$

$(35)_-$ $(26)_-$

$(6,5,2,1,-3,...)$ $(6,4,3,1,-2,-5,...)$ $(5,3,-1,-2,...)$

$(6,4,2,1,-3,...)$ $(5,2,-1,-3,...)$ $(4,3,-1,-2,...)$
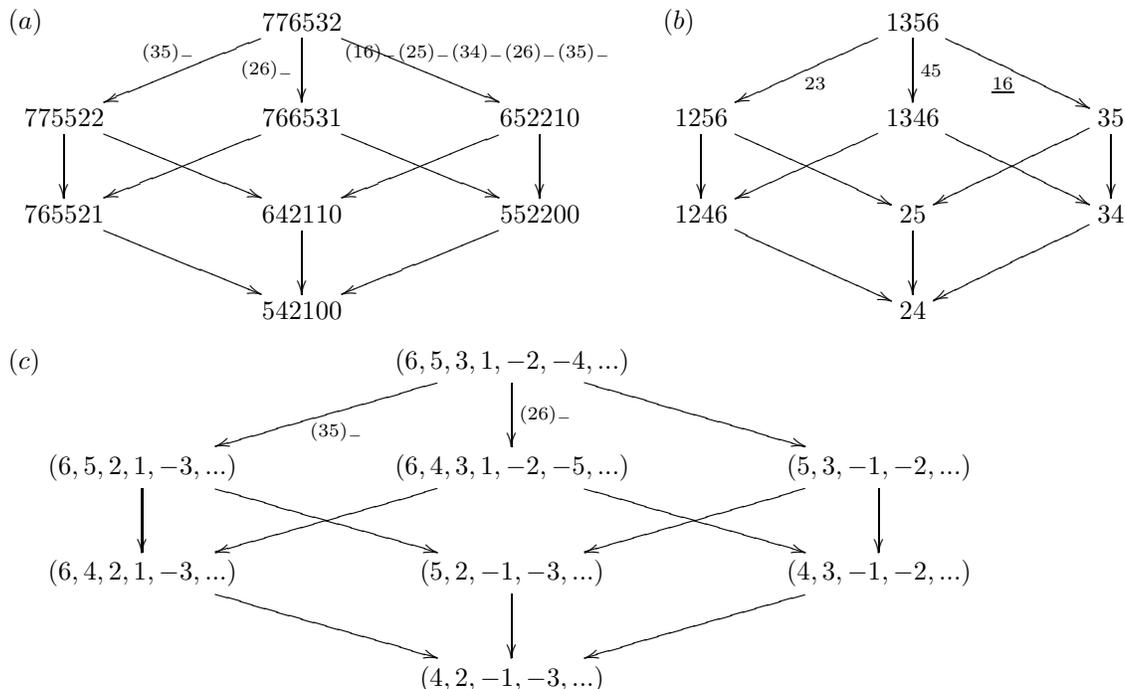
$(4,2,-1,-3,...)$

FIGURE 6.   Three labellings of the same hypercube in case $\delta = 2$:
(a) partition labelling; (b) $P(\mathbb{N})$ labelling; (c) descending sequence
labelling.

It follows from the construction and Theorem 4.19 that if a vertex of some
hypercube $h_\delta(\tau)$ is $o_\delta(\lambda)$ for some λ, then a vertex beneath it down an α or $\underline{12}$-edge
is $o_\delta(\mu)$ with λ/μ a δ-pair.

(**5.6**) Figure 6 gives an example of $h_\delta(\lambda)$ with $\delta = 2$. We take $\lambda = (7,7,6,5,3,2)$
so $\lambda + \rho_2 = (6,5,3,1,-2,-4,-7,-8,...)$ giving $o_2(\lambda) = \{1,3,5,6\}$ and hence $\Gamma_\delta^\lambda = \{\{2,3\},\{4,5\},\{1,6\}\}$. Similarly $\lambda - e_4 = (7,7,6,4,3,2) \overset{e_2}{\leadsto} (6,5,3,0,-2,-4,-7,...) \overset{o_2}{\leadsto} \{1,3,5,6\}$. Note here that $h_2(\lambda - e_4)$ is isomorphic to $h_2(\lambda)$ above. That is, both
are images of h^{1356}.

(**5.7**) Note from the construction that no two vertices of h^a (or $h_\delta(\mu)$) have the
same label. We shall write $h_\delta(\mu)_\nu = 1$ if ν appears in $h_\delta(\mu)$, and $= 0$ otherwise.
Note that for any given block $[\lambda]_\delta$ we have asigned a hypercube to each partition in
the block. The vertices in this hypercube then correspond to partitions in the same
block. In this way we can use the hypercubes to determine, for each δ, a matrix
(of almost all 0s, and some 1s), with rows and columns labelled by partitions. The
μ-th row is $(h_\delta(\mu)_\nu)_{\nu \in \Lambda}$.
We will see in Theorem 7.1 that the resultant matrix gives our block decomposition
matrix.
It will also be useful to record the *depth i* of each entry in the hypercube, by
writing v^i (v a formal parameter) instead of 1 in the appropriate position. (Thus

	0	12	13	14	23	15	24	16	25	34	17	26	35	18	27	36	45	1234	19	28	37	46	1235	1X	29	38	47	56	1236	1245	1 11	2 10	39	48	57	1237	1246	1345	1 12
0	1																																						
12	v	1																																					
13		v	1																																				
14			v	1																																			
23				v	1																																		
15					v	1																																	
24			v²		v	v	1																																
16							v	1																															
25						v²	v	v	1																														
34	v²	v							v	1																													
17								v			1																												
26						v²		v	v			1																											
35							v²		v	v			1																										
18												v		1																									
27						v²						v	v		1																								
36							v²						v	v		1																							
45					v²		v						v				1																						
1234	v²	v											v					1																					
19														v					1																				
28												v²		v	v					1																			
37													v²		v	v					1																		
46															v²	v	v					1																	
1235															v²		v						1																
1X																	v							1															
29																v²			v	v					1														
38															v²				v	v						1													
47																v²			v	v							1												
56																v²				v		v						1											
1236																v²			v			v							1										
1245																v²				v		v								1									
1 11																						v									1								
2 10																				v²					v	v						1							
39																				v²						v	v						1						
48																				v²							v	v						1					
57																				v²							v	v							1				
1237																				v²							v		v							1			
1246																v³			v²	v²							v	v²	v	v							1		
1345								v²					v															v	v									1	
1 12																																	v						1

FIGURE 7.   Table encoding array of polynomials in the G_{even} labelling scheme (every non-zero polynomial is of form v^i). Here 12 denotes $\{1,2\}$, and so on.

this polynomial version evaluates to the decomposition matrix at $v = 1$.) The first few vertices of this form are shown in Figure 7, using the $P(\mathbb{N})$ labelling scheme.

(5.8) For $\alpha \in \mathbb{N}$ define 'bump' map $\vec{\alpha} : \mathbb{N} \to \mathbb{N}$ by $\vec{\alpha}(i) = i$ for $i < \alpha$ and $\vec{\alpha}(i) = i+2$ otherwise. Let $b = (b_1, b_2, ...)$ be a binary sequence, and $\alpha \in \mathbb{N}$. Then $\hat{\alpha}b$ (resp. $\check{\alpha}b$) is the sequence obtained from b by inserting 01 (resp. 10) in the $\alpha, \alpha + 1$ positions (i.e. so that this pair become the elements in the α and $\alpha + 1$ positions in the sequence, with any terms at or above these positions in b bumped two places

further up in $\hat{\alpha}b$). Examples: $\hat{2}01 = 0011$, $\check{2}01 = 0101$. Define $\hat{\alpha} : P(\mathbb{N}) \to P(\mathbb{N})$ similarly. In the sense indicated by Fig.5(b) we have an 'exact sequence'

$$h(\check{\alpha}a, \vec{\alpha}\Gamma^a) \hookrightarrow h^{\hat{\alpha}a} \twoheadrightarrow h(\hat{\alpha}a, \vec{\alpha}\Gamma^a). \tag{9}$$

6. Embedding properties of δ-blocks in Λ

Here we consider how the embeddings in $\mathbb{R}^{\mathbb{N}}$ of the different block graphs relate to each other. Via (3.14), this will allow us to pass information between blocks.

(**6.1**) Suppose $w \in \mathcal{D}$ such that $w\mathsf{e}_\delta(\lambda) = \mathsf{e}_\delta(\mu)$. *When δ is fixed* we may write $w.\lambda$ for μ. Also if λ is a vertex of $G_\delta(\mu)$ and α is the label (inherited from G_{even}) on an edge touching λ we write $\langle \alpha \rangle \lambda$, or simply $\alpha\lambda$, for the vertex at the other end.

(**6.2**) The isomorphism implicit in Theorem 4.19 between any pair of block graphs $G_\delta(\lambda)$ and $G_\delta(\lambda')$ defines a pairing of each vertex in $G_\delta(\lambda)$ with the corresponding vertex in $G_\delta(\lambda')$. A pair of block graphs is *adjacent* if they have the same singularity, and every such pair of vertices is adjacent as a pair of partitions. (If λ, λ' are adjacent partitions in the same $\mathcal{D}$-facet then $G_\delta(\lambda)$ and $G_\delta(\lambda')$ are adjacent, since the same reflection group elements serve to traverse these graphs [8]. We shall need to show adjacency of a more general pairing of graphs.)

For given λ, if $\lambda' = \lambda - e_i$ we write $f_i : G_\delta(\lambda) \to G_\delta(\lambda - e_i)$ for the direct graph isomorphism. (Strictly speaking f_i depends on λ too, but we suppress this for brevity.)

(**6.3**) Fix δ and suppose $\lambda \in \Lambda$ has a removable box e_i. Suppose that $\lambda/\alpha\lambda$ is a δ-pair containing e_i. Write π_α for the π-reflection fixing this δ-pair. Then note that $\pi_\alpha(e_i)$ is an addable box of $\alpha\lambda$.

(**6.4**) **Lemma**. *Fix δ. Suppose $\lambda \in \Lambda$ has a removable box e_i such that $s_\delta(\lambda) = s_\delta(\lambda - e_i)$. Then*
(I) $o_\delta(\lambda) = o_\delta(\lambda - e_i)$;
(II) There does not exist a $\lambda - e_i - e_{i'} \in [\lambda]_\delta$; nor a $(\lambda - e_i) + e_i + e_{i'} \in [\lambda - e_i]_\delta$.

Proof. (I) Write x for $(\lambda + \rho_\delta)_i$. Thus, for some $y < x - 1$:

$$\lambda + \rho_\delta \sim (..., \underbrace{x}_{i}, y, ...), \qquad \lambda + \rho_\delta - e_i \sim (..., \underbrace{x-1}_{i}, y, ...) \tag{10}$$

If $p_\delta(\lambda) = p_\delta(\lambda - e_i)$ then one can readily check that the changed row i appears in the magnitude order in both cases, and in the same position. In case $x = 1/2$ there is a sign change, but by the toggle rule $o_\delta(\lambda) = o_\delta(\lambda - e_i)$. If $p_\delta(\lambda) \neq p_\delta(\lambda - e_i)$ then from (10) we see firstly that $-x$ occurs in $\lambda + \rho_\delta$ and $1 - x$ occurs in $\lambda + \rho_\delta - e_i$ (for if neither occurs then p_δ does not change between them; while if only one occurs then s_δ changes). It follows immediately that $1 - x, -x$ occur (and are adjacent) in both. Secondly, $y < x - 1$ so $x - 1$ does not occur in $\lambda + \rho_\delta$. In computing o_δ we discount the $\pm x$ pair in $\lambda + \rho_\delta$ and the $\pm(x-1)$ pair in $\lambda + \rho_\delta - e_i$. The discrepancy is thus now a $1 - x$ in $\lambda + \rho_\delta$ compared to a $-x$ in $\lambda + \rho_\delta - e_i$. But if $1 - x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta$ then $-x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta - e_i$, with all else equal, so o_δ is unchanged.

(II) Suppose $\lambda/\lambda - e_i - e_{i'}$ *is* a δ-pair. Here removing the last box in row i means that rows i, i' become a singular pair in $\lambda - e_i$, where they were not before, so $p_\delta(\lambda - e_i) \neq p_\delta(\lambda)$. Thus $s_\delta(\lambda - e_i) = s_\delta(\lambda) + 1$ unless we also *lost* a singular pair i, i''. This would have to be with $i'' = i' + 1$ directly under the last box in
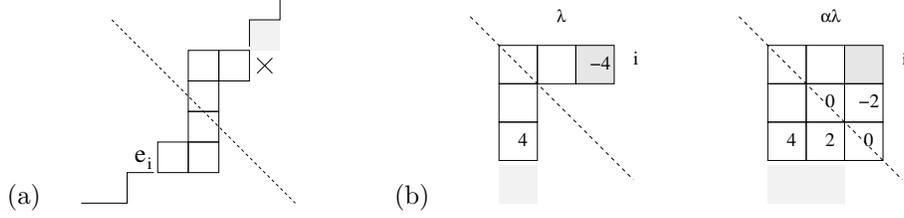
FIGURE 8. Examples (a), (b) for Lemma 6.6.

row i'. But this cannot happen since that box is removable. (The argument in case $i' = i + 1$ requires a slight modification.) The last claim is proved similarly. $\square$

(**6.5**) **Lemma**. *Fix δ and suppose $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ as before. Suppose λ has an edge down labelled α, i.e. $\lambda/\alpha\lambda$ is a δ-pair; and let w be the product of commuting reflections such that $w\mathsf{e}_\delta(\lambda) = \mathsf{e}_\delta(\alpha\lambda)$, as in Lemma (4.11). Then   (I) $w\mathsf{e}_\delta(\lambda - e_i)$ is dominant (i.e. $w.(\lambda - e_i) \in \Lambda$);*
(II) $w\mathsf{e}_\delta(\lambda - e_i) = \mathsf{e}_\delta(\alpha(\lambda - e_i))$.

Proof. Note that $o_\delta(\lambda - e_i) = o_\delta(\lambda)$ by Lemma 6.4, so $\alpha(\lambda - e_i)$ makes sense; and $o_\delta(\alpha(\lambda - e_i)) = o_\delta(\alpha\lambda)$ (since both are equal to the formal set $\alpha o_\delta(\lambda)$). Suppose $w.(\lambda - e_i)$ *is* in Λ. Then it is in $[\lambda - e_i]_\delta$ by [8, Th.5.2], adjacent to $\alpha\lambda$ with the same singularity, and by Lemma (6.4) (applied appropriately) $o_\delta(w.(\lambda - e_i)) = o_\delta(\alpha\lambda)$. Thus it is enough to show (I).

We split into two cases. (A) If e_i intersects $\lambda/\alpha\lambda$ then $\pi_\alpha(e_i)$ is addable to $\alpha\lambda$ as noted in (6.3). That is $\mathsf{e}_\delta(\alpha\lambda + \pi_\alpha(e_i)) = w\mathsf{e}_\delta(\lambda - e_i)$ is dominant. (B) If e_i does not intersect $\lambda/\alpha\lambda$ then $w\mathsf{e}_\delta(\lambda - e_i)$ is the same as $w\mathsf{e}_\delta(\lambda)$ everywhere except in row i: $w\mathsf{e}_\delta(\lambda - e_i) = w\mathsf{e}_\delta(\lambda) - e_i$. Since $\lambda - e_i$ is dominant, $\lambda_i > \lambda_{i+1}$, but $(\alpha\lambda)_i = \lambda_i$ in this case, and $(\alpha\lambda)_{i+1} \leq \lambda_{i+1}$, so $(\alpha\lambda)_i > (\alpha\lambda)_{i+1}$, so $\alpha\lambda - e_i$ is dominant, so $\mathsf{e}_\delta(\alpha\lambda - e_i) = w\mathsf{e}_\delta(\lambda - e_i)$ is dominant. $\square$

(**6.6**) **Lemma**. *Fix δ and suppose $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ as before. Suppose $\alpha\lambda/\lambda$ is a δ-pair (i.e. α is an edge up from λ). Then there is a reflection group element w such that $w.\lambda = \alpha\lambda$ (so $w.\alpha\lambda = \lambda$) and $w.(\lambda - e_i)$ is dominant; whereupon $w.(\lambda - e_i) = \alpha(\lambda - e_i)$.*

Proof. As above it is enough to show $w.(\lambda - e_i) \in \Lambda$. Given that $w.\lambda$ is dominant, a *failure* of dominance of $w.(\lambda - e_i)$ must be either: (case A) a row with which row-i is paired in w (j, say) is longer than row-$(j-1)$ in $w.(\lambda - e_i)$; or (B) the i-th row itself is shorter than row-$(i+1)$ in $w.(\lambda - e_i)$ (i.e. row-$(i+1)$ intersects the δ-pair). We need to eliminate these.

Case (A): Suppose first that e_i is 'behind' other than the last row of the skew. Then there is a box of the skew immediately to its right and one immediately below it. The π-rotation images of these are behind and above the image of e_i, so $w.(\lambda - e_i) \in \Lambda$. On the other hand, suppose e_i is behind the last row of the skew. For example see Fig.8(a) (the box $\pi_\alpha(e_i)$ is marked $\times$). Here $w.(\lambda - e_i)$ is dominant unless the box above $\pi_\alpha(e_i)$ is missing from λ. But if this is missing then this row and the i-row are a singular pair in $\lambda - e_i$. Neither row can be in a singular pair in λ so this contradicts the hypothesis.

(B) If the i-th row is not 'moved' by w then the failure would have to be that the skew $\alpha\lambda/\lambda$ includes a box directly under e_i. But in that case a δ-balanced box to

e_i given by $\pi_\alpha(e_i)$ is directly to the left of the skew, and we have a setup something like Fig.8(b) (the δ-balanced box is the box marked 4). If there is no box below the $\pi_\alpha(e_i)$ in λ then row-i is not in a singular pair in λ, and row-i and the row containing the $\pi_\alpha(e_i)$ are a singular pair in $\lambda - e_i$, thus $s_\delta(\lambda) \neq s_\delta(\lambda - e_i)$ so we can exclude this. If there is a box below the $\pi_\alpha(e_i)$ in λ then this row and row-i are a singular pair in λ, and row-i and the row containing the $\pi_\alpha(e_i)$ are a singular pair in $\lambda - e_i$. In this case, a w which also has a factor acting on the i-th and undrawn row has the same effect on λ as one which does not. Its effect on $\lambda - e_i$ is to restore the box e_i and to add a box in the undrawn row. This $w.(\lambda - e_i)$ is dominant since the added box is under a box added in the original skew. $\square$

Since the block graph $G_\delta(\lambda)$ is connected we may use Lemmas 6.5 and 6.6 to show:

(**6.7**) **Theorem**. *If $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then $G_\delta(\lambda)$ is adjacent to $G_\delta(\lambda - e_i)$.* $\square$

(**6.8**) Lemma (6.4)(I) says that if the partitions $\lambda, \lambda - e_i$ have the same singularity then they pass to the same point on the block graph G_{even}. That is $f_i(\lambda) = \lambda - e_i$ and so on. Thus for $\mu \in [\lambda]_\delta$

$$h_\delta(\lambda)_\mu = h_\delta(\lambda - e_i)_{f_i(\mu)}$$

(**6.9**) **Lemma**. *If $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then for all pairs $(\mu, f_i(\mu)) \in [\lambda]_\delta \times [\lambda - e_i]_\delta$*

$$Proj_\lambda Ind\, \bar\Delta_n(f_i(\mu)) = \bar\Delta_{n+1}(\mu)$$

$$Proj_{f_i(\lambda)} Ind\, \bar\Delta_n(\mu) = \bar\Delta_{n+1}(f_i(\mu)) \tag{11}$$

Proof. Note that the pair $(\mu, f_i(\mu))$ are adjacent by Theorem 6.7. For any ν Prop.3.14 gives $Ind\, \bar\Delta(\nu) = \left(+_j \bar\Delta(\nu + e_j)\right) + \left(+_k \bar\Delta(\nu - e_k)\right)$. For $\nu = f_i(\mu)$ adjacent to μ, one of these summands is $\bar\Delta(\mu)$. Specifically either (i) $\mu = \nu + e_l$ (some l); or (ii) $\mu = \nu - e_l$ (some l). In case (i) other summands are of form $\mu - e_l + e_j$, $\mu - e_l - e_k$. By Prop.(4.5) the former are not in $[\mu]_\delta$, and since $s_\delta(\lambda) = s_\delta(\lambda - e_i)$, Lemma (6.4)(II) excludes the latter. The other case is similar. $\square$

7. THE DECOMPOSITION THEOREM

(**7.1**) **Theorem**. *For each $\delta \in \mathbb{Z}$ the Brauer algebra Δ-decomposition matrix D over $\mathbb{C}$ is given by*

$$(\bar P_n^\delta(\lambda) : \bar\Delta_n^\delta(\mu)) = h_\delta(\lambda)_\mu$$

or equivalently

$$\bar P_n^\delta(\lambda) = +_{\mu \in h_\delta(\lambda)}\, \bar\Delta_n^\delta(\mu).$$

(Recall we omit $\lambda = \emptyset$ in case $\delta = 0$.)
This data determines the Cartan decomposition matrix C for any finite n by (3.16).

Proof. We prove for a fixed but arbitrary δ, working by induction on n. The base cases are $n = 0, 1$, which are trivial (and $n = 2$ for $\delta = 0$, which is straightforward). We assume the theorem holds up to level $n - 1$, and consider $\lambda \vdash n$. (For $|\lambda| < n$ the result holds by (3.16) and the inductive assumption.)   Note that if λ is at the bottom of its block then the claim is trivially true by (3.16). If λ is not at the bottom of its block then there is at least one edge down, $\lambda \xrightarrow{\alpha} \mu$ say, in the block graph. Choose one such edge, and choose e_i a removable box of greatest magnitude charge from the removable boxes of $\lambda/\mu = \lambda/\alpha\lambda$. The next step depends on $|\lambda/\alpha\lambda|$.

(**7.2**) We call a removable box of largest magnitude charge (among those removable in the given skew) a *rim-end removable box*. Examples are shown in Figure 2. The rim-end removable boxes (as labelled by charge) in the figure are (i) 22; (ii) -16; (iii) 8.

(**7.3**) **Proposition**. *Fix δ. Pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. Then*

$$s_\delta(\lambda - e_i) = \begin{cases} s_\delta(\lambda) + 1 & \text{if } |\lambda/\alpha\lambda| = 2 \\ s_\delta(\lambda) & \text{otherwise} \end{cases}$$

Proof: If $|\lambda/\alpha\lambda| = 2$ the charges in the two boxes are (say) x and $-x$. Removing x (from row i) we get a row ending in charge $x + 2$, giving $(\lambda + \rho_\delta)_i = -\frac{x+2}{2} + \frac{1}{2} = -\frac{x+1}{2}$. The row ending in $-x$ has $(\lambda + \rho_\delta)_j = -\frac{-x}{2} + \frac{1}{2} = \frac{x+1}{2}$ thus these two rows are now a singular pair.

For $|\lambda/\alpha\lambda| \neq 2$ there is a unique rim-end removable box. The case $\lambda/\alpha\lambda = (2^2)$ is elementary. We split the remainder into two cases. If the upper end of a rim in $\lambda/\alpha\lambda$ ends in a row of length greater than 1 then the removable box is at the upper end. Write $-x$ for its charge and i for its row. (Cf. the upper rim in Figure 2(ii), which ends in $-x = -16$.) Note that in this case there cannot be a row in λ ending in a box with charge $x + 2$ or x. Note that a pair of rows is singular if the sum of charges in their end boxes is 2. It follows that neither the i-th row of λ nor that of $\lambda - e_i$ is in a singular pair. Thus λ, $\lambda - e_i$ have the same set of singular pairs. (Indeed we remain in the same facet.)
If the lower end of a rim in $\lambda/\alpha\lambda$ ends in a column of length greater than 1 then the removable box is at the lower end. Write x for its charge and i for its row. (Cf. the lower rim in Figure 2(i), which ends in $x = 22$; and Fig.2(iii) which ends in $x = 8$.) Note that in this case there is a row in λ ending in a box with charge $-x$, and one with $-x + 2$. It follows that both the i-th row of λ and that of $\lambda - e_i$ is in a singular pair (albeit each with a different partner). Thus $\lambda - e_i$ has a different set of singular pairs, but the same *number* of pairs: $s_\delta(\lambda - e_i) = s_\delta(\lambda)$. $\square$

(**7.4**) **Proposition**. *Fix δ. For $\lambda \in \Lambda$ pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. In case $|\lambda/\alpha\lambda| \neq 2$*
(i) the Δ-decomposition data for $\bar{P}(\lambda)$ is the 'translate' of that for $\bar{P}(\lambda - e_i)$:

$$(\bar{P}(\lambda) : \bar{\Delta}(\mu)) = (\bar{P}(\lambda - e_i) : \bar{\Delta}(f_i(\mu))) \qquad \forall \mu \in [\lambda]_\delta$$

(ii) This verifies the inductive step for the main theorem in such cases. That is, $h_\delta(\lambda) \cong h_\delta(\lambda - e_i)$.

Proof: By Proposition 3.18 the 'translation' $\text{Proj}_\lambda \text{Ind } \bar{P}(\lambda - e_i) = \bar{P}(\lambda) \oplus Q$ with $Q = \text{Proj}_\lambda Q$ some projective, possibly zero. In case $|\lambda/\alpha\lambda| \neq 2$ each standard module occuring in $\bar{P}(\lambda - e_i)$ induces precisely one standard module after projection onto the block of λ, by Lemma 6.9 (noting Proposition 7.3). More specifically, suppose $\bar{\Delta}(\nu(1)), \bar{\Delta}(\nu(2)), ..., \bar{\Delta}(\nu(l))$ is a Δ-filtration series for $\bar{P}(\lambda - e_i)$ (i.e. $\bar{P}(\lambda - e_i) \simeq //_j \bar{\Delta}(\nu(j))$). Then by (6.2) there is a sequence $\mu(1), \mu(2), ...$ such that $\nu(j) = f_i(\mu(j))$; and (using Prop.3.14(i), 3.16 and exactness of Res$-$ and Proj$_\lambda-$)

$$\bar{P}(\lambda) \oplus Q = \text{Proj}_\lambda \text{Ind } \bar{P}(\lambda - e_i) \simeq //_j \text{Proj}_\lambda \text{Ind } \bar{\Delta}(f_i(\mu(j))) = //_j \bar{\Delta}(\mu(j))$$

On inducing again and projecting back to the block of $\lambda - e_i$, by (11) we have

$$\text{Proj}_{\lambda - e_i} \text{Ind } (\bar{P}(\lambda) \oplus Q) \simeq //_j \bar{\Delta}(f_i(\mu(j)))$$

That is, each standard module occuring in $(\bar{P}(\lambda) \oplus Q)$ induces precisely one standard module after projection onto the block of $\lambda - e_i$. It follows that this second 'translation' may be identified with $\bar{P}(\lambda - e_i)$ again. Since this is indecomposable, the *first* translation cannot be split, and hence is precisely $\bar{P}(\lambda)$ — with the same decomposition pattern. For the last part use (6.8). $\square$

(**7.5**) **Proposition**. *Fix δ. Pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. In case $|\lambda/\alpha\lambda| = 2$*
(I) $\mathbf{b}_\delta(\lambda) = \hat{\alpha}\mathbf{b}_\delta(\lambda - e_i)$, $\mathbf{b}_\delta(\alpha\lambda) = \check{\alpha}\mathbf{b}_\delta(\lambda - e_i)$.
(II) If $(\bar{P}(\lambda - e_i) : \bar{\Delta}(\nu)) = h_\delta(\lambda - e_i)_\nu$ (all ν) then $(\bar{P}(\lambda) : \bar{\Delta}(\mu)) = h_\delta(\lambda)_\mu$ (all μ).

Proof: (I) As shown in the proof of Prop. 7.3, removing e_i from λ makes that row part of a singular pair with the row containing the box with opposite charge. Thus a pair which contributed an 01 sequence in $\mathbf{b}_\delta(\lambda)$ does not contribute to $\mathbf{b}_\delta(\lambda - e_i)$ — i.e. $\mathbf{b}_\delta(\lambda) = \hat{\beta}\mathbf{b}_\delta(\lambda - e_i)$ for some β. It remains to confirm the *position* of the modification.   For some $x \geq 0$ and some i' we have

$$
\begin{array}{lll}
\mathbf{e}_\delta(\lambda) & = (\lambda_1 - \frac{\delta}{2}, ..., & \overbrace{x+1}^{i-th} \quad ,..., \quad \overbrace{-x}^{i'-th} \quad ,...) \\
\mathbf{e}_\delta(\lambda - e_i) & = (\lambda_1 - \frac{\delta}{2}, ..., & x \quad ,..., \quad -x \quad ,...) \\
\mathbf{e}_\delta(\alpha\lambda) = \mathbf{e}_\delta(\lambda - e_i - e_{i'}) & = (\lambda_1 - \frac{\delta}{2}, ..., & x \quad ,..., \quad -x-1 \quad ,...)
\end{array}
$$

Altogether the i, i'-pair contribute an 01 (resp.10) in $\mathbf{b}_\delta(\lambda)$ (resp. $\mathbf{b}_\delta(\alpha\lambda)$).   Since the α action on λ manifests (by definition) as $10 \leftrightarrow 01$ in the $\alpha, \alpha + 1$ position of $\mathbf{b}_\delta(\lambda)$ we see that $\beta = \alpha$.

(II) Applying $\mathrm{Proj}_\lambda-$ to Proposition 3.14(ii) here we get a short exact sequence

$$
0 \rightarrow \bar{\Delta}(\lambda - e_i - e_{i'}) \rightarrow \mathrm{Proj}_\lambda \mathrm{Ind}\, \bar{\Delta}(\lambda - e_i) \rightarrow \bar{\Delta}(\lambda) \rightarrow 0 \tag{12}
$$

(non-split, by [7, Lemma 4.10]).   Translating $\bar{P}(\lambda - e_i)$ away from and then back to $\lambda - e_i$ therefore produces a projective whose dominating content is two copies of $\bar{\Delta}(\lambda - e_i)$ (one from each of the factors in (12)).   Hence, by (3.17), $\mathrm{Proj}_{\lambda-e_i} \mathrm{Ind}\ (\mathrm{Proj}_\lambda \mathrm{Ind}\ \bar{P}(\lambda - e_i)) = \bar{P}(\lambda - e_i) \oplus \bar{P}(\lambda - e_i)$. It follows that $\mathrm{Proj}_\lambda \mathrm{Ind}\ \bar{P}(\lambda - e_i) = \bar{P}(\lambda)$. Now assume $(\bar{P}(\lambda - e_i) : \bar{\Delta}(-)) = h_\delta(\lambda - e_i)$. It remains to show that $(\mathrm{Proj}_\lambda \mathrm{Ind} \bar{P}(\lambda - e_i) : \bar{\Delta}(-)) = h_\delta(\lambda)$.

For each $\bar{\Delta}(\mu)$ in $\bar{P}(\lambda - e_i)$ one sees readily that $\mathrm{Proj}_\lambda \mathrm{Ind} \bar{\Delta}(\mu) = \bar{\Delta}(\hat{\alpha}\mu) + \bar{\Delta}(\check{\alpha}\mu)$. That the collection thus engendered overall is $h_\delta(\lambda)$ now follows directly from (I) and Equation(9). Indeed we have (non-split [7, Lemma 4.10]) $0 \rightarrow \bar{\Delta}(\check{\alpha}\mu) \rightarrow \mathrm{Proj}_\lambda \mathrm{Ind}\ \bar{\Delta}(\mu) \rightarrow \bar{\Delta}(\hat{\alpha}\mu) \rightarrow 0$. $\square$

Proposition 7.5 completes the cases for the main inductive step, establishing the Theorem. $\square$

(**7.6**) Example for Proposition 7.5: $\delta = 1$, computing for $\lambda = 4422$ via $\lambda - e_2 = 4322$. We have $\mathbf{e}_1(4322) = (7/2, 3/2, -1/2, -3/2, ...)$ so $o_1(4322) = \mathrm{toggle}(\{2\}) = \{1, 2\}$. By the inductive hypothesis we have

$$
(\bar{P}(4322) : \bar{\Delta}(-)) = h_1(4322)_- = \quad
\begin{array}{ccccc}
4322 & & 12 & & 01 \\
& \searrow & & \searrow & \\
& 221 & \cong \underline{12} & \emptyset & \cong & 10
\end{array}
$$

Here the first form of the hypercube is in partition labelling; the second form is in $P(\mathbb{N})$ labelling (having applied the toggle); and the last is the untoggled binary

representation. Note that we have reverted to the untoggled form at the last since we will be inserting an 01 subsequence (removing the need for the toggle) at the next step. Translating off the wall we get $4322+221 \rightarrow (4422+4321)+(321+22)$. In binary this corresponds to $01 \rightarrow 0**1 \rightarrow 0101+0011$ and $10 \rightarrow 1**0 \rightarrow 1100+1010$. These four sequences therefore encode the content of P_{4422}.

   The Theorem is verified in this case, since:

$$
h_1(4422) = \quad 4321 \underset{22}{\overset{4422}{\diamondsuit}} 321 \;\cong\; 24 \underset{12}{\overset{34}{\underset{14}{\diamondsuit}}}{}_{23} 13 \;\cong\; 0101 \underset{11}{\overset{0011}{\diamondsuit}} 101
$$

Note how the insertion of a binary pair in the α position, and action of α on that pair, transforms $h_1(4322)$ to produce $h_1(4422)$. The effect is (i) to extend the hypercube by a new generating direction (labelled by α); (ii) the generating edge inherited from $h_1(4322)$ changes label from 12 to 14 due to the bump (which illustrates how such non-G_{even} edge labels arise in this construction).

8. On parabolic Kazhdan–Lusztig polynomials

   Associated to each Coxeter system (W', S') and parabolic (W, S), acting as reflection groups on space V, is an array $P = P(W'/W)$ of Kazhdan–Lusztig polynomials — one for each ordered pair of alcoves. (Deodhar's recursive formula [11] computes these polynomials in principle. However it generally tells us little about them in practice.) These polynomials play analogous roles to h^a in certain cases in representation theory (see [22, 20] and references therein). Finally, then, we explain where the combinatorial *idea* for the form of h^a comes from, by computing $P(\mathcal{D}/\mathcal{D}_+)$.

8.1. **The recursion for array $P(W'/W)$.** Let (W', S') be a Coxeter system, containing (W, S) as a parabolic subsystem. Let G_a be the equivalent of G_{alc} in this case, and write $(\mathcal{A}^+, <)$ for the poset defined by this acyclic digraph. The array $P = P(W'/W)$ is a (generally semiinfinite) lower unitriangular matrix, with row and column positions indexed by $\mathcal{A}^+$. Write $P = (p_{AB})_{A,B \in \mathcal{A}^+}$. It is natural to organise this data into rows (although it is also of interest to organise into columns). The rows are thus of finite support.

   The recursion for rows of P above the root in the poset order is as follows (see [22] for equivalent constructions). To compute the row p_A we first compute another polynomial for each alcove D, p'_{AD}, also denoted $p'_A(D)$ as follows. (Actually $p'_A(D)$ can depend on the choice made next in the computation, but p_A does not and we supress this dependence in notation.)
Pick an edge (B, A) in G_a ending at A (so p_B is known). For each alcove D let $\Gamma^{\pm}_D$ be the set of alcoves D' of G_a such that (D', D) (resp. (D, D')) is an edge in the orbit of the edge (B, A). (By the Cayley property (4.20) we can express $(B, A) = (B, Bs)$, $s \in S'$, whereupon any such D' must obey $(D', D) = (D', D's) = (Ds, D)$ (respectively $(D, D') = (D, Ds)$).) Then

$$
p'_A(D) = \sum_{D' \in \Gamma^+_D} (v^{-1} p_B(D) + p_B(D')) + \sum_{D' \in \Gamma^-_D} (v p_B(D) + p_B(D')) \qquad (13)
$$

(As noted there is at most one edge in the orbit of (B, A) involving any alcove D. Thus at most one of these sums is non-trivial, and that contains only one entry. In

particular (B, A) is in its own orbit, so $p'_A(A) = v^{-1}p_B(A) + p_B(B) = 1$.) Finally,

$$p_A = p'_A - \sum_{D<A} p'_A(D)(v=0)\, p_D \tag{14}$$

8.2. **Hypercubes revisited.** For $i < j \in \mathbb{N}$ recall the 'operator' $\langle ij\rangle-$. This has action defined on $a \in P_{even}(\mathbb{N})$ in case one of i, j is in a, whereupon it swaps it for the other. Otherwise $\langle ij\rangle a$ is undefined. Example: $\langle 36\rangle 56 = 35$. In particular set $\langle \alpha\rangle := \langle \alpha \ \ \alpha+1\rangle$. Note that if $\langle\alpha\rangle a$ is defined then $(a, \langle\alpha\rangle a)$ is an edge in G_{even} with label α. Where defined, each $\langle ij\rangle$ acts involutively; and takes a to $\langle ij\rangle a$ comparable to a in the G_{even} order. Each $\langle ij\rangle$ has the same effect on the given a as some (strictly descending (or ascending)) sequence of $\langle\alpha\rangle$'s. In our example $56 \xrightarrow{4} 46 \xrightarrow{3} 36 \xrightarrow{5} 35$. Similarly operator $\langle \underline{ij}\rangle-$ has action defined in case both or neither of i, j are in a, and toggles this state. Example: $\langle\underline{16}\rangle 1456 = 45$ which expands, for example, as $1456 \xrightarrow{3} 1356 \xrightarrow{2} 1256 \xrightarrow{4} 1246 \xrightarrow{5} 1245 \xrightarrow{12} 45$.

(8.1) Lemma. *Suppose $\{\alpha, \alpha+1\} \in \Gamma_a$ (so $\langle\alpha\rangle a < a$). Let $\{\alpha\} \cup X$, $\{\alpha+1\} \cup Y$ be parts in $\mathcal{T}(\langle\alpha\rangle a)$ (X, Y could contain a vertex or be empty). Then $\mathcal{T}(a)$ differs from $\mathcal{T}(\langle\alpha\rangle a)$ in that these parts are replaced by $\{\alpha, \alpha+1\}$, $X \cup Y$ ($X \cup Y$ may be empty).*

Proof: It is clear that $\{\alpha, \alpha+1\}$ is in $\mathcal{T}(a)$, so it remains to consider X, Y; and to show that all other pairs agree between $\mathcal{T}(a)$ and $\mathcal{T}(\langle\alpha\rangle a)$. If $X \cup Y = \emptyset$ then $\alpha, \alpha+1$ singletons in $\langle\alpha\rangle a$ and there are no pairs bridging over them, so no other pair is changed between $\langle\alpha\rangle a$ and a.
If $X = \{i\}$, $Y = \{j\}$ say, then $j \in \langle\alpha\rangle a$ (since $\alpha+1 \notin \langle\alpha\rangle a$ by construction). Suppose $j > \alpha+1$ and $i < \alpha$. Then we are in a situation like
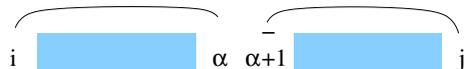
By construction there are no 11 pairs in the i, α or $\alpha+1, j$ intervals. The algorithm for extracting the sequences in the shaded regions will thus operate in the same way for each sequence. In a the algorithm generates a pair at $\alpha, \alpha+1$ as already noted, so we may pass to an iteration where these and both shaded parts have been dealt with. Vertex i is not involved in a pair from below (else it would be in $\langle\alpha\rangle a$), and $j \in a$, so we get a pair $\{i, j\}$ as required.
Suppose $j > \alpha+1$ and $i > j$. Then we are in a situation like
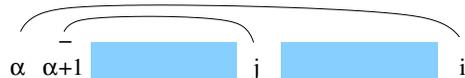
The same argument goes through until noting that $\alpha, i \in \langle\alpha\rangle a$, so that there is an even number of 1s in the remainder sequence (algorithm stage 5) left of α. This even property still holds for a, so j is not involved in a pair from below. Again we have the required outcome. The other cases are similar. $\square$

8.3. **Kazhdan–Lusztig polynomial Theorem.** We continue to use labels $a \subset \mathbb{N}$ for alcoves, and hence the rows (and columns) of $P(\mathcal{D}/\mathcal{D}^+)$. That is, there is a polynomial $p_a(b) = p_{a,b}$, in formal variable v, for each pair $a, b \in P_{even}(\mathbb{N})$. We write $p_a = \{p_{a,b}\}_{b \in P_{even}(\mathbb{N})}$ for the complete row of the array labelled by a.

Following (5.7) we define polynomial h_b^a by $h_b^a = v^i$ if b appears in hypercube h^a at depth i; and $h_b^a = 0$ if b does not appear in h^a.

(8.2) Theorem. *Let $a, b \subset \mathbb{N}$ label alcoves for $P(\mathcal{D}/\mathcal{D}^+)$ via ϕ_+. Then $p_{a,b} = h_b^a$.*

Proof: We work by induction on the graph order on a. We can then get p_a by looking at $p_{\langle\alpha\rangle a}$, where α labels one of the edges in the 'shoulder' of h^a. For any $\langle\alpha\rangle$ and any $b \in P(\mathbb{N})$ let

$$\langle\alpha\rangle h^b := \{\langle\alpha\rangle c \mid c \in h^b; \ \langle\alpha\rangle c \text{ defined}\} \qquad \text{and} \qquad \langle\alpha\rangle^2 h^b := \{c \mid c \in h^b; \ \langle\alpha\rangle c \text{ defined}\}$$

For example $\langle\alpha\rangle h^{\langle\alpha\rangle a} \ni a$ since $\langle\alpha\rangle\langle\alpha\rangle a = a$. Note that $\zeta : \langle\alpha\rangle h^{\langle\alpha\rangle a} \rightarrow \langle\alpha\rangle^2 h^{\langle\alpha\rangle a}$ given by $c \mapsto \langle\alpha\rangle c$ is a bijection between disjoint sets. By eq.(13) and Theorem 4.22, $p_{a,c} \neq 0$ for alcove c if there is a c' in $p_{\langle\alpha\rangle a}$ that, as a vertex of G_{even}, has an edge labelled α attached to it, and either $c = c'$ or $c = \langle\alpha\rangle c'$ (eq.(14) not withstanding). The vertices occuring in p_a will thus be those in $\langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$. By the bijection ζ and the inductive hypothesis every alcove appears in at most one way, so every non-zero polynomial will be of form v^i. We need to check that p_a agrees with h^a. For any $b \in P(\mathbb{N})$ define

$$\Gamma^b \setminus \alpha = \Gamma^b \setminus \{\alpha, \alpha{+}1\} \qquad \text{and} \qquad \Gamma^b(\alpha) = \{\{i, j\} \in \Gamma^b \mid \{i, j\} \cap \{\alpha, \alpha{+}1\} = \emptyset\}$$

Consider the 'ideal' $I_{\langle\alpha\rangle a}$ with vertices $c \leq \langle\alpha\rangle a$ in hypercube h^a (Figure 5). Note that

$$I_{\langle\alpha\rangle a} = h(\langle\alpha\rangle a, \Gamma^a \setminus \alpha) \tag{15}$$

and that the quotient of h^a by this ideal has the same shoulder set. Note also that this quotient $h^a / I_{\langle\alpha\rangle a}$ consists of the images under α of the vertices in $I_{\langle\alpha\rangle a}$.

It follows from Lemma 8.1 that $\Gamma^a \setminus \alpha$ agrees with $\Gamma^{\langle\alpha\rangle a}(\alpha)$ *except* that if there are pairs α, i and $\alpha{+}1, j$ in $\Gamma^{\langle\alpha\rangle a}$ then there will be a pair i, j in $\Gamma^a \setminus \alpha$ (that obviously does not appear in $\Gamma^{\langle\alpha\rangle a}$). That is $\Gamma^{\langle\alpha\rangle a}(\alpha) \subseteq \Gamma^a \setminus \alpha$. From (15) we then have that $h(\langle\alpha\rangle a, \Gamma^{\langle\alpha\rangle a}(\alpha))$ is a subgraph of $I_{\langle\alpha\rangle a}$ and hence of h^a (albeit one layer down from the 'head'), and also of $h^{\langle\alpha\rangle a}$.

It follows that all the vertices in $h(\langle\alpha\rangle a, \Gamma^{\langle\alpha\rangle a}(\alpha))$ have α-images (and these images are above in the graph order). Thus all these vertices and images appear in p_a, by (13). The power of v for each image vertex is inherited from the original vertex (for example $p_a(a) = p_a(\langle\alpha\rangle\langle\alpha\rangle a) = p_{\langle\alpha\rangle a}(\langle\alpha\rangle a) = v^0$), while the power of v for the original vertex is raised by 1 (example: $p_a(\langle\alpha\rangle a) = v p_{\langle\alpha\rangle a}(\langle\alpha\rangle a) = v v^0 = v^1$). Thus *all these vertices have exponent in agreement with h^a.*

The other vertices in the shoulder of $h^{\langle\alpha\rangle a}$ (the ones, if any, at the end of edges of form α, i and $\alpha{+}1, j$) do not have α-images. Thus we have agreement between h^a and $p_a \sim \langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$ except for the ideal generated by $\langle ij\rangle a$ as above (if any) in h^a on the one hand; and the possible descendents of $\langle\alpha, i\rangle\langle\alpha\rangle a$ and $\langle\alpha{+}1, j\rangle\langle\alpha\rangle a$ in $h^{\langle\alpha\rangle a}$ that *do* have α-images on the other.

If there is no such $\langle ij\rangle a$ then there are no descendents of $\langle\alpha, i\rangle\langle\alpha\rangle a$ and $\langle\alpha{+}1, j\rangle\langle\alpha\rangle a$ in $h^{\langle\alpha\rangle a}$ with α-images and we are done. So let us suppose there is $\langle ij\rangle a$ in h^a. Then for our a we have

$$\langle ij\rangle a = \langle\alpha, i\rangle\langle\alpha{+}1, j\rangle\langle\alpha\rangle a \tag{16}$$

Example: $a = \{1, 5, 8, 10, 11, 12\} \overset{\langle 45\rangle}{\rightarrow} 1 \ 4 \ 8 \ 10 \ 11 \ 12 \overset{\langle 5 \ 12\rangle}{\rightarrow} 1 \ 4 \ 5 \ 8 \ 10 \ 11 \overset{\langle 3 \ 4\rangle}{\rightarrow}$ $1 \ 3 \ 5 \ 8 \ 10 \ 11 = \langle 3 \ 12\rangle a$. A similar version works for $\underline{\langle ij\rangle}$ operators.

The $\langle ij \rangle a$ in $h^{\langle \alpha \rangle a}$ is in level 2 (v^2) by (16), and has a hypercube $h(\langle ij \rangle a, \Gamma^{\langle \alpha \rangle a}(\alpha))$ below it. All the elements of this hypercube have α-images, since $\langle \alpha \rangle, \langle ij \rangle$ commute. Note for example that $\langle ij \rangle a$ itself has an α-image (although $\langle ij \rangle a$ is below $\langle \alpha + 1, j \rangle \langle \alpha \rangle a$, which does not have an α-image, in the *graph* order), and that its α-image $\langle \alpha \rangle \langle ij \rangle a$ is *below* it in the graph order. The other labels in the ideal behave similarly. Thus the polynomials asigned by Equation(13) to the relevant part of $p_a \sim \langle \alpha \rangle^2 h^{\langle \alpha \rangle a} \cup \langle \alpha \rangle h^{\langle \alpha \rangle a}$ are, for v^k the relevant polynomial from $p_{\langle \alpha \rangle a}$, v^k (for the α-image) and v^{k-1} (the vertex 'left behind') respectively. The -1 compensates for the vertex appearing in $h^{\langle \alpha \rangle a}$ one layer lower than in h^a (where it appears in the shoulder in the case of $\langle ij \rangle a$ itself for example), so subject to the working assumptions we verify $p_a \equiv h^a$. Note finally that this -1 increment only occurs for the vertex $\langle ij \rangle a$ and those below it, and thus for polynomials v^k with exponent $k \geq 2$. Thus we never have an increment of form $v^1 \to v^{1-1} = v^0$, so no subtraction (14). $\square$

Concluding remarks. A planned application of this work is as a base for corresponding calculations over fields of finite characteristic (cf. [8, §6]). A *physically* motivated application is in computing eigenvectors of the Young matrix (the adjacency matrix of the Young graph [18]), which are involved in quantum spin chain computations (see e.g. [6]). We note that formal connections between parabolic Kazhdan–Lusztig polynomials and Brauer algebra decomposition matrices can be constructed in principle by other approaches [21]. However such formal approaches do not give the specific decomposition numbers that we compute here (and which are required for the applications mentioned). Finally we note that [14, 3] include formulations of 'inverse' Kazhdan–Lusztig polynomials related to the $\mathcal{D}/\mathcal{D}_+$ case, considered from a different perspective.

References

1. F W Anderson and K R Fuller, *Rings and categories of modules*, Springer, 1974.
2. D J Benson, *Representations and cohomology I*, Cambridge, 1995.
3. B D Boe, *Kazhdan–lusztig polynomials for Hermitian symmetric spaces*, Tr. A.M.S. **309** (1988), 279–294.
4. R Brauer, *On algebras which are connected with the semi–simple continuous groups*, Annals of Mathematics **38** (1937), 854–872.
5. W P Brown, *An algebra related to the orthogonal group*, Michigan Math. J. **3** (1955-56), 1–22.
6. C Candu and H Saleur, *A lattice approach to the conformal $OSp(2s+2|2s)$ supercoset sigma model. part I: Algebraic structures in the spin chain. the Brauer algebra*, Nucl.Phys. **B808** (2009), 441–486.
7. A G Cox, M De Visscher, and P P Martin, *The blocks of the Brauer algebra in characteristic zero*, Representation Theory **13** (2009), 272–308, (math.RT/0601387).
8. _____, *A geometric characterisation of the blocks of the Brauer algebra*, JLMS **80** (2009), 471–494, (math.RT/0612584).
9. _____, *Alcove geometry and a translation principle for the Brauer algebra*, J Pure Appl. Algebra **215** (2011), 335–367, (arXiv:0807.3892v1).
10. C W Curtis and I Reiner, *Methods of representation theory with applications to finite groups and orders*, vol. 1, Wiley, New York, 1990.
11. V V Deodhar, *On some geometric aspects of Bruhat orderings II. the parabolic analogue of Kazhdan-Lusztig polynomials*, J Algebra **111** (1987), 483–506.

12. S Donkin, *The q-Schur algebra*, LMS Lecture Notes Series, vol. 253, Cambridge University Press, 1998.
13. W F Doran, D B Wales, and P J Hanlon, *On semisimplicity of the Brauer centralizer algebras*, J Algebra **211** (1999), 647–685.
14. T J Enright and B Shelton, *Categories of highest weight modules: applications to classical Hermitian symmetric pairs*, Memoirs AMS **67** (1987), no. 367.
15. J A Green, *Polynomial representations of GL_n*, Springer-Verlag, Berlin, 1980.
16. J E Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.
17. G D James and A Kerber, *The representation theory of the symmetric group*, Addison-Wesley, London, 1981.
18. S V Kerov, *Asymptotic representation theory of the symmetric group and its applications in analysis*, Translations, vol. 219, AMS, 2003.
19. P P Martin, *Temperley–Lieb algebras for non–planar statistical mechanics — the partition algebra construction*, Journal of Knot Theory and its Ramifications **3** (1994), no. 1, 51–82.
20. P P Martin and D Woodcock, *Generalized blob algebras and alcove geometry*, LMS J Comput Math **6** (2003), 249–296, (math.RT/0205263).
21. R Orellana and A Ram, *Affine braids, Markov traces and the category O*, Newton Institute preprint NI01032-SFM, 2001.
22. W. Soergel, *Kazhdan-Lusztig polynomials and a combinatoric for tilting modules*, Representation Theory **1** (1997), 83–114.
23. H Wenzl, *On the structure of Brauer's centralizer algebras*, Ann. Math. **128** (1988), 173–193.

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