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INDEFINITENESS IN SEMI-INTUITIONISTIC SET THEORIES: ON A CONJECTURE OF FEFERMAN

MICHAEL RATHJEN

Abstract. The paper proves a conjecture of Solomon Feferman concerning the indefiniteness of the continuum hypothesis relative to a semi-intuitionistic set theory.

§1. Introduction. Frege in [13, Section 68] wrote: *Ich setze voraus, dass man wisse, was der Umfang eines Begriffes sei.*[1] Dummett's diagnosis of the failure of Frege's logicist project in the final chapter of [8] focusses on the adoption of classical quantification over domains comprised of objects falling under an indefinitely extensible concept. He repudiates the classical view as illegitimate and puts forward reasons in favor of an intuitionistic interpretation of quantification. Solomon Feferman, in recent years, has argued that the Continuum Hypothesis (CH) might not be a definite mathematical problem (see [10–12][2]).

> My reason for that is that the concept of arbitrary set essential to its formulation is vague or underdetermined and there is no way to sharpen it without violating what it is supposed to be about. In addition, there is considerable circumstantial evidence to support the view that CH is not definite. ([10, p.1]).

In particular the power set, $\mathcal{P}(A)$, of a given set A may be considered to be an indefinite collection whose members are subsets of A, but whose exact extent is indeterminate (open-ended). In [10], Feferman proposed a logical framework for what's definite and for what's not.

> One way of saying of a statement φ that it is definite is that it is true or false; on a deflationary account of truth that's the same as saying that the Law of Excluded Middle (LEM) holds of φ, i.e., one has $\varphi \vee \neg\varphi$. Since LEM is rejected in intuitionistic logic as a basic principle, that suggests the slogan, "What's definite is the domain of classical logic, what's not is that of intuitionistic logic." [...] And in the case of set theory, where every set is conceived to be a definite totality,

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[1] Translation: *I assume that it is known what the extension of a concept is.*

[2] Incidentally, the paper [10] was written for Peter Koellner's *Exploring the frontiers of incompleteness* (EFI) Project, Harvard 2011-2012.

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> we would have classical logic for bounded quantification while intuitionistic logic is to be used for unbounded quantification. ([10, p. 23])

At the end of [10] he made that idea more precise by suggesting semi-intuitionistic set theories as frameworks for formulating questions of definiteness and studying the definiteness of specific set-theoretic statements. In relation to CH, he conjectured that this statement is not definite in the specific case of a semi-intuitionistic set theory **T**, in the sense that **T** does not prove CH $\vee$ $\neg$CH. The set-theoretical point of view expressed by **T** accepts the definiteness of the continuum in its guise as the arithmetical/geometric structure of the real line, but does not allow the powerset operation to be applied to arbitrary sets.

The objective of this paper is to prove Feferman's conjecture. In this sense it is a technical paper. It lays out new evidence for the reader to consider. However, as far as the ongoing discussions of the foundational status of CH are concerned, readers will have to form their own conclusions.

A chief technique applied in this article is realizability over relativized constructible hierarchies combined with forcing. More widely the impression is that CH is not an isolated case in that other statements could be proved to be indefinite relative to semi-intuitionistic set theories in this way. At any rate, it appears that the paper adds a hitherto unexplored tool to the weaponry earmarked for engineering specific realizability models and proving independence results.

An outline of the paper reads as follows: Section 2 introduces formal systems of semi-intuitionistic set theory and in particular the theory **T**. Section 3 is devoted to the relativized constructible hierarchy $L[A]$ and its properties. In Section 4, $L[A]$ features as a domain of computation which gets utilized in Section 5 as a realizability universe for **T**. By carefully designing sets of ordinals C and E and employing results from forcing, realizability of **T** over $L[C]$ and $L[C \cup E]$ yields conflicting information that leads to a contradiction, and thus provides a proof of the desired conjecture.

§2. Semi-intuitionistic set theory.

The study of subsystems of **ZF** formulated in intuitionistic logic with Bounded Separation was apparently initiated by Pozsgay [21, 22] and then pursued more systematically by Tharp [29], Friedman [14] and Wolf [30]. These systems are actually semi-intuitionistic as they contain the law of excluded middle for bounded formulae.

Classical Kripke–Platek set theory, **KP**, is an important theory that accommodates a great deal of set theory. Its transitive models, called admissible sets, have been a major source of interaction between model theory, recursion theory and set theory (cf. [3]). **KP** arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to bounded formulae. Here we are interested in its intuitionistic cousin.

Definition 2.1. *Intuitionistic Kripke–Platek set theory*, **IKP**, is formulated in the usual language of set theory containing $\in$ as the only non-logical symbol besides $=$. Formulae are built from prime formulae $a \in b$ and $a = b$ by use of propositional connectives and quantifiers $\forall x, \exists x$. Quantifiers of the forms $\forall x \in a$, $\exists x \in a$ are called *bounded*. *Bounded* or Δ_0-*formulae* are the formulae wherein all quantifiers are bounded. **IKP** is based on intuitionistic logic and has the following non-logical axioms: *Extensionality, Pair, Union, Infinity* (in the specific version that there is a smallest set containing the empty set 0 and closed under the successor operation, $x' = x \cup \{x\}$), *Bounded Separation*

$$\exists x\, \forall u\, \left[u \in x \leftrightarrow (u \in a\, \wedge\, \varphi(u))\right]$$

for all bounded formulae $\varphi(u)$, *Bounded Collection*

$$\forall x \in a\, \exists y\, \psi(x, y)\, \rightarrow\, \exists z\, \forall x \in a\, \exists y \in z\, \psi(x, y)$$

for all bounded formulae $\psi(x, y)$, and *Set Induction*

$$\forall x\, \left[(\forall y \in x\, \theta(y)) \rightarrow \theta(x)\right] \rightarrow\, \forall x\, \theta(x)$$

for all formulae $\theta(x)$.

Feferman in [9] proceeded to add several further schemata to the axioms of **IKP**. The most basic principle that he added follows from the idea that in semi-constructive set theory each set is considered to be a definite totality. As a consequence of Δ_0 separation one obtains a restricted LEM:

$$(\Delta_0\text{−LEM})\quad \varphi \vee \neg\varphi,\ \text{for all } \Delta_0\text{-formulae } \varphi.$$

Markov's principle in the form

$$(\mathbf{MP})\quad \neg\neg\exists x\varphi \rightarrow \exists x\varphi,\ \text{for all } \Delta_0\text{-formulae } \varphi$$

is another principle that is frequently added in this context.

Some further principles that are considered in [9] are (BOS) (Bounded Omniscience Scheme) and $\mathrm{AC}_{\mathrm{Set}}$ (Axiom of Choice).

$$(\mathrm{BOS})\quad \forall x \in a\, [\varphi(x) \vee \neg\varphi(x)] \rightarrow [\forall x \in a\, \varphi(x)\ \vee\ \exists x \in a\, \neg\varphi(x)]$$

for all formulae $\varphi(x)$.

$$(\mathrm{AC}_{\mathrm{Set}})\quad \forall x \in a\, \exists y\, \psi(x, y) \rightarrow \exists f\, [\mathrm{Fun}(f)\, \wedge\, \mathrm{dom}(f) = a\, \wedge\, \forall x \in a\, \varphi(x, f(x))]$$

for all formulae $\psi(x, y)$, where $\mathrm{Fun}(f)$ expresses in the usual set-theoretic form that f is a function, and $\mathrm{dom}(f) = a$ expresses that the domain of f is the set a.

Feferman [9, Theorem 6] shows that **SCS** := **IKP** + (Δ_0−LEM) + (MP) + (BOS) + ($\mathrm{AC}_{\mathrm{Set}}$) has the same proof-theoretic strength as **KP** (and therefore the same as **IKP**). His proof uses a functional interpretation. The same result can be obtained via a realizability interpretation using codes for Σ_1 partial recursive set functions as realizers along the lines of Tharp's 1971 arcticle [29].

REMARK 2.2.

(i) **SCS** proves the full replacement schema of **ZF**. Moreover, **SCS** proves strong collection, i.e., all formulae

$$\forall x \in a\, \exists y\, \varphi(x, y) \rightarrow \exists z\, [\forall x \in a\, \exists y \in z\, \varphi(x, y)\, \wedge\, \forall y \in z\, \exists x \in a\, \varphi(x, y)],$$

where $\varphi(x, y)$ is an arbitrary formula.

Strong collection is an axiom schema of Constructive Zermelo-Fraenkel set theory, **CZF** (cf. [1, 2]) and also of Tharp's set theory [29].

(ii) **SCS** is a subtheory of Tharp's semi-intuitionistic set theory **IZF** [29], for if $\forall x \in a\, \exists y\, \varphi(x, y)$ holds, then there is a set d such that $\forall x \in a\, \exists z \in d\, \exists y[z = \langle x, y\rangle \wedge \varphi(x, y)]$ and $\forall z \in d\, \exists x \in a\, \exists y[z = \langle x, y\rangle \wedge \varphi(x, y)]$ using (strong) collection, and by axiom 6 of **IZF**, d is the surjective image of an ordinal, i.e., there is an ordinal α and a function g with domain α and range d. Note that d is a set of ordered pairs. Now define a function f with domain a by letting $f(x)$ be the second projection of $g(\xi)$ where ξ is the least ordinal $< \alpha$ such that the first projection of $g(\xi)$ equals x.

As it turns out, some of the axioms of **SCS** are redundant.

PROPOSITION 2.3. **IKP** $+ (\mathrm{AC}_{\mathrm{Set}})$ *proves* $(\Delta_0{-}\mathrm{LEM})$ *and* (BOS).

PROOF. First we prove $(\Delta_0{-}\mathrm{LEM})$, using Diaconescu's old constructions [7]. Let 0 be the empty set, $1 := \{0\}$ and $A = \{0, 1\}$. Note that (intuitionistically) $\forall x, y \in A\,[x = y \vee x \neq y]$ (where $x \neq y$ abbreviates $\neg x = y$) since $0 \neq 1$ as $0 \in 1$ and $0 \notin 0$. Suppose φ is Δ_0. Define

$$a := \{n \in A \mid n = 0 \vee [n = 1 \wedge \varphi]\}$$

and

$$b := \{n \in A \mid n = 1 \vee [n = 0 \wedge \varphi]\}.$$

a and b are sets by Δ_0 separation. Obviously we have

$$\forall z \in \{a, b\}\, \exists k \in A\; k \in z$$

since $0 \in a$ and $1 \in b$. So we may apply $(\mathrm{AC}_{\mathrm{Set}})$ to obtain a function f with domain $\{a, b\}$ such that $f(a), f(b) \in A$. We thus have $f(a) = f(b)$ or $\neg(f(a) = f(b))$. In the first case, we can infer that φ. In the second case, we have $a \neq b$. As φ implies $a = b$, we get $\neg\varphi$.

To show (BOS) assume

$$\forall x \in a\,[\psi(x) \vee \neg\psi(x)],$$

where $\psi(x)$ is an arbitrary formula. Thus,

$$\forall x \in a\, \exists y\,[(\psi(x) \wedge y = 0) \vee (\neg\psi(x) \wedge y = 1)].$$

With the help of $(\mathrm{AC}_{\mathrm{Set}})$ there is a function f with domain a such that

$$\forall x \in a\,[(\psi(x) \wedge f(x) = 0) \vee (\neg\psi(x) \wedge f(x) = 1)] \tag{1}$$

and hence $\forall x \in a\,[f(x) = 0) \vee f(x) = 1]$. Using $(\Delta_0{-}\mathrm{LEM})$ we have $\exists x \in a\; f(x) = 1$ or $\forall x \in a\; f(x) = 0$. In the former case we deduce $\exists x \in a\, \neg\psi(x)$ from (1), whereas in the latter case we infer that $\forall x \in a\; \psi(x)$. ⊣

DEFINITION 2.4. Let **T** be the theory

$$\mathbf{SCS} + \text{‘}\mathbb{R} \text{ is a set’},$$

where **SCS** is from Definition 2.1 and ‘$\mathbb{R}$ is a set’ asserts that the reals, $\mathbb{R}$, form a set. Since **SCS** has classical logic for Δ_0-formulae it is not necessary to pay much attention to the question of how the reals are actually formalized as is so often the case in intuitionistic contexts. Thus, any of the following equivalent statements could be used to formalize the existence of $\mathbb{R}$ as a set:

- The collection of all functions from $\mathbb{N}$ to $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, is a set.
- The collection of all subsets of $\mathbb{N}$ is a set.

Remark 2.5. The proof-theoretic strength of **T** resides strictly between full classical second order arithmetic and Zermelo set theory. In particular all theorems of classical second order arithmetic are theorems of **T**.

CH is the statement that every infinite set of reals is either in one-one correspondence with $\mathbb{N}$ or with $\mathbb{R}$. More formally, this can be expressed as follows:

$$\forall x \subseteq \mathbb{R}\,[x \neq \emptyset \to (\exists f\ f : \omega \twoheadrightarrow x \vee \exists f\ f : x \twoheadrightarrow \mathbb{R})],$$

where $f : y \twoheadrightarrow z$ signifies that f is a surjective function with domain y and co-domain z.

Conjecture 2.6 (Feferman). **T** *does not prove* CH $\vee$ ¬CH.

When one ponders how to prove the conjecture one of the first ideas that comes to mind is that intuitionistic set theories S very often have the disjunction property, i.e., if $S \vdash \psi \vee \theta$ then $S \vdash \psi$ or $S \vdash \theta$ (cf. [25, 26]). If this property held for **T** it would certainly settle the conjecture in the affirmative. However, **T** being semi-intuitionistic, the disjunction property does not hold for it. The technique of realizability certainly springs to mind when tackling such problems and consequently one would like to show that there is a realizability interpretation of **T** that has no realizer for CH $\vee$ ¬CH. There are several essentially different forms of realizability for set theories to choose from (cf. [4, 5, 14, 19, 20, 23–27, 29]). Moreover, what should the realizers be and how should the realizability universe be defined?

§3. The relativized constructible hierarchy.

Later we shall look at realizability interpretations in the relativized constructible hierarchy. The latter comes in two versions: For a set A we have $L(A)$ and $L[A]$. $L(A)$ is the smallest inner model that contains A. In $L(A)$, the transitive closure of A is added at level 0 and for higher levels the definition is the same as for L, whereas in $L[A]$, A acts as an additional predicate for defining sets. The two hierarchies can be quite different. E.g., in general $L(A)$ is not a model of the axiom of choice, AC, whereas $L[A]$ is always a model of AC. Another difference is that $L[\mathbb{R}] = L$ whereas $L \neq L(\mathbb{R})$ when $\mathbb{R} \notin L$.[3] Only $L[A]$ is interesting for the purposes of this paper.

Definition 3.1. Let $\mathcal{L}_\in$ be the language of set theory and $\mathcal{L}_\in(P)$ be its augmentation by a unary predicate symbol P. Let A be a set. Any set X gives rise to a structure $\langle X, \in, A \cap X\rangle$ for $\mathcal{L}_\in(P)$ with domain X where the elementhood symbol is interpreted by the elementhood relation restricted to $X \times X$ and P is interpreted as $A \cap X$. Thereby A acts as a unary predicate on X. A subset Y of X is said to be definable in $\langle X, \in, A \cap X\rangle$ if there is a formula $\varphi(x, y_1, \dots, y_r)$ of $\mathcal{L}_\in(P)$ with all free variables exhibited and $b_1, \dots, b_r \in X$ such that for all $a \in X$,

$$a \in Y \quad \text{iff} \quad \langle X, \in, A \cap X\rangle \models \varphi(a, b_1, \dots, b_r),$$

where, of course, $\langle X, \in, A \cap X\rangle \models \varphi(a, b_1, \dots, b_r)$ signifies that φ holds in the structure under the variable assignment $x \mapsto a$ and $y_i \mapsto b_i$.

[3]Note that in the buildup of $L[\mathbb{R}]$, $\mathbb{R}$ is just used as a predicate. By identifying $\mathbb{R}$ with the set of all functions from $\mathbb{N}$ to $\mathbb{N}$, this is merely the predicate of being such a function, which is Δ_0 in $\mathbb{N}$, hence absolute. Thus nothing outside of L can be generated in this way.

The sets $L_\alpha[A]$ are defined by recursion on α as follows:

(i) $\mathrm{Def}^A(X) := \{Y \subseteq X \mid Y \text{ definable in } \langle X, \in, A \cap X\rangle\}$.
(ii) $L_0[A] = \emptyset$.
(iii) $L_{\alpha+1}[A] = \mathrm{Def}^A(L_\alpha[A])$.
(iv) $L_\lambda = \bigcup_{\xi<\lambda} L_\xi[A]$ for limits λ.
(v) $L[A] = \bigcup_\alpha L_\alpha[A]$.

The next proposition lists some important properties of $L_\alpha[A]$. Bounded quantifiers are of the form $\forall x \in y$ and $\exists x \in y$. A bounded or Δ_0-formula of $\mathcal{L}_\in(P)$ is a formula in which all quantifiers appear bounded. A formula of $\mathcal{L}_\in(P)$ of the form $\exists z\varphi(z)$ ($\forall z\varphi(z)$) with φ bounded is said to be Σ_1 (Π_1). Let $\alpha > 0$. A relation on $L_\alpha[A]$ is said to be $\Sigma_1^{L_\alpha[A]}$ ($\Pi_1^{L_\alpha[A]}$) if it is definable (with parameters) on the structure $\langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$ via a Σ_1 (Π_1) formula of $\mathcal{L}_\in(P)$. A relation on $L_\alpha[A]$ is $\Delta_1^{L_\alpha[A]}$ if it is both $\Sigma_1^{L_\alpha[A]}$ and $\Pi_1^{L_\alpha[A]}$.

For a set X, $|X|$ denotes the cardinality of X. For further unexplained notions and proofs see [6, II. pp. 102–104] or [15, 18].

Proposition 3.2.

1. $\alpha \leq \beta \Rightarrow L_\alpha[A] \subseteq L_\beta[A]$.
2. $\alpha < \beta \Rightarrow L_\alpha[A] \in L_\beta[A]$.
3. $L_\alpha[A]$ *is transitive.*
4. $L[A] \cap \alpha = L_\alpha[A] \cap \alpha = \alpha$.
5. *For* $\alpha \geq \omega$, $|L_\alpha[A]| = |\alpha|$.
6. $L[A] \models \mathbf{ZF}$.
7. $\nu \mapsto L_\nu[A]$ *is uniformly* $\Delta_1^{L_\lambda[A]}$ *for limits* $\lambda > \omega$.
8. $B = A \cap L[A] \Rightarrow L[A] = L[B] \wedge (V = L[B])^{L[A]}$.
9. *There is a* Σ_1 *formula* $\mathrm{wo}(x, y, z)$ *such that*
$$\mathbf{KP} \vdash \text{“}\{\langle x, y\rangle \mid \mathrm{wo}(x, y, a)\} \textit{ is a wellordering of } L[a]\text{”}$$
and if $<_{L[A]}$ *denotes the wellordering of* $L[A]$ *determined by* wo, *then for any limit* $\lambda > \omega$,
$$<_{L[A]} \cap (L[A] \times L[A]) \textit{ is } \Sigma_1^{L_\lambda[A]}.$$
10. $L[A]$ *is model of* **AC**.
11. $\lambda > \omega$ *limit* $\wedge$ $B = A \cap L_\lambda[A] \Rightarrow L_\lambda[A] = L_\lambda[B]$.

§4. Computability over $L[A]$.

In this section we develop the recursion theory of partial $\Sigma_1^{L[A]}$ functions, that is functions (not necessarily everywhere defined) whose graphs are $\Sigma_1^{L[A]}$. Below we shall write $L_\alpha[A] \models \varphi$ rather than the more correct $\langle L_\alpha[A], \in, L_\alpha[A] \cap A\rangle \models \varphi$. Likewise, $\langle L[A], \in, L[A] \cap A\rangle \models \varphi$ will be shortened to $L[A] \models \varphi$.

Definition 4.1. $\langle a, b\rangle$ denotes the ordered pair of two sets a and b. If c is an ordered pair $\langle a, b\rangle$ let $(c)_0 = a$ and $(c)_1 = b$. If c is not an ordered pair let $(c)_0 = (c)_1 = 0$. We also define ordered n-tuples via $\langle a_1\rangle := a_1$ and $\langle a_1, \ldots, a_n, a_{n+1}\rangle := \langle\langle a_1, \ldots, a_n\rangle, a_{n+1}\rangle$.

It is a standard procedure to assign to each formula ψ of $\mathcal{L}_\in(P)$ a Gödel number $\ulcorner\psi\urcorner$ such that $\ulcorner\psi\urcorner$ is a hereditarily definable set, for instance by using the pairing function $a, b \mapsto \langle a, b\rangle$. There is a formula $\mathrm{Sat}(v, w)$ of $\mathcal{L}_\in(P)$ such that for all Δ_0 formulae $\theta(x_1, \dots, x_n)$ of $\mathcal{L}_\in(P)$, not involving other free variables, the following holds for any limit $\lambda > \omega$ and all $\vec{a} = a_1, \dots, a_n \in L_\lambda[A]$:

$$L_\lambda \models \theta(\vec{a}) \quad \text{iff} \quad L_\lambda \models \mathrm{Sat}(\ulcorner\theta\urcorner, \langle\vec{a}\rangle). \tag{2}$$

Moreover, Sat is uniformly $\Delta_1^{L_\lambda[A]}$ for limits $\lambda > \omega$ (see [6, II]).

Now let λ be a limit $> \omega$. For $e, a_1, \dots, a_n, b \in L_\lambda[A]$ define

$$[e]_n^{L_\lambda[A]}(a_1, \dots, a_n) \simeq b \tag{3}$$

if e is an ordered pair $\langle\ulcorner\psi\urcorner, c\rangle$ with ψ being a Δ_0-formula of $\mathcal{L}_\in(P)$, not involving free variables other than $x_1, \dots, x_{n+2}$, such that

$$L_\lambda[A] \models \mathrm{Sat}(\ulcorner\psi\urcorner, \langle a_1, \dots, a_n, c, d\rangle) \tag{4}$$

and $(d)_0 = b$, where d is the $<_{L[A]}$-least ordered pair satisfying (4).

Likewise, $[e]_n^{L[A]}(a_1, \dots, a_n) \simeq b$ is defined by replacing $L_\lambda[A]$ by $L[A]$ in the foregoing definition.

Lemma 4.2. *Let $\tau > \omega$ be a limit of limits, i.e.,*

$$\forall\xi < \tau\, \exists\lambda < \tau\, [\xi < \lambda \wedge \lambda \textit{ limit}].$$

(i) *For $e \in L_\tau[A]$, the partial function f on $L_\tau[A]$ given by*

$$f(a_1, \dots, a_n) = b \quad \textit{iff} \quad [e]_n^{L_\tau[A]}(a_1, \dots, a_n) \simeq b$$

is $\Sigma_1^{L_\tau[A]}$ (uniformly for all such τ).

(ii) *For every n-ary partial $\Sigma_1^{L_\tau[A]}$ function f there exists an index $e \in L_\tau[A]$ such that, for all $a_1, \dots, a_n \in L_\tau[A]$,*

$$f(a_1, \dots, a_n) = b \quad \textit{iff} \quad [e]_n^{L_\tau[A]}(a_1, \dots, a_n) \simeq b.$$

(iii) (i) *and* (ii) *hold with $L[A]$ in place of $L_\tau[A]$.*

(iv) *$[e]_n^{L_\tau[A]}(a_1, \dots, a_n) \simeq b$ implies $[e]_n^{L[A]}(a_1, \dots, a_n) \simeq b$ and $[e]_n^{L_\lambda[A]}(a_1, \dots, a_n) \simeq b$ for all limits $\lambda > \tau$.*

(v) *If $[e]_n^{L[A]}(a_1, \dots, a_n) \simeq b$ then $[e]_n^{L_\lambda[A]}(a_1, \dots, a_n) \simeq b$ for some limit λ.*

Proof. (i) First note that by Proposition 3.2 the relation $<_{L[A]}$ restricted to $L_\lambda[A]$ is $\Sigma_1^{L_\lambda[A]}$ for all limits $\lambda > \omega$. Thus the $<_{L[A]}$-leastness of d with respect to (4) can be expressed by

$$\begin{aligned}\exists\lambda < \tau\, [\lambda \text{ limit} > \omega \wedge\ & a_1, \dots, a_n, c, d \in L_\lambda[A] \\ & \wedge L_\lambda[A] \models \mathrm{Sat}(\ulcorner\vartheta\urcorner, \langle a_1, \dots, a_n, c, d\rangle) \\ & \forall u \in L_\lambda[A](u <_{L[A]} d \;\rightarrow\; L_\lambda[A] \models \mathrm{Sat}(\ulcorner\neg\vartheta\urcorner, \langle a_1, \dots, a_n, c, u\rangle))],\end{aligned}$$

which is clearly $\Sigma_1^{L_\tau[A]}$.

(ii) Since f is $\Sigma_1^{L_\tau[A]}$ there is a Σ_1-formula $\exists x_{n+3}\vartheta_0(x_1, \dots, x_{n+3})$ of $\mathcal{L}_\in(P)$ and a parameter $c \in L_\tau[A]$ (several parameters can be coded as one) such that

$$f(a_1, \dots, a_n) = b \quad \text{iff} \quad L_\tau[A] \models \exists x_{n+3}\vartheta_0(a_1, \dots, a_n, c, b, x_{n+3}).$$

Now let

$$\vartheta(x_1, \dots, x_{n+2}) \equiv \vartheta_0(x_1, \dots, x_{n+1}, (x_{n+2})_0, (x_{n+2})_1).$$

Then

$$f(a_1,\ldots,a_n)=b \quad \text{iff} \quad L_\tau[A]\models \vartheta_0(a_1,\ldots,a_n,c,(d)_0,(d)_1] \text{ and } (d)_0=b,$$

where d is the $<_{L[A]}$-least u such that $L_\tau[A]\models \vartheta_0[a_1,\ldots,a_n,c,(u)_0,(u)_1]$. Hence, with $e=\langle \ulcorner\vartheta\urcorner, c\rangle$, we have $f(a_1,\ldots,a_n)=b$ iff $[e]_n^{L_\tau[A]}(a_1,\ldots,a_n)\simeq b$.

(iii) is proved in the same way as (i) and (ii).

(iv) follows since Σ_1 statements are upward persistent.

(v) follows since the statement is of Σ_1 form. ⊣

In several respects the recursion theory of partial $\Sigma_1^{L[A]}$ functions and partial $\Sigma_1^{L_\tau[A]}$ functions (for τ being a limit of limits) shares similarities with ordinary recursion theory over ω. In particular, the analogues of the S-m-n theorem and the recursion theorem hold.

§5. Realizability over $L[A]$.

$L[A]$ will be employed as a realizability universe. There is a germane notion of realizability where realizers are indices of partial $\Sigma_1^{L[A]}$ functions.

Definition 5.1. For $d\in L[A]$ and set-theoretic sentences ψ with parameters from $L[A]$ we define the realizability relation $d\Vdash_A \psi$.

Below we shall write $[e]^{L[A]}(\vec{a})\Vdash_A\psi$ rather than the more accurate

$$\exists u\in L[A]\,(\,[e]_n^{L[A]}(\vec{a})\simeq u\ \wedge\ u\Vdash_A\psi\,),$$

where $\vec{a}=a_1,\ldots,a_n$. It will also be assumed that all quantifiers range over $L[A]$.

$$\begin{aligned}
e\Vdash_A c\in d \quad &\text{iff}\quad c\in d,\\
e\Vdash_A c=d \quad &\text{iff}\quad c=d,\\
e\Vdash_A \varphi\wedge\psi \quad &\text{iff}\quad (e)_0\Vdash_A\varphi \text{ and } (e)_1\Vdash_A\psi,\\
e\Vdash_A \varphi\vee\psi \quad &\text{iff}\quad [(e)_0=0\wedge(e)_1\Vdash_A\varphi] \text{ or } [(e)_0=1\wedge(e)_1\Vdash_A\psi],\\
e\Vdash_A \varphi\rightarrow\psi \quad &\text{iff}\quad \forall a\,[a\Vdash_A\varphi\ \Rightarrow\ [e]^{L[A]}(a)\Vdash_A\psi],\\
e\Vdash_A \exists x\theta(x) \quad &\text{iff}\quad (e)_1\Vdash_A\theta((e)_0),\\
e\Vdash_A \forall x\theta(x) \quad &\text{iff}\quad \forall a\,[e]^{L[A]}(a)\Vdash_A\theta(a).
\end{aligned}$$

Occasionally we shall write $\Vdash_A\psi$ for $\exists e\in L[A]\ e\Vdash_A\psi$.

Theorem 5.2 (Realizability Theorem). *Let $\mathbb{R}^{L[A]}$ be the set of real numbers in the sense of $L[A]$. If $\psi(x_1,\ldots,x_n)$ is a formula of set theory, with all free variables among the exhibited, and $\mathcal{D}$ is a proof of $\psi(x_1,\ldots,x_n)$ in $\mathbf{T}$, then one can effectively construct a hereditarily finite set $e_{\mathcal{D}}$ which only depends on $\mathcal{D}$ (and not on A) such that for all $a_1,\ldots,a_n\in L[A]$,*

$$[e_{\mathcal{D}}]^{L[A]}(a_1,\ldots,a_n,\mathbb{R}^{L[A]})\Vdash_A\psi(a_1,\ldots,a_n). \tag{5}$$

Proof. With little modification, the proof of Tharp's realizability theorem [29] carries over to show this realizability theorem. It can also be gleaned from the proofs of the realizability theorems [27, Theorems 3.7–3.9], using considerable simplifications of the proofs brought about by the fact that there is uniform $\Sigma_1^{L[A]}$ selection function, i.e., there exists a hereditarily finite set e_{ac} such that for all A and nonempty sets a, $[e_{ac}]^{L[A]}(a)\in a$. ⊣

§6. Designing $L[C]$. In order to show that CH $\vee \neg$CH is not deducible in **T** we intend to employ Theorem 5.2. Aiming at a contradiction, we assume we have a derivation $\mathcal{D}$ of CH $\vee \neg$CH in **T** and thus a hereditarily finite set $e_{\mathcal{D}}$ such that

$$[e_{\mathcal{D}}](\mathbb{R}^{L[A]}) \Vdash_A \mathrm{CH} \vee \neg\mathrm{CH} \tag{6}$$

holds for all sets A. To refute this, we intend to carefully design a counterexample C.[4] We shall start from a set-theoretic universe V_0 such that

$$V_0 \models \mathbf{ZFC} + 2^{\aleph_0} = \aleph_2.$$

V_0 can be obtained from any universe V' such that $V' \models \mathbf{ZFC} + \mathrm{GCH}$ (e.g. L) by forcing with $\mathrm{Fn}(\kappa \times \omega, 2)$, where the latter denotes the set of all finite functions with domain $\subset \kappa \times \omega$ and range 2 and $\kappa = (\aleph_2)^{V'}$, i.e., κ is $\aleph_2$ in the sense of V' (see [17, VII.5.14]). Now let $\mathbb{R}^{V_0}$ be the reals in the sense of V_0. We would like to pick a set $C \in V_0$ such that $\mathbb{R}^{V_0} \in L[C]$.[5] Since V_0 satisfies AC there is an injective function F in V_0 with domain $\mathbb{R}^{V_0}$ whose range is a set of ordinals. Identifying $\mathbb{R}^{V_0}$ with the set $\{g \in V_0 \mid g : \mathbb{N} \to \mathbb{N}\}$, let

$$C = \{\omega^{F(g)+2} + \omega \cdot g(n) + n \mid g \in \mathbb{R}^{V_0}\}. \tag{7}$$

Then C is a set of ordinals in V_0 and, owing to the uniqueness of the Cantor normal form, $\mathbb{R}^{V_0}$ is definable from C in $L[C]$. The latter entails that $\mathbb{R}^{V_0} \in L[C]$ and thus

$$\mathbb{R}^{V_0} = \mathbb{R}^{L[C]}. \tag{8}$$

As a result, $L[C] \not\models$ CH and therefore

$$\text{for all } d \in L[C],\ d \not\Vdash_A \mathrm{CH}. \tag{9}$$

The assumption (6) implies that there exists $b \in L[C]$ such that $L[C] \models [e_{\mathcal{D}}](\mathbb{R}^{L[C]}) \simeq b$. Moreover, (6) and (9) entail that

$$(b)_0 = 1. \tag{10}$$

We can now pick a sufficiently large limit ordinal ρ such that $C \in L_\rho[C]$, $\mathbb{R}^{V_0} \in L_\rho[C]$ and $b \in L_\rho[C]$. By Lemma 4.2(v) we can also arrange that

$$L_\rho[C] \models [e_{\mathcal{D}}](\mathbb{R}^{L[C]}) \simeq b. \tag{11}$$

Moreover, from Lemma 4.2(iv) and Proposition 3.2(11) it follows that for every set of ordinals B with $B \cap \rho = \emptyset$ we have

$$L[C \cup B] \models [e_{\mathcal{D}}](\mathbb{R}^{L[C]}) \simeq b. \tag{12}$$

The next step consists in taking a forcing extension V_1 of V_0 which does not pick up new real numbers but satisfies $V_1 \models 2^{\aleph_0} = \aleph_1$, i.e.,

$$\mathbb{R}^{V_0} = \mathbb{R}^{V_1} \ \wedge\ (\aleph_1)^{V_0} = (\aleph_1)^{V_1} \ \wedge\ V_1 \models \mathrm{CH}. \tag{13}$$

---

[4] Note that there are sets A, A' and hereditarily finite sets e, e' such that $[e]^{L[A]}(\mathbb{R}^{L[A]}) \Vdash_A$ CH as well as $[e']^{L[A']}(\mathbb{R}^{L[A']}) \Vdash_{A'} \neg$CH, and a fortiori there exists a set A such that $[e'']^{L[A]}(\mathbb{R}^{L[A]}) \Vdash_A$ CH $\vee \neg$CH for some hereditarily finite e''.

[5] We cannot choose C to be $\mathbb{R}^{V_0}$ since $L[\mathbb{R}^{V_0}] = L$ (cf. footnote 3) and therefore $\mathbb{R}^{V_0} \notin L[\mathbb{R}^{V_0}]$ as the size of the reals in V_0 is assumed to be $\aleph_2$ whereas it's always $\aleph_1$ in L.

The latter can be arranged by forcing with

$$\mathbb{P} := (\mathrm{Fn}(\aleph_1, \aleph_2, \aleph_1))^{V_0},$$

i.e., the set of functions $f \in V_0$ which are countable in V_0 with domain contained in $(\aleph_1)^{V_0}$ and range contained in $(\aleph_2)^{V_0}$. That (13) holds follows, e.g., from [17, Ch.VII 6.13, 6.14, 6.15, 6.2].

Next we would like to engineer a set $E \in V_1$ of ordinals all of whose members are greater than ρ such that $L[C \cup E] \models \mathrm{CH}$. Since V_1 is a model of the axiom of choice, there are functions G and H with domains $\{\alpha \mid \omega \leq \alpha < (\aleph_1)^{V_1}\}$ and $\{\beta \mid (\aleph_1)^{V_1} \leq \beta < (\aleph_2)^{V_1}\}$, respectively, such that for each $\alpha \in \mathrm{dom}(G)$, $G_\alpha := G(\alpha)$ is a bijection between α and ω, and for each $\beta \in \mathrm{dom}(H)$, $H_\beta := H(\beta)$ is a bijection between β and $(\aleph_1)^{V_1}$. Let κ and π be fixed points of the function $\xi \mapsto \omega^\xi$ such that $\kappa < \pi$ and $\rho, (\aleph_1)^{V_1}, (\aleph_2)^{V_1} < \kappa$. Now define

$$\begin{aligned} E_1 &:= \{\kappa^\alpha \cdot (1 + \xi) + G_\alpha(\xi) \mid \alpha \in \mathrm{dom}(G) \wedge \xi < \alpha\}, \\ E_2 &:= \{\pi^\beta \cdot (1 + \gamma) + H_\beta(\gamma) \mid \beta \in \mathrm{dom}(H) \wedge \gamma < \beta\}, \\ E &:= E_1 \cup E_2, \end{aligned}$$

where of course κ^α and π^β refer to the operation of ordinal exponentiation. Then $E_1 \cap E_2 = \emptyset$. Moreover, owing to the uniqueness of Cantor normal forms (e.g. [28, Theorem 8.4.4]), for each $\alpha \in \mathrm{dom}(G)$, G_α is definable from $C \cup E$ in $L[C \cup E]$ (using the parameter κ), and likewise, for each $\beta \in \mathrm{dom}(H)$, H_β is definable from $C \cup E$ in $L[C \cup E]$ (using the parameter π). To elaborate on this, suppose $\alpha \in \mathrm{dom}(G)$. Then for $\xi < \alpha$ search for the least ordinal δ such that $\kappa^\alpha \cdot (1 + \xi) + \delta \in E$. Necessarily, $\delta = G_\alpha(\xi)$.

As a consequence of the above, we have

$$(\aleph_1)^{V_1} = (\aleph_1)^{L[C \cup E]} \wedge (\aleph_2)^{V_1} = (\aleph_2)^{L[C \cup E]} \wedge L[C \cup E] \models \mathrm{CH}. \tag{14}$$

To see the latter, suppose that $x \in L[C \cup E]$ and x is an infinite set of reals. As $L[C \cup E]$ is a model of AC, there is an ordinal η and a bijection $\ell \in L[C \cup E]$ between x and η. Since $L[C \cup E] \subseteq V_1$, $\eta < (\aleph_2)^{V_1}$ must obtain, and hence there is a bijection in $L[C \cup E]$ either between ω and x or between $(\aleph_1)^{V_1} = (\aleph_1)^{L[C \cup E]}$ and x. From (8) and (13), we also conclude that

$$\mathbb{R}^{L[C]} = \mathbb{R}^{L[C \cup E]}. \tag{15}$$

Utilizing the wellordering $<_{L[C \cup E]}$ and (14), there exists a $\Sigma_1^{L[C \cup E]}$ partial function g that finds for each non-empty set of reals either a surjection of ω onto x or a surjection of x onto $\mathbb{R}^{L[C]}$ since being such a mapping f is a Δ_0 property of f in the parameters x, ω and $\mathbb{R}^{L[C]}$. Thus there is a realizer $d \in L[C \cup E]$ such that $d \Vdash_{C \cup E} \mathrm{CH}$. From (12) and (6) it then follows that $(b)_0 = 0$, contradicting (10). In sum, a contradiction has been inferred from (6). On account of Theorem 5.2, this implies that $\mathrm{CH} \vee \neg\mathrm{CH}$ is not provable in **T**.

§7. Extensions.

There are several ways in which the theory **T** can be strengthened without forfeiting the unprovability of $\mathrm{CH} \vee \neg\mathrm{CH}$. For statements θ of second order arithmetic, i.e., those expressible in the language of the structure

$$(\mathbb{R}; \in; \mathbb{N}; <, +, \cdot, 0, 1), \tag{16}$$

T proves $\theta \vee \neg\theta$. This for instance applies if θ expresses Π^1_n-determinacy. If θ is true then also $\mathbf{T}+\theta$ does not prove CH $\vee$ ¬CH. To see this note first that the Realizability Theorem 5.2 also works for $\mathbf{T}+\theta$ if θ holds in $L[A]$. Subsequently one can employ the same proof as in the previous section, starting with a universe $V_0 \models \mathbf{ZFC}+2^{\aleph_0}=\aleph_2+\theta$. Note that since V_0 and V_1 share the same reals, θ will also be true in V_1.

It is also possible to go a bit beyond this level. Let PD be the statement of projective determinacy (see e.g. [16]). In $\mathbf{T}$ one can define a satisfaction relation for the structure in (16) which is Δ_1 in $\mathbb{R}$ and actually a set computable from $\mathbb{R}$ in the sense of Section 4. As a result one obtains a variant of Theorem 5.2.

PROPOSITION 7.1 (Realizability Theorem PD). *Suppose* $L[A] \models$ PD. *Let* $\mathbb{R}^{L[A]}$ *be the set of real numbers in the sense of* $L[A]$. *If* $\psi(x_1,\dots,x_n)$ *is a formula of set theory, with all free variables among the exhibited, and* $\mathcal{D}$ *is a proof of* $\psi(x_1,\dots,x_n)$ *in* $\mathbf{T}+$ PD, *then one can effectively construct a hereditarily finite set* $e_{\mathcal{D}}$ *which only depends on* $\mathcal{D}$ *(and not on* A*) such that for all* $a_1,\dots,a_n \in L[A]$,

$$[e_{\mathcal{D}}]^{L[A]}(a_1,\dots,a_n,\mathbb{R}^{L[A]}) \Vdash_A \psi(a_1,\dots,a_n). \tag{17}$$

PROOF. We only need to concern ourselves with PD. First note that the satisfaction predicate for the structure of the reals is computable from the parameter $\mathbb{R}^{L[A]}$. Moreover, if a set is realizably a projective set it is indeed a projective set (and vice versa) and thus a winning strategy (which is a real) exists and thus can be searched for (and found) in the computable universe $L[A]$. Whence PD is realizable. ⊣

THEOREM 7.2.

(i) $\mathbf{T} \vdash$ PD $\vee$ ¬PD.

(ii) *Assuming* $\mathbf{ZFC}+$ PD *in the background*, $\mathbf{T}+$ PD $\nvdash$ CH $\vee$ ¬CH.

PROOF. (i) PD is just the statement that Π^1_n-determinacy holds for all n. Since in $\mathbf{T}$ one has excluded middle for the satisfaction predicate pertaining to the structure (16) and $\forall n$ is a bounded quantifier, (i) follows.

(ii) We just sketch a proof. To commence one starts with a universe $V_0 \models \mathbf{ZFC}+$ PD $+\, 2^{\aleph_0}=\aleph_2$. Then one carries out the same construction as in Section 6. As V_0 and V_1 have the same reals they share the same projective sets of reals, and hence PD also holds in V_1. ⊣

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