Article:

Ramirez, Felipe A. (2017) Counterexamples, covering systems, and zero-one laws for inhomogeneous approximation. International Journal of Number Theory. pp. 633-654. ISSN 1793-0421

https://doi.org/10.1142/S1793042117500324

COUNTEREXAMPLES, COVERING SYSTEMS, AND ZERO-ONE LAWS FOR INHOMOGENEOUS APPROXIMATION

FELIPE A. RAMÍREZ

Abstract. We develop the inhomogeneous counterpart to some key aspects of the story of the Duffin–Schaeffer Conjecture (1941). Specifically, we construct counterexamples to a number of candidates for a sans-monotonicity version of Szüsz's inhomogeneous (1958) version of Khintchine's Theorem (1924). For example, given any real sequence $\{y_i\}$, we build a divergent series of non-negative reals $\psi(n)$ such that for any $y \in \{y_i\}$, almost no real number is inhomogeneously ψ-approximable with inhomogeneous parameter y. Furthermore, given any second sequence $\{z_i\}$ not intersecting the rational span of $\{1, y_i\}$, and assuming a dynamical version of Erdős' Covering Systems Conjecture (1950), we can ensure that almost every real number is inhomogeneously ψ-approximable with any inhomogeneous parameter $z \in \{z_i\}$. Next, we prove a positive result that is near optimal in view of the limitations that our counterexamples impose. This leads to a discussion of natural analogues of the Duffin–Schaeffer Conjecture and Duffin–Schaeffer Theorem (1941) in the inhomogeneous setting. As a step toward these, we prove versions of Gallagher's Zero-One Law (1961) for inhomogeneous approximation by reduced fractions.

Contents

1. Introduction and results

The basic question in (homogeneous) Diophantine approximation is about approximating real numbers by rational numbers. Given a real number x, how small can $|x - a/n|$ be as a function of n, where a/n is rational? This is the same as asking how small we can make $\|nx\|$, the distance from nx to the nearest integer. In *inhomogeneous* Diophantine approximation we have some other real number y—our inhomogeneous parameter—and we try to minimize $\|nx + y\|$.

There are of course innumerable questions one can ask for homogeneous and inhomogeneous approximation. It often happens that the homogeneous theory is mirrored in the inhomogeneous setting. For an example of this, take the homogeneous and inhomogeneous

The author was supported by the EPSRC Programme Grant EP/J018260/1.

versions of Khinthine's Theorem, below. Sometimes, techniques for homogeneous approximation can lead to results for inhomogeneous approximation. (See, for example, [Cas57, Chapter V].) The reverse may also happen: inhomogeneous considerations can illuminate facts in the homogeneous world. A famous example of this is Kurzweil's Theorem [Kur55], where badly (homogeneously) approximable real numbers x are characterized by their behavior with respect to all possible inhomogeneous expressions $\|nx + y\|$.

Our goal here is to develop the inhomogeneous counterpart to some of the narrative underlying one of Diophantine approximation's most vexing open problems: the Duffin–Schaeffer Conjecture.

1.1. Homogeneous theory. The following theorem is the foundation of *metric Diophantine approximation*. Note that we use the term 'approximating function' to mean a non-negative real-valued function of the natural numbers.

Khintchine's Theorem ([Khi24])**.** *Let ψ be a non-increasing approximating function. Then almost every or almost no real number x satisfies the inequality $\|nx\| < \psi(n)$ with infinitely many integers n, according as $\sum_n \psi(n)$ diverges or converges.*

A great deal of effort has been devoted to finding a suitable Khintchine-like statement that would not require the approximating function to be monotonic. In 1941, Duffin and Schaeffer [DS41] showed that one cannot simply remove the word 'non-increasing' from Khintchine's Theorem. Specifically, they produced an approximating function ψ that would serve as counterexample to the divergence part of the resulting statement. (Theorems 1/2 and 3 generalize this to the inhomogeneous setting.) Still, the *Duffin–Schaeffer Counterexample* has the property that $\sum_n \varphi(n)\psi(n)/n$ converges, where φ is Euler's φ-function. This led to their formulating what is now one of the foremost open problems of Diophantine approximation.

Duffin–Schaeffer Conjecture ([DS41])**.** *If ψ is an approximating function such that the sum $\sum_n \varphi(n)\psi(n)/n$ diverges, then almost every x satisfies the inequality $|nx - a| < \psi(n)$ with infinitely many coprime[1] integer pairs (a, n).*

The Duffin–Schaeffer Conjecture continues to be actively pursued, and has only been neared by partial and related results. In the original paper, Duffin and Schaeffer proved the first such partial result.

Duffin–Schaeffer Theorem ([DS41])**.** *If ψ is an approximating function such that $\sum_n \psi(n)$ diverges and*

$$\limsup_{N\to\infty} \left(\sum_{n=1}^{N} \frac{\varphi(n)\psi(n)}{n}\right)\left(\sum_{n=1}^{N} \psi(n)\right)^{-1} > 0,$$

then almost every x satisfies the inequality $|nx - a| < \psi(n)$ with infinitely many coprime integer pairs (a, n).

Many others have followed. For example, Erdős verified the conjecture for approximating functions that take the value ε/n on their support, for some $\varepsilon > 0$ [Erd70]; Vaaler improved this to functions of the form $\psi(n) = O(n^{-1})$ [Vaa78]; Pollington and Vaughan proved that the

[1]One might also consider a version of this conjecture where the word 'coprime' has been removed. This is of course weaker than the stated conjecture. Whether it is easier to prove seems to be an unexplored question.

Duffin–Schaeffer Conjecture holds in higher dimensions [PV90]; and recently, Beresnevich, Haynes, Harman, Pollington, and Velani have proved the conjecture under "extra divergence" assumptions, and Aistleitner has proved it for "slow divergence" [HPV12, BHHV13, Ais14].

An important tool for attacks on the Duffin–Schaeffer Conjecture, and indeed in many other problems in Diophantine approximation, is the "zero-one law"—a statement precluding that the measure of a set be anything other than zero or one. The following zero-one law tells us that the set that is predicted to be full in the Duffin–Schaeffer Conjecture is either null or full.

Gallagher's Zero-One Law ([Gal61])**.** *Let ψ be an approximating function. Then almost every or almost no x satisfies the inequality $|nx - a| < \psi(n)$ with infinitely many coprime integer pairs (a, n).*

Modifications of Gallagher's proof yield Theorems 5 and 6—inhomogeneous versions where we consider countably many inhomogeneous parameters simultaneously.

1.2. Inhomogeneous theory.

It was later proved by Szüsz that an *inhomogeneous* version of Khintchine's Theorem also holds.

Inhomogeneous Khintchine Theorem ([Szü58])**.** *Let y be a real number and ψ a non-increasing approximating function. Then almost every or almost no real number x satisfies the inequality $\|nx+y\| < \psi(n)$ with infinitely many integers n, according as $\sum_n \psi(n)$ diverges or converges.*

Again, one can ask about the possibility of removing the word 'non-increasing' from this theorem. Our first result, Theorem 1, shows that this is impossible by giving a counterexample to the resulting statement. In fact, it is a counterexample to much more.

The following theorem shows that if one lets the inhomogeneous parameter vary, then monotonicity *can* be removed.

Doubly Metric Inhomogeneous Khintchine Theorem ([Cas57])**.** *Let ψ be an approximating function. Then almost every or almost no real pair (x, y) satisfies the inequality $\|nx + y\| < \psi(n)$ with infinitely many integers n, according as $\sum_n \psi(n)$ diverges or converges.*

One may therefore be tempted to suspect that if we restrict the inhomogeneous parameter only slightly (instead of letting it vary over all real numbers as in the above theorem) then we will still retain a similar statement to the one above, without the need for monotonicity. For example, we may require the inhomogeneous parameter to lie on an equidistributed sequence of real numbers, and hope that a "doubly metric"-style statement will hold for a "density-one" subsequence (the idea being that this equidistributed sequence would be a generic sampling from the full-measure set provided by the Doubly Metric Inhomogeneous Khintchine Theorem). But this is also ruled out by our first result. We construct counterexamples for *any* given sequence of inhomogeneous parameters.

Theorem 1. *For any sequence $\{y_i\}$ of real numbers there is an approximating function ψ such that the sum $\sum_n \psi(n)$ diverges and such that for any $y \in \{y_i\}$, there are* ***almost no*** *real numbers x for which the inequality $\|nx + y\| < \psi(n)$ is satisfied by infinitely many integers $n \geq 1$.*

In fact, the counterexamples we construct are automatically counterexamples for $y = 0$, as well as *any* rational combination of 1 with finitely many elements of the sequence $\{y_i\}$, which leads to the question of whether this can be avoided. (See Remark 8 and Questions 9, 10, and 11.) As for inhomogeneous parameters that are *not* in the rational span of 1 with $\{y_i\}$, we have the following continuation of Theorem 1.

Theorem 2 (To be read as a continuation of Theorem 1)**.** *Moreover, if Conjecture 14 is true, and $\{z_i\}$ is a second sequence of real numbers no element of which lies in the rational span of 1 with finitely many elements of $\{y_i\}$, then we may take ψ to have the additional property that for any $z \in \{z_i\}$,* ***almost every*** *real number x satisfies the inequality $\|nx + z\| < \psi(n)$ with infinitely many $n \geq 1$.*

Remark. Conjecture 14 is a dynamical version of Erdős' famous "covering systems" conjecture [Erd50, Erd95] and recent progress on it [FFK+07, Hou15]. We have left a full discussion of this for §3 so as not to disrupt the present narration. Suffice it to say that [FFK+07, Hou15] provide overwhelming evidence in favor of Conjecture 14.

Notice that by the Borel–Cantelli Lemma, the conclusion of Theorem 2 forces $\sum_n \psi(n)$ to diverge, and so when Theorems 1 and 2 are read together the divergence condition is redundant. We list and prove the two theorems separately, both for the readers' convenience, and because Theorem 1 has a simpler proof that does not require Conjecture 14.

It is natural to ask whether points in the rational span of $\{1, y_i\}$ are the *only* ones for which the counterexamples in Theorem 1/2 work. Rather than answer this directly, we construct counterexamples that work for an *uncountable* set of possible inhomogeneous parameters (as well as their rational combinations with 1).

Theorem 3. *There is an uncountable "Cantor-type" set C of real numbers and an approximating function ψ such that the sum $\sum_n \psi(n)$ diverges and such that for any $y \in C$ there are almost no real numbers x for which the inequality $\|nx + y\| < \psi(n)$ is satisfied by infinitely many integers $n \geq 1$.*

Remark. From the construction in the proof, it will be clear that there are uncountably many such Cantor sets, each with its own approximating function.

Theorems 1 and 3 imply that a non-monotonic inhomogeneous version of Khintchine's Theorem has absolutely no hope of being true, even if we relax the requirement that the inhomogeneous part be fixed and instead let it come from some set of *permitted* real numbers. Still, if we take the permitted set to be an equidistributed sequence, then we can prove the following.

Theorem 4. *Let $\{y_m\}$ be an equidistributed sequence* $\bmod 1$*. Suppose ψ is an approximating function such that $\sum_n \psi(n)$ diverges. Then for every $R, \varepsilon > 0$ there is a density-one set of integers $m \geq 1$ with the property that the set of real x for which the inequality $\|nx + y_m\| < \psi(n)$ has at least R integer solutions n, has measure at least $1 - \varepsilon$.*

Theorem 1 prevents us from doing too much better than this. On the other hand, the approximating functions ψ in Theorems 1 and 3 have the property that $\sum_n \varphi(n)\psi(n)/n$ converges. So it becomes natural to ask about inhomogeneous versions of the Duffin–Schaeffer Conjecture, where we seek approximations by *reduced* fractions, and use an accordingly modified divergence condition. There is, of course, the direct inhomogeneous translation:

Inhomogeneous Duffin–Schaeffer Conjecture. *Let y be a real number. If ψ is an approximating function such that $\sum_n \varphi(n)\psi(n)/n$ diverges, then for almost every real number x there are infinitely many coprime integer pairs (a, n) such that $|nx - a + y| < \psi(n)$.*

But this is much stronger than the original Duffin–Schaeffer Conjecture, and probably therefore harder. It makes sense to explore related questions. For example, in the spirit of Questions 9 and 10, are there any dependencies between the Inhomogeneous Duffin–Schaeffer Conjecture for one inhomogeneous parameter versus another? And, in the spirit of Theorems 1–4, can we make progress on an inhomogeneous version of the Duffin–Schaeffer Conjecture where we do not fix the inhomogeneous parameter, but instead let it come from some predetermined sequence? We discuss these questions in §5, and in §6 we make some progress in the form of the following zero-one laws, inspired by Gallagher's and deduced by modifying his proof.

Theorem 5. *Let y be a real number and ψ an approximating function.*

- *For almost every or almost no real number x there exists an integer $m \geq 1$ such that $|nx - a + my| < \psi(n)$ has infinitely many coprime integer solutions (a, n).*
- *For almost every or almost no real number x there exist* infinitely many *such integers $m \geq 1$.*

Remark. Notice that Gallagher's Zero-One Law [Gal61] is the same statement when $y = 0$. (Of course, in that case, the integer m has no role to play.) Our proof follows his.

Theorem 6. *Let y be a real number and ψ an approximating function.*

- ***Either*** *for every integer $m \geq 1$, there are almost no real x for which*

$$|nx - a + my| < \psi(n) \text{ for infinitely many } (a, n) = 1, \tag{1}$$

- ***Or*** *at least one of the following holds:*
  - *There is some $m \geq 1$ such that (1) holds for almost every real x.*
  - *For any $\varepsilon > 0$ there are arbitrarily many $m \geq 1$ such that the set of x for which (1) holds has measure greater than $1 - \varepsilon$.*

These leave open the more obvious zero-one law, where we have a single fixed inhomogeneous parameter. We will visit this in a future project.

2. Proofs of Theorems 1 and 3: Counterexamples

In this section we prove Theorems 1 and 3. We begin with the following lemma.

Lemma 7. *For any $\varepsilon > 0$, integer $\ell \geq 1$, and real numbers $y_1, \dots, y_\ell$, the sum*

$$\sum_{\max_i \|ny_i\| < \varepsilon} \frac{1}{n+1}$$

diverges.

Proof. The numbers $n \geq 1$ such that $\max_i \|ny_i\| < \varepsilon$ comprise a set of positive density in the natural numbers. Therefore, the sum of reciprocals diverges. □

Proof of Theorem 1. Let $\{y_i\}$ be a sequence of real numbers. Let $\{n_m^{(1)}\}_m$ be the sequence of times when $\|n_m^{(1)} y_1\| \le 2^{-2}$, and let

$$K_1 = \prod_{m=1}^{M_1} \left(n_m^{(1)} + 1\right)$$

where $M_1 > 0$ is chosen so that

$$\sum_{m=1}^{M_1} \frac{1}{n_m^{(1)} + 1} \ge 2^2.$$

Such an M_1 exists by Lemma 7.

Inductively, let $\{n_m^{(j)}\}_m$ be the sequence of times $> K_{j-1}$ when

$$\max_{i=1,\dots,j} \|n_m^{(j)} y_i\| \le 2^{-j-1}, \tag{2}$$

and let

$$K_j = \prod_{m=1}^{M_j} \left(n_m^{(j)} + 1\right)$$

where $M_j > 0$ is chosen so that

$$\sum_{m=1}^{M_j} \frac{1}{n_m^{(j)} + 1} \ge 2^{j+1}. \tag{3}$$

Again, Lemma 7 allows us to choose such an M_j.

Let $k_m^{(j)} = K_j/(n_m^{(j)} + 1)$. Notice that the $k_m^{(j)}$ are pairwise distinct because the K_j form a strictly increasing sequence of positive integers, and for any fixed j, we have

$$K_{j-1} < k_{M_j}^{(j)} < k_{M_j-1}^{(j)} < \cdots < k_2^{(j)} < k_1^{(j)} < K_j,$$

by our construction.

Define

$$\psi(k) = \begin{cases} \dfrac{k}{K_j} 2^{-j-1} = \dfrac{2^{-j-1}}{n_m^{(j)} + 1} & \text{for } k = k_m^{(j)}, m = 1, \dots, M_j \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Let $y \in \{y_i\}$, and let us set the notation

$$E_n^y(\psi) = \frac{\mathbb{Z} + y}{n} + \left(-\frac{\psi(n)}{n}, \frac{\psi(n)}{n}\right)$$

and

$$E_n^y(\varepsilon) = \frac{\mathbb{Z} + y}{n} + \left(-\frac{\varepsilon}{n}, \frac{\varepsilon}{n}\right)$$

when the argument is a real constant.

The rest of this proof follows from three claims that are proved separately below, because they will be used again later. Claim 1 states that if y_j comes after $y \in \{y_i\}$, then $E^y_{k_m^{(j)}}(\psi) \subset E^y_{K_j}(2^{-j})$ for all $m = 1, \dots, M_j$. This implies that

$$\limsup_{n\to\infty} E_n^y(\psi) \subset \limsup_{j\to\infty} E^y_{K_j}(2^{-j}).$$

Claim 2 shows that

$$\left|\limsup_{j\to\infty} E^y_{K_j}(2^{-j})\right| = 0.$$

And Claim 3 shows that $\sum_n \psi(n) = \infty$. This proves the theorem. □

Claim 1. *If y_j comes after $y \in \{y_i\}$, then $E^y_{k_m^{(j)}}(\psi) \subset E^y_{K_j}(2^{-j})$ for all $m = 1, \dots, M_j$.*

Proof. First, we show that

$$\frac{\mathbb{Z}+y}{k_m^{(j)}} \subset \frac{\mathbb{Z}+y}{K_j} + \left[-\frac{2^{-j-1}}{K_j}, \frac{2^{-j-1}}{K_j}\right].$$

For this, it is enough to show that $y/k_m^{(j)}$ is within a distance of $2^{-j-1}/K_j$ from an element $(\ell+y)/K_j$ of $(\mathbb{Z}+y)/K_j$. Then all elements of $(\mathbb{Z}+y)/k_m^{(j)}$ will also be, because they will just be shifts of $y/k_m^{(j)}$ by integer multiples of $1/k_m^{(j)}$, which are of course integer multiples of $1/K_j$. But

$$\min_\ell \left|\frac{y}{k_m^{(j)}} - \frac{\ell+y}{K_j}\right| = \frac{1}{K_j}\min_\ell \left|n_m^{(j)} y - \ell\right| \le \frac{2^{-j-1}}{K_j},$$

which proves it.

Now we have

$$\begin{aligned} E^y_{k_m^{(j)}}(\psi) &= \frac{\mathbb{Z}+y}{k_m^{(j)}} + \left(-\frac{\psi(k_m^{(j)})}{k_m^{(j)}}, \frac{\psi(k_m^{(j)})}{k_m^{(j)}}\right) \\ &\subset \frac{\mathbb{Z}+y}{K_j} + \left(-\frac{\psi(k_m^{(j)})}{k_m^{(j)}} - \frac{2^{-j-1}}{K_j}, \frac{\psi(k_m^{(j)})}{k_m^{(j)}} + \frac{2^{-j-1}}{K_j}\right) \\ &= \frac{\mathbb{Z}+y}{K_j} + \left(-\frac{2^{-j}}{K_j}, \frac{2^{-j}}{K_j}\right) = E^y_{K_j}(2^{-j}), \end{aligned}$$

which is what we wanted to prove. □

Claim 2. $|\limsup_{j\to\infty} E^y_{K_j}(2^{-j})| = 0$.

Proof. The sum

$$\sum_j \left|E^y_{K_j}(2^{-j}) \cap [0,1)\right| = \sum_j 2^{-j+1}$$

converges, therefore by the Borel–Cantelli Lemma we have $|\limsup_{j\to\infty} E^y_{K_j}(2^{-j}) \cap [0,1)| = 0$. The claim follows because $\limsup_{j\to\infty} E^y_{K_j}(2^{-j})$ is 1-periodic. □

Claim 3. $\sum_n \psi(n) = \infty$.

***Proof*.** We compute the sum:

$$\begin{aligned}\sum_n \psi(n) &= \sum_j \sum_{m=1}^{M_j} \psi(k_m^{(j)}) \\ &= \sum_j \sum_m \frac{k_m^{(j)}}{K_j} 2^{-j-1} \\ &= \sum_j 2^{-j-1} \sum_{m=1}^{M_j} \frac{1}{n_m^{(j)}+1} \\ &\overset{(5)}{\geq} \sum_j 1,\end{aligned}$$

which diverges. □

***Proof of Theorem 3*.** Let $\{n_m^{(0)}\}_{m=1}^{M_0}$ be such that

$$\sum_{m=1}^{M_0} \frac{1}{n_m^{(0)}+1} \geq 2$$

and let

$$K_0 = \prod_{m=1}^{M_0} \big(n_m^{(0)}+1\big).$$

Let C_0 be the set of all $y \in [0,1]$ such that $\|n_m^{(0)} y\| \leq 2^{-1}$ for $m = 1, \ldots, M_0$. Then $C_0 = \bigcup_{i=1}^{i_0} I_i^{(0)}$ is a finite union of subintervals of $[0,1]$. (Actually, since $\|\cdot\|$ *always* takes values $\leq 2^{-1}$, we have $C_0 = [0,1]$. This 0th step is really only here to seed the inductive process.)

We proceed inductively. For $j \geq 1$, let $L_j > K_{j-1}$ be such that $[0,1) \subset L_j I_i^{(j-1)}$ for all $i = 1, \ldots, i_{j-1}$. Pick an integer $\ell_j \geq 1$ and real numbers $y_1^{(j)}, \ldots, y_{\ell_j}^{(j)} \in C_{j-1}$ and let $\{n_m^{(j)}\}_{m=1}^{M_j}$ be such that $n_m^{(j)} \geq L_j$, $\max_i \|n_m^{(j)} y_i^{(j)}\| \leq 2^{-j-1}$, and

$$\sum_{m=1}^{M_j} \frac{1}{n_m^{(j)}+1} \geq 2^{j+1}. \tag{5}$$

It is guaranteed that we can do this by Lemma 7. Let

$$K_j = \prod_{m=1}^{M_j} \big(n_m^{(j)}+1\big).$$

Let C_j be the set of $y \in C_{j-1}$ with $\|n_m^{(j)} y\| \leq 2^{-j-1}$ for all $m = 1, \ldots, M_j$. Then $C_j = \bigcup_{i=1}^{i_j} I_i^{(j)}$ is a finite union of closed subintervals of C_{j-1}. (These subintervals are non-empty because we have constructed them to contain $y_1^{(j)}, \ldots, y_{\ell_j}^{(j)}$. That they are *contained* in C_{j-1} is ensured by the condition $n_m^{(j)} \geq L_j$.) Our uncountable "Cantor-type" set is $C = \bigcap_{j \geq 0} C_j$.

Let $k_m^{(j)} = K_j/(n_m^{(j)}+1)$. Define ψ by (4). Let $y \in C$, and define $E_n^y(\psi)$ and $E_{K_j}^y(2^{-j})$ as in the proof of Theorem 1.

A simple modification of Claim 1 shows that

$$\limsup_{n\to\infty} E_n^y(\psi) \subset \limsup_{j\to\infty} E_{K_j}^y(2^{-j}),$$

Claim 2 shows that

$$\left|\limsup_{j\to\infty} E_{K_j}^y\left(2^{-j}\right)\right| = 0,$$

and Claim 3 shows that $\sum_n \psi(n) = \infty$. This proves the theorem. □

Remark 8. Notice that the counterexamples in Theorems 1 and 3 also serve as counterexamples for the homogeneous case, because Claim 1 holds for $y = 0$. More strikingly, they work for any rational combination of 1 with elements of the sequence $\{y_i\}$ (in the case of Theorem 1, and the set C in the case of Theorem 3), as we will now sketch.

Suppose $w_1, \dots, w_\ell \in \{y_i\}$ and let $z = a_1w_1 + \cdots + a_\ell w_\ell + b$, were $a_1, \dots, a_\ell, b$ are rational. We leave as an exercise to show that there exist positive integers c_1, c_2 depending on $a_1, \dots, a_\ell, b$ such that

$$E^z_{k_m^{(j)}}(\psi) \subset E^z_{c_1K_j}\left(c_2 2^{-j}\right)$$

for all sufficiently large j and $m = 1, \dots, M_j$, where ψ is the counterexample constructed in the proof of Theorem 1. (One can take c_1 as a common denominator for $a_1, \dots, a_\ell, b$, while c_2 can be taken sufficiently large.) Therefore,

$$\limsup_{n\to\infty} E_n^z(\psi) \subset \limsup_{j\to\infty} E^z_{c_1K_j}\left(c_2 2^{-j}\right).$$

But an application of the Borel–Cantelli Lemma shows that

$$\limsup_{j\to\infty} E^z_{c_1K_j}\left(c_2 2^{-j}\right)$$

has measure zero.

What is striking in the above remark is not the fact that one can find a counterexample for all of these rational combinations. (After all, Theorem 1 already guarantees this, since the rational span of $\{1, y_i\}$ is itself a denumerable set.) It is the fact that all these rational combinations come for free from the construction, without having to enumerate them as a new sequence. This leads us to ask whether this is just a side-effect of our particular construction, or if it is unavoidable.

Question 9. Suppose ψ is an approximating function such that for any y among a fixed set $\{y_1, \dots, y_\ell\}$ of real numbers, almost no real number x satisfies $\|nx + y\| < \psi(n)$ with infinitely many integers n. Does the same necessarily hold for all rational combinations of the y_i's with 1?

A 'yes' would imply in particular that the homogeneous situation is all we need to study. In other words, we can ask the following *a priori* weaker question.

Question 10. Suppose that almost every real x satisfies $\|nx\| < \psi(n)$ with infinitely many integers $n \geq 1$. Does this imply that for any real y, almost every x satisfies $\|nx + y\| < \psi(n)$ with infinitely many integers $n \geq 1$? That is, if almost every real number is homogeneously ψ-approximable, does this imply that almost every real number is *inhomogeneously* ψ-approximable with any inhomogeneous parameter?

It seems reasonable to think so. After all, it is already known to be true for monotonic approximating functions, by Khintchine's Theorem and its inhomogeneous analogue. Still, one may pursue the following counter-question, perhaps in search of a contradiction.

Question 11. Let $\{y_i\}$ and $\{z_i\}$ be disjoint sequences of real numbers. Is there an approximating function ψ such that

- for any $y \in \{y_i\}$, **almost no** real numbers x satisfy the inequality $\|nx + y\| < \psi(n)$ with infinitely many integers $n \geq 1$, while
- for any $z \in \{z_i\}$, **(almost) all** real numbers x satisfy the inequality $\|nx+z\| < \psi(n)$ with infinitely many integers $n \geq 1$?

Remark. Note that the second condition forces $\sum_n \psi(n)$ to diverge.

If the answers to Questions 9 and 10 are 'yes'—and we suspect they are—we can still ask Question 11 with the additional stipulation that every $z \in \{z_i\}$ lies outside the rational span of 1 and $\{y_i\}$. This leads to Theorem 2, which we handle in §3.

3. Proof of Theorem 2: Dynamically defined covering systems

For the integers $T := \{1 < T_1 < \cdots < T_M\}$ and $R := \{R_1, \ldots, R_M\}$, consider the *system of residues*

$$\mathcal{S}(T, R) := \{n \in \mathbb{Z} : n \equiv R_m \pmod{T_m} \text{ for some } m = 1, \ldots, M\}.$$

We say $\mathcal{S}(T, R)$ is a *covering system* if it contains *all* the integers. Erdős asked the following question in 1950: *Do there exist covering systems with arbitrarily large T_1?* He conjectured that there did exist such covering systems [Erd50], and even offered a cash prize for a proof [Erd95]. However, the conjecture was *disproved* by Hough in 2015; he showed that there is no covering system with $T_1 > 10^{16}$ [Hou15, Theorem 1]. The proof builds on work of Filaseta, Ford, Konyagin, Pomerance, and Yu where among other things they studied the asymptotic density of the *complement* of a given system of residues ([FFK+07], 2007).

Notice that a system of residues $\mathcal{S}(T, R)$ (whether it is covering or not) is a periodic subset of the integers. If we center an interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ around each point of the system, then the resulting union of intervals is also periodic. Then the asymptotic density of the system of residues can be interpreted as the proportion of the real line that is occupied by this union of intervals. Since this proportion is invariant under scaling, we can contract by a multiple of the period, and interpret the proportion of the real line occupied by our union of intervals as exactly the measure of the contracted set intersected with the unit interval $[0, 1)$.

We now ask a related question. Let us start with an M-tuple $\boldsymbol{t} := (t_1, \ldots, t_M)$ of integers such that $1 \leq t_1 < \cdots < t_M$, and a real number $\varepsilon > 0$, and let us consider translations of $t_m\mathbb{Z}$ by *any* real numbers $\boldsymbol{r} := (r_1, \ldots, r_M)$. Again, the union of these is periodic. How much of the real line can we occupy by centering an interval $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ at each point of this set? What we are really asking is: how much of $[0, 1)$ is occupied by

$$\bigcup_{m=1}^{M} \frac{t_m\mathbb{Z} + r_m}{K} + \left(-\frac{\varepsilon}{2K}, \frac{\varepsilon}{2K}\right) \cap [0, 1), \tag{6}$$

where $K = \prod t_i$ (or, alternatively, $K = \operatorname{lcm}(t_1, \ldots, t_M)$, or any multiple thereof)? Hough's result suggests that if $t_1/\varepsilon > 10^{16}$, then the set (6) cannot equal $[0, 1)$. But what about the *measure* of the set (6)? It will be convenient to denote by $\mu(\varepsilon, \boldsymbol{t}, \boldsymbol{r})$ the measure of its complement, and define

$$\alpha(\varepsilon, \boldsymbol{t}) := \prod_{m=1}^{M} \Big(1 - \frac{\varepsilon}{t_m}\Big).$$

Now we are asking how *small* we can make $\mu(\varepsilon, \boldsymbol{t}, \boldsymbol{r})$. For this, we can derive some answers from [FFK+07]. For example, the following lemma shows that the expected value of μ is α.

Lemma 12 (Version of [FFK+07, Lemma 5.1] for real residues)**.** *For any fixed M-tuple $\boldsymbol{t} = (t_1, \dots, t_M)$ of integers with $1 \le t_1 < \cdots < t_M$ and any $\varepsilon > 0$, the expected value of $\mu(\varepsilon, \boldsymbol{t}, \boldsymbol{r})$ is $\alpha(\varepsilon, \boldsymbol{t})$.*

Proof. Letting

$$E(m, r) := \frac{t_m \mathbb{Z} + r}{K} + \Big(-\frac{\varepsilon}{2K}, \frac{\varepsilon}{2K}\Big),$$

the expectation is the integral

$$\frac{1}{K^M} \int_{[0,K)^M} \Big| \widehat{E}(1, r_1) \cap \cdots \cap \widehat{E}(M, r_M) \Big| \, dr_1 \, \dots \, dr_M = \prod_{m=1}^{M} \Big(1 - \frac{\varepsilon}{t_m}\Big),$$

which proves the lemma. □

Moreover, Filaseta *et al.* show (in the context of the original problem of Erdős) that deviations from this expected value are on average extremely small for large t_1.

Theorem 13 ([FFK+07, Theorem 7])**.** *For any fixed M-tuple $\boldsymbol{t} = (t_1, \dots, t_M)$ of integers with $1 \le t_1 < \cdots < t_M$ and $\varepsilon = 1/k$ for some integer $k \ge 1$, the variance of $\mu(\varepsilon, \boldsymbol{t}, \boldsymbol{r})$ over integer values of $\boldsymbol{r}$ is $\ll \frac{\alpha(\varepsilon, \boldsymbol{t})^2 \log(t_1/\varepsilon)}{(t_1/\varepsilon)^2}$.*

Remark. It is a computation to verify that Theorem 13 is also true for general $\varepsilon > 0$ and with variance over all real M-tuples $\boldsymbol{r}$.

In the following we conjecture that given any irrational number z, one can find positive integers $t_1 < \cdots < t_M$ for which the M-tuple $\boldsymbol{r} = \boldsymbol{t}z = (t_1 z, \dots, t_M z)$ is "typical," in the sense that $\mu(\varepsilon, \boldsymbol{t}, \boldsymbol{t}z)$ is close to what Lemma 12 and Theorem 13 lead us to expect. Furthermore, we can choose the t_m's from certain dynamically defined positive-density subsets of the integers, and such that $\alpha(\varepsilon, \boldsymbol{t})$ is arbitrarily small. (Note that in the notation, $n_m + 1$ will play the role of t_m.)

Conjecture 14 (Dynamically defined covering systems)**.** *Suppose $y_1, \dots, y_\ell, z$ are real numbers such that z is not in the rational span of $1, y_1, \dots, y_\ell$, and let $\varepsilon, \delta > 0$. Then for any $n_0 > 0$ there are positive integers $\{n_0 < n_1 < n_2 < \cdots < n_M\}$ and real numbers $w_1, \dots, w_\ell$ such that the following hold:*

(1) *We have $\max_i \|n_m y_i - w_i\| \le \varepsilon/2$ for all $m = 1, \dots, M$.*
(2) *The measure of*

$$\Bigg(\bigcup_{m=1}^{M} \frac{\mathbb{Z} + z}{k_m} + \Big(-\frac{\varepsilon}{2K}, \frac{\varepsilon}{2K}\Big)\Bigg) \cap [0, 1) \tag{7}$$

is at least $1 - \delta$, where $K = \prod_{m=1}^{M} (n_m + 1)$ and $k_m = K/(n_m + 1)$.

There is good reason to believe Conjecture 14. (In fact, it is even reasonable to believe that the w's are superfluous.) In the following lemma we show that we can find arbitrarily many progressions $\{n_1 < \cdots < n_M\}$, with n_1 arbitrarily large, such that the *expected* measure of (7), as calculated in Lemma 12, is as close to 1 as we want. And Theorem 13 suggests that the actual measure of (7) is extremely likely to be near its expected value.

Lemma 15. *Suppose $y_1, \ldots, y_\ell$ are real numbers and $\varepsilon, \delta > 0$. Suppose that $(w_1, \ldots, w_\ell)$ is an accumulation point for the orbit of 0 by $(y_1, \ldots, y_\ell)$-translations of the ℓ-dimensional torus. Then for any $n_0 > 0$ there are integers $\{n_0 < n_1 < \cdots < n_M\}$ such that $\max_i \|n_m y_i - w_i\| \leq \varepsilon/2$ for all $m = 1, \ldots, M$, and such that $\prod_{m=1}^{M}\left(1 - \frac{\varepsilon}{n_m+1}\right) < \delta$.*

Proof. Let $\{\bar{n}_m\}$ be the sequence of all positive integers such that $\max_i \|\bar{n}_m y_i - w_i\| \leq \varepsilon/2$. We will show that $\{\bar{n}_m\}$ contains an infinite generalized arithmetic subsequence. That is, there is a set $S = \{s_j : j = 1, \ldots, 2^\ell\}$ of positive integers and an infinite subsequence $\{n_m\} \subset \{\bar{n}_m\}$ such that $n_{m+1} - n_m \in S$ for all m.

We provide a simple construction. Let the s_j be such that $\mathcal{O}_j + s_j(y_1, \ldots, y_\ell)$ lies within $B := (w_1, \ldots, w_\ell) + (-\varepsilon/2, \varepsilon/2)^\ell$, where $\mathcal{O}_j$ is the jth hyper-octant of B. Then the sequence $\{n_m\}$ can be constructed in the following way: Fix an initial term n_1 such that $n_1(y_1, \ldots, y_\ell)$ lies in B. The terms thereafter are set by the rule $n_m = n_{m-1} + s_j$ whenever $n_{m-1}(y_1, \ldots, y_\ell) \in \mathcal{O}_j$.

Now a simple calculation shows that by making M large we can make $\prod_{m=1}^{M}\left(1 - \frac{\varepsilon}{n_m+1}\right)$ arbitrarily small. □

Proof of Theorem 2. Let $\{y_i\}, \{z_i\}$ be as in the theorem statement, and assume Conjecture 14 to be true. Let us define the new sequence

$$\{\bar{z}_i\} = \{z_1, z_2, z_1, z_2, z_3, z_1, z_2, z_3, z_4, \ldots\},$$

and let $\{\delta_j\} \subset (0, 1)$ be a sequence decreasing to 0.

Let $\{n_1^{(1)} < \cdots < n_{M_1}^{(1)}\}$ and $w_1^{(1)}$ be the progression and number guaranteed by Conjecture 14, with $y_1, w_1^{(1)}, \bar{z}_1, 2^{-1}, \delta_1$ playing the roles of $y_1, w_1, z, \varepsilon, \delta$ respectively. Let

$$K_1 = \prod_{m=1}^{M_1}\left(n_m^{(1)} + 1\right).$$

Inductively, let $\{n_1^{(j)} < \cdots < n_{M_j}^{(j)}\}$ and $w_1^{(j)}, \ldots, w_j^{(j)}$ be the progression and real numbers guaranteed by Conjecture 14, with $y_1, \ldots, y_j, w_1^{(j)}, \ldots, w_j^{(j)}, \bar{z}_j, 2^{-j}, \delta_j$ playing the roles of $y_1, \ldots, y_\ell, w_1, \ldots, w_\ell, z, \varepsilon, \delta$, respectively, such that $n_1^{(j)} \geq K_{j-1}$. Let

$$K_j = \prod_{m=1}^{M_j}\left(n_m^{(j)} + 1\right).$$

We may now define ψ by (4), as in the proof of Theorem 1. Our construction guarantees that for every $z \in \{z_i\}$ we have $|\limsup_n E_n^z(\psi)| = 1$. A simple modification of Claim 1 shows that for any i we will have

$$\limsup_{n\to\infty} E_n^{y_i}(\psi) \subset \limsup_{j\to\infty} E_{K_j}^{y_i + w_i^{(j)}}\left(2^{-j}\right).$$

And the proof of Claim 2 shows that

$$\left|\limsup_{j\to\infty} E_{K_j}^{y_i+w_i^{(j)}}\left(2^{-j}\right)\right| = 0,$$

so we have $|\limsup_n E_n^y(\psi)| = 0$ for any $y \in \{y_i\}$. This proves Theorem 2. □

4. Proof of Theorem 4: Equidistributed inhomogeneous parameters

For every positive integer m let $f(x,m) \geq 0$ be an integrable function of the real variable x. Let

$$\underline{\mu}(f) := \liminf_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\int_0^1 f(x,m)\,dx$$

and

$$\overline{\mu}(f) := \limsup_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\int_0^1 f(x,m)\,dx.$$

If the two coincide, denote $\mu(f) = \underline{\mu}(f) = \overline{\mu}(f)$. For a set A of pairs (x,m) such that $A_k = \{(x,m) \in A \mid m = k\}$ is measurable for all k, denote $\mu(A) := \mu(\mathbf{1}_A)$, and similarly for $\underline{\mu}(A)$ and $\overline{\mu}(A)$. Notice that we will always have $\underline{\mu}(A) \leq \overline{\mu}(A) \leq 1$.

The goal of this section is to prove the following "doubly metric" statement, from which will follow Theorem 4.

Theorem 16. *Let $\{y_m\}$ be an equidistributed sequence* mod 1*. Suppose ψ is an approximating function such that $\sum_n \psi(n)$ diverges. Let $R > 0$ and let F denote the set of pairs (x,m) for which the inequality $\|nx + y_m\| < \psi(n)$ has at least R integer solutions $n \geq 1$. Then $\mu(F) = 1$.*

Remark. Theorem 16 should be compared with the Doubly Metric Inhomogeneous Khintchine Theorem, where it is shown under the same condition on ψ that for almost every pair (x,y) the inequality $\|nx+y\| < \psi(n)$ has infinitely many integer solutions $n \geq 1$. Since we are taking the sequence $\{y_m\}$ to be equidistributed, we may naturally expect that for almost every fixed x, the y_m's are a generic sampling from the corresponding full set of y's. What we prove is similar but weaker. The proof follows [Cas57, Page 121, Theorem II].

The proof of Theorem 16 is based on the following analogue of the Paley–Zygmund Lemma.

Lemma 17. *Suppose that for $f(x,m) \geq 0$, the quantities $\mu(f)$ and $\mu(f^2)$ exist and are finite. Suppose $\mu(f) \geq a\sqrt{\mu(f^2)}$ and $0 \leq b \leq a$, and let*

$$A = \left\{(x,m) : f(x,m) \geq b\sqrt{\mu(f^2)}\right\}.$$

Then $\underline{\mu}(A) \geq (a-b)^2$.

Proof. For every M we have

$$\left(\frac{1}{M}\sum_{m=1}^{M}\int_0^1 \mathbf{1}_A(x,m)f(x,m)\,dx\right)^2 \leq \left(\frac{1}{M}\sum_{m=1}^{M}\int_0^1 \mathbf{1}_A(x,m)\,dx\right)\left(\frac{1}{M}\sum_{m=1}^{M}\int_0^1 f(x,m)^2\,dx\right),$$

by the Cauchy–Schwarz Inequality, and therefore in the limit as $M \to \infty$ we will have $\underline{\mu}(\mathbf{1}_A \cdot f)_1^2 \le \underline{\mu}(A)\, \mu(f^2)$. Now, since on the complement of A we have $f(x,m) \le b\sqrt{\mu(f^2)}$, we therefore have for every fixed M

$$\begin{aligned}\frac{1}{M}\sum_{m=1}^{M}\int_0^1 &\mathbf{1}_A(x,m) f(x,m)\,dx \\ &= \frac{1}{M}\sum_{m=1}^{M}\int_0^1 f(x,m)\,dx - \frac{1}{M}\sum_{m=1}^{M}\int_0^1 \mathbf{1}_{A^c}(x,m) f(x,m)\,dx \\ &\ge \frac{1}{M}\sum_{m=1}^{M}\int_0^1 f(x,m)\,dx - b\sqrt{\mu(f^2)},\end{aligned}$$

and in the limit as $M \to \infty$ we will find $\underline{\mu}(\mathbf{1}_A \cdot f) \ge \mu(f) - b\sqrt{\mu(f^2)} \ge (a-b)\sqrt{\mu(f^2)}$. Combining, we have that $\underline{\mu}(A) \ge (a-b)^2$. □

Now, let $\{y_m\}$ and ψ be as in the statement of Theorem 16, and let $\Delta_N(x,m)$ denote the number of integer solutions of $\|nx + y_m\| < \psi(n)$ with $0 < n \le N$. Then

$$\Delta_N(x,m) = \sum_{n=1}^{N} \mathbf{1}_n(nx + y_m) \quad \text{where} \quad \mathbf{1}_n := \mathbf{1}_{\left(-\frac{\psi(n)}{n}, \frac{\psi(n)}{n}\right)}.$$

Notice that

$$\mu(\Delta_N) = \sum_{n=1}^{N} 2\psi(n),$$

so our divergence assumption can be stated as $\mu(\Delta_N) \to \infty$ as $N \to \infty$.

Lemma 18. *For $n \ne k$, we will have*

$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\int_0^1 \mathbf{1}_n(nx + y_m)\mathbf{1}_k(kx + y_m)\,dx = 4\psi(n)\psi(k).$$

Proof. The function

$$\int_0^1 \mathbf{1}_n(nx + y)\mathbf{1}_k(kx + y)\,dx$$

is continuous in the real variable y. Therefore, since $\{y_m\}$ is equidistributed, we have that

$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\int_0^1 \mathbf{1}_n(nx + y_m)\mathbf{1}_k(kx + y_m)\,dx = \int_0^1\int_0^1 \mathbf{1}_n(nx + y)\mathbf{1}_k(kx + y)\,dx\,dy,$$

which is equal to $4\psi(n)\psi(k)$. □

Corollary 19. *For any $\varepsilon > 0$, we will have $\mu(\Delta_N) \ge (1-\varepsilon)\sqrt{\mu(\Delta_N^2)}$ for all sufficiently large N.*

Proof. We calculate

$$\begin{aligned}
\mu\big(\Delta_N^2\big) &= \lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\int_0^1 \Delta_N(x,m)^2\,dx \\
&= \lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\int_0^1 \sum_{n,k\le N} \mathbf{1}_n(nx+y_m)\mathbf{1}_k(kx+y_m)\,dx \\
&= \lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\int_0^1 \left(\sum_{n\le N}\mathbf{1}_n(nx+y_m)^2 + \sum_{\substack{n,k\le N\\ n\ne k}} \mathbf{1}_n(nx+y_m)\mathbf{1}_k(kx+y_m)\right)dx \\
&\overset{\text{Lem. 18}}{=} \mu(\Delta_N) + \sum_{\substack{n,k\le N\\ n\ne k}} 4\psi(n)\psi(k) \\
&\le \mu(\Delta_N) + \mu(\Delta_N)^2 \\
&\le (1-\varepsilon)^{-2}\mu(\Delta_N)^2
\end{aligned}$$

for N sufficiently large, since $\mu(\Delta_N)\to\infty$ as $N\to\infty$. $\square$

Proof of Theorem 16. For an arbitrary small $\varepsilon>0$, let $a=1-\varepsilon$ and $b=\varepsilon$. Corollary 19 and our divergence assumption tell us that for N sufficiently large we have

$$\mu(\Delta_N)\ge(1-\varepsilon)\sqrt{\mu(\Delta_N^2)} \quad\text{and}\quad \varepsilon\mu(\Delta_N)\ge R.$$

For these N, Lemma 17 implies that we will have $\Delta_N(x,m)\ge\varepsilon\mu(\Delta_N)\ge R$ on a set F_N with $\underline{\mu}(F_N)\ge(1-2\varepsilon)^2$. Notice that $F_N\subset F$. Since $\varepsilon>0$ was arbitrary, this shows $\mu(F)=1$. $\square$

Before stating the proof of Theorem 4, we prove the following simple lemma.

Lemma 20. *If $0\le a_m\le 1$ and $\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}a_m=1$, then for every $\varepsilon>0$, the set of integers $m\ge1$ for which $a_m\ge1-\varepsilon$ has asymptotic density* 1.

Proof. Suppose $0\le a_m\le1$ and that there is some $\varepsilon>0$ for which the set of integers $m\ge1$ with $a_m\ge1-\varepsilon$ has lower asymptotic density $A<1$. There is some sequence $M_j\to\infty$ of integers "realizing" this lower density, and so

$$\frac{1}{M_j}\sum_{m=1}^{M_j}a_m < A+(1-A)(1-\varepsilon)+o(1) \quad\text{as}\quad j\to\infty.$$

But $A+(1-A)(1-\varepsilon)=1-\varepsilon(1-A)<1$, so $\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}a_m\ne1$. $\square$

Proof of Theorem 4. Theorem 16 tells us that $\mu(F)=1$, that is,

$$\lim_{m\to\infty}\frac{1}{M}\sum_{m=1}^{M}\int_0^1\mathbf{1}_F(m,x)\,dx=1.$$

The theorem follows by applying Lemma 20 with $a_m=\int_0^1\mathbf{1}_F(x,m)\,dx$. $\square$

5. Discussion of inhomogeneous versions of the Duffin–Schaeffer Conjecture

For the approximating functions ψ coming from Theorems 1/2 and 3, notice that

$$\begin{aligned}\sum_n \frac{\varphi(n)\psi(n)}{n} &= \sum_j \sum_{m=1}^{M_j} \frac{\varphi(k_m)k_m}{k_m K_j} 2^{-j} \\ &\leq \sum_j \frac{1}{2^j K_j} \sum_{d|K_j} \varphi(d) \\ &= \sum_j 2^{-j},\end{aligned}$$

which converges. It therefore makes sense to formulate inhomogeneous versions of the Duffin–Schaeffer Conjecture that take this into account.

In §1.2 we have stated the direct translation of the Duffin–Schaeffer Conjecture to the inhomogeneous setting, the *Inhomogeneous Duffin–Schaeffer Conjecture* (IDSC). Obviously, it would be ideal to prove the IDSC, but since it is *a priori* stronger than the original conjecture, it may make more sense to aim for more modest goals first. Can we prove the IDSC for a particular inhomogeneous parameter y? Can we prove it for a family of inhomogeneous parameters, perhaps badly approximable y's? Can we prove it for *almost every* y? Or prove that it is a "zero-one" situation with respect to inhomogeneous parameters? In the spirit of Question 10, does the DSC imply the IDSC? Any of these would be nice.

In §1.2 we remarked on the temptation to remove monotonicity from the Inhomogeneous Khintchine Theorem by allowing the inhomogeneous part to vary among a (say, equidistributed) sequence. Theorem 1 shows that this is impossible, but Theorem 4 shows that taking an equidistributed sequence still results in a best-case scenario where we can say *something*. This leads us to the following question.

Question 21 (Countably inhomogeneous Duffin–Schaeffer Conjecture)**.** Let $\{y_i\}$ be some sequence of real numbers, and suppose ψ is an approximating function such that the sum $\sum_n \varphi(n)\psi(n)/n$ diverges. Does this imply that for almost every real number x there exists an integer $m \geq 1$ such that infinitely many coprime integer pairs (a, n) satisfy the inequality $|nx - a + y_m| < \psi(n)$?

Since we are inclined to believe that the Inhomogeneous Duffin–Schaeffer Conjecture is true, we must therefore also expect an affirmative answer to Question 21, regardless of the given sequence $\{y_i\}$. But in order to attack Question 21 directly, it makes more sense to restrict our attention to certain kinds of sequences. In view of Theorem 4, the most natural ones to consider are equidistributed.

In fact, one could modify Question 21 in a number of ways. Instead of a "countably inhomogeneous" version of the Duffin–Schaeffer Conjecture, one may also seek such a version of the Duffin–Schaeffer *Theorem*, where we make the additional assumption that

$$\limsup_{N\to\infty} \left(\sum_{n=1}^{N} \frac{\varphi(n)\psi(n)}{n} \right) \left(\sum_{n=1}^{N} \psi(n) \right)^{-1} > 0.$$

In any case, it would be helpful to have zero-one laws analogous to Gallagher's. We provide some in §6.

6. Proof of Theorem 5: Inhomogeneous zero-one laws

The aim in this section is to prove the following restatement of Theorem 5.

Theorem 22 (Theorem 5)**.** *Let y be a real number. Let W^y denote the set of real numbers x for which*

$$|nx - a + y| < \psi(n), \quad (a, n) = 1,$$

for infinitely many integers a, n. Let $W = \bigcup_m W^{my}$ and $\bar{W} = \limsup_{m\to\infty} W^{my}$. Then $|W| \in \{0, 1\}$ and $|\bar{W}| \in \{0, 1\}$.

Remark. W is the set of real numbers x for which there exists an integer m such that $|nx - a + my| < \psi(n)$ has infinitely many solutions $(a, n) = 1$, and $\bar{W}$ is the set of real numbers x for which there exist *infinitely many* such integers $m \geq 1$.

The proof of Theorem 22 follows Gallagher [Gal61], which in turn relies partly on the following lemma.

Lemma 23 (Cassels, [Cas50])**.** *Let $\{I_k\}$ be a sequence of intervals and let $\{U_k\}$ be a sequence of measurable sets such that for some positive $\varepsilon < 1$,*

$$U_k \subset I_k, \quad |U_k| \geq \varepsilon |I_k|, \quad |I_k| \to 0.$$

Then $|\limsup_{k\to\infty} I_k| = |\limsup_{k\to\infty} U_k|$.

Proof of Theorem 22. This proof follows Gallagher [Gal61]. The first reduction is to the case $\psi(n) = o(n)$, justified by the following fact:

> *The length L_n of the longest interval of consecutive integers not coprime to n satisfies $L_n = o(n)$.*

In the proof we will express $W = A \cup B \cup C$, and show that there is a 0-1 law for each of the sets A, B, C. For this, we will show that each of these is invariant under certain ergodic transformations.

For a prime number p and integers m, ν with $\nu \geq 1$, consider the inequality

$$|nx - a + my| < p^{\nu-1}\psi(n), \quad (a, n) = 1. \tag{8}$$

We define the sets

$$\begin{aligned} A(p^\nu, m) &= \{x \text{ satisfying (8) infinitely often with } p \nmid n\} \\ B(p^\nu, m) &= \{x \text{ satisfying (8) infinitely often with } p \parallel n\} \\ C(p^\nu, m) &= \left\{x \text{ satisfying (8) infinitely often with } p^2 \mid n\right\}, \end{aligned}$$

and

$$\begin{aligned} A(p^\nu) &= \bigcup_m A(p^\nu, m) & \bar{A}(p^\nu) &= \limsup_{m\to\infty} A(p^\nu, m) \\ B(p^\nu) &= \bigcup_m B(p^\nu, m) & \bar{B}(p^\nu) &= \limsup_{m\to\infty} B(p^\nu, m) \\ C(p^\nu) &= \bigcup_m C(p^\nu, m) & \bar{C}(p^\nu) &= \limsup_{m\to\infty} C(p^\nu, m). \end{aligned}$$

Notice that $W^{my} = A(p, m) \cup B(p, m) \cup C(p, m)$ for any prime p, and therefore that $W = A(p) \cup B(p) \cup C(p)$ for any prime p. Notice also that $\bar{W} = \bar{A}(p) \cup \bar{B}(p) \cup \bar{C}(p)$ for any prime p.

It is clear that $A(p,m) \subseteq A(p^\nu, m)$ for any $\nu \geq 1$. Since we are assuming that $\psi(n) = o(n)$, we may use Lemma 23 to conclude that $|A(p,m)| = |A(p^\nu, m)|$. It is therefore clear that $A(p) \subseteq A(p^\nu)$ and $|A(p)| = |A(p^\nu)|$, and therefore that $\bigcup_{\nu\geq 1} A(p^\nu)$ has the same measure. Also, we have that $\bar{A}(p) \subseteq \bar{A}(p^\nu)$, and the Borel–Cantelli Lemma implies that $\left|\bar{A}(p)\right| = \left|\bar{A}(p^\nu)\right|$. Therefore, $\bigcup_{\nu\geq 1} \bar{A}(p^\nu)$ has the same measure. This paragraph holds also after replacing A's with B's.

(In the remaining paragraphs of this proof, all instances of A, B, C, W can be replaced with $\bar{A}, \bar{B}, \bar{C}, \bar{W}$ to prove the 0-1 law for $\bar{W}$.)

Notice that if x satisfies (8) for some m and $p \nmid n$, then

$$|n(px) - pa + mpy| < p^\nu \psi(n), \quad (pa, n) = 1,$$

which shows that multiplication by p carries $A(p^\nu)$ into $A(p^{\nu+1})$, and therefore $\bigcup_{\nu\geq 1} A(p)$ is taken into itself. Since multiplication by p is an ergodic transformation of the circle, $\bigcup_{\nu\geq 1} A(p)$ must therefore have measure 0 or 1, hence $A(p)$ has measure 0 or 1.

As for B, notice that if x satisfies (8) with some m and $p \parallel n$, then

$$\left| n\left(px + \frac{1}{p}\right) - pa - \frac{n}{p} + mpy \right| < p^\nu \psi(n), \quad \left(pa + \frac{n}{p}, n\right) = 1,$$

and the same arguments will show that $B(p)$ has measure 0 or 1, this time using that $x \mapsto px + \frac{1}{p}$ is ergodic.

Now we know that if either $A(p)$ or $B(p)$ have positive measure, then W is full. So let us assume that $|A(p)| = |B(p)| = 0$ for all p, so that $|W| = |C(p)|$ and in fact $|W \triangle C(p)| = 0$ for all p. If m, a, n satisfy (8) with $p^2 \mid n$ and $\nu = 1$, then

$$\left| n\left(x \pm \frac{1}{p}\right) - a \pm \frac{n}{p} + my \right| < \psi(n), \quad \left(a \pm \frac{n}{p}, n\right) = 1,$$

which shows that $C(p)$ is periodic with period $1/p$. This means in particular that if I is any interval of length $1/p$, then $|C(p) \cap I| = |C(p)| \cdot |I|$. And since $|W \triangle C(p)| = 0$, we have $|W \cap I| = |W| \cdot |I|$.

Now suppose that $|W| > 0$, and let x_0 be a density point of W. Let

$$I_p = \left(x_0 - \frac{1}{2p}, x_0 + \frac{1}{2p}\right).$$

Then by Lebesgue's density theorem $|W \cap I_p| \sim |I_p|$ as $p \to \infty$. Therefore $|W| = 1$, finishing the proof. (Also, for $\bar{W}$, after taking the above parenthetical into account.) $\square$

By inspecting the proof of Theorem 22 we can deduce the following.

Theorem 24 (Theorem 6)**.** *Let y be a real number. Let W^y denote the set of real numbers x for which*

$$|nx - a + y| < \psi(n), \quad (a, n) = 1,$$

for infinitely many integers, a, n. Then at least one of the following holds:

- *W^{my} is null for every m.*
- *There is some m for which W^{my} is full.*
- *For any $\varepsilon > 0$ there are arbitrarily many $m \geq 1$ with $|W^{my}| > 1 - \varepsilon$.*

Proof. By Theorem 5 we know that the measure of $W = \bigcup_m W^{my}$ is either 0 or 1. If it is 0, then of course W^{my} is also null for every m. So suppose that $|W| = 1$. This implies that some W^{my} must have positive measure. But $W^{my} = A(p,m) \cup B(p,m) \cup C(p,m)$ for any prime p, so one of these three sets must have positive measure. If for infinitely many primes p we have $|A(p,m)| = |B(p,m)| = 0$, then our argument in the proof of Theorem 5 shows that W^{my} must be full. On the other hand if $|A(p,m)| > 0$, then $|p^\nu A(p,m)| > 1 - \varepsilon$ if ν is large enough. And our arguments in the previous proof show that $|p^\nu A(p,m)| \leq |A(p^{\nu+1}, p^\nu m)| = |A(p, p^\nu m)|$. Therefore, $|W^{p^\nu my}| > 1 - \varepsilon$ for all ν large enough. □

Acknowledgments. This project was inspired and encouraged by numerous conversations with Sanju Velani. I am grateful to both him and Victor Beresnevich for welcoming me into the Number Theory Study Group at the University of York, UK, and for initiating me into the field of metric Diophantine approximation. I also thank the anonymous referee for a careful reading.

References

[Ais14] Christoph Aistleitner, *A note on the duffinschaeffer conjecture with slow divergence*, Bulletin of the London Mathematical Society **46** (2014), no. 1, 164–168. 3

[BHHV13] Victor Beresnevich, Glyn Harman, Alan Haynes, and Sanju Velani, *The Duffin-Schaeffer conjecture with extra divergence II*, Math. Z. **275** (2013), no. 1-2, 127–133. 3

[Cas50] J. W. S. Cassels, *Some metrical theorems in Diophantine approximation. I*, Proc. Cambridge Philos. Soc. **46** (1950), 209–218. 17

[Cas57] ______, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, Cambridge University Press, New York, 1957. 2, 3, 13

[DS41] R. J. Duffin and A. C. Schaeffer, *Khintchine's problem in metric Diophantine approximation*, Duke Math. J. **8** (1941), 243–255. 2

[Erd50] P. Erdős, *On integers of the form $2^k + p$ and some related problems*, Summa Brasil. Math. **2** (1950), 113–123. 4, 10

[Erd70] ______, *On the distribution of the convergents of almost all real numbers*, J. Number Theory **2** (1970), 425–441. 2

[Erd95] Paul Erdős, *Some of my favourite problems in number theory, combinatorics, and geometry*, Resenhas **2** (1995), no. 2, 165–186, Combinatorics Week (Portuguese) (São Paulo, 1994). 4, 10

[FFK+07] Michael Filaseta, Kevin Ford, Sergei Konyagin, Carl Pomerance, and Gang Yu, *Sieving by large integers and covering systems of congruences*, J. Amer. Math. Soc. **20** (2007), no. 2, 495–517. 4, 10, 11

[Gal61] Patrick Gallagher, *Approximation by reduced fractions*, J. Math. Soc. Japan **13** (1961), 342–345. 3, 5, 17

[Hou15] Bob Hough, *Solution of the minimum modulus problem for covering systems*, Ann. of Math. (2) **181** (2015), no. 1, 361–382. 4, 10

[HPV12] Alan K. Haynes, Andrew D. Pollington, and Sanju L. Velani, *The Duffin-Schaeffer conjecture with extra divergence*, Math. Ann. **353** (2012), no. 2, 259–273. 3

[Khi24] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. **92** (1924), no. 1-2, 115–125. 2

[Kur55] J. Kurzweil, *On the metric theory of inhomogeneous diophantine approximations*, Studia Math. **15** (1955), 84–112. 2

[PV90] A. D. Pollington and R. C. Vaughan, *The k-dimensional Duffin and Schaeffer conjecture*, Mathematika **37** (1990), no. 2, 190–200. 3

[Szü58] P. Szüsz, *Über die metrische Theorie der Diophantischen Approximation*, Acta Math. Sci. Hungar **9** (1958), 177–193. 3

[Vaa78] Jeffrey D. Vaaler, *On the metric theory of Diophantine approximation*, Pacific J. Math. **76** (1978), no. 2, 527–539. 2

Wesleyan University, Middletown, CT, USA
E-mail address: framirez@wesleyan.edu