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NEARLY MORITA EQUIVALENCES AND RIGID OBJECTS

ROBERT MARSH AND YANN PALU

Abstract. If T and T' are two cluster-tilting objects of an acyclic cluster category related by a mutation, their endomorphism algebras are nearly-Morita equivalent [BMR07], i.e. their module categories are equivalent "up to a simple module". This result has been generalised by D. Yang, using a result of P-G. Plamondon, to any simple mutation of maximal rigid objects in a 2-Calabi–Yau triangulated category. In this paper, we investigate the more general case of any mutation of a (non-necessarily maximal) rigid object in a triangulated category with a Serre functor. In that setup, the endomorphism algebras might not be nearly-Morita equivalent and we obtain a weaker property that we call pseudo-Morita equivalence. Inspired by [BM12, BM13], we also describe our result in terms of localisations.

Keywords: triangulated category, pseudo-Morita equivalence, rigid object, localisation, adjunction

Contents

Introduction and main results

In this paper, our aim is to prove a weak form of nearly-Morita equivalence for mutations of (non-maximal) rigid objects in triangulated categories. Before recalling the case of cluster-tilting objects [BMR07], we first give an example.

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Let Q be a linear orientation of the Dynkin diagram of type A_3. The Auslander–Reiten quiver of the acyclic cluster category $\mathcal{C}_Q$, defined in [BMR$^+$06], is as follows:

$$\begin{tikzcd}
 & & T_3 \arrow[dr] & & \Sigma T_1 \arrow[dr] & & T_1 \arrow[dr] & & \Sigma T_2^* \\
 & T_2 \arrow[ur] \arrow[dr] & & {} \arrow[ur] \arrow[dr] & & \Sigma T_2 \arrow[ur] \arrow[dr] & & T_2 \arrow[ur] \arrow[dr] & \\
T_1 \arrow[ur] & & \Sigma T_2^* \arrow[ur] & & T_2^* \arrow[ur] & & \Sigma T_3 \arrow[ur] & & T_3
\end{tikzcd}$$

The object $T = T_1 \oplus T_2 \oplus T_3$ is cluster-tilting. Its mutation at T_2 is the cluster-tilting object $T' = T_1 \oplus T_2^* \oplus T_3$. We write Γ for the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ and Γ' for $\operatorname{End}_{\mathcal{C}}(T')^{\mathrm{op}}$. Then the two algebras Γ and Γ' are related as follows.

On the one hand, the functor $\mathcal{C}(T, -)$ induces an equivalence of categories $\mathcal{C}/(\Sigma T) \simeq \operatorname{mod} \Gamma$, where $\operatorname{mod} \Gamma$ is the category of finitely generated left modules, and the Auslander–Reiten quiver of $\operatorname{mod} \Gamma$ is thus:

$$\begin{tikzcd}
 & & {} \arrow[dr] & & \\
 & {} \arrow[ur] \arrow[dr] & & {} \arrow[dr] & \\
{} \arrow[ur] & & S_2 \arrow[ur] & & {}
\end{tikzcd}$$

where $S_2 = \mathcal{C}(T, \Sigma T_2^*)$ is the simple top of the projective indecomposable $\mathcal{C}(T, T_2)$.

On the other hand, the functor $\mathcal{C}(T', -)$ induces an equivalence of categories $\mathcal{C}/(\Sigma T') \simeq \operatorname{mod} \Gamma'$ and the Auslander–Reiten quiver of $\operatorname{mod} \Gamma'$ is thus:

$$\begin{tikzcd}
 & & & {} \arrow[dr] \arrow[d, phantom, "\cdots"] & & & & {} \\
S_2^* \arrow[dr] & & {} \arrow[ur] & {} & {} \arrow[dr] & & S_2^* \arrow[ur] & \\
 & {} \arrow[ur] \arrow[u, phantom, "\cdots"] & & & & {} \arrow[ur] \arrow[u, phantom, "\cdots"] & &
\end{tikzcd}$$

where $S_2^* = \mathcal{C}(T', \Sigma T_2)$ is the simple top of the projective indecomposable $\mathcal{C}(T', T_2^*)$, where the two arrows starting at S_2^* are identified, and where dots indicate zero relations.

The two Auslander–Reiten quivers are not isomorphic, therefore Γ and Γ' are not Morita equivalent. But they are not very far from being so: The difference in the Auslander–Reiten quivers comes from the simples S_2 and S_2^*.

The common Auslander-Reiten quiver of the categories $\operatorname{mod} \Gamma/(\operatorname{add} S_2)$ and $\operatorname{mod} \Gamma/(\operatorname{add} S_2^*)$ is thus:

$$\begin{tikzcd}
 & & {} \arrow[dr] \arrow[d, phantom, "\cdots"] & & \\
 & {} \arrow[ur] & {} & {} \arrow[dr] & \\
{} \arrow[ur] & & & & {}
\end{tikzcd}$$

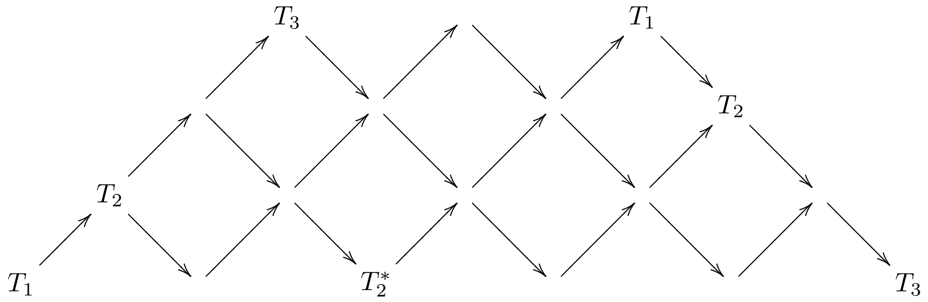
This phenomenon, proved in [BMR07], has been called “nearly Morita equivalence” by C. M. Ringel. Let us state the precise result.

Let Q be an acyclic quiver, and let T be a cluster-tilting object in the cluster category $\mathcal{C}_Q$. Let $T' = T/T_k \oplus T_k^*$ be the mutation of T at an indecomposable summand T_k; then T' is also a cluster-tilting object. Let Γ (respectively, Γ') be the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}_Q}(T)^{\mathrm{op}}$ (respectively, $\operatorname{End}_{\mathcal{C}_Q}(T')^{\mathrm{op}}$) and S_k (respectively, S_k^*) be the simple top of the projective indecomposable Γ-module $\mathcal{C}_Q(T, T_k)$ (respectively, the simple top of the Γ'-module $\mathcal{C}_Q(T', T_k^*)$).

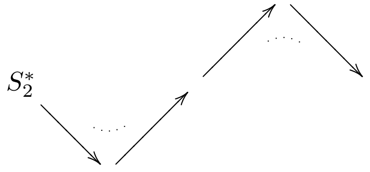
Then, by a result of [BMR07], the categories $\operatorname{mod}\Gamma/\operatorname{add}S_k$ and $\operatorname{mod}\Gamma'/\operatorname{add}S_k^*$ are equivalent. By [Yan12, Corollary 4.3], nearly-Morita equivalence, in the more general setup of simple, 2-periodic mutations of rigid objects (or rigid, Krull–Schmidt subcategories) in any triangulated category, follows from [Pla11, Proposition 2.7].

Our main aim in this paper is to prove an analogous result for any mutation of (non-maximal) rigid objects. Before explaining our results, let us have a look at an example which shows that one cannot expect these mutations to induce a nearly-Morita equivalence in general.

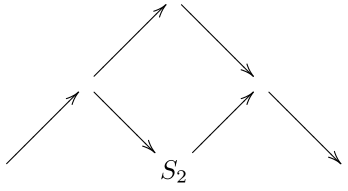
Let $T = T_1 \oplus T_2 \oplus T_3$ be the rigid object of the acyclic cluster category $\mathcal{C} = \mathcal{C}_{A_4}$ given by:



and let $T' = T_1 \oplus T_2^* \oplus T_3$ be the rigid object obtained by mutating T at the summand T_2. This means that ΣT_2^* is the cone of a minimal right $\operatorname{add}T/T_2$-approximation of T_2. In the example, there is a triangle $T_2^* \to T_1 \to T_2 \to \Sigma T_2^*$. Let Λ (respectively, Λ') be the algebra $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ (respectively, $\operatorname{End}_{\mathcal{C}}(T')^{\mathrm{op}}$). Using results in [BM12], [BM13] (see also [KR07]), we can easily compute the AR quivers of $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda'$:

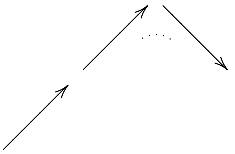


$\operatorname{mod}\Lambda'$

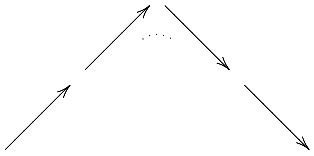


$\operatorname{mod}\Lambda$

The algebras Λ and Λ' are not nearly-Morita equivalent. On factoring out by S_2 (respectively, S_2^*), we obtain the following Auslander-Reiten quivers:



$\operatorname{mod}\Lambda'/\operatorname{add}S_2^*$



$\operatorname{mod}\Lambda/\operatorname{add}S_2$

However, these algebras are not very far from being nearly-Morita equivalent. Indeed, the Auslander–Reiten quivers differ by only one arrow. The corresponding morphism can be characterised in $\operatorname{mod}\Lambda$ as being surjective with kernel in the subcategory $\operatorname{add} S_2$.

Let $\mathcal{C}$ be an acyclic cluster category, and let T be a rigid object in $\mathcal{C}$. Let $T' = T/T_k \oplus T_k^*$ be the mutation of T at the summand T_k. Let Λ (respectively, Λ') be the algebra $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ (respectively, $\operatorname{End}_{\mathcal{C}}(T')^{\mathrm{op}}$), and let S_k (respectively, S_k^*) be the simple top of the projective indecomposable Λ-module $\mathcal{C}(T, T_k)$ (respectively, the Λ'-module $\mathcal{C}(T', T_k^*)$).

As suggested by the example above, let us consider the class $\mathcal{R}$ of epimorphisms in $\operatorname{mod}\Lambda$ with kernels in $\operatorname{add} S_k$, and the class $\mathcal{R}^*$ of monomorphisms in $\operatorname{mod}\Lambda'$ with cokernels in $\operatorname{add} S_k^*$.

Theorem A. *There is an equivalence of categories:*

$$(\operatorname{mod}\Lambda)_{\mathcal{R}} \simeq (\operatorname{mod}\Lambda')_{\mathcal{R}^*}.$$

This result is not completely satisfactory since it does not resemble nearly-Morita equivalence. The following remark will help in restating the Theorem in a form which looks more like nearly-Morita equivalence.

Let $M \in \operatorname{mod}\Lambda$. If there is a short exact sequence $0 \to S_k \to L \xrightarrow{f} M \to 0$, the morphism f belongs to $\mathcal{R}$. Therefore the objects L and M become isomorphic in the localisation $(\operatorname{mod}\Lambda)_{\mathcal{R}}$. This suggests that the objects having non-split extensions with S_k can be removed from $\operatorname{mod}\Lambda$ without changing the localisation. We thus define $\mathcal{E}$ to be the full subcategory of $\operatorname{mod}\Lambda$ whose objects M satisfy $\operatorname{Ext}^1_\Lambda(M, S_k) = 0$. Dually, let $\mathcal{E}'$ be the full subcategory of $\operatorname{mod}\Lambda'$ whose objects N satisfy $\operatorname{Ext}^1_{\Lambda'}(S_k^*, N) = 0$.

Note that $\mathcal{E}$ and $\mathcal{E}'$ are extension-closed in $\operatorname{mod}\Lambda$ (respectively, $\operatorname{mod}\Lambda'$) and are thus exact categories.

Theorem B. *There is an equivalence of categories:*

$$(\operatorname{mod}\Lambda)_{\mathcal{R}} \simeq \mathcal{E}/\operatorname{add} S_k.$$

Dually, there is an equivalence of categories:

$$(\operatorname{mod}\Lambda')_{\mathcal{R}^*} \simeq \mathcal{E}'/\operatorname{add} S_k^*.$$

Combining the two theorems gives the following.

Corollary. *There is an equivalence of categories:*

$$\mathcal{E}/\operatorname{add} S_k \simeq \mathcal{E}'/\operatorname{add} S_k^*.$$

This resembles nearly-Morita equivalence except that, unlike in the cluster-tilting case, one has to restrict to an exact subcategory before killing the simple.

Unfortunately, these statements do not specialise to a nearly-Morita equivalence in the cluster-tilting case: In the setup of [BMR07], we obtain a weaker statement.

The proofs of Theorems A and B are in Subsection 3.1 (but note that the proofs appear in reverse order to the above). In fact, we will prove more general results than those mentioned above. First, we only assume the triangulated category $\mathcal{C}$ to be Krull–Schimdt, with a Serre functor. Second, we allow mutations at non-indecomposable summands. Our results hold, in particular, in any triangulated category in the following list (whose items overlap):

- Hom-finite generalised higher cluster categories ([Ami09], [Guo11]);
- stable categories of maximal Cohen–Macaulay modules over an odd dimensional isolated hypersurface singularity ([BIKR08]);
- cluster tubes ([BKL08], [BMV10]...);
- (higher) cluster categories of type A_∞ ([HJ12], [HJ13]);
- the triangulated orbit categories listed in [Ami07];
- stable categories constructed from preprojective algebras in [GLS]...

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1. SETUP AND NOTATION

We fix a field k, and a Krull–Schmidt, k-linear, Hom-finite, triangulated category $\mathcal{C}$, with suspension functor Σ. An object X in $\mathcal{C}$ is called *rigid* if $\operatorname{Ext}^1_{\mathcal{C}}(X,X)=0$, where we write $\operatorname{Ext}^1_{\mathcal{C}}(X,Y)$ for $\mathcal{C}(X,\Sigma Y)$. We write $X^{\perp}$ for the right Hom-perp of X, i.e. the subcategory of $\mathcal{C}$ on objects Y such that $\mathcal{C}(X,Y)=0$. Note that this notation differs from that used in [BM13], which we often cite, but here the Hom-perpendicular categories play a key role so we use a different notation.

Let $T\in\mathcal{C}$ be a basic rigid object. Let R be a direct summand of T and write $T=\overline{T}\oplus R$. Let T' be the object obtained from T by replacing R by the negative shift R^* of the cone of a minimal right $\operatorname{add}\overline{T}$-approximation of R. We have a triangle $R^*\to B\to R\to\Sigma R^*$, with $B\in\operatorname{add}\overline{T}$, $B\to R$ a minimal right $\operatorname{add}\overline{T}$-approximation, and $T'=\overline{T}\oplus R^*$. By [BMR$^+$06, Lemma 6.7], $\Sigma R^*\in\overline{T}^{\perp}$. We assume that T' is rigid. More precisely, we assume that R^* is rigid and that R^* belongs to ${}^{\perp}\Sigma\overline{T}$. By [IY08, Proposition 2.6(1)] and [BMR$^+$06, Lemma 6.5], R and R^* are basic and have the same number of indecomposable direct summands. We keep these assumptions throughout the paper.

Remark: We note that, if $\mathcal{C}$ is 2-Calabi–Yau, then T' is automatically rigid (see [BMR$^+$06, Section 6]). However, WuZhong Yang kindly warned us that this is not true in general: Example 2.20 in [YZZ15] shows that R^* might not be rigid. Moreover, even when R^* is rigid it might not belong to ${}^{\perp}\Sigma\overline{T}$. An example illustrating this latter phenomenon can be found in [YZ15, Example 2.15]. In that example, $\mathcal{C}$ is the 3-cluster category of type A_3, which contains a cluster-tilting object T. If R is any indecomposable summand of T, the (left) mutation of T at R gives an indecomposable rigid object R^* which belongs to $\Sigma^{-1}\overline{T}^{\perp}$, but not to ${}^{\perp}\Sigma\overline{T}$.

In some statements, we will assume additionally that $\mathcal{C}$ has a Serre functor.

We also need some more notation. If X is an object in $\mathcal{C}$, we write (X) for the ideal of morphisms factoring through the additive subcategory $\operatorname{add}X$ generated by X. All modules considered are left modules.

We denote by $\mathcal{C}(T)$ the full subcategory of $\mathcal{C}$ whose objects are the cones of morphisms $T_1\to T_0$, where $T_0,T_1\in\operatorname{add}T$, and by $\overline{\mathcal{C}}(T)$ the full subcategory of $\mathcal{C}$ whose objects are the cones of morphisms $\overline{T_1}\to T_0$, where $T_0\in\operatorname{add}T$ and $\overline{T_1}\in\operatorname{add}\overline{T}$.

More generally, for any two full subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{C}$, we use the notation $\mathcal{A}*\mathcal{B}$ for the full subcategory whose objects X are extensions of an object in $\mathcal{B}$ by an object in $\mathcal{A}$ (i.e. X appears in a triangle $A\to X\to B\to\Sigma A$ with $A\in\mathcal{A}$ and $B\in\mathcal{B}$). It follows from the octahedral axiom that the operation $*$ is associative. By abuse of notation, if A,B are objects in $\mathcal{C}$, we will write $A*B$ for $\operatorname{add}A*\operatorname{add}B$.

Thus one could also define $\mathcal{C}(T)$ and $\overline{\mathcal{C}}(T)$ by: $\mathcal{C}(T)=T*\Sigma T$ and $\overline{\mathcal{C}}(T)=T*\Sigma\overline{T}$.

Remark: Our results hold in the more general setup of rigid subcategories: replace $\operatorname{add} T$ by a rigid subcategory $\mathcal{T}$, with the following additional assumptions: $\mathcal{T}$ is contravariantly finite, $\overline{\mathcal{T}}$ is functorially finite and $\mathcal{T}'$ is covariantly finite. This requires changing the functors of the form $\mathcal{C}(T,-)$ taking values in the category $\operatorname{mod}\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ into functors of the form $\mathcal{C}(?,-)|_{\mathcal{T}}$, taking values in $\operatorname{mod}\mathcal{T}$, and all references to [BM13] by references to [Bel13].

2. Pseudo-Morita equivalence

2.1. **Adjunctions.** The methods used in this subsection are inspired by [Bel13, BM13, BM12], and much resemble results in [Nak13, Section 3]. Indeed, [Nak13, Corollary 3.8] applied to the twin cotorsion pair $(\Sigma\overline{T}, \overline{T}^{\perp}), (\Sigma T', T'^{\perp})$ (where we use the notation from Subsection 2.2) gives the existence of a right adjoint to the fully faithful functor $\overline{\mathcal{C}}(T)/(\Sigma T') \longrightarrow \mathcal{C}/(\Sigma T')$ from which it is possible to deduce our Proposition 2.5. For convenience of the reader, we nonetheless include a complete proof.

The subcategory $\mathcal{C}(T)$ is known to be contravariantly finite, by [BM13, Lemmas 3.3 and 3.6]. An analogous proof gives Lemma 2.2 below. We first need a definition.

Definition 2.1. Let $\mathcal{S}$ be the set of morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$ such that for any triangle $Z \to X \xrightarrow{f} Y \xrightarrow{g} \Sigma Z$, we have $Z \in \overline{T}^{\perp}$ and $g \in (T^{\perp})$.

Lemma 2.2. (a) *Let $X' \xrightarrow{s} X$ be a morphism in $\mathcal{S}$ with $X' \in \overline{\mathcal{C}}(T)$. Then s is a right $\overline{\mathcal{C}}(T)$ approximation of X.*
(b) *Each object X in $\mathcal{C}$ has a right $\overline{\mathcal{C}}(T)$-approximation $R_0X \xrightarrow{\eta_X} X$ lying in $\mathcal{S}$.*
(c) *The category $\overline{\mathcal{C}}(T)$ is a contravariantly finite subcategory of $\mathcal{C}$.*

Proof. Suppose that $X' \xrightarrow{s} X$ is a morphism in $\mathcal{S}$ with $X' \in \overline{\mathcal{C}}(T)$. Thus, we may complete s to a triangle:

$$X' \xrightarrow{s} X \xrightarrow{g} \Sigma Z \to \Sigma X'$$

where g factors through $T^{\perp}$ and ΣZ lies in $(\Sigma T)^{\perp}$

Let $X'' \xrightarrow{u} X$ be a morphism in $\mathcal{C}$. Assume X'' lies in $\overline{\mathcal{C}}(T)$, so that there is a triangle $U_0 \xrightarrow{p} X'' \to \Sigma\overline{U}_1 \to \Sigma U_0$, with $U_0 \in \operatorname{add} T$ and $\overline{U}_1 \in \operatorname{add}\overline{T}$. Since g factors through $T^{\perp}$ and $U_0 \in \operatorname{add} T$, we have $gup = 0$ and therefore have the following commutative diagram whose rows are triangles:

$$\begin{tikzcd}
U_0 \arrow[r, "p"] \arrow[d, dashed, "w"'] & X'' \arrow[r, "\eta"] \arrow[d, "u"] \arrow[dl, dashed, "u'"'] & \Sigma\overline{U}_1 \arrow[r] \arrow[d, dashed, "v"] & \Sigma U_0 \arrow[d, dashed, "\Sigma w"] \\
X' \arrow[r, "s"'] & X \arrow[r, "g"'] & \Sigma Z \arrow[r] & \Sigma X'
\end{tikzcd}$$

Moreover, ΣZ lies in $(\Sigma\overline{T})^{\perp}$ and $\Sigma\overline{U}_1$ is in $\operatorname{add}\Sigma\overline{T}$, so the composition $gu = v\eta$ is zero. Thus, there is a morphism u' such that $u = su'$. Part (a) is shown.

For part (b), let $X \in \mathcal{C}$. Let $T_0^X \to X$ be a minimal right $\operatorname{add} T$-approximation of X. Complete it to a triangle $Y \to T_0^X \to X \to \Sigma Y$. Let $\overline{T}_1^Y \to Y$ be a minimal right $\operatorname{add}\overline{T}$-approximation of Y. Applying the octahedral axiom, we obtain the following diagram:

$$\begin{tikzcd}
\overline{T}_1^Y \arrow[r, equal] \arrow[d] & \overline{T}_1^Y \arrow[d] & & \\
Y \arrow[r] \arrow[d] & T_0^X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & \Sigma Y \arrow[d] \\
Z \arrow[r, "f"] \arrow[d] & R_0X \arrow[r, "\eta_X"] \arrow[d] & X \arrow[r, "g"] & \Sigma Z \\
\Sigma\overline{T}_1^Y \arrow[r, equal] & \Sigma\overline{T}_1^Y & &
\end{tikzcd}$$

Applying the functors $\mathcal{C}(T,-)$ and $\mathcal{C}(\overline{T},-)$ to the triangles above shows that $\Sigma Y \in T^{\perp}$ and $Z \in \overline{T}^{\perp}$. Note that $R_0X \in \overline{\mathcal{C}}(T)$. Then, by part (a), η_X is a right $\overline{\mathcal{C}}(T)$-approximation of X, and part (b) is shown. Part (c) follows immediately from part (b). $\surd$

The following remark is stated as a lemma since it will be used several times in the paper.

Lemma 2.3. *Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}$ with $X \in \overline{\mathcal{C}}(T)$ and assume that f factors through $\overline{T}^{\perp}$ in $\mathcal{C}$. Then f factors through $\overline{T}^{\perp} \cap \overline{\mathcal{C}}(T)$.*

Proof. Let $\overline{T}_0 \xrightarrow{u} X$ be a minimal right $\overline{T}$-approximation of X in $\mathcal{C}$. Complete the morphism u to a triangle $\overline{T}_0 \overset{u}{\to} X \overset{v}{\to} Z \to \Sigma\overline{T}_0$ in $\mathcal{C}$. As shown in [BMR$^+$06] (apply the functor $\mathcal{C}(\overline{T},-)$ to the triangle above) the cone Z belongs to $\overline{T}^{\perp}$. Moreover, the composition fu vanishes since f factors through $\overline{T}^{\perp}$ and it follows that the morphism f factors through v. It remains to be checked that the object Z lies in $\overline{\mathcal{C}}(T)$. The triangle above shows that $Z \in \overline{\mathcal{C}}(T) * \operatorname{add}\Sigma\overline{T}$, and we have:

$$\begin{aligned}
\overline{\mathcal{C}}(T) * \operatorname{add}\Sigma\overline{T} &= (\operatorname{add}T * \operatorname{add}\Sigma\overline{T}) * \operatorname{add}\Sigma\overline{T} \\
&= \operatorname{add}T * (\operatorname{add}\Sigma\overline{T} * \operatorname{add}\Sigma\overline{T}) \\
&= \operatorname{add}T * \operatorname{add}\Sigma\overline{T},
\end{aligned}$$

where the last equality holds since $\Sigma\overline{T}$ is rigid. $\surd$

The following lemma, which is used in the proof of Proposition 2.5, is a particular case of [ML98, IV.1 Theorem 2 (ii)].

Lemma 2.4. *Let $\mathcal{B}$ be a category, and let $\mathcal{A}$ be a full subcategory of $\mathcal{B}$. Suppose that, for any $B \in \mathcal{B}$, there is an object $G_0B \in \mathcal{A}$ and a morphism $G_0B \xrightarrow{\eta_B} B$ such that for all $A \xrightarrow{f} B$ with $A \in \mathcal{A}$, the morphism f lifts uniquely through η_B. Then the inclusion $\mathcal{A} \subseteq \mathcal{B}$ has a right adjoint $G : \mathcal{B} \to \mathcal{A}$ such that, for all $B \in \mathcal{B}$, $GB = G_0B$.*

The functor G of the previous lemma is defined on arrows as follows: For any $B \xrightarrow{b} B'$ in $\mathcal{B}$, Gb is the unique lift through $\eta_{B'}$ of the composition $b\eta_B$:

$$\begin{tikzcd}
G_0B \arrow[r, "\eta_B"] \arrow[d, dashed, "Gb"'] & B \arrow[d, "b"] \\
G_0B' \arrow[r, "\eta_{B'}"'] & B'.
\end{tikzcd}$$

The following proposition is inspired by [Bel13]:

Proposition 2.5. *The inclusion of* $\overline{\mathcal{C}}(T)$ *into* $\mathcal{C}$ *induces a fully faithful functor* $\frac{\overline{\mathcal{C}}(T)}{\overline{T}^{\perp}\cap\overline{\mathcal{C}}(T)} \xrightarrow{I} \frac{\mathcal{C}}{\overline{T}^{\perp}}$. *Moreover, the functor* I *admits an additive right adjoint* R*, such that, for all* X *in* $\mathcal{C}$*,* $RX = R_0X$*, in the notation of Lemma 2.2.*

Proof. The inclusion of $\overline{\mathcal{C}}(T)$ in $\mathcal{C}$ induces a full functor:

$$\begin{tikzcd}
\overline{\mathcal{C}}(T) \arrow[r, hook] \arrow[d, two heads, "\overline{Q}"'] & \mathcal{C} \arrow[d, two heads, "Q"] \\
\overline{\mathcal{C}}(T)/\left(\overline{T}^{\perp}\cap\overline{\mathcal{C}}(T)\right) \arrow[r, dashed, "I"] & \mathcal{C}/\left(\overline{T}^{\perp}\right).
\end{tikzcd}$$

We first check that the functor I is faithful. This amounts to proving that if a morphism in $\overline{\mathcal{C}}(T)$ factors through $\overline{T}^{\perp}$ in $\mathcal{C}$, then it already factors through $\overline{T}^{\perp}$ in $\overline{\mathcal{C}}(T)$. This follows from Lemma 2.3. In what follows, we will identify $\overline{\mathcal{C}}(T)/(\overline{T}^{\perp}\cap\overline{\mathcal{C}}(T))$ with the image of $\overline{\mathcal{C}}(T)$ in $\mathcal{C}/(\overline{T}^{\perp})$.

Next, we prove the existence of a right adjoint. For this, we use the particular case of [ML98, IV-1 Theorem 2 (ii)], stated in Lemma 2.4.

Let $X \in \mathcal{C}$. Consider the morphism $R_0X \xrightarrow{\eta_X} X$ constructed in Lemma 2.2. We claim that $Q\eta_X$ is universal from I to X, in the sense of MacLane, i.e. any morphism in $\mathcal{C}/(\overline{T}^{\perp})$ from an object in $\overline{\mathcal{C}}(T)$ to X factors uniquely through $Q\eta_X$ in $\mathcal{C}/(\overline{T}^{\perp})$. Since η_X is a right $\overline{\mathcal{C}}(T)$-approximation of X in $\mathcal{C}$, its image $Q\eta_X$ is a right $\overline{\mathcal{C}}(T)/(\overline{T}^{\perp}\cap\overline{\mathcal{C}}(T))$-approximation of X in $\mathcal{C}/(\overline{T}^{\perp})$, so that we only have to prove uniqueness.

Let $Y \in \overline{\mathcal{C}}(T)$ and let $Y \xrightarrow{u} R_0X$ be a morphism in $\mathcal{C}$ such that $Q(\eta_X u) = 0$. Since the kernel of Q is the ideal $(\overline{T}^{\perp})$ of $\mathcal{C}$, this means that the composition $\eta_X u$ factors through $\overline{T}^{\perp}$. Since its source belongs to $\overline{\mathcal{C}}(T)$, Lemma 2.3 shows that $\eta_X u$ factors through $\overline{T}^{\perp}\cap\overline{\mathcal{C}}(T)$. Let $Y' \in \overline{T}^{\perp}\cap\overline{\mathcal{C}}(T)$ be such that the square:

$$\begin{tikzcd}
 & Y \arrow[r, "a"] \arrow[d, "u"'] & Y' \arrow[d, "b"] & \\
Z \arrow[r, "\alpha"] & R_0X \arrow[r, "\eta_X"] & X \arrow[r] & \Sigma Z
\end{tikzcd}$$

commutes. Since η_X is a right $\overline{\mathcal{C}}(T)$-approximation, there exists a morphism $Y' \xrightarrow{c} R_0X$ with $b = \eta_X c$. We have $\eta_X(u - ca) = 0$ so that the morphism $u - ca$ factors through α. By construction, $Z \in \overline{T}^{\perp}$, therefore we have $u \in (\overline{T}^{\perp})$, which proves uniqueness.

Finally, we note that the functor R is additive since it is the right adjoint of the additive functor I. $\surd$

If the category admits a Serre functor S, then a dual version of Proposition 2.5 will be of interest to us. We first note that applying to ST' the construction dual to that of R_0 gives, for any $X \in \mathcal{C}$, a triangle $Z \xrightarrow{\alpha} X \xrightarrow{\varepsilon_X} L_0X \longrightarrow \Sigma Z$, where L_0X belongs to $\operatorname{add}\Sigma^{-1}S\overline{T} * \operatorname{add} ST'$, ε_X is a minimal left $\operatorname{add}\Sigma^{-1}S\overline{T} * \operatorname{add} ST'$-approximation, α factors through ${}^{\perp}(ST') = (T')^{\perp}$, and ΣZ belongs to $\overline{T}^{\perp}$.

Proposition 2.6. *Assume that the category* $\mathcal{C}$ *has a Serre functor* S *and let* $\underline{\mathcal{C}}(T')$ *be the full subcategory* $\operatorname{add}\Sigma^{-1}S\overline{T} * \operatorname{add} ST'$ *of* $\mathcal{C}$. *Then the inclusion of* $\underline{\mathcal{C}}(T')$ *into* $\mathcal{C}$ *induces a fully faithful functor* $\underline{\mathcal{C}}(T')/(\overline{T}^{\perp}) \xrightarrow{J} \mathcal{C}/(\overline{T}^{\perp})$. *Moreover, the functor* J *admits an additive left adjoint* L*, such that* $LX = L_0X$ *for all* $X \in \mathcal{C}$.

The only reason why we assume the existence of a Serre functor here is that it converts a left perpendicular subcategory into a right perpendicular subcategory. This allows us to view both categories in Propositions 2.5 and 2.6 as subcategories of the same category $\mathcal{C}/(\overline{T}^{\perp})$.

2.2. **Main result.** Our aim in this section is to prove that if $\mathcal{C}$ has a Serre functor then the categories $\overline{\mathcal{C}}(T)/(\Sigma T')$ and $\overline{\mathcal{C}}(T)/(T)$ are equivalent (Theorem 2.9). This will then be used in the next section in order to compare the module categories over the endomorphism algebras of T and T'.

We need the following key lemma, which will often be used throughout the paper.

Lemma 2.7. *We have:*

(a) $\overline{\mathcal{C}}(T) = T * \Sigma\overline{T} = \overline{T} * \Sigma T'$;

(b) $\overline{\mathcal{C}}(T) \cap \overline{T}^{\perp} = \operatorname{add}\Sigma T'$;

(c) *if* $\mathcal{C}$ *has a Serre functor* S, *then* $(\Sigma^{-1}S\overline{T} * ST') \cap \overline{T}^{\perp} = \operatorname{add}\Sigma^{-1}ST$.

Proof. (a) The exchange triangle shows that $T \in \overline{T} * \Sigma T'$. We thus have

$$\begin{aligned} T * \Sigma\overline{T} &\subseteq (\overline{T} * \Sigma T') * \Sigma\overline{T} \\ &= \overline{T} * (\Sigma T' * \Sigma\overline{T}) \\ &= \overline{T} * \Sigma T'. \end{aligned}$$

The reverse inclusion is obtained by applying this inclusion to $\Sigma T'$ (instead of T) in the opposite category.

(b) immediately follows from (a).

(c) also follows from (a):

$$\begin{aligned} (\Sigma^{-1}S\overline{T} * ST') \cap \overline{T}^{\perp} &= (\Sigma^{-1}ST * S\overline{T}) \cap {}^{\perp}S\overline{T} \text{ (by (a))} \\ &= \Sigma^{-1}ST. \end{aligned}$$

$\surd$

Assume that $\mathcal{C}$ has a Serre functor S. Recall that we write $\overline{\mathcal{C}}(T)$ (respectively, $\underline{\mathcal{C}}(T')$) for the full subcategory $T * \Sigma\overline{T}$ (respectively, $\Sigma^{-1}S\overline{T} * ST'$) of $\mathcal{C}$. By Proposition 2.5, Proposition 2.6 and Lemma 2.7, we have a pair of adjoint functors (G, H), where $G = LI$ and $H = RJ$. Since I, J, L and R are additive, so are G and H.

$$\begin{tikzcd}
& \mathcal{C}/(\overline{T}^{\perp}) \arrow[dl, shift left, "R"] \arrow[dr, shift left, "L"] & \\
\overline{\mathcal{C}}(T)/(\Sigma T') \arrow[ur, shift left, "I"] \arrow[rr, shift left, "G"] & & \underline{\mathcal{C}}(T')/(\Sigma^{-1}ST) \arrow[ul, shift left, "J"'] \arrow[ll, shift left, "H"]
\end{tikzcd}$$

Remark 2.8. We write τ for the Auslander–Reiten translation $\tau = S\Sigma^{-1}$ (see [RvdB02, §I.2]). Then, by Lemma 2.7 we have that $\underline{\mathcal{C}}(T') = \tau\overline{\mathcal{C}}(T)$.

Theorem 2.9. *Assume that* $\mathcal{C}$ *has a Serre functor* S. *Then the functors* G *and* H *are quasi-inverse equivalences of categories. In particular, the categories* $\overline{\mathcal{C}}(T)/(\Sigma T')$ *and* $\overline{\mathcal{C}}(T)/(T)$ *are equivalent.*

Proof. The construction would be simplified if we had that, if X belongs to $T * \Sigma\overline{T}$, then the left $\operatorname{add}\Sigma^{-1}S\overline{T} * \operatorname{add}ST'$-approximation $X \xrightarrow{\varepsilon_X} L_0X$ of X (see Proposition 2.6 and the paragraph before it) is also a minimal right $T * \Sigma\overline{T}$-approximation of L_0X. However, this cannot be expected to hold in general (take X to be $\Sigma T'$, for instance).

We can modify this approach in the following way. First, since the functors G and H are additive, we may assume that X is indecomposable. This will help in proving that X is a summand of R_0L_0X. Second, we will add to X a minimal right add$(\Sigma T')$ approximation $\Sigma T'_0$ of L_0X. This will be needed in order to get a right approximation of L_0X, while being harmless since the objects X and $X \oplus \Sigma T'_0$ are isomorphic in $\overline{\mathcal{C}}(T)/(\Sigma T')$.

So, take an indecomposable object $X \in T * \Sigma\overline{T}$ and assume that X does not belong to $\operatorname{add}\Sigma T'$ (otherwise, X would be isomorphic to 0 in $\overline{\mathcal{C}}(T)/(\Sigma T')$. Consider the triangle $Z \xrightarrow{\alpha} X \xrightarrow{\varepsilon_X} L_0X \xrightarrow{\beta} \Sigma Z$ in $\mathcal{C}$, constructed in the paragraph before Proposition 2.6, where $\alpha \in \big((T')^{\perp}\big)$ and $\Sigma Z \in \overline{T}^{\perp}$. Let $\Sigma T'_0 \xrightarrow{p} L_0X$ be a minimal right $\operatorname{add}\Sigma T'$-approximation of L_0X in $\mathcal{C}$. We claim that $X \oplus \Sigma T'_0 \xrightarrow{[\varepsilon_X\, p]} L_0X$ is a right $T * \Sigma\overline{T}$-approximation of L_0X in $\mathcal{C}$. Let $X' \xrightarrow{f} L_0X$ be a morphism in $\mathcal{C}$, with $X' \in T * \Sigma\overline{T}$. By assumption, there is a triangle $T'_1 \xrightarrow{a} \overline{T}_0 \xrightarrow{b} X' \xrightarrow{c} \Sigma T'_1$ in $\mathcal{C}$, with $T'_1 \in \operatorname{add} T'$ and $\overline{T}_0 \in \operatorname{add}\overline{T}$. Since $\overline{T}_0$ is in $\operatorname{add}\overline{T}$ and ΣZ is in $\overline{T}^{\perp}$, the composition $\beta f b$ vanishes and f induces a morphism of triangles:

$$\begin{tikzcd}
\overline{T}_0 \arrow[r, "b"] \arrow[d, "\Sigma^{-1}h"'] & X' \arrow[r, "c"] \arrow[d, "f"] \arrow[dl, dashed, "v"'] & \Sigma T'_1 \arrow[r, "-\Sigma a"] \arrow[d, "g"] \arrow[dl, dashed, "u"'] \arrow[dr, bend left, "0"] & \Sigma\overline{T}_0 \arrow[d, "h"] \\
X \arrow[r, "\varepsilon_X"'] & L_0X \arrow[r, "\beta"'] & \Sigma Z \arrow[r, "-\Sigma\alpha"'] & \Sigma X
\end{tikzcd}$$

Since α factors through $(T')^{\perp}$, we have $(-\Sigma\alpha)g = 0$ and there exists $\Sigma T'_1 \xrightarrow{u} L_0X$ such that $g = \beta u$. This implies $\beta(f - uc) = 0$ so that there exists $X' \xrightarrow{v} X$ such that $f = uc + \varepsilon_X v$. The composition uc is in the ideal $(\Sigma T')$ and thus factors through p, i.e. there exists w making the following square commute

$$\begin{tikzcd}
X' \arrow[r, "c"] \arrow[d, dashed, "w"'] & \Sigma T'_1 \arrow[d, "u"] \\
\Sigma T'_0 \arrow[r, "p"] & L_0X
\end{tikzcd}$$

We thus have $f = [\varepsilon_X\ p]\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ and therefore the morphism $[\varepsilon_X\ p]$ is a right $T * \Sigma\overline{T}$-approximation of L_0X in $\mathcal{C}$.

Since also $R_0L_0X \xrightarrow{\eta_{L_0X}} L_0X$ is a right $T * \Sigma\overline{T}$-approximation of L_0X in $\mathcal{C}$, we can write R_0L_0X as a direct sum $X' \oplus X''$, and $\eta_{L_0X} = [\eta'\ 0] : X' \oplus X'' \to L_0X$, where $X' \xrightarrow{\eta'} L_0X$ is a minimal right $T * \Sigma\overline{T}$-approximation. Moreover, we have $X'' \in \operatorname{add}\Sigma T'$, since in the triangle $Z' \to X' \oplus X'' \to L_0X \to$ given by Lemma 2.2, Z' belongs to $\overline{T}^{\perp}$, and X'' is a summand of Z'. Thus X'' belongs to $\in \overline{T}^{\perp} \cap \overline{\mathcal{C}}(T)$ which is $\operatorname{add}\Sigma T'$ by Lemma 2.7.

Now X' is a summand of the approximation $X \oplus \Sigma T'_0$. Moreover, X' contains X as a summand. Otherwise, we would have $R_0L_0X \in \operatorname{add}\Sigma T'$, which implies $L_0X \in \overline{T}^{\perp} \cap \underline{\mathcal{C}}(T') = \operatorname{add}\Sigma^{-1}ST'$ (by applying the functor $\mathcal{C}(\overline{T}, -)$ to the triangle $Z' \to R_0L_0X \to L_0X \to$), which dually implies $X \in \operatorname{add}\Sigma T'$ (note that we assumed $X \notin \operatorname{add}\Sigma T'$).

As a consequence, given a lift

$$\begin{tikzcd}
 & X \arrow[dl, dashed, "\widetilde{\varphi}_X"'] \arrow[d, "\varepsilon_X"] \\
R_0L_0X \arrow[r, "\eta_{L_0X}"] & L_0X
\end{tikzcd}$$

, the image φ_X of $\widetilde{\varphi}_X$ in $\overline{\mathcal{C}}(T)/(\Sigma T')$, which is independent of the choice of $\widetilde{\varphi}_X$ by Proposition 2.5, is an isomorphism (and ε_X is a minimal right $\overline{\mathcal{C}}(T)$-approximation of L_0X in $\mathcal{C}/(\Sigma T')$).

Let us check that we have defined a natural isomorphism $\varphi : 1 \to HG$.

Let $X \xrightarrow{f} Y$ be a morphism in $\overline{\mathcal{C}}(T)$. By construction, there is a diagram in $\mathcal{C}$

$$\begin{tikzcd}
X \arrow[rr, "\widetilde{\varphi}_X"] \arrow[dr] \arrow[ddd, "f"'] & & R_0L_0X \arrow[dl] \arrow[ddd, "HGf"] \\
& L_0X \arrow[d, "Gf"] & \\
& L_0Y & \\
Y \arrow[ur] \arrow[rr, "\widetilde{\varphi}_Y"'] & & R_0L_0Y \arrow[ul, "\eta"']
\end{tikzcd}$$

where we write η for η_{L_0Y} and where the inner two triangles and the inner two squares commute. We thus have $\eta(HGf \circ \widetilde{\varphi}_X - \widetilde{\varphi}_Y \circ f) = 0$, and $HGf \circ \widetilde{\varphi}_X - \widetilde{\varphi}_Y \circ f$ factors through $\overline{Z} \to R_0L_0Y$ in the triangle $\overline{Z} \to R_0L_0Y \xrightarrow{\eta} L_0Y \to$, where $\overline{Z} \in \overline{T}^{\perp}$. This shows that $HGf \circ \widetilde{\varphi}_X - \widetilde{\varphi}_Y \circ f$ factors through $\overline{T}^{\perp}$. Since X lies in $\overline{\mathcal{C}}(T)$, we can apply Lemma 2.3. The morphism $HGf \circ \widetilde{\varphi}_X - \widetilde{\varphi}_Y \circ f$ factors through $\overline{T}^{\perp} \cap \overline{\mathcal{C}}(T)$, which is $\operatorname{add} \Sigma T'$ by Lemma 2.7. As a consequence, φ is a natural transformation.

By duality, there is a natural isomorphism $GH \to 1$; and the functors G and H are quasi-inverse equivalences of categories. √

2.3. **A module-theoretic interpretation.** In this section we assume that $\mathcal{C}$ has a Serre functor S. In this case, the assumptions of functorial finiteness (see Section 1) are automatically satisfied for all rigid objects (but have to be added in the case of rigid subcategories). We write D for the duality functor $\operatorname{Hom}_k(-, k)$. Recall that $T \in \mathcal{C}$ is a basic rigid object, and R is a direct summand of T, with $T = \overline{T} \oplus R$. We write $\overline{T} = T_1 \oplus \cdots \oplus T_n$ and $R = T_{n+1} \oplus \cdots \oplus T_m$, where the T_i are indecomposable. Recall also that ΣR^* is the cone of a minimal right $\operatorname{add} \overline{T}$-approximation of R. We have $T' = \overline{T} \oplus R^* = T'_1 \oplus \cdots \oplus T'_m$ where $T'_i = T_i$ if $i \leq n$. Define Λ (respectively, Λ'), to be the endomorphism algebra $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, (respectively, $\operatorname{End}_{\mathcal{C}}(T')^{\mathrm{op}}$).

Let S_j be the simple top of the indecomposable projective Λ-module $\mathcal{C}(T, T_j)$, and let S'_j be the simple socle of the indecomposable injective Λ'-module $D\mathcal{C}(\Sigma T'_j, \Sigma T')$. We consider the exact categories $\mathcal{E}$ and $\mathcal{E}'$ defined as follows. The category $\mathcal{E}$ (respectively, $\mathcal{E}'$) is the full subcategory of $\operatorname{mod} \Lambda$, (respectively, $\operatorname{mod} \Lambda'$) whose objects M (respectively, N), satisfy $\operatorname{Ext}^1_{\Lambda}(M, S_j) = 0$, (respectively, $\operatorname{Ext}^1_{\Lambda'}(S'_j, N) = 0$) for all $j > n$.

Remark 2.10. For each indecomposable summand R_i of R, let $R_i^* \to \overline{U}_i \to R_i \to \Sigma R_i^*$ be a triangle in $\mathcal{C}$, with $\overline{U}_i \to R_i$ a minimal right $\overline{T}$-approximation. Then (as in [BMR+06]) the object R_i^* is indecomposable. Moreover, $\oplus_i \overline{U}_i \to \oplus_i R_i$ is a minimal right $\overline{T}$-approximation of R. As a consequence, R^* is isomorphic to $\oplus_i R_i^*$. This shows that the basic objects R and R^* have the same number of indecomposable summands.

We can now restate Theorem 2.9 in module-theoretic terms:

Theorem 2.11. *Suppose that $\mathcal{C}$ has a Serre functor. Then there is an equivalence of categories:*

$$\mathcal{E} / \operatorname{add} \mathcal{C}(T, \Sigma R^*) \simeq \mathcal{E}' / \operatorname{add} D\mathcal{C}(R, \Sigma T').$$

The proof will be given later in this section. We note that, if $\mathcal{C}$ is 2-Calabi–Yau, then the modules $D\mathcal{C}(R, \Sigma T')$ and $\mathcal{C}(T', \Sigma R)$ are isomorphic. We also note that although the statement of the equivalence does not need a Serre functor, the existence is needed in the proof, in order to apply Theorem 2.9.

In order to prove Theorem 2.11, we will need the following two lemmas.

Lemma 2.12. *The functor* $\mathcal{C}(T,-)$ *induces a fully faithful functor*

$$\overline{\mathcal{C}}(T)/(\Sigma\overline{T}) \longrightarrow \operatorname{mod}\Lambda.$$

Its essential image is $\mathcal{E}$.

Proof. Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}$ factoring through $T^{\perp}$. Recall that $\overline{\mathcal{C}}(T) = T * \Sigma\overline{T}$. Assume that X belongs to $\overline{\mathcal{C}}(T)$, and let $\overline{V}_1 \to V_0 \to X \to \Sigma\overline{V}_1$ be a triangle in $\mathcal{C}$ with $V_0 \in \operatorname{add} T$ and $\overline{V}_1 \in \operatorname{add}\overline{T}$. Since f factors through $T^{\perp}$, the composition $V_0 \to X \to Y$ vanishes, and f factors through $\Sigma\overline{V}_1$. This implies the first part of the lemma (the fullness of $\mathcal{C}(T,-)$ follows from [IY08, Prop. 6.2]; see also [BM13, Lemma 4.3]).

For any M in $\operatorname{mod}\Lambda$, let $X \in \mathcal{C}(T)$ be such that X has no summands in $\operatorname{add}\Sigma T$ and $\mathcal{C}(T,X) \simeq M$. Let $U_\beta \to U_\alpha \to X \to \Sigma U_\beta$ be a triangle with $U_\alpha, U_\beta \in \operatorname{add} T$ and $U_\alpha \to X$ right-minimal. Then $\mathcal{C}(T,U_\beta) \to \mathcal{C}(T,U_\alpha) \to \mathcal{C}(T,X) \to 0$ is a minimal projective presentation of $\mathcal{C}(T,X)$, and the dimension of $\operatorname{Ext}^1_\Lambda(M,S_j)$ is the multiplicity of P_j in $\mathcal{C}(T,U_\beta)$. $\surd$

Dually, we obtain the following:

Lemma 2.13. *The functor* $D\mathcal{C}(-,\Sigma T')$ *induces a fully faithful functor*

$$\overline{\mathcal{C}}(T)/(\overline{T}) \longrightarrow \operatorname{mod}\Lambda'.$$

Its essential image is $\mathcal{E}'$.

Proof. The proof is dual to that of Lemma 2.12. We use the description $\overline{\mathcal{C}}(T) = \overline{T} * \Sigma T'$ from Lemma 2.7, and note that any triangle $U_1' \to \overline{U}_0 \to X \to \Sigma U_1'$, where U_1' belongs to $\operatorname{add} T'$ and $\overline{U}_0$ belongs to $\operatorname{add}\overline{T}$, gives rise to an injective co-presentation $0 \to D\mathcal{C}(X,\Sigma T') \to D\mathcal{C}(\Sigma U_1',\Sigma T') \to D\mathcal{C}(\Sigma\overline{U}_0,\Sigma T')$. $\surd$

Proof of Theorem 2.11: By Lemma 2.12, the functor $\mathcal{C}(T,-)$ induces an equivalence of categories from $\overline{\mathcal{C}}(T)/(\Sigma\overline{T})$ to $\mathcal{E}$. Since $\overline{\mathcal{C}}(T)/(\Sigma T')$ is isomorphic to $\big(\overline{\mathcal{C}}(T)/(\Sigma\overline{T})\big)/(\Sigma R^*)$, the functor $\mathcal{C}(T,-)$ induces an equivalence of categories from $\overline{\mathcal{C}}(T)/\operatorname{add}\Sigma T'$ to $\mathcal{E}/\operatorname{add}\mathcal{C}(T,\Sigma R^*)$. Dually, one can use Lemma 2.13 to notice that the functor $D\mathcal{C}(-,\Sigma T')$ induces an equivalence of categories from $(\overline{T} * T')/\operatorname{add} T$ to $\mathcal{E}'/\operatorname{add} D\mathcal{C}(R,\Sigma T')$. The statement now follows from Theorem 2.9. $\surd$

There are two particular cases of Theorem 2.11 that are worth noting. They are weak forms of nearly-Morita equivalences that we call pseudo-Morita equivalences. They occur in the case where R is indecomposable, i.e. $m = n+1$, and we make this assumption for the rest of the section. Note that $R = T_m$ and $R^* = T_m'$.

Let Q_m be the Λ-module $\mathcal{C}(T,\Sigma R^*) = \mathcal{C}(T,\Sigma T_m')$ appearing in Theorem 2.11. Similarly, we have the Λ'-modules $Q_m' = D\mathcal{C}(R,\Sigma T') = D\mathcal{C}(T_m,\Sigma T')$. Then we have the following:

Lemma 2.14. *Suppose that* R *is indecomposable. Let* e *is the idempotent for* Λ *corresponding to* T_m*. Then we have the isomorphism*

$$Q_m \simeq \Lambda/\Lambda(1-e)\Lambda.$$

Furthermore, Q_m *is a simple object of* $\mathcal{E}$*. Dually, let* e' *be the idempotent for* Λ' *corresponding to* $\Sigma T_m'$*. Then we have the isomorphism*

$$Q_m' \simeq \Lambda'/\Lambda(1-e')\Lambda'.$$

Furthermore, Q_m' *is an indecomposable* Λ'*-module and a simple object of* $\mathcal{E}'$.

Proof. The long exact sequences associated with the exchange triangle $T'_m \to \overline{U}_m \to T_m \to \Sigma T'_m$ of Remark 2.10 show that the functor $\mathcal{C}(-, \Sigma T'_m)$ vanishes on $\operatorname{add}(\overline{T})$ and that the Λ-module $\mathcal{C}(T_m, \Sigma T'_m)$ is isomorphic to the module $\mathcal{C}/(\operatorname{add}\overline{T})(T_m, T_m)$. We thus have an isomorphism of Λ-modules:

$$\frac{\Lambda}{\Lambda(1-e)\Lambda} \simeq \frac{\mathcal{C}(T,T)}{(\overline{T})} \simeq \frac{\mathcal{C}}{(\operatorname{add}\overline{T})}(T_m, T_m) \simeq \mathcal{C}(T_m, \Sigma T'_m) \simeq Q_m.$$

Similarly, using the exchange triangle as above, we obtain an isomorphism between the Λ'-modules $D\mathcal{C}(T_m, \Sigma T')$ and $D\mathcal{C}/(\operatorname{add}\Sigma\overline{T})(\Sigma T', \Sigma T')$, the latter being isomorphic to Q'_m.

Since Q_m is projective over $\Lambda/\Lambda(1-e)\Lambda$, we have

$$\operatorname{Ext}^1_{\Lambda}(Q_m, S_m) \simeq \operatorname{Ext}^1_{\Lambda/\Lambda(1-e)\Lambda}(Q_m, S_m) = 0,$$

and therefore Q_m lies in $\mathcal{E}$. If N is a non-trivial submodule of Q_m lying in $\mathcal{E}$, then N is also a $\Lambda/\Lambda(1-e)\Lambda$-module satisfying $\operatorname{Ext}^1_{\Lambda(1-e)\Lambda}(N, S_m) = 0$. Since S_m is the only simple $\Lambda(1-e)\Lambda$-module, N is projective over $\Lambda(1-e)\Lambda$, so it must equal Q_m. It follows that Q_m is a simple object of the exact category $\mathcal{E}$. Since $\mathcal{E}$ is closed under direct summands, it now follows that Q_m is an indecomposable Λ-module. The proofs of the duals of these last two statements are similar. $\surd$

Corollary 2.15. *Suppose that $\mathcal{C}$ satisfies the assumptions in Section 1 and that it has a Serre functor. Suppose further that R is indecomposable, Then there is an equivalence of categories:*

$$\mathcal{E}/\operatorname{add} Q_m \simeq \mathcal{E}'/\operatorname{add} Q'_m.$$

Proof. This follows from Theorem 2.11 and Lemma 2.14. $\surd$

Corollary 2.16. *Suppose that the assumptions in Corollary 2.15 hold, and, in addition, that the Gabriel quiver of Λ has no loop at the vertex corresponding to R. Then there is an equivalence of categories:*

$$\mathcal{E}/\operatorname{add} S_m \simeq \mathcal{E}'/\operatorname{add} S'_m.$$

Proof. This is a particular case of Corollary 2.15. Indeed, by [IY08, Proposition 2.6 (1)] R^* is also indecomposable and has no loop. This implies that Q_m and Q'_m are isomorphic to S_m and S'_m respectively. $\surd$

3. LOCALISATION

3.1. Notation and statement of main results.

We continue with the assumptions and notation from Section 1. We do not assume here that $\mathcal{C}$ has a Serre functor, except in Corollary 3.5. Also, contrary to [BM13], we do not make any skeletal smallness assumption. This is because all the localisations that we consider are shown to be equivalent to a subquotient of $\mathcal{C}$. Therefore no set-theoretic difficulties arise, and the localisations we consider are all categories without passing to a higher universe.

Recall that, by [KR07, BM13], the functor $\mathcal{C}(T, -)$ induces an equivalence of categories from $\mathcal{C}(T)/\Sigma T$ to $\operatorname{mod}\Lambda$. In particular, it is dense and full when restricted to $\mathcal{C}(T)$.

Definition 3.1. Let $\mathcal{B}$ be the full subcategory of $\operatorname{mod}\Lambda$ given by the (essential) image of $\overline{T}^{\perp}$ under $\mathcal{C}(T, -)$. Let $\mathcal{S}_{\mathcal{B},0}$ be the class of all epimorphisms $f \in \operatorname{mod}\Lambda$ whose kernel belongs to $\mathcal{B}$. Dually, we let $\mathcal{B}'$ be the full subcategory of $\operatorname{mod}\Lambda'$ given by the (essential) image of ${}^{\perp}\Sigma\overline{T}$ under $D\mathcal{C}(-, \Sigma T')$ and set $\mathcal{S}_{0,\mathcal{B}'}$ to be the class of all monomorphisms $g \in \operatorname{mod}\Lambda'$ whose cokernel belongs to $\mathcal{B}'$.

Let F be the composition of the fully faithful functor $\overline{\mathcal{C}}(T)/\Sigma\overline{T} \to \mathcal{C}(T)/\Sigma T \to \operatorname{mod}\Lambda$ and the localisation functor $\operatorname{mod}\Lambda \xrightarrow{L_{\mathcal{S}_{\mathcal{B},0}}} (\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}$. Then, since $\mathcal{C}(T,\Sigma R^*)$ belongs to $\mathcal{B}$, we have that $F(\Sigma R^*) \simeq 0$ in $(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}$. Hence, F induces a functor $\overline{F}$ as in the following diagram:

(3.1)
$$\begin{tikzcd}
\overline{\mathcal{C}}(T)/(\Sigma\overline{T}) \arrow[r, hook] \arrow[d, two heads] \arrow[drr, "F"] & \mathcal{C}(T)/(\Sigma T) \arrow[r, "\simeq"] & \operatorname{mod}\Lambda \arrow[d, "L_{\mathcal{S}_{\mathcal{B},0}}"] \\
\overline{\mathcal{C}}(T)/(\Sigma T') \arrow[rr, dashed, "\overline{F}"] & & (\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}.
\end{tikzcd}$$

Our main aim in this section is to show that the following holds:

Theorem 3.2. *The functor $\overline{F} : \overline{\mathcal{C}}(T)/(\Sigma T') \longrightarrow (\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}$ is an equivalence of categories. Dually, there is an equivalence $\overline{\mathcal{C}}(T)/(T) \longrightarrow (\operatorname{mod}\Lambda')_{\mathcal{S}_{0,\mathcal{B}'}}$.*

This has two key corollaries, which we state below, after a lemma needed in the proof of the first one.

Lemma 3.3. *For any object $M \in \mathcal{B}$, there exists $X \in \mathcal{C}(T) \cap \overline{T}^{\perp}$ such that $\mathcal{C}(T,X) \simeq M$.*

Proof. For any object $M \in \mathcal{B}$, there exists an object $Y \in \overline{T}^{\perp}$ such that $\mathcal{C}(T,Y) \simeq M$. By [BM13, Lemma 3.3], there is a triangle $Z \to X \to Y \xrightarrow{\varepsilon} \Sigma Z$ in $\mathcal{C}$, where $X \in \mathcal{C}(T)$, $Z \in T^{\perp}$ and $\varepsilon \in (T^{\perp})$. Then we have $\mathcal{C}(T,X) \simeq \mathcal{C}(T,Y) \simeq M$, which can be seen by applying the functor $\mathcal{C}(T,-)$ to the triangle above. Moreover, X belongs to $\overline{T}^{\perp}$, since both Z and Y belong to $\overline{T}^{\perp}$. √

Corollary 3.4. *There is an equivalence of categories*

$$(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}} \simeq \mathcal{E}/\operatorname{add}\mathcal{C}(T,\Sigma R^*)$$

and, dually, an equivalence of categories

$$(\operatorname{mod}\Lambda')_{\mathcal{S}_{0,\mathcal{B}'}} \simeq \mathcal{E}'/\operatorname{add}D\mathcal{C}(R,\Sigma T').$$

Proof. For the first statement, combine Theorem 3.2 with Lemma 2.12, and for the second statement, combine Theorem 3.2 with Lemma 2.13. √

Proof of Theorem B. We set $\mathcal{C}$ to be an acyclic cluster category and T a rigid object in $\mathcal{C}$. We consider the case $m = n+1$ and $R = T_m$ is indecomposable. As in the proofs of Corollaries 2.15 and 2.16, $\mathcal{C}(T,\Sigma R^*) \simeq Q_m \simeq S_m$ in this case. In particular, there are no loops in the quiver of $\operatorname{End}_{\mathcal{C}}(T)$ at the vertex corresponding to S_m.

By Lemma 3.3, we have $\mathcal{B} = \mathcal{C}(T,\overline{T}^{\perp}) = \mathcal{C}(T,\mathcal{C}(T)\cap\overline{T}^{\perp})$. For $j = 1,\ldots,n+1$, let $P_j = \mathcal{C}(T,T_j)$ be the jth indecomposable projective in $\operatorname{mod}\Lambda$. Then an object X in $\mathcal{C}(T)$ lies in $\overline{T}^{\perp}$ if and only if $\operatorname{Hom}_{\Lambda}(P_j,\mathcal{C}(T,X)) = 0$ for $j = 1,2,\ldots,n$, which holds if and only if $\mathcal{C}(T,X)$ lies in $\operatorname{add}(S_m)$. It follows that $\mathcal{B} = \operatorname{add}(S_m)$ in this case, and hence $\mathcal{S}_{\mathcal{B},0}$ coincides with the class $\mathcal{R}$ of morphisms considered in the introduction. We see that the first statement in Theorem B follows from the first statement in Corollary 3.4. The second statement in Theorem B follows from the second statement in Corollary 3.4. √

Corollary 3.5. *If the category $\mathcal{C}$ admits a Serre functor, then there is an equivalence of categories:*

$$(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}} \simeq (\operatorname{mod}\Lambda')_{\mathcal{S}_{0,\mathcal{B}'}}.$$

Proof. Combine Theorem 3.2 with Theorem 2.9. √

Proof of Theorem A. Since an acyclic cluster category has a Serre functor, Theorem A follows from Corollary 3.5 and the observations in the proof of Theorem B above. $\surd$

We shall also use Theorem 3.2 to show that the categories $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{C}_{\widetilde{\mathcal{S}}}$ are isomorphic (Theorem 3.19). We also remark that Lemma 3.8 may be of independent interest.

3.2. **Proof of Theorem 3.2.**

We show the first statement of the Theorem. The second statement follows from a dual argument. In order to prove that $\overline{F}$ is full and dense, it is enough to prove that F is full and dense. The functor F is easily seen to be dense (Proposition 3.13). Showing that it is full requires a bit more work (Lemmas 3.8 and 3.9), and in order to do so we describe, in Lemma 3.11, the category $(\operatorname{mod}\Lambda)_{\mathcal{S}_{B,0}}$ as a localisation of $\mathcal{C}$. We then show that the functor $\operatorname{Hom}_\Lambda(U,-)$ induces a functor $(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}} \longrightarrow \operatorname{mod}\overline{\Lambda}$ (Lemma 3.15). Composing F with this induced functor and applying results from [BM13] then gives us the faithfulness of $\overline{F}$ (Proposition 3.16).

Lemma 3.6. *The full subcategory $\mathcal{B}$ of* $\operatorname{mod}\Lambda$ *is closed under taking images and submodules.*

Proof. Let $M \xrightarrow{u} N$ be a morphism in $\operatorname{mod}\Lambda$. Then there are objects $X, Y \in \mathcal{C}(T)$ such that $\mathcal{C}(T,X) \simeq M$ and $\mathcal{C}(T,Y) \simeq N$, and a morphism $f : X \to Y$ such that $\mathcal{C}(T,f) \simeq u$. We complete f to a triangle

$$Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z.$$

If M lies in $\mathcal{B}$ and u is an epimorphism then, by Lemma 3.3, we may take X in $\overline{T}^{\perp}$ and, by [BM13, Lemma 2.5], h factors through $T^{\perp}$. If N lies in $\mathcal{B}$ and u is a monomorphism then, by Lemma 3.3, we may take Y in $\overline{T}^{\perp}$ and, by [BM13, Lemma 2.5], f factors through $T^{\perp}$.

In either case, the result follows from applying the functor $\mathcal{C}(\overline{T},-)$ to this triangle. $\surd$

Proposition 3.7. *The functor F is dense.*

Proof. For any module $M \in \operatorname{mod}\Lambda$, let X be an object in $\mathcal{C}(T)$ such that $\mathcal{C}(T,X) \simeq M$. In Lemma 2.2, we constructed a triangle $Z \longrightarrow R_0X \xrightarrow{\eta_X} X \xrightarrow{g} \Sigma Z$, with $R_0X \in \overline{\mathcal{C}}(T)$, $Z \in \overline{T}^{\perp}$, and $g \in (T^{\perp})$. We claim that the morphism η_X is inverted in $(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}$. There is an exact sequence in $\operatorname{mod}\Lambda$:

$$\mathcal{C}(T,Z) \to \mathcal{C}(T,R_0X) \to \mathcal{C}(T,X) \xrightarrow{0} \mathcal{C}(T,\Sigma Z).$$

Therefore, the morphism $\mathcal{C}(T,\eta_X)$ is surjective and $\mathcal{C}(T,Z)$ surjects onto its kernel. Since $\mathcal{B}$ is closed under images (Lemma 3.6), we may conclude that $\mathcal{C}(T,\eta_X)$ belongs to $\mathcal{S}_{\mathcal{B},0}$, and the claim is shown. This shows that $M \simeq FR_0X$ in $(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}$ and we are done. $\surd$

Lemma 3.8. *Let $Z \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{\varepsilon} \Sigma Z$ be a triangle in $\mathcal{C}$ with $X, Y \in \mathcal{C}(T)$ and $\varepsilon \in (T^{\perp})$. Then Z belongs to $\mathcal{C}(T)$.*

Proof. Let $T^Z \xrightarrow{f} Z$ be a minimal right $\operatorname{add}T$-approximation. Complete it to a triangle $U \to T^Z \xrightarrow{f} Z \xrightarrow{\delta} \Sigma U$. We note that, since f is an approximation and T is rigid, we have $\Sigma U \in T^{\perp}$, as can be seen by applying the functor $\mathcal{C}(T,-)$ to the triangle above, in a manner similar to that of [BMR$^+$06, Lemma 6.3] (a more general version of this assertion can be found in [Jør09, Lemma 2.1]).

Since Y belongs to $\mathcal{C}(T)$, there is a triangle $T_1^Y \rightarrow T_0^Y \xrightarrow{a} Y \xrightarrow{\eta} \Sigma T_1^Y$. By assumption, the composition εa vanishes, so that there is a morphism $T_0^Y \xrightarrow{b} X$ such that $a = vb$. We thus have a morphism of triangles

$$\begin{tikzcd}
T^Z \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] \arrow[d, "f"] & T^Z \oplus T_0^Y \arrow[r, "{[0\,1]}"] \arrow[d, "{[uf\ b]}"] & T_0^Y \arrow[r, "0"] \arrow[d, "a"] & \Sigma T^Z \arrow[d, "\Sigma f"] \\
Z \arrow[r, "u"] & X \arrow[r, "v"] & Y \arrow[r, "\varepsilon"] & \Sigma Z
\end{tikzcd}$$

that we complete to a nine diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & T_1^Y \arrow[r] \arrow[d] & \Sigma U \arrow[d] \\
T^Z \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] \arrow[d, "f"] & T^Z \oplus T_0^Y \arrow[r, "{[0\,1]}"] \arrow[d, "{[uf\ b]}"] & T_0^Y \arrow[r, "0"] \arrow[d, "a"] & \Sigma T^Z \arrow[d, "\Sigma f"] \\
Z \arrow[r, "u"] \arrow[d, "\delta"] & X \arrow[r, "v"] \arrow[d] & Y \arrow[r, "\varepsilon"] \arrow[d, "\eta"] & \Sigma Z \\
\Sigma U \arrow[r] & \Sigma V \arrow[r] & \Sigma T_1^Y &
\end{tikzcd}$$

Since $\Sigma U \in T^{\perp}$, the morphism $T_1^Y \rightarrow \Sigma U$ vanishes and the top triangle splits. Thus U is a summand of V and it is enough to prove that V belongs to $\operatorname{add} T$. Since $X \in \mathcal{C}(T)$, this amounts to proving that the morphism $T^Z \oplus T_0^Y \rightarrow X$ is an $\operatorname{add} T$-approximation. Let us thus prove the latter statement. Let $W \xrightarrow{g} X$ be a morphism in $\mathcal{C}$ with $W \in \operatorname{add} T$ (the morphisms are illustrated in the diagram below). Then the composition $\eta v g$ is zero so that there is a morphism $W \xrightarrow{c} T_0^Y$ with $vg = ac$. This implies $vg = vbc$, and there is a morphism $W \xrightarrow{d} Z$ such that $g - bc = ud$. Now f is an $\operatorname{add} T$-approximation so that there is a morphism $W \xrightarrow{e} T^Z$ satisfying $d = fe$. The last two equalities give $g = ufe + bc = [uf\ b] \left[\begin{smallmatrix}e\\c\end{smallmatrix}\right]$ and we have shown that $[uf\ b]$ is an $\operatorname{add} T$-approximation. $\surd$

The following diagram shows the morphisms g, b, c, d and e.

$$\begin{tikzcd}
 & & & & W \arrow[dllll, dotted, bend right, "e"'] \arrow[ddllll, dashed, "d"'] \arrow[dll, dashed, "c"'] \arrow[ddlll, bend left, "g"] \\
T^Z \arrow[r, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] \arrow[d, "f"'] & T^Z \oplus T_0^Y \arrow[r, "{[0\,1]}"] \arrow[d, "{[uf\ b]}"'] & T_0^Y \arrow[r, "0"] \arrow[d, "a"'] \arrow[dl, dashed, "b"'] & \Sigma T^Z \arrow[d, "\Sigma f"] & \\
Z \arrow[r, "u"'] & X \arrow[r, "v"'] & Y \arrow[r, "\varepsilon"'] \arrow[d, "\eta"] & \Sigma Z & \\
 & & \Sigma T_1^Y & &
\end{tikzcd}$$

Lemma 3.9. *Let $Z \xrightarrow{f} X$ be a morphism in $\mathcal{C}$ with $Z \in \mathcal{C}(T)$. Then $\mathcal{C}(T, f)$ factors through $\mathcal{B}$ if and only if f factors through $\overline{T}^{\perp}$.*

Proof. If f belongs to the ideal $(\overline{T}^{\perp})$, then $\mathcal{C}(T, f)$ factors through $\mathcal{B}$ by the definition of $\mathcal{B}$. Let us prove the converse. Since $Z \in \mathcal{C}(T)$, there is a triangle $T_1 \rightarrow T_0 \xrightarrow{g} Z \rightarrow \Sigma T_1$ in $\mathcal{C}$ with $T_0, T_1 \in \operatorname{add} T$. Assume that $\mathcal{C}(T, f)$ belongs to

($\mathcal{B}$). Then there exists $U \in \overline{T}^{\perp}$, and there exist maps $\mathcal{C}(T,Z) \xrightarrow{b} \mathcal{C}(T,U)$ and $\mathcal{C}(T,U) \xrightarrow{a} \mathcal{C}(T,X)$ such that $\mathcal{C}(T,f) = a \circ b$.

We would like to lift a and b to morphisms in the category $\mathcal{C}$. This cannot be done in general, since the functor $\mathcal{C}(T,-)$ is not full. Fortunately, it is full when restricted to $\mathcal{C}(T)$. We thus use [BM13, Lemma 3.3] in order to replace the object U by an object U' whose image under $\mathcal{C}(T,-)$ is isomorphic to that of U, but with the additional property that U' is in $\mathcal{C}(T)$. Let us therefore apply [BM13, Lemma 3.3] so as to get triangles $Y_U \to U' \to U \xrightarrow{\varepsilon} \Sigma Y_U$ and $Y_X \to X' \xrightarrow{u} X \xrightarrow{\eta} \Sigma Y_X$ in $\mathcal{C}$, where U', X' belong to $\mathcal{C}(T)$, where Y_U, Y_X belong to $T^{\perp}$, and where the morphisms ε and η factor through $T^{\perp}$. Since U is in $\overline{T}^{\perp}$ and Y_U in $T^{\perp}$, U' is in $\overline{T}^{\perp}$ as well.

The modules $\mathcal{C}(T,U)$ and $\mathcal{C}(T,U')$ are isomorphic and $\mathcal{C}(T,u)$ is an isomorphism so that there are morphisms $\mathcal{C}(T,Z) \xrightarrow{b'} \mathcal{C}(T,U')$ and $\mathcal{C}(T,U') \xrightarrow{a'} \mathcal{C}(T,X')$ satisfying $\mathcal{C}(T,u) \circ a' \circ b' = \mathcal{C}(T,f)$. Now, the objects Z, U' and X' all belong to $\mathcal{C}(T)$ so that there exist morphisms α, β in $\mathcal{C}$ with $\mathcal{C}(T,\alpha) = a'$ and $\mathcal{C}(T,\beta) = b'$. We thus have the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
T_0 \arrow[d, "g"] & U' \arrow[r, "\alpha"] & X' \arrow[d, "u"] \\
Z \arrow[ur, "\beta"] \arrow[rr, "f"] \arrow[d] & & X \\
\Sigma T_1 \arrow[urr, dashed] & &
\end{tikzcd}$$

where the square $f - u\alpha\beta$ commutes up to a summand in $T^{\perp}$. Since $T_0 \in \operatorname{add} T$, the composition $(f - u\alpha\beta)g$ vanishes and $f - u\alpha\beta$ factors through ΣT_1. This shows that f factors through $U' \oplus \Sigma T_1$ which belongs to $\overline{T}^{\perp}$, and we are done. $\surd$

Definition 3.10. Let $\widetilde{\mathcal{S}}$ be the class of morphisms $X \xrightarrow{s} Y$ in $\mathcal{C}$ such that, for any triangle $Z \xrightarrow{f} X \xrightarrow{s} Y \xrightarrow{g} \Sigma Z$ we have $f \in (\overline{T}^{\perp})$ and $g \in (T^{\perp})$. Note that this is a weaker property than that defining $\mathcal{S}$ (where instead of the property $f \in (\overline{T}^{\perp})$ we had $Z \in \overline{T}^{\perp}$). Therefore $\mathcal{S} \subseteq \widetilde{\mathcal{S}}$.

Let $\mathcal{C} \xrightarrow{L_{\widetilde{\mathcal{S}}}} \mathcal{C}_{\widetilde{\mathcal{S}}}$ be the localisation functor with respect to the class $\widetilde{\mathcal{S}}$.

Lemma 3.11. *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathcal{C}(T{,}-)"] \arrow[d, "L_{\widetilde{\mathcal{S}}}"'] & \operatorname{mod}\Lambda \arrow[d, "L_{\mathcal{S}_{\mathcal{B},0}}"] \\
\mathcal{C}_{\widetilde{\mathcal{S}}} \arrow[r, "G'"] & (\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}},
\end{tikzcd}$$

where G' is an equivalence of categories.

Proof. It is proved in [BM13] that the functor $\mathcal{C}(T,-) : \mathcal{C} \to \operatorname{mod}\Lambda$ is a localisation functor for the class $\mathcal{S}_T$ of morphisms $X \xrightarrow{f} Y$ such that, when completed to a triangle $Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z$, we have $g, h \in (T^{\perp})$. Since this class is contained in the class $\widetilde{\mathcal{S}}$, it is enough to prove $\mathcal{C}(T, \widetilde{\mathcal{S}}) = \mathcal{S}_{\mathcal{B},0}$. Let s be in $\widetilde{\mathcal{S}}$. There is a triangle $Z \xrightarrow{g} X \xrightarrow{s} Y \xrightarrow{h} \Sigma Z$ in $\mathcal{C}$ with $g \in (\overline{T}^{\perp})$ and $h \in (T^{\perp})$. Applying the functor $\mathcal{C}(T,-)$ gives an exact sequence in $\operatorname{mod}\Lambda$:

$$\mathcal{C}(T,Z) \longrightarrow \mathcal{C}(T,X) \longrightarrow \mathcal{C}(T,Y) \xrightarrow{0} \mathcal{C}(T,\Sigma Z),$$

where $\mathcal{C}(T,g) : \mathcal{C}(T,Z) \to \mathcal{C}(T,X)$ factors through some $B \in \mathcal{B}$. Thus $\mathcal{C}(T,s)$ is an epimorphism and its kernel is isomorphic to a quotient of a submodule of B (see Remark 3.12 below). By Lemma 3.6 the subcategory $\mathcal{B}$ is stable under taking images and submodules, so that $\mathcal{C}(T,s)$ belongs to $\mathcal{S}_{\mathcal{B},0}$.

Conversely, let $0 \to B \longrightarrow M \xrightarrow{f} N \to 0$ be a short exact sequence in $\operatorname{mod}\Lambda$, with $B \in \mathcal{B}$. There is a morphism $X \xrightarrow{s} Y$ in $\mathcal{C}$, with $X, Y \in \mathcal{C}(T)$ such that $\mathcal{C}(T,s) \simeq f$. Complete it to a triangle $Z \xrightarrow{u} X \xrightarrow{s} Y \xrightarrow{v} \Sigma Z$ in $\mathcal{C}$. Then $v \in (T^{\perp})$ since f is an epimorphism, and $\mathcal{C}(T,u)$ factors through B since $su = 0$. Lemma 3.8 shows that Z lies in $\mathcal{C}(T)$ and we can apply Lemma 3.9 to conclude that u factors through $\overline{T}^{\perp}$. √

Remark 3.12. Let $L \xrightarrow{g} M \xrightarrow{f} N$ be exact in an abelian category. Assume that the morphism g factors as $u \circ v$ through some object B. Then the kernel of f is isomorphic to a quotient of a subobject of B.

Proof. Let $K \xrightarrow{i} M$ be a kernel for f. Since the sequence is exact, there is an epimorphism $L \xrightarrow{g'} K$ such that $ig' = g$. The morphism v factors as in the following diagram:

$$\begin{tikzcd}
B' \arrow[r, tail, "j"] & B \arrow[dr, "u"] & & \\
L \arrow[u, two heads, "p"] \arrow[ur, "v"] \arrow[rr, "g"] \arrow[dr, two heads, "g'"'] & & M \arrow[r, "f"] & N \\
& K \arrow[ur, tail, "i"'] & &
\end{tikzcd}$$

The composition $fujp = fg$ vanishes so that $fuj = 0$ and there is some $B' \xrightarrow{q} K$ so that $iq = uj$. It remains to show that q is an epimorphism. Since i is a monomorphism, the equalities $iqp = ujp = g = ig'$ imply $qp = g'$. Since the morphism g' is an epimorphism, q is an epimorphism also. √

Proposition 3.13. *The functor F is full.*

Proof. Let $X \in \overline{\mathcal{C}}(T)$. Then there is a triangle $\overline{T}_1 \xrightarrow{\beta} T_0 \xrightarrow{\alpha} X \xrightarrow{\gamma} \Sigma\overline{T}_1$ in $\mathcal{C}$. Consider a hook diagram

$$\begin{tikzcd}
& \mathcal{C}(T,X) \arrow[d, "\mathcal{C}(T,f)"] \\
\mathcal{C}(T,U) \arrow[r, "\mathcal{C}(T,s)"'] & \mathcal{C}(T,V)
\end{tikzcd}$$

in $\operatorname{mod}\Lambda$, with $U, V \in \mathcal{C}(T)$ and $\mathcal{C}(T,s) \in \mathcal{S}_{\mathcal{B},0}$. Let us prove that the morphism $\mathcal{C}(T,f)$ lifts through the morphism $\mathcal{C}(T,s)$. The proof of Lemma 3.11 shows that s belongs to $\widetilde{\mathcal{S}}$. We thus have a triangle $W \xrightarrow{g} U \xrightarrow{s} V \xrightarrow{h} \Sigma U$ in $\mathcal{C}$ with $g \in (\overline{T}^{\perp})$ and $h \in (T^{\perp})$. The composition $hf\alpha$ vanishes, so that f induces a morphism of triangles

$$\begin{tikzcd}
& \overline{T}_1 \arrow[r] \arrow[dl, dashed, "\Sigma^{-1}b"'] \arrow[d, "a"] & T_0 \arrow[r, "\alpha"] \arrow[d] & X \arrow[r, "\gamma"] \arrow[dl, dotted, "c"'] \arrow[d, "f"] & \Sigma\overline{T}_1 \arrow[dl, dashed, "b"'] \arrow[d, "v"] \\
\Sigma^{-1}V \arrow[r] & W \arrow[r, "g"] & U \arrow[r, "s"] & V \arrow[r, "h"] & \Sigma W.
\end{tikzcd}$$

The morphism g factors through $\overline{T}^{\perp}$ so that the composition ga is zero, giving the existence of a morphism b such that $hb = v$. The equalities $hf = v\gamma = hb\gamma$ imply the existence of a morphism c such that $f = b\gamma + sc$. Therefore $\mathcal{C}(T, s) \circ \mathcal{C}(T, c) = \mathcal{C}(T, f)$. We can conclude by induction on the number of hooks in a morphism from $\mathcal{C}(T, X)$ to $\mathcal{C}(T, Y)$. $\surd$

We write U for $\mathcal{C}(T, \overline{T})$. Define $\overline{\Lambda}$ to be the endomorphism algebra of U in $\operatorname{mod}\Lambda$. Then $\overline{\Lambda} \simeq \operatorname{End}_{\mathcal{C}}(\overline{T})$.

Lemma 3.14. *The diagram*

$$\begin{tikzcd}
\overline{\mathcal{C}}(T)/(\Sigma\overline{T}) \arrow[r, "\mathcal{C}(T,-)"] \arrow[dr, "\mathcal{C}(\overline{T},-)"'] & \operatorname{mod}\Lambda \arrow[d, "\operatorname{Hom}_{\Lambda}(U,-)"] \\
 & \operatorname{mod}\overline{\Lambda}
\end{tikzcd}$$

commutes up to a natural isomorphism.

Proof. For any $X \in \overline{\mathcal{C}}(T)$, define a map $\varphi_X : \mathcal{C}(\overline{T}, X) \longrightarrow \operatorname{Hom}_{\Lambda}\big(\mathcal{C}(T, \overline{T}), \mathcal{C}(T, X)\big)$ by $\varphi_X(\alpha) = \mathcal{C}(T, \alpha)$. Then φ_X is $\overline{\Lambda}$-linear, since $\mathcal{C}(T, -)$ is a covariant functor, and it is an isomorphism, since $\mathcal{C}(T, -) : \overline{\mathcal{C}}(T)/(\Sigma\overline{T}) \to \operatorname{mod}\Lambda$ is fully faithful. The transformation φ is easily seen to be natural: Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}$. Write f_* for the image of f under the functor $\operatorname{Hom}_{\Lambda}(U, -) \circ \mathcal{C}(T, -)$. We have to check that $\varphi_Y \circ \mathcal{C}(\overline{T}, f) = f_* \circ \varphi_X$. The left-hand side of this equality sends a morphism $\overline{T} \xrightarrow{\alpha} X$ to the map sending $u \in \mathcal{C}(T, \overline{T})$ to $(f \circ \alpha) \circ u$, while the right-hand side sends α to $u \mapsto f \circ (\alpha \circ u)$. $\surd$

Lemma 3.15. *The functor* $\operatorname{Hom}_{\Lambda}(U, -)$ *induces a functor* $(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}} \longrightarrow \operatorname{mod}\overline{\Lambda}$.

Proof. Suppose that the morphism $f = \mathcal{C}(T, s)$ lies in $\mathcal{S}_{\mathcal{B},0}$. Then s belongs to $\widetilde{\mathcal{S}}$ by Lemma 3.11. In particular, s is part of a triangle (r, s, t) with $r, t \in (\overline{T}^{\perp})$, so that $\mathcal{C}(\overline{T}, s)$ is an isomorphism (as proved in [BM13, Lemma 2.5]). Hence, by Lemma 3.14, $\operatorname{Hom}_{\Lambda}(U, f)$ is an isomorphism. $\surd$

Proposition 3.16. *The functor $\overline{F}$ is faithful.*

Proof. Assume that $Fu = Fv$ for some $u, v : X \to Y$ in $\overline{\mathcal{C}}(T)$. Then Lemmas 3.14 and 3.15 imply $\mathcal{C}(\overline{T}, u) = \mathcal{C}(\overline{T}, v)$ and [BM13, Lemma 2.3] implies $u - v \in (\overline{T}^{\perp})$. Since X belongs to $\overline{\mathcal{C}}(T) = \overline{T} * \Sigma T'$, there is a triangle $\overline{T}_{\alpha} \xrightarrow{w} X \to \Sigma T'_{\beta} \to$, with $\overline{T}_{\alpha} \in \operatorname{add}\overline{T}$ and $T'_{\beta} \in \operatorname{add} T'$. The composition $(u - v)w$ vanishes so that $u - v \in (\Sigma T')$ and the functor $\overline{F}$ is faithful. $\surd$

Proof of Theorem 3.2. By Proposition 3.7, F is dense, and by Proposition 3.13, F is full. Hence $\overline{F}$ is also full and dense. By Proposition 3.16, $\overline{F}$ is faithful, and Theorem 3.2 follows. $\surd$

3.3. **More localisations.** In this section, we prove, under the assumptions as in Section 1, that the localisations $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{C}_{\widetilde{\mathcal{S}}}$ are isomorphic. We note that this result does not seem to follow easily from Lemma 3.11, as one would expect by analogy with [BM13, Section 4].

Lemma 3.17. *The full subcategory $\overline{\mathcal{C}}(T)$ of $\mathcal{C}$ is stable under taking direct summands.*

Proof. Let $X, X' \in \mathcal{C}$ be so that $X \oplus X'$ belongs to $\overline{\mathcal{C}}(T)$. Let $U_0 \to X$, $V_0 \to X'$ be minimal right $\operatorname{add} T$-approximations. Then $U_0 \oplus V_0 \to X \oplus X'$ is a minimal right $\operatorname{add} T$-approximation. When completing it to a triangle $W \to U_0 \oplus V_0 \to X \oplus X' \to$, we thus have $W \in \operatorname{add} \overline{T}$. The nine lemma gives a commutative diagram whose rows and columns are triangles in $\mathcal{C}$:

$$\begin{tikzcd}
\Sigma^{-1}X \arrow[r] \arrow[d] & \Sigma^{-1}(X \oplus X') \arrow[r] \arrow[d] & \Sigma^{-1}X' \arrow[r, "0"] \arrow[d] & X \arrow[d] \\
U_1 \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & V_1 \arrow[r] \arrow[d] & \Sigma U_1 \arrow[d] \\
U_0 \arrow[r] \arrow[d] & U_0 \oplus V_0 \arrow[r] \arrow[d] & V_0 \arrow[r, "0"] \arrow[d] & \Sigma U_0 \arrow[d] \\
X \arrow[r] & X \oplus X' \arrow[r] & X' \arrow[r, "0"] & \Sigma X
\end{tikzcd}$$

We want to show that the triangle in the second row splits, which would then imply that U_1 and V_1, being summands of W, belong to $\operatorname{add} \overline{T}$. The composition $\Sigma^{-1}X' \to V_1 \to \Sigma U_1$ is zero so that the morphism $V_1 \to \Sigma U_1$ factors through $V_1 \to V_0$. But there are no non-zero morphisms from V_0 to ΣU_1 since $V_0 \in \operatorname{add} T$ and ΣU_1 is the cone of the right $\operatorname{add} T$-approximation $U_0 \to X$. $\surd$

Lemma 3.18. *Let X and Y be objects in $\overline{\mathcal{C}}(T)$, and let $X \xrightarrow{s} Y$ be a morphism in $\widetilde{\mathcal{S}}$. Then there exist $\overline{U} \in \operatorname{add} \overline{T}$, and morphisms $\Sigma\overline{U} \xrightarrow{c} Y$, $Y \xrightarrow{a} \Sigma\overline{U}$, and $Y \xrightarrow{d} X$ in $\mathcal{C}$ such that:*

1. *The morphism $X \oplus \Sigma\overline{U} \xrightarrow{[s\ c]} Y$ is in $\mathcal{S}$;*
2. *The image in $\mathcal{C}_{\mathcal{S}}$ of the morphism $Y \xrightarrow{\left[\begin{smallmatrix} d \\ a \end{smallmatrix}\right]} X \oplus \Sigma\overline{U}$ is inverse to $[s\ c]$.*

In particular, all morphisms in $\overline{\mathcal{C}}(T)$ which belong to $\widetilde{\mathcal{S}}$ are inverted by the localisation functor $L_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$.

Proof. Let $X \xrightarrow{s} Y$ be a morphism in $\widetilde{\mathcal{S}}$, with X and Y in $\overline{\mathcal{C}}(T)$. Complete it to a triangle $X \xrightarrow{s} Y \xrightarrow{v} \Sigma Z \xrightarrow{u} \Sigma X$. We first show how to define the object $\overline{U}$ and the morphisms a, c, d. By assumption, the morphisms v and u factor through $T^{\perp}$ and $(\Sigma\overline{T})^{\perp}$, respectively. There is a triangle $\overline{U} \to U \xrightarrow{\alpha} Y \xrightarrow{a} \Sigma\overline{U}$, with $U \in \operatorname{add} T$ and $\overline{U} \in \operatorname{add} \overline{T}$. Since v is in $(T^{\perp})$, the composition $v\alpha$ vanishes and there is a morphism b as in the diagram below, such that $v = ba$.

$$\begin{tikzcd}
 & U \arrow[d, "\alpha"'] & & \\
X \arrow[r, shift left, "s"] & Y \arrow[l, dashed, shift left, "d"] \arrow[r, "v"] \arrow[d, "a"'] & \Sigma Z \arrow[r, "u"] & \Sigma X \\
 & \Sigma\overline{U} \arrow[u, dashed, bend right, "c"'] \arrow[ur, dashed, "b"'] & &
\end{tikzcd}$$

The composition ub also vanishes since u factors through $(\Sigma\overline{T})^{\perp}$. Therefore, there is a morphism c such that $b = vc$. Moreover, there is a morphism d such that $1_Y = sd + ca$. Indeed, we have the following equalities: $vca = ba = v$ so that $1_Y - ca$ factors through s. Hence the morphism $[s\ c] : X \oplus \Sigma\overline{U} \to Y$ is a retraction.

Applying the octahedral axiom to the composition

$$X \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} X \oplus \Sigma\overline{U} \xrightarrow{[s\ c]} Y$$

yields the following commutative diagram whose rows and columns are triangles in $\mathcal{C}$.

$$\begin{tikzcd}
 & \overline{U} \arrow[r, equal] \arrow[d, "h"] & \overline{U} \arrow[d, "0"] & \\
\Sigma^{-1}Y \arrow[r, "f"] \arrow[d, equal] & Z \arrow[r, "g"] \arrow[d, "k"] & X \arrow[r, "s"] \arrow[d, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & Y \arrow[d, equal] \\
\Sigma^{-1}Y \arrow[r, "0"] & Z' \arrow[r] \arrow[d] & X \oplus \Sigma\overline{U} \arrow[r, "{[s\ c]}"] \arrow[d] & Y \\
 & \Sigma\overline{U} \arrow[r, equal] & \Sigma\overline{U} &
\end{tikzcd}$$

Since $[s\ c]$ is a retraction, the triangle in the lower row splits, and the morphism $\Sigma^{-1}Y \to Z'$ is zero. Let p be a morphism from $\overline{T}$ to Z. Since $s \in \widetilde{\mathcal{S}}$, g factors through $\overline{T}^{\perp}$, so $gp = 0$ and therefore p factors through f. Since $kf = 0$, f factors through h and therefore p factors through h. Applying the functor $\mathcal{C}(\overline{T}, -)$ to the triangle $\overline{U} \to Z \to Z' \to \Sigma\overline{U}$ gives $Z' \in \overline{T}^{\perp}$. We have thus constructed a triangle

$$Z' \to X \oplus \Sigma\overline{U} \xrightarrow{[s\ c]} Y \xrightarrow{0} \Sigma Z'$$

in $\mathcal{C}$, where Z' belongs to $\overline{T}^{\perp}$. This implies (1).

We now check (2). We have $[s\ c]\left[\begin{smallmatrix}d\\a\end{smallmatrix}\right] = 1_Y$ in $\mathcal{C}$. Since $[s\ c]$ lies in $\mathcal{S}$ by (1), it is invertible in $\mathcal{C}_{\mathcal{S}}$ and (2) follows.

Finally, we check the last part of the statement. Let $\pi : X \oplus \Sigma\overline{U} \to X$ be the first projection. Extending π to a triangle in $\mathcal{C}$, we have:

$$\Sigma\overline{U} \to X \oplus \Sigma\overline{U} \xrightarrow{\pi} X \xrightarrow{0} \Sigma^2\overline{U}.$$

Since $\Sigma\overline{U} \in \overline{T}^{\perp}$ and the zero map factors through $T^{\perp}$, we see that $\pi \in \mathcal{S}$. Furthermore, $s\pi = [s\,0] : X \oplus \Sigma\overline{U} \to Y$, so $s\pi - [s\,c] = [0, -c]$ factors through $\Sigma\overline{U}$, where $\Sigma\overline{U}$ lies in $\Sigma T'$. Morphisms of the form $A \oplus V \xrightarrow{[1\,0]} A$, with $V \in \overline{T}^{\perp}$, lie in $\mathcal{S}$. Hence, as in [BM13, Lemma 3.5], $L_{\mathcal{S}}(s)L_{\mathcal{S}}(\pi) = L_{\mathcal{S}}([s\,c])$. Since $\pi, [s\,c]$ both lie in $\mathcal{S}$, their images under $L_{\mathcal{S}}$ are invertible in $\mathcal{C}_{\mathcal{S}}$, and it follows that the image $L_{\mathcal{S}}(s)$ is also invertible in $\mathcal{C}_{\mathcal{S}}$, as required. $\surd$

For a morphism f which is part of a triangle $Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z$, recall that f belongs to the collection $\mathcal{S}$ if and only if Z belongs to $\overline{T}^{\perp}$ and h factors through $T^{\perp}$; and that f belongs to $\widetilde{\mathcal{S}}$ if and only if g factors through $\overline{T}^{\perp}$ and h factors through $T^{\perp}$.

Theorem 3.19. *There is an isomorphism of categories*

$$\mathcal{C}_{\mathcal{S}} \simeq \mathcal{C}_{\widetilde{\mathcal{S}}}.$$

Proof. As proved in Lemma 3.11, the categories $\mathcal{C}_{\widetilde{\mathcal{S}}}$ and $(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}$ are equivalent. By Theorem 3.2, the category $(\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}$ is equivalent to $\overline{\mathcal{C}}(T)/(\Sigma T')$.

It is easy to check that any morphism of the form $X \oplus U \xrightarrow{[1\,0]} X$, with $U \in \overline{T}^{\perp}$, lies in $\mathcal{S}$. Hence, arguing as in [BM13, Lemma 3.5], if u, v are any morphisms in $\mathcal{C}$ such that v factors through $\overline{T}^{\perp}$, then $L_{\mathcal{S}}(u) = L_{\mathcal{S}}(u + v)$. It follows that $L_{\mathcal{S}}$ induces a functor from $\overline{\mathcal{C}}(T)/(\operatorname{add}\Sigma T')$ to $\mathcal{C}_{\mathcal{S}}$, which we also denote by $L_{\mathcal{S}}$. Since $\widetilde{\mathcal{S}}$ contains $\mathcal{S}$, the same argument applies to $L_{\widetilde{\mathcal{S}}}$. Furthermore, by the universal property of localization, the left hand side of the diagram

$$\begin{tikzcd}
 & \overline{\mathcal{C}}(T)/(\operatorname{add}\Sigma T') \arrow[dl] \arrow[d] \arrow[dr, "\simeq"] & \\
\mathcal{C}_{\mathcal{S}} \arrow[r] & \mathcal{C}_{\widetilde{\mathcal{S}}} \arrow[r, "\simeq"] & (\operatorname{mod}\Lambda)_{\mathcal{S}_{\mathcal{B},0}}
\end{tikzcd}$$

commutes, where the functor $\mathcal{C}_{\mathcal{S}} \to \mathcal{C}_{\widetilde{\mathcal{S}}}$ is the identity on objects. The right hand triangle commutes by Lemma 3.11 and (3.1). It is thus enough to show that the functor $L_{\mathcal{S}} : \overline{\mathcal{C}}(T)/(\Sigma T') \to \mathcal{C}_{\mathcal{S}}$ is an equivalence of categories.

The functor $L_{\mathcal{S}}$ is dense by Lemma 2.2.

We next check that $L_{\mathcal{S}}$ is full. Let X, Y be objects of $\overline{\mathcal{C}}(T)$ and let $s : X \to Y$ be a morphism in $\mathcal{S}$. By part (2) of Lemma 3.18, $sd = 1_Y - ca$, where ca factors through $\overline{T}^{\perp}$. Arguing as above, we have that $L_{\mathcal{S}}(s)L_{\mathcal{S}}(d) = L_{\mathcal{S}}(1_Y)$, so $L_{\mathcal{S}}(d) = L_{\mathcal{S}}(s)^{-1}$. It follows that $L_{\mathcal{S}}$ (on $\overline{\mathcal{C}}(T)/(\operatorname{add}\Sigma T')$) is full.

It thus remains to prove that $L_{\mathcal{S}}$ is faithful. Via the use of the functor $\mathcal{C}(\overline{T}, -)$ and of the category $\operatorname{mod}\overline{\Lambda}$, this follows from results in [BM13]. Recall that we write $\overline{\Lambda}$ for the endomorphism algebra of $\overline{T}$ in $\mathcal{C}$. The functor $\mathcal{C}(\overline{T}, -)$ from $\overline{\mathcal{C}}(T)/(\Sigma T')$ to $\operatorname{mod}\overline{\Lambda}$ inverts all morphisms in $\mathcal{S}$, as proved in [BM13, Lemma 2.4]. By the universal property of localisations, there is a (unique) functor $\mathcal{C}_{\mathcal{S}} \xrightarrow{F'} \operatorname{mod}\overline{\Lambda}$ such that $\mathcal{C}(\overline{T}, -) = F'L_{\mathcal{S}}$. Assume that the image under $L_{\mathcal{S}}$ of a morphism f in $\overline{\mathcal{C}}(T)$ is zero. Then $F'L_{\mathcal{S}}(f) = 0$ so that $\mathcal{C}(\overline{T}, f)$ is zero in $\operatorname{mod}\overline{\Lambda}$. By [BM13, Lemma 2.3], this implies that f factors through $\overline{T}^{\perp}$. Since X is in $\overline{\mathcal{C}}(T)$, this implies, by Lemma 2.3, that f factors through $\overline{\mathcal{C}}(T) \cap \overline{T}^{\perp}$, which is $\operatorname{add}\Sigma T'$ by Lemma 2.7. Therefore f is zero in $\overline{\mathcal{C}}(T)/(\Sigma T')$ and the functor $L_{\mathcal{S}}$ is faithful. $\surd$

Remark 3.20. The reader might wonder why our proof makes a detour through the category $\operatorname{mod}\overline{\Lambda}$. One might think of a more direct proof as follows. Since we have an inclusion $\mathcal{S} \subseteq \widetilde{\mathcal{S}}$, it is enough to prove that every morphism in $\widetilde{\mathcal{S}}$ is inverted in $\mathcal{C}_{\mathcal{S}}$. This should easily follow from lemma 3.18: Let $X \xrightarrow{f} Y$ be a morphism in $\widetilde{\mathcal{S}}$. Then there is a commutative diagram

$$\begin{tikzcd}
R_0X \arrow[r, "f'"] \arrow[d, "\eta_X"'] & R_0Y \arrow[d, "\eta_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where R_0X, R_0Y are in $\overline{\mathcal{C}}(T)$ and η_X, η_Y in $\mathcal{S}$. It thus only remains to be checked that the morphism f' can be chosen in $\widetilde{\mathcal{S}}$. If so, Lemma 3.18 would imply that f' is inverted by $L_{\mathcal{S}}$. Since η_X and η_Y are in $\mathcal{S}$, f would also be inverted by $L_{\mathcal{S}}$. The problem here is that even though it is easily checked that $\widetilde{\mathcal{S}}$ is stable under composition, $\widetilde{\mathcal{S}}$ does not seem to satisfy the 2-out-of-3 property.

4. Examples

4.1. **Mutating a cluster-tilting object at a loop.** We consider the triangulated [Kel05] orbit category $D^{\mathrm{b}}(\operatorname{mod} \mathrm{A}_9)/\tau^3[1]$. Its Auslander–Reiten quiver is depicted in Figure 1. Copies of a fundamental domain are indicated by dashed lines. Let T be the direct sum $a \oplus b \oplus c$. Then T is a cluster-tilting object with a loop at c.

Let T' be the cluster-tilting object obtained by mutating T at c. Since there is a triangle $s \to b \oplus b \to c \to$, we have $T' = a \oplus b \oplus s$. The indecomposable objects lying in $\overline{\mathcal{C}}(T)$ are encircled (recall that $\overline{\mathcal{C}}(T) = T * \Sigma\overline{T} = \overline{T} * \Sigma T'$). The objects a, b, c

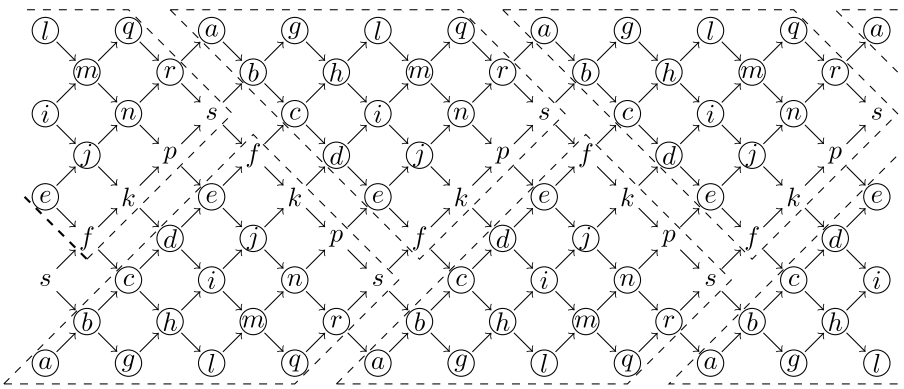

FIGURE 1. The AR-quiver of $D^{\mathrm{b}}(\operatorname{mod} \mathrm{A}_9)/\tau^3[1]$; Example 4.1

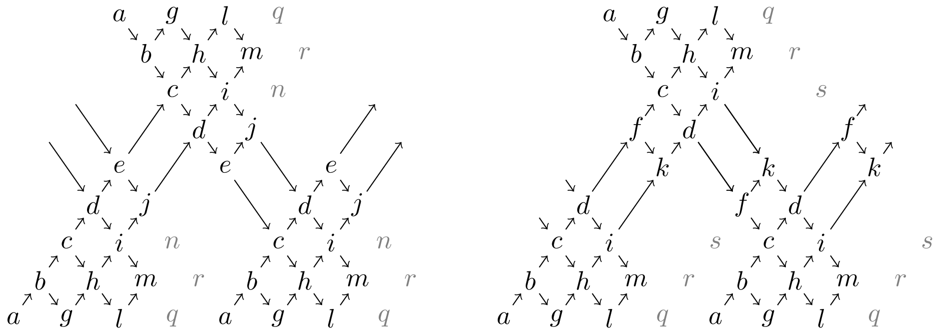

FIGURE 2. The categories $\overline{\mathcal{C}}(T)/(\Sigma T')$ and $\tau\overline{\mathcal{C}}(T)/(\tau T)$; Example 4.1

(respectively, n, q, r) belong to $\overline{\mathcal{C}}(T)$ since they are in $\operatorname{add} T$ (respectively $\operatorname{add} \Sigma T'$). The remaining encircled objects are in $\overline{\mathcal{C}}(T)$ since there are triangles $a \to c \to d \to$; $b \to c \to e \to$; $a \to b \to g \to$; $a \to c \to h \to$; $a \oplus a \to c \to i \to$; $a \oplus b \to c \to j \to$; $b \to c \to l$ and $a \oplus b \to c \to m \to$. The other four indecomposable objects are not in $\overline{\mathcal{C}}(T)$ since s is the shift of c, and there are triangles $c \to c \to f \to$, $a \oplus c \to c \to k \to$ and $b \oplus c \to c \to p \to$.

By Theorem 2.9, the categories $\overline{\mathcal{C}}(T)/(\Sigma T')$ and $\tau\overline{\mathcal{C}}(T)/(\tau T)$ are equivalent. These two categories are illustrated in Figure 2.

4.2. **Mutating a rigid object at a loop.** In the same category $D^{\mathrm{b}}(\operatorname{mod} \mathrm{A}_9)/\tau^3[1]$ as in the previous example, we now consider the rigid object $T = a \oplus c$. There is a triangle $n \to a \oplus a \to c \to$ (see Figure 3), so that we can choose T' to be $a \oplus n$. Indecomposable objects in $\overline{\mathcal{C}}(T)$ are encircled. The shift of T' is $i \oplus q$. Deleting either vertices labelled a and c, or vertices i and q, in the encircled part of the AR-quiver yields isomorphic quivers, as depicted in Figure 4.

4.3. **Mutating at a non-indecomposable summand.** In this example, we consider the triangulated [Kel05] orbit category $D^{\mathrm{b}}(\operatorname{mod} \mathrm{A}_5)/\tau^{-2}[1]$. It was shown in [BMR$^+$06] that this category is a Krull–Schmidt, Hom-finite category with a Serre functor and its Auslander–Reiten quiver is depicted in Figure 5. As for the previous examples, we have not drawn the arrows. The two subquivers in inside the dotted boxes have to be identified so as to match the two copies of a, b, c and d.

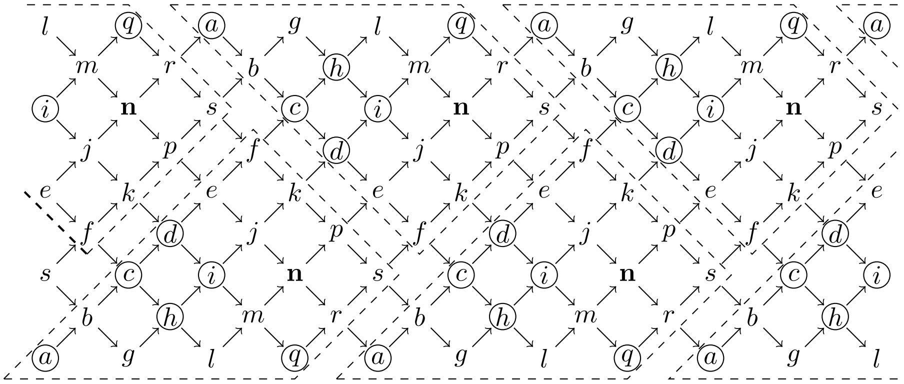

FIGURE 3. The AR-quiver of $D^{\mathrm{b}}(\operatorname{mod} \mathrm{A}_9)/\tau^3[1]$; Example 4.2

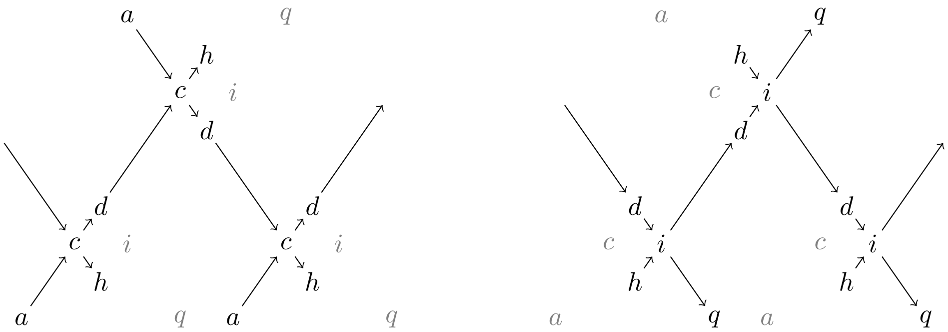

FIGURE 4. The categories $\overline{\mathcal{C}}(T)/(\Sigma T')$ and $\overline{\mathcal{C}}(T)/(T)$; Example 4.2

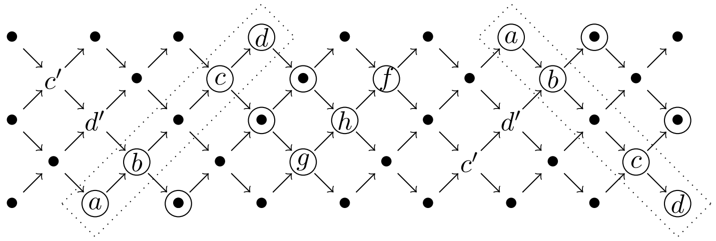

FIGURE 5. The AR-quiver of $D^{\mathrm{b}}(\operatorname{mod} \mathrm{A}_5)/\tau^{-2}[1]$; Example 4.3

We choose rigid objects is $T = a \oplus b \oplus c \oplus d$, and $T' = a \oplus b \oplus c' \oplus d'$. Indecomposable objects in $\overline{\mathcal{C}}(T)$ are encircled. The indecomposable objects labelled e, f, g, h are the shifts of a, b, c', d', respectively. As predicted by Theorem 2.9, one obtains isomorphic quivers by deleting either vertices a, b, c, d or vertices e, f, g, h.

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