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Physics of day-to-day network flow dynamics

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Abstract. This paper offers a new look at the network flow dynamics from the viewpoint of physics by demonstrating that the traffic system, in terms of the aggregate effects of human behaviors, may exhibit like a physical system. Specifically, we look into the day-to-day evolution of network flows that arises from travelers' route choices and their learning behavior on perceived travel costs. We show that the flow dynamics is analogous to a damped oscillatory system. The concepts of energies are introduced, including the potential energy and the kinetic energy. The potential energy, stored in each link, increases with the traffic flow on that link; the kinetic energy, generated by travelers' day-to-day route swapping, is proportional to the square of the path flow changing speed. The potential and kinetic energies are converted to each other throughout the whole flow evolution, and the total system energy keeps decreasing owing to travelers' tendency to stay on their current routes, which is analogous to the damping of a physical system. Finally, the system will approach the equilibrium state with minimum total potential energy and zero kinetic energy. We prove the stability of the day-to-day dynamics and provide numerical experiments to elucidate the interesting findings.

Keywords: Day-to-day dynamics; Network flow; User learning; Potential energy; Kinetic energy

1. Introduction

The notion of user equilibrium (UE), as the norm for transportation system analysis, describes the ideal static state of the transportation networks as a result of the aggregate behavior of road users when they are all rational utility-maximizers. It was first proposed by and then further extended to stochastic user equilibrium (SUE) (Daganzo and Sheffi, 1977), boundedly rational user equilibrium (BRUE) (Mahmassani and Chang, 1987) and so on, attempting to predict traffic flows with more realistic assumptions on travelers' behavior.

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The UE traffic assignment problem was formulated as a mathematical programming problem in Beckmann et al. (1956), known as the “Beckmann’s transformation”. Although the formulation reflects some inherent properties of the traffic network, it is in a long time viewed as no intuitive economic or behavioral interpretation but merely a mathematical construct to solve UE (Sheffi, 1984). In economics, traffic assignment problem with separable link cost functions is modeled as the potential game (Rosenthal, 1973). In a potential game, all players’ strategies are mapped into one global potential function. The difference in values for the potential function has the same value as the payoff of each player when changing one’s strategy ceteris paribus. The equilibrium is reached at the local optima of the potential function. From this point of view, the Beckmann’s transformation can be endowed with a physical-like meaning - the “potential” of the transportation network. The equilibria are obtained when the system achieves the lowest potential (Monderer and Shapley, 1996; Sandholm, 2001).

Most studies on static traffic assignment examine the final equilibrium state when travelers have no incentive for route swapping. However, in a real traffic network, it is always observed that traffic flows fluctuate from time to time (Guo and Liu, 2011; He and Liu, 2012), due to the interference of external factors and change of the network itself. Furthermore, a disequilibrated transportation network would incline to approach the equilibrium (Guo and Liu, 2011; He and Liu, 2012) through travelers’ route swapping. To explain the mechanism of network flow fluctuation and attainment of UE states, a substantial stream of research has been developed to look into the “day-to-day” flow dynamics.

The flow dynamics is sometimes described by a set of deterministic ordinary differential equations (ODEs) (Cho and Hwang, 2005; Han and Du, 2012; He et al., 2010; He and Liu, 2012; Smith and Mounce, 2011). A “rational behavior adjustment process” (RBAP) with fixed demand was proposed by Zhang et al. (2001) and Yang and Zhang (2009), assuming that “the aggregated travel cost in the system based on the previous day’s route travel costs will decrease when the route flows change from day to day” (Yang and Zhang, 2009). Many typical models follow RBAP, such as those proposed by Smith (1984), Friesz et al. (1994) and Nagurney and Zhang (1997). An equivalent link-based “discrete rational adjustment process” was proposed by Guo et al. (2013, 2015). The BRUE-based day-to-day dynamics was investigated recently in Di et al. (2015), Guo and Liu (2011), Guo (2013) and Wu et al. (2013).

In another branch of the literature, fluctuation of network flows is examined as a result of travelers’ perception and day-to-day learning on route travel costs (Bie and Lo, 2010; Cantarella and Cascetta, 1995; Cascetta and Cantarella, 1993; Horowitz, 1984; Watling, 1999; Xiao and Lo, 2015; Ye and Yang, 2013). Travelers are assumed to possess their own perception on future traffic conditions and choose routes based on their perception. The perceived cost is updated according to new experience or real-time traffic information, and the route choice is modeled as a stochastic network loading process, given the new perceived costs. The corresponding stationary state of these models is SUE.

The above two types of models are based on different assumptions. The first type deals with traffic flow evolution by updating traffic flows based on actual traffic conditions, but it ignores the impact of historical traffic information on travelers' route choice decisions. In contrast, the second type treats the flow swapping as a result of cognition changing from a more intrinsic aspect, but it is more difficult to verify since the cognition is more difficult to measure and estimate than flows.

In this paper, we explore the mechanism of travelers' learning behavior and route swapping behavior in an integrated manner. Road users on the routes with higher costs will tend to switch to the ones with lower costs, while taking history into account. As a result, their route swapping speeds depend on the route travel cost differences both currently and in the past. Such behavior is defined as the "inertia" in day-to-day flow dynamics in this study. When introducing "inertia" into the continuous route swapping model, the day-to-day dynamics can be described by a set of second-order ODEs, which is similar to the physical motion equation of a harmonic oscillator. By analogy to the physical system, we are able to identify the "damping factor", "restoring force", "potential energy" and "kinetic energy" of the network during the day-to-day evolution. The difference between the actual costs on each route-pair acts like "restoring force" in the transportation network, while the route swapping brings "damping". The Beckmann's transformation is translated into the "potential energy", as it was treated in the previous literature (Jin, 2007; Peeta and Yang, 2003; Sandholm, 2001), and the "kinetic energy" is defined to be associated with the flow changing. Total energy of the transportation network now comprises both the potential energy and the kinetic energy. The system keeps losing energy due to the "friction" caused by travelers' tendency to stay on their current routes and eventually reaches UE. The UE state is also the minimum potential energy state with zero kinetic energy, which is consistent with the minimum total potential energy principle (Hashin and Shtrikman, 1963).

The rest of this paper is organized as follows. Section 2 develops a continuous-time day-to-day dynamical model by considering travelers' learning process and route swapping behavior. The day-to-day flow dynamics is formulated as a set of second-order ODEs, which possesses a similar form to the motion equation of a harmonic oscillator. Section 3 briefly introduces the dynamics and energies of a damped oscillatory system. By analogy to a damped oscillatory system, formulae of kinetic and potential energies of the dynamical traffic network are developed and relationship between network energy and traffic equilibrium is discussed. In Section 4, stability analysis is provided by LaSalle's theorem. The total mechanical energy function is chosen to be the Lyapunov function. Section 5 examines some interesting properties of the day-to-day model by numerical examples. The last section concludes the study and highlights some future research directions.

2. The second-order day-to-day dynamics

Consider a directed traffic network $G = (N, A)$ consisting of a set N of nodes and a set A of links. Let W denote the set of origin-destination (OD) pairs and R_w the set of all paths connecting OD pair $w \in W$. Let $d_w > 0$ be the traffic demand between OD pair $w \in W$, f_{rw} the flow on path $r \in R_w$ between OD pair $w \in W$ and v_a the flow on link $a \in A$. Let $|R_w|$ represent the cardinality of set R_w, i.e., there are $|R_w|$ number of paths connecting OD pair $w \in W$. Each link has a separable cost function $c_a(v_a)$, which is assumed to be nonnegative, differentiable, convex and strictly increasing. Define $\Delta = [\delta_{ar}]$ and $\Lambda = [\lambda_{rw}]$ as the link-path and OD-path incidence matrices, respectively, where δ_{ar} equals 1 if path r uses link a and 0 otherwise, and λ_{rw} equals 1 if path r connects OD pair w and 0 otherwise.

Let $\mathbf{f} = (f_{rw}, r \in R_w, w \in W)^{\mathrm{T}}$, $\mathbf{d} = (d_w, w \in W)^{\mathrm{T}}$ and $\mathbf{v} = (v_a, a \in A)^{\mathrm{T}}$ be the vectors of path flows, OD demands and link flows, respectively, where superscript “T” represents the transpose operation. The relationship between link flows, path flows and OD demands can be expressed by

$$\mathbf{v} = \Delta\mathbf{f}, \mathbf{d} = \Lambda\mathbf{f} \tag{1}$$

The actual path travel cost c_{rw} along a path $r \in R_w$ between OD pair $w \in W$ is given to be the sum of travel costs on all the links constituting this path, i.e., $c_{rw} = \sum_{a\in A} c_a \delta_{ar}$, where c_a is the travel cost on link a.

2.1. *Wardrop's user equilibrium*

To begin with, we introduce the mathematical formulation of Wardrop's UE. Let u_w denote the minimum path travel cost between OD pair $w \in W$ and $\mathbf{u} = (u_w, w \in W)^{\mathrm{T}}$. Following Wardrop (1952) and Beckmann et al. (1956), the fixed-demand UE condition is given as follows.

Definition 1. A path flow vector $\mathbf{f}^*$ satisfying $\Lambda\mathbf{f}^* = \mathbf{d}$ is said to be a user equilibrium path flow pattern if the following condition holds:

$$c_{rw}(f_{rw}^*)\begin{cases} = u_w, & f_{rw}^* > 0 \\ \geq u_w, & f_{rw}^* = 0 \end{cases} \quad \forall r \in R_w, w \in W \tag{2}$$

Condition (2) can be obtained from the first-order optimality conditions of the following minimization problem:

$$\min_{\mathbf{v}} \sum_{a\in A} \int_0^{v_a} c_a(\omega)\mathrm{d}\omega \tag{3}$$

subject to $\mathbf{v} = \Delta\mathbf{f}$, $\mathbf{d} = \Lambda\mathbf{f}$, $\mathbf{f} \geq \mathbf{0}$

Eq. (3) is the mathematical form of the “Beckmann’s transformation”. Since the link travel cost functions are strictly increasing and convex, the UE link flow pattern $\mathbf{v}^*$ is unique, while the UE path flow pattern is generally not unique, and any path flow vector contained in the set $\{\mathbf{f}^*|\mathbf{f}^* \geq \mathbf{0}, \Delta\mathbf{f}^* = \mathbf{v}^*, \Lambda\mathbf{f}^* = \mathbf{d}\}$ is a UE path flow pattern.

2.2. Dynamical network flow evolution model

Consider a discrete-time day-to-day flow evolution model with learning process, where the calendar time is denoted by t and the time step is δt. Travelers are assumed to possess their own perception on travel costs, denoted by C_{rw}, for all paths $r \in R_w$, $w \in W$. The perceived path travel costs may not be exactly the same as the actual travel costs they experienced. When the perception differs from the real condition, travelers may try to “correct” their perception by some rule. We call this behavior the travelers’ “learning behavior”. As assumed in many previous studies such as Bie and Lo (2010), Horowitz (1984), Watling (1999), Xiao and Lo (2015), Yang et al. (1993) and Ye and Yang (2013), the day-to-day learning behavior is simply modeled by the exponential moving average as follows:

$$C_{rw}(t) = \varphi_w c_{rw}(t-\delta t) + (1-\varphi_w)C_{rw}(t-\delta t),\ \ 0 < \varphi_w \leq 1 \tag{4}$$

Eq. (4) states that the new perceived path travel cost is a linear combination of the previous actual and perceived path travel costs; φ_w is an OD-dependent weighting factor.

It is reasonable to assume that, the longer the time step is, the less certain the travelers will be about their prior perception (Berman et al., 2009). As a result, travelers will put less weight on their pervious perception when updating it. Therefore, it is reasonable to consider a general learning process with time-step-dependent weighting parameter, which can be formulated as follows,

$$C_{rw}(t) = \varphi_w(\delta t) c_{rw}(t-\delta t) + \big(1-\varphi_w(\delta t)\big)C_{rw}(t-\delta t) \tag{5}$$

where $\varphi_w(\delta t)$ is an increasing function of δt satisfying

$$\varphi_w(0) = 0 \tag{6}$$

$$\lim_{\delta t\to\infty} \varphi_w(\delta t) = 1 \tag{7}$$

One typical form of $\varphi_w(\delta t)$ is shown in Eq. (8), if we assume that the memory of the past information decays at a rate proportional to its current value.

$$\varphi_w(\delta t) = 1 - e^{-\gamma_w \delta t},\ \gamma_w > 0 \tag{8}$$

Once travelers update their perceptions on travel costs, the next step is to reconsider their route choices. We assume that the path swapping rate from path $r \in W$ to path $i \in W$ is proportional to the cost difference between these two paths with a constant coefficient η_w, which can be interpreted as travelers' sensitivity to the cost difference. We further assume that travelers between the same OD pair are homogenous and have identical sensitivity for all pairs of paths in the same OD pair, i.e., the coefficient η_w is only OD-dependent. Now the aggregate flow swapping rate on path $r \in R_w$ can be expressed by Eq. (9) as follows:

$$f_{rw}(t) - f_{rw}(t - \delta t) = -\eta_w \delta t \sum_{i \in R_w} \left(C_{rw}(t) - C_{iw}(t) \right) \tag{9}$$

As we show in Appendices A and B, under certain conditions, the day-to-day flow dynamics proposed in (9) is a degenerated case of both the network tatonnement process (NTP) in Friesz et al. (1994) and the projected dynamical system (PDS) in Nagurney and Zhang (1997).

Integrating the learning behavior in Eq. (5) with the route swapping behavior in Eq. (9) and then pushing the time step to zero, we have the following continuous form of the network flow dynamics (the derivation is provided in Appendix C),

$$\frac{1}{\eta_w \theta_w |R_w|} \ddot{f}_{rw} + \frac{1}{\eta_w |R_w|} \dot{f}_{rw} - \frac{1}{|R_w|} \sum_{i \in R_w} (c_{iw} - c_{rw}) = 0, \theta_w > 0, \eta_w > 0 \tag{10}$$

where $\theta_w = \dot{\varphi}_w(0)$ represents the initial decay rate of the memory on previous perceptions. It is interesting that, from the learning model, we obtain a second-order dynamical process with respect to path flows. In previous literature (Friesz et al., 1994; Nagurney and Zhang, 1997; Smith, 1984; Yang and Zhang, 2009), the flow swapping rate $\dot{f}_{rw}$ is directly determined by actual travel costs on all paths. However, Eq. (10) shows that, by considering travelers' learning process, the path flow swapping rate $\dot{f}_{rw}$ is dependent not only on actual path travel costs but also on the second-order derivative $\ddot{f}_{rw}$ of the path flow. When $\theta_w \to \infty$, indicating that travelers have no prior perception on travel costs but just rely on the latest traffic information to adjust routes, Eq. (10) will degenerate to the first-order model as follows,

$$\dot{f}_{rw} = -\eta_w \sum_{i \in R_w} (c_{rw} - c_{iw}), \eta_w > 0 \tag{11}$$

For the flow dynamics, let $\mathbf{x}_1 = \mathbf{f}$, $\mathbf{x}_2 = \dot{\mathbf{f}}$, $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$. Also denote $\mathbf{g}(\mathbf{x}_1) = (g_{rw}, r \in$

$R_w, w \in W)^{\mathrm{T}}$, where $g_{rw} = \sum_{i\in R_w}(c_{iw} - c_{rw})$. Then the dynamics in Eq. (10) can be rewritten into the following system of first-order ODEs,

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_2 \\ \boldsymbol{\theta}\boldsymbol{\eta}\mathbf{g}(\mathbf{x}_1) - \boldsymbol{\theta}\mathbf{x}_2 \end{bmatrix} \tag{12}$$

where $\boldsymbol{\theta} = \mathrm{diag}\{\theta_w I_{|R_w|}, w \in W\}$ with $I_{|R_w|}$ being an identity matrix of size $|R_w|$, and $\boldsymbol{\eta} = \mathrm{diag}\{\eta_w I_{|R_w|}, w \in W\}$.

Since

$$\sum_{r\in R_w} \dot{x}_{2,rw} = \sum_{r\in R_w}\sum_{i\in R_w} \theta_w \eta_w (c_{iw} - c_{rw}) - \theta_w \sum_{r\in R_w} x_{2,rw} = -\theta_w \sum_{r\in R_w} x_{2,rw} \tag{13}$$

then solving the ODE above yields

$$\sum_{r\in R_w} x_{2,rw} = a e^{-\theta_w t} \tag{14}$$

where a is constant. If the initial state of $\mathbf{x}_2$, denoted by $\mathbf{x}_2^0$, satisfies

$$\sum_{r\in R_w} x_{2,rw}^0 = 0, \quad \forall w \in W \tag{15}$$

Then $a = 0$ and $\sum_{r\in R_w} \dot{x}_{1,rw} = \sum_{r\in R_w} x_{2,rw} = 0$ for all calendar time, thus the flow dynamics always satisfies the flow conservation condition. We further make the following assumption to guarantee that the path flows are always positive during the evolution process. In other words, we assume that the path flows only evolve in the interior of the feasible path flow set.

Assumption 1. Define

$K = \{\mathbf{x}_1 | \mathbf{x}_1 > \mathbf{0}, \Lambda\mathbf{x}_1 = \mathbf{d}\}$

where $\mathbf{x}_1 > \mathbf{0}$ means that all the elements of $\mathbf{x}_1$ are positive, and

$H = \{\mathbf{x}_2 | x_{2,rw} \in \mathbb{R}, \sum_{r\in R_w} x_{2,rw} = 0, \forall r \in R_w, w \in W\}$.

We assume

(i) Each UE path flow pattern $\mathbf{x}_1^*$ is in K;

(ii) Starting with any point in $K \times H$, the trajectories of $\mathbf{x}$ under dynamical process (12) will always stay in $K \times H$.

Remark 1. In a deterministic network with arbitrary initial conditions, Assumption 1 can be easily violated. However, if we consider moderate path flow fluctuation in the neighborhood of the equilibrium points with no degenerate path, this assumption is generally satisfied. Here a degenerate path is the minimum-cost path whose flow must be zero under the equilibrium condition (Yang and Bell, 2007).

Theorem 1. With Assumption 1, all paths considered between the same OD pair have identical travel cost under UE. Then $\mathbf{x}_1^* \in K$ states UE if and only if $\mathbf{x}^* = \begin{bmatrix} \mathbf{x}_1^* \\ \mathbf{0} \end{bmatrix}$ is a stationary point of the dynamical system (12).

Proof. At the stationary point of Eq. (12), we have $\mathbf{x}_2^* = \mathbf{0}$ and $\mathbf{g}(\mathbf{x}_1^*) = \mathbf{0}$, which leads to

$$\sum_{i \in R_w} (c_{iw} - c_{rw}) = \sum_{i \in R_w} c_{iw} - |R_w| c_{rw} = 0, \qquad \forall r \in R_w, w \in W \tag{16}$$

Eq. (16) holds if and only if $c_{rw} = c_{iw}$ for all $r, i \in R_w$, $w \in W$, which is also the UE condition since $\mathbf{x}_1^* \in K$. ■

3. Physics of the day-to-day dynamics

To understand our day-to-day model better, we borrow the physical concepts and make analogy of our dynamical system to a damped oscillatory system. Availability of this analogy lies in the similarity of the formulations of both systems in representing dynamics or motion.

3.1. Dynamics and energy of a damped oscillatory system

We first briefly illustrate the dynamics of a damped oscillator. Suppose a rigid body of mass M is connected to the wall by a spring. A restoring force $F(x)$ is applied on the mass by the spring, where x is the displacement. Let $E_p(x)$ be the potential energy of the mass at position x. Without loss of generality, set $E_p(0) = 0$ and then

$$E_p(x) = \int_0^x -F(x) dx \tag{17}$$

Kinetic energy of the mass at a speed s is defined as

$$E_k = \frac{1}{2} M s^2 \tag{18}$$

In real oscillators, friction always exists to resist the motion of the mass. The frictional force F_f is usually assumed to be proportional to the velocity s of the object, i.e.,

$$F_f = -\vartheta s = -\vartheta \dot{x} \tag{19}$$

where ϑ is the damping coefficient. By Newton's Second Law,

$$M\ddot{x} + \vartheta \dot{x} - F(x) = 0 \tag{20}$$

Eq. (20) is called the equation of motion of the oscillatory system. Especially, if the restoring force is proportional to the displacement, i.e. $F(x) = -kx$, where k is the spring constant, then the system is called a damped harmonic oscillator. Motion equation of the damped harmonic oscillator can be rewritten into the following form,

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \tag{21}$$

where $\omega_0 = \sqrt{k/M}$ is the "undamped angular frequency" and $\zeta = \vartheta/\left(2\sqrt{Mk}\right)$ the "damping ratio". Damping ratio determines the oscillation pattern of the system as listed below:

i) Undamped ($\zeta = 0$): the system keeps oscillating and is unstable;
ii) Underdamped ($0 < \zeta < 1$): the system oscillates with the amplitude decreasing to zero. The angular frequency is given by

$$\omega_1 = \omega_0\sqrt{1-\zeta^2} \tag{22}$$

iii) Critically damped ($\zeta = 1$): the system converges at the fastest speed without oscillating;
iv) Overdamped ($\zeta > 1$): the system approaches to equilibrium without oscillating. The larger the damping ratio ζ is, the slower the system converges.

Figure 1 shows typical trajectories of a harmonic oscillator under different damping ratios.

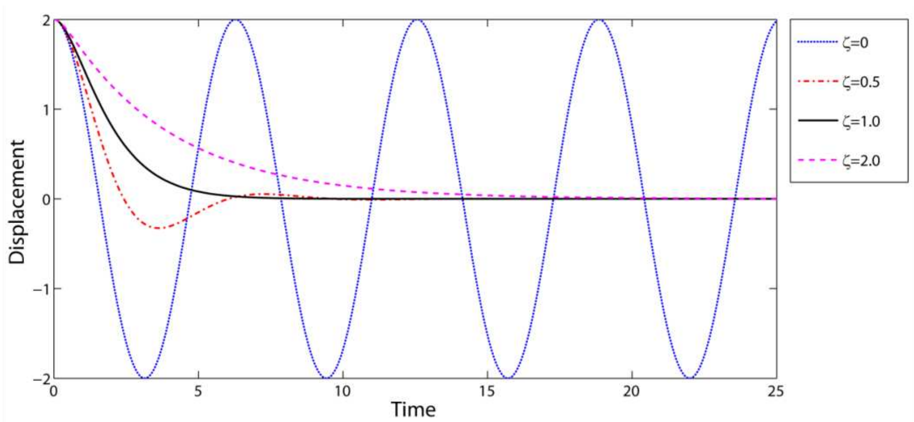

Figure 1. Harmonic oscillation under different damping ratios

3.2. Energies of the traffic network with flow dynamics

In a traffic network, travelers make daily route choices based on the perceived path travel costs and also update the perceptions according to previous travel experience; as a result, path flows may fluctuate from day to day. It is interesting to see that the dynamics of such a traffic network and that of a damped oscillatory system have some characteristics in common. First of all, they both have "inertia". A rigid mass system tends to keep its former motion state because of inertia; while in a traffic network, travelers tend to keep their former route choices because of the impact of historical information on their perception of travel costs. Secondly, both systems have "damping". For the physical system, damping is caused by frictions, with which the system gradually loses energy before equilibrating. In a traffic network, damping is caused by travelers' route swapping behavior to reduce individual cost. In this subsection, we will explore the similarity between the traffic network under day-to-day flow dynamics and the damped oscillatory system, and then discuss the system behavior from the perspective of energy. Alike to the damped oscillatory system, potential energy and kinetic energy of a traffic network will be defined and the evolution of energy will be investigated.

Enlightened by the oscillatory system, we regard the link flow $v_a = \sum_{w \in W} \sum_{r \in R_w} \delta_{ar} f_{rw}$ as the total "deformation" or "displacement" of link $a \in A$, and the link travel cost $c_a(v_a)$ as the "restoring force". Thus it's natural to define $\int_0^{v_a} c_a(\omega)\mathrm{d}\omega$ as the potential energy on link $a \in A$, which is a scalar and thus additive. Then the total potential energy on the network, denoted by E_p, is equal to

$$E_p = \sum_{a \in A} \int_0^{v_a} c_a(\omega)\mathrm{d}\omega \tag{23}$$

which is exactly taking the form of the "Beckmann's transformation" in Eq. (3) and is also similar to that in Eq. (17).

Similar to Eq. (18), define $\frac{1}{2} m_{rw} x_{2,rw}^2$ to be the kinetic energy of path $r \in R_w$, and then the kinetic energy of the network can be written as

$$E_k = \sum_{w \in W} \sum_{r \in R_w} \frac{1}{2} m_{rw} x_{2,rw}^2 \tag{24}$$

where m_{rw} acts as the mass of path $r \in R_w, w \in W$. Assume that the travel time c_a is in time unit h (i.e. hour) and traffic flow v_a in veh/h, then the dimension of potential energy E_p in Eq. (23) is veh. Furthermore, the unit of flow changing speed $x_{2,rw}$ is veh/h^2. As it will be specified later in this subsection, m_{rw} is in h^4/veh, then the dimension of kinetic energy E_k in Eq. (24) is $h^4/veh \cdot (veh/h^2)^2 = veh$. Therefore, the potential energy E_p and kinetic energy E_k have the same dimension. Total mechanical energy of the system at any calendar time is the summation of its potential energy and kinetic energy, i.e.,

$$E(\mathbf{x}) = \sum_{a \in A} \int_0^{v_a(\mathbf{x}_1)} c_a(\omega)\mathrm{d}\omega + \sum_{w \in W} \sum_{r \in R_w} \frac{1}{2} m_{rw} x_{2,rw}^2 \tag{25}$$

where $\mathbf{x}_1 > \mathbf{0}$, $\Lambda \mathbf{x}_1 = \mathbf{d}$ and $x_{2,rw} \in \mathbb{R}$, for all $r \in R_w$, $w \in W$.

Then we have the following theorem on the relationship between system energy and UE.

Theorem 2. $E(\mathbf{x})$ is strictly convex and thus has the unique minimum at $\mathbf{x}^* = \begin{bmatrix} \mathbf{x}_1^* \\ \mathbf{x}_2^* \end{bmatrix}$, where

$\mathbf{x}_1^*$ is the user equilibrium path flow pattern and $\mathbf{x}_2^* = \mathbf{0}$.

Proof. By assumption, $c_a(v_a)$ is strictly increasing for all $a \in A$, thus $E(\mathbf{x})$ is strictly convex, which implies that $E(\mathbf{x})$ has a unique minimum. To minimize Eq. (25), the Lagrangian is

$$L(\mathbf{x}, \mathbf{u}) = E(\mathbf{x}) + \sum_{w \in W} u_w \left(d_w - \sum_{r \in R_w} x_{1,rw} \right) \tag{26}$$

Therefore the first-order optimality conditions are

$$x_{1,rw}^* \frac{\partial L}{\partial x_{1,rw}^*} = x_{1,rw}^* (c_{rw}^* - u_w^*) = 0, \quad \forall\, r \in R_w, w \in W \tag{27}$$

$$c_{rw}^* - u_w^* \geq 0\,, \ \ x_{1,rw}^* \geq 0, \ \forall\, r \in R_w, w \in W \tag{28}$$

$$\frac{\partial L}{\partial x_{2,rw}^*} = m_{rw} x_{2,rw}^* = 0, \quad \forall\, r \in R_w, w \in W \tag{29}$$

Conditions (27) and (28) together indicate UE, therefore $\mathbf{x}_1^*$ is the UE path flow pattern. By Eq. (29), $x_{2,rw}^* = 0$ for all $r \in R_w$, $w \in W$, i.e., $\mathbf{x}_2^* = \mathbf{0}$. This completes the proof. ■

Since the potential energy in Eq. (25) is the same as the objective function in Eq. (3), thus when at UE, the system potential energy and system total energy reach minimum simultaneously, while the kinetic energy is zero and also minimum.

By comparing the counterparts of Eqs. (10) and (20), $m_w \triangleq \frac{1}{\eta_w \theta_w |R_w|}$ can be interpreted as the "mass" of each path between OD pair $w \in W$. Since we assume that all the travelers between the same OD pair are homogeneous, characterized by the identical η_w and θ_w, then all the paths between the same OD pair will have the same "mass", i.e., $m_{rw} = m_w$ for all $r \in R_w$. Further we define $M_w = |R_w| m_w = \frac{1}{\eta_w \theta_w}$ as the total mass of OD pair $w \in W$. In reality, the smaller θ_w is, the more slowly travelers' memory fades; and the smaller η_w is, the less sensitive the travelers are to the cost differences. Both situations will make travelers more prone to stay on their original routes. By defining the "mass", such phenomenon can be interpreted from the angle of the physical system: smaller θ_w or η_w means larger "mass" of each path. The larger the mass is, the more difficultly the system can change its state. Furthermore, analogously to the oscillatory system, $\frac{1}{\eta_w |R_w|}$ can be interpreted as the "damping coefficient", f_{rw} the "displacement" and $\frac{1}{|R_w|} \sum_{i \in R_w} (c_{iw} - c_{rw})$ the "restoring force" on path $r \in R_w$. It can be observed that the "friction force" (reflected by the damping coefficient $\frac{1}{\eta_w |R_w|}$) is inversely proportional to the parameter η_w, which reflects travelers' tendency to stay on their current routes. A smaller η_w means that travelers are more willing to stick to their current choices, which just looks like that there is a larger friction force resisting the path flows to change. Also, the "restoring force" is larger when path travel costs have greater difference.

Assume that the link/path travel time is in unit h and link/path flow in veh/h. Then from Eq. (5), θ_w has dimension $1/h$, and from Eq. (9), η_w has dimension veh/h^3; therefore, M_w, as well as m_w, has dimension $1/(1/h \cdot veh/h^3) = h^4/veh$.

Remark 2. The analogy above is coarse since the day-to-day dynamical system is far more complicated than the nonlinear oscillatory system defined by Eq. (21). The "restoring force" cannot be directly calculated from the flow in a single path but is dependent on all path flows on the network.

Remark 3. The second-order flow dynamics under path-based tolls can be analogous to an oscillatory system with external forces. Here the tolls are acting just like the "external force" applied to a driven harmonic oscillator. Specifically, assuming the toll on path $r \in R_w$ at time t to be $\tau_{rw}(t)$, then Eq. (5) is modified to the following form:

$$C_{rw}(t) = \varphi_w(\delta t)\big(c_{rw}(t-\delta t) + \tau_{rw}(t-\delta t)\big) + \big(1-\varphi_w(\delta t)\big)C_{rw}(t-\delta t) \qquad (30)$$

Following the procedure in Appendix C, we can express the second-order day-to-day dynamics with dynamic path-based tolls (with unity value of time) as

$$\begin{aligned}\frac{1}{\theta_w \eta_w |R_w|}\ddot{f}_{rw} + \frac{1}{\eta_w |R_w|}\dot{f}_{rw} + \frac{1}{|R_w|}\sum_{i\in R_w}(c_{rw}-c_{iw}) + \frac{1}{|R_w|}\sum_{i\in R_w}(\tau_{rw}-\tau_{iw}) \\ + \frac{1}{\theta_w |R_w|}\sum_{i\in R_w}(\dot{\tau}_{rw}-\dot{\tau}_{iw}) = 0, r\in R_w, w\in W\end{aligned} \qquad (31)$$

We can see that the "external forces" consist of not only the tolls but also the derivatives of the tolls, and they together affect the flow evolution. Moreover, understanding the mechanism of the network flow evolution can help design more efficient toll schemes, and one interesting issue is to find the optimal dynamic tolls that minimize convergence time or toll revenue. By applying the Pontryagin's maximum principle, a bang-bang toll scheme switching among two or more discrete toll levels would be promising.

4. Stability analysis

Regarding the day-to-day dynamics, stability is always a major concern in the literature. The stability analysis of a dynamical system resorts to various stability theorems (Khalil, 2002) by constructing the Lyapunov functions. In this section, we examine the stability of our day-to-day dynamics in Eq. (12) by LaSalle's theorem. The Lyapunov's second theorem is not applicable in this case because the non-unique UE path flow patterns are usually not isolated points but constituting a convex set.

Definition 2. Consider an autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{32}$$

where $\mathbf{F}: M \to \mathbb{R}^m$ is a locally Lipschitz map from a domain $M \subset \mathbb{R}^m$ into $\mathbb{R}^m$. A set Π is said to be an invariant set with respect to (32) if

$$\mathbf{x}(0) \in \Pi \Rightarrow \mathbf{x}(t) \in \Pi, \ \forall\, t \in \mathbb{R}$$

A set Π is said to be a positively invariant set if

$$\mathbf{x}(0) \in \Pi \Rightarrow \mathbf{x}(t) \in \Pi, \ \forall\, t \geq 0$$

With this definition, we have the following LaSalle's theorem (Khalil, 2002).

Theorem 3. Let Ω be a compact (closed and bounded) positively invariant set of the dynamical system (32) and $V: \Omega \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(\mathbf{x}) \leq 0$ in Ω. Let Π be the set of all points in Ω where $\dot{V}(\mathbf{x}) = 0$. Let G be the largest invariant set in Π. Then every solution starting in Ω approaches G as $t \to \infty$.

$V(\mathbf{x})$ is called the Lyapunov function of the dynamical system represented by Eq. (32). In the literature of day-to-day dynamics, Lyapunov functions could be of the quadratic form (Smith, 1984), the 2-norm distance (Friesz et al., 1994; Nagurney and Zhang, 1997) or the Beckmann's transformation (Jin, 2007; Peeta and Yang, 2003). Here by analogy with a physical system, the total system energy could be an ideal Lyapunov function since if the system keeps losing its mechanical energy over time, it must eventually reach some final equilibrium state with the lowest mechanical energy. Based on Theorem 3, we obtain the following theorem.

Theorem 4. With Assumption 1, if $M_w = \frac{1}{\theta_w \eta_w}$ for all $w \in W$, then the function

$$V(\mathbf{x}) = E(\mathbf{x}) - min\, E(\mathbf{x}) \tag{33}$$

is a Lyapunov function of the dynamical system in Eq. (12). If the initial state of $\mathbf{x}_2$ satisfies Eq. (15), then under the dynamical process in Eq. (12), the path flow pattern will approach the UE path flow set, and as a result, the link flows will approach the unique UE link flow pattern.

Proof. With Assumption 1, the set $K \times H$ is an invariant set of the dynamical system in Eq. (12). The gradient of Eq. (33) can be calculated as follows,

$$\frac{\partial V(\mathbf{x})}{\partial x_{1,rw}} = \frac{\partial}{\partial x_{1,rw}} \sum_{a \in A} \int_0^{v_a(\mathbf{x}_1)} c_a(\omega) \mathrm{d}\omega = c_{rw} \tag{34}$$

$$\frac{\partial V(\mathbf{x})}{\partial x_{2,rw}} = \frac{\partial}{\partial x_{2,rw}} \sum_{w \in W} \sum_{r \in R_w} \frac{1}{2} m_w x_{2,rw}^2 = m_w \, x_{2,rw} \tag{35}$$

Then we have

$$\begin{aligned}
&\dot{V}(\mathbf{x}) \\
&= \nabla V(\mathbf{x})^{\mathrm{T}} \dot{\mathbf{x}} \\
&= \sum_{w \in W} \sum_{r \in R_w} c_{rw} \dot{x}_{1,rw} + \sum_{w \in W} \sum_{r \in R_w} m_w x_{2,rw} \dot{x}_{2,rw} \\
&= \sum_{w \in W} \sum_{r \in R_w} c_{rw} x_{2,rw} + \sum_{w \in W} \sum_{r \in R_w} m_w x_{2,rw} \left(\sum_{i \in R_w} \eta_w \theta_w (c_{iw} - c_{rw}) - \theta_w x_{2,rw} \right) \\
&= \sum_{w \in W} \sum_{r \in R_w} c_{rw} x_{2,rw} + \sum_{w \in W} \theta_w \eta_w m_w \sum_{r \in R_w} x_{2,rw} \left(\sum_{i \in R_w} c_{iw} - |R_w| c_{rw} \right) \\
&\quad - \sum_{w \in W} \theta_w m_w \sum_{r \in R_w} x_{2,rw}^2 \\
&= \sum_{w \in W} \sum_{r \in R_w} c_{rw} x_{2,rw} + \sum_{w \in W} \theta_w \eta_w m_w \sum_{r \in R_w} x_{2,rw} \sum_{i \in R_w} c_{iw} \\
&\quad - \sum_{w \in W} \theta_w \eta_w m_w |R_w| \sum_{r \in R_w} x_{2,rw} c_{rw} - \sum_{w \in W} \theta_w m_w \sum_{r \in R_w} x_{2,rw}^2 \\
&= \sum_{w \in W} (1 - \theta_w \eta_w m_w |R_w|) \sum_{r \in R_w} c_{rw} x_{2,rw} + \sum_{w \in W} \theta_w \eta_w m_w \sum_{i \in R_w} c_{iw} \sum_{r \in R_w} x_{2,rw} \\
&\quad - \sum_{w \in W} \theta_w m_w \sum_{r \in R_w} x_{2,rw}^2
\end{aligned} \tag{36}$$

Substituting $m_w = \frac{1}{\theta_w \eta_w |R_w|}$ into Eq. (36) yields

$$\dot{V}(\mathbf{x}) = \sum_{w \in W} \frac{1}{|R_w|} \sum_{i \in R_w} c_{iw} \sum_{r \in R_w} x_{2,rw} - \sum_{w \in W} \frac{1}{\eta_w |R_w|} \sum_{r \in R_w} x_{2,rw}^2 \tag{37}$$

If the initial state of $\mathbf{x}_2$ satisfies Eq. (15), then $\sum_{r \in R_w} x_{2,rw} = 0$ for all $w \in W$. Thus we have

$$\dot{V}(\mathbf{x}) = -\sum_{w \in W} \frac{1}{\eta_w |R_w|} \sum_{r \in R_w} x_{2,rw}^2 \leq 0 \tag{38}$$

Equality in Eq. (38) holds if and only if $x_{2,rw} = 0$ for all $r \in R_w$, $w \in W$, i.e., $\mathbf{x}_2 = \mathbf{0}$. Thus the set Π in Theorem 3 reads $\Pi = \{\mathbf{x} | \dot{V}(\mathbf{x}) = 0\} = \{\mathbf{x} | \mathbf{x}_2 = \mathbf{0}\}$. Furthermore, if at some time t, we have a point in Π with $\mathbf{x}_2(t) = \mathbf{0}$ and $\mathbf{x}_1(t) \neq \mathbf{x}_1^*$, then according to Eq. (12) and Theorem 1, $\dot{\mathbf{x}}_2(t) \neq 0$. As a result, the trajectory will not stay in Π. Therefore, the points that can stay in Π for all time are only those states with $\mathbf{x}_1 = \mathbf{x}_1^*$, i.e., those $\mathbf{x}^*$ defined in Theorem 2. As a result, the largest invariant set in Π is $G = \{\mathbf{x}^*\}$, and by Theorem 3, every trajectory starting in $K \times H$ will approach $\{\mathbf{x}^*\}$. From Theorem 1, $\mathbf{x}_1^*$ states UE, hence the path flow pattern will approach the UE path flow set. ■

We conclude this section by highlighting the analogy between the traffic network and the physical system. Starting from some initial state, the traffic network gradually loses its energy until it stops at the equilibrium with zero kinetic energy and minimum potential energy, which is consistent with the famous minimum total potential energy principle.

5. Numerical experiment

In this section, we provide examples to illustrate the characteristics of our new day-to-day model and analyze the physical and dynamical effects of the parameters.

5.1. Experiment settings

The original network structure is shown in Figure 2(a), consisting of four nodes and four directed links. For each link, the performance function takes the BPR (Bureau of Public Roads, 1964) form as follows,

$$c_a(v_a) = c_a^0 \left[1 + 0.15 \left(\frac{v_a}{y_a}\right)^4\right], a \in \{1,2,3,4,5\}$$

where c_a^0 is the free flow travel cost and y_a is the capacity of link $a \in A$. The free flow travel costs and capacities of all the four links are listed in Table 1. There is only one OD pair with a fixed demand of 10 and served by the following two paths:

Path 1: $O \to 1 \to 2 \to D$; Path 2: $O \to 3 \to 4 \to D$

The UE path flow pattern is $(f_1^*, f_2^*) = (2.55, 7.45)$.

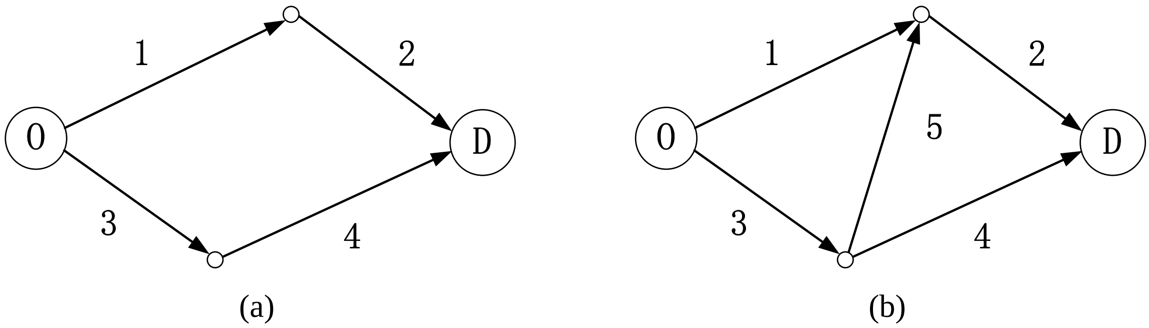

Figure 2. Network structures: (a) original network, (b) new network.

Table 1. Parameters of the link performance functions

Link no. a	1	2	3	4	5
Free flow cost (c_a^0)	2	1	0.5	2.5	1
Link capacity (y_a)	2.5	2.5	5	10	2.5

Connecting the two intermediate nodes by a new link (link 5) will generate an additional path (Path 3: $O \rightarrow 3 \rightarrow 5 \rightarrow 2 \rightarrow D$), as shown in Figure 2(b). The performance function of link 5 is also the BPR function with the free flow travel cost $c_5^0 = 1$ and link capacity $y_5 = 2.5$, as listed in Table 1. The new UE path flow pattern is $(f_1^{\#}, f_2^{\#}, f_3^{\#}) = (1.78, 6.56, 1.66)$.

5.2. Comparison of the first-order and second-order day-to-day models

We compare the performance of our proposed second-order model with some classic first-order models in the literature, on the simple two-path network in Figure 2(a). The first-order models include the proportional-switch adjustment process (PSAP) in Smith (1984),

$$\dot{f}_{rw} = \alpha \sum_{s \in R_w} \left(f_{sw}[c_{sw} - c_{rw}]_+ - f_{rw}[c_{rw} - c_{sw}]_+ \right), r \in R_w, w \in W, \alpha > 0$$

where $[\cdot]_+ = \max\{\cdot, 0\}$; the NTP in Friesz et al. (1994),

$$\dot{\mathbf{f}} = \alpha[P_{\mathcal{F}}(\mathbf{f} - \beta\mathbf{c}) - \mathbf{f}],\ \alpha > 0,\ \beta > 0 \tag{39}$$

where $\mathcal{F} = \{\mathbf{f} | \mathbf{f} \geq \mathbf{0}, \Lambda\mathbf{f} = \mathbf{d}\}$, $P_{\mathcal{F}}(\mathbf{z}) = \arg\min_{\mathbf{f} \in \mathcal{F}} \|\mathbf{f} - \mathbf{z}\|^2$, and $\mathbf{c} = (c_{rw}, r \in R_w, w \in W)^{\mathrm{T}}$; the PDS in Nagurney and Zhang (1997),

$$\dot{\mathbf{f}} = \alpha \lim_{\varepsilon \to 0} \frac{P_{\mathcal{F}}(\mathbf{f} - \varepsilon\mathbf{c}) - \mathbf{f}}{\varepsilon}, \alpha > 0$$

and the first-in-first-out (FIFO) dynamics in Jin (2007),

$$\dot{f}_{rw} = -\alpha d_w f_{rw} \left(c_{rw} - \frac{1}{d_w} \sum_{s \in R_w} f_{sw} c_{sw} \right), r \in R_w, w \in W, \alpha > 0$$

For all the above-mentioned first-order continuous-time dynamics in the two-path network, if the path flow pattern oscillates around UE, then the evolution trajectory must pass some path flow pattern $\mathbf{f}$ twice, implying that the corresponding derivative $\dot{\mathbf{f}}$ have two different values. This is impossible, because $\dot{\mathbf{f}}$ is uniquely determined by $\mathbf{f}$. Thus, the path flow pattern must evolve to UE monotonically without oscillation. In other words, the above-mentioned first-order continuous-time day-to-day models will exhibit only the overdamped-like patterns in the two-path network. In comparison, the second-order model will exhibit both overdamping and underdamping, depending on the value of parameters. Our statement is further illustrated by Figure 3.

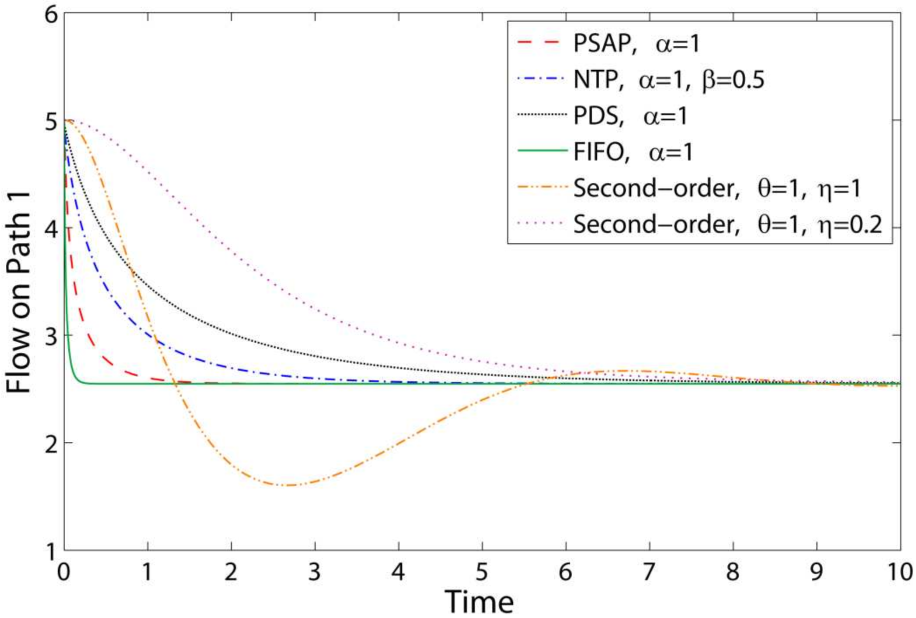

Figure 3. Comparison of the first- and second-order continuous-time models.

It is worth mentioning that, the oscillation in path flows can appear in the first-order discrete-time models when the time step is relatively large. Under this circumstance, the fluctuation is due to travelers' oversensitivity to the difference of travel costs, which is different from the reason of the oscillation in the second-order continuous-time model, and the latter will be further discussed in the subsequent subsections.

5.3. Flow evolution under different η

In the following subsections, we will focus on the second-order dynamics and investigate the path flow evolution from the original UE pattern $(f_1^*, f_2^*, 0)$ to the new one $(f_1^\#, f_2^\#, f_3^\#)$, under the dynamical process given by Eq. (12). The initial flow swapping speeds between any two paths are all assumed to be zero.

First, we fix $\theta = 1.0$ and set η to be 0.4, 1.0 and 10.0, respectively. Evolution trajectories of the path flows are depicted in Figure 4.

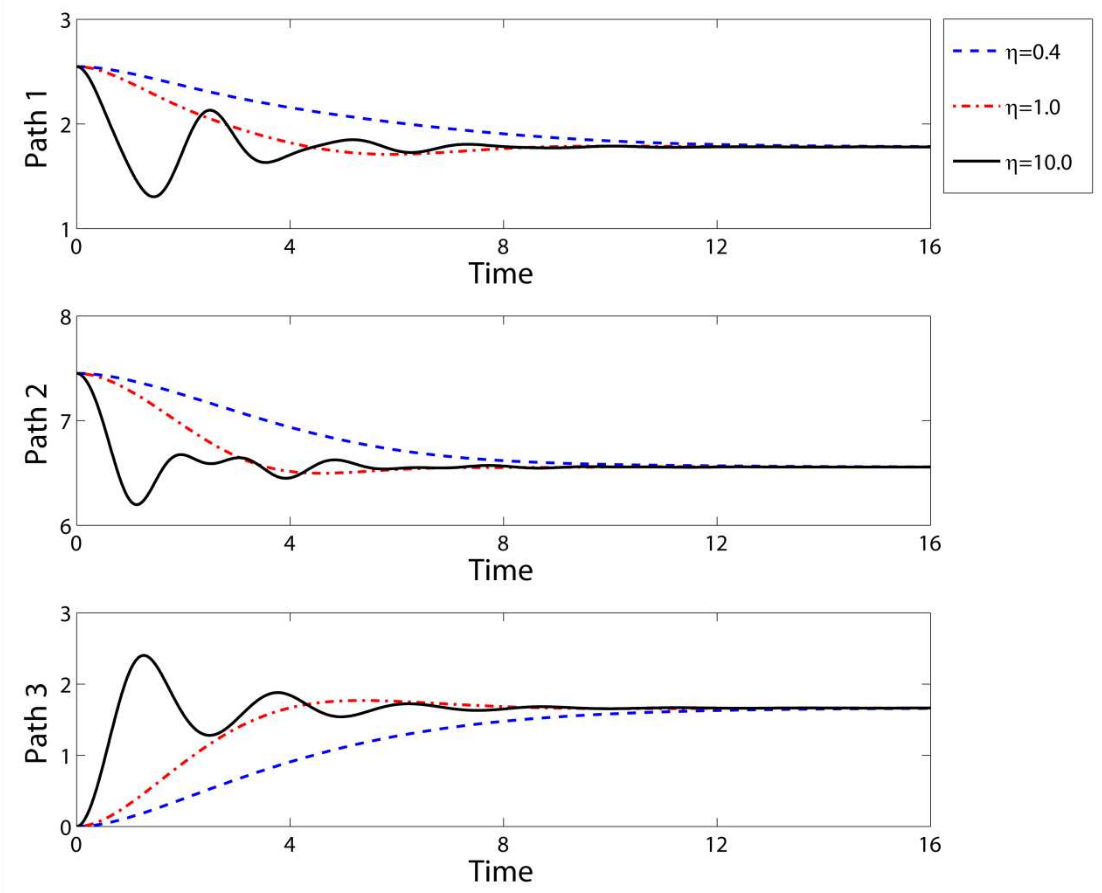

Figure 4. Evolution of path flows with fixed $\theta = 1.0$ and different $\eta = 0.4, 1.0, 10.0$.

η represents travelers' sensitivity to the differences of perceived travel costs between different paths. A greater η means that the travelers are more sensitive so that the swapping speeds are higher. It is interesting to see that, when η is large (e.g., 10.0), the path flows will oscillate relatively dramatically. The system experiences several periods of oscillation with a relatively long convergence time, which is similar to the "underdamped" case of the damped harmonic oscillator as shown in Figure 1. When η is small and less than one (e.g. 0.4), the path flows converge to the equilibrium slowly without oscillation. This situation corresponds to the "overdamped" case of a damped harmonic oscillator. When η is chosen to be a proper value (e.g. 1.0, which may not be the exact critical value as that in the case of "critically damping" for a damped harmonic oscillator), the system converges faster than it does in the case of $\eta = 0.4$ while oscillates less than it does in the case of $\eta = 10.0$. This example shows that, there may exist a critical value of η as that in the "critically damped" case for a damped harmonic oscillator, under which the system converges fastest without oscillation. But due to the complexity of the traffic network, it is difficult to sort out this critical value analytically.

5.4. Flow evolution under different θ

In this case, we fix $\eta = 1.0$ and look at the flow evolution under different θ, where $\theta = 0.2, 1.0, 5.0$.

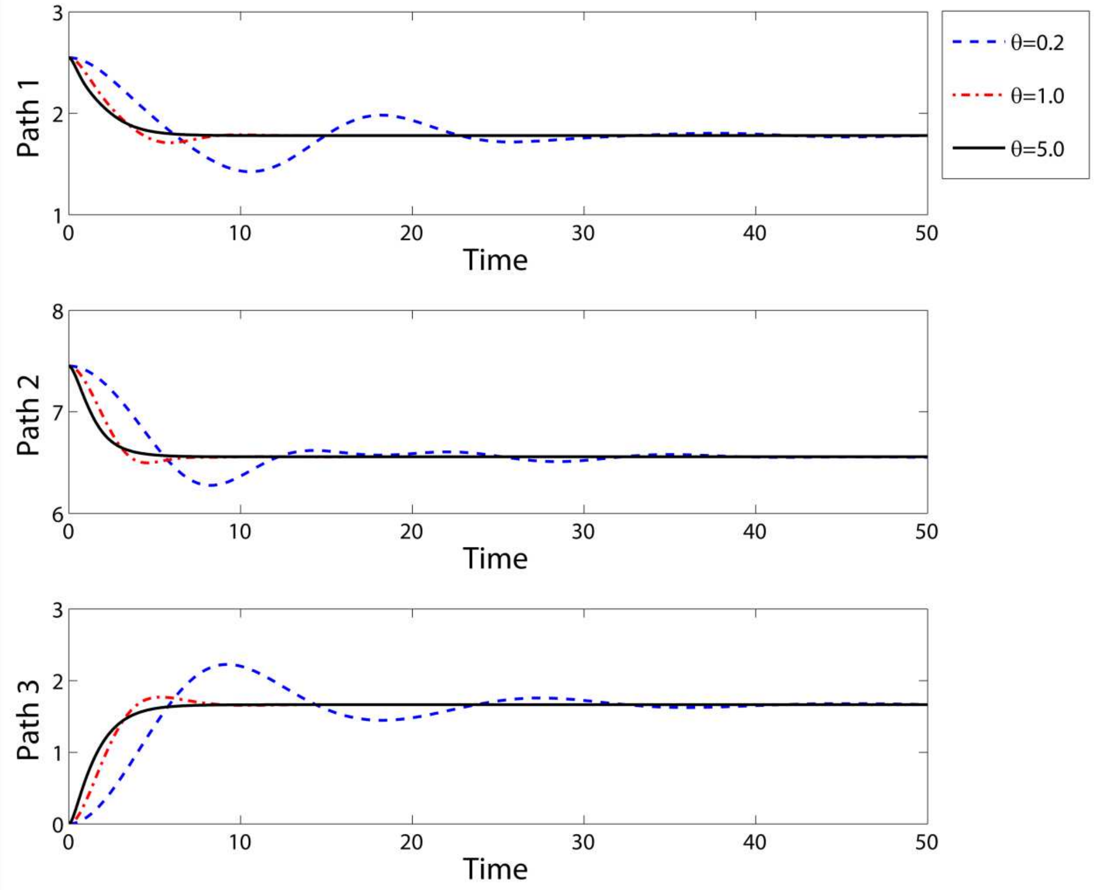

Figure 5. Evolution of path flows with fixed $\eta = 1.0$ and different $\theta = 0.2, 1.0, 5.0$.

θ reflects the initial decay rate of the memory, which will affect the weight travelers put on their experienced travel costs when updating their perceived travel costs. The larger θ is, the faster the memory fades and the larger weight travelers put on the experienced travel cost. As shown in Figure 5, the relationship between convergence time and θ is monotone. The larger θ is, the less the system oscillates and the faster the system converges.

The phenomenon can be interpreted by analogy with the physical system. From Eq. (10), given η_w, θ_w will affect the “mass”; the larger the θ_w, the smaller the “mass”. Moreover, referring to Eq. (21), decreasing “mass” will increase “damping ratio”. Therefore, when θ_w increases, the “damping ratio” will increase, and consequently, the system will evolve from “underdamped” to “overdamped”.

In the following two subsections, we attempt to investigate two critical characteristics of the network. For the sake of convenience, we borrow the terms "damping ratio" and "angular frequency" from the harmonic oscillator. The analogy here is really rough, but the findings we obtain through the analogy are still interesting.

5.5. Synonymous damping ratio

Comparing Eqs. (10) and (21), intuitively, we define the synonymous damping ratio of the traffic system as $\zeta = \frac{1}{2}\sqrt{\theta/\eta}$. Since the synonymous damping ratio is determined by θ/η, then by choosing appropriate values of θ and η in Table 2, we can investigate the effect of the θ/η ratio on the system's damping behavior. Flow trajectories on Path 1 are shown in Figure 6, and we adjust the horizontal axes on purpose to exhibit the shapes of the trajectories. It is clearly shown that, shapes of the trajectories with a same value of θ/η (marked by the same color and line style) are quite similar, but are quite different when θ/η are different. It provides some clue that the ratio of θ/η may determine the oscillation pattern of the system. However, whether the critical damping occurs at $\theta/\eta = 4$, i.e., $\zeta = 1$, is still unclear.

Table 2. Values of θ and η

θ/η		η				
		1/4	1/2	1	2	4
θ	1/4	1	1/2	1/4	1/2	1
	1/2	2	1	1/2	1/4	1/2
	1	4	2	1	1/2	1/4
	2	8	4	2	1	1/2
	4	16	8	4	2	1

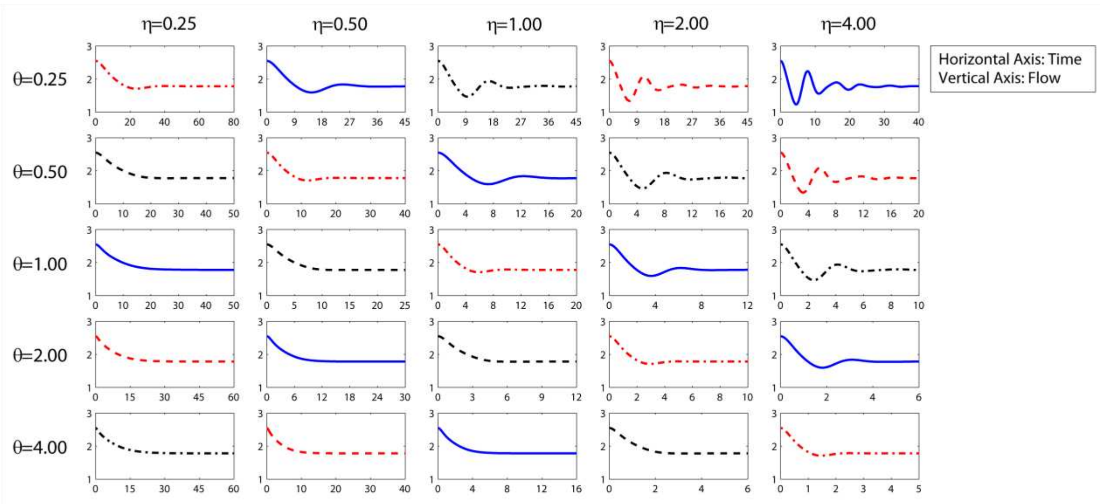

Figure 6. Evolution of flow on Path 1 with different θ and η

5.6. Synonymous angular frequency

Similar to the physical system, we define the synonymous undamped angular frequency $\omega_0 = \sqrt{\eta\theta}$ and the synonymous angular frequency $\omega_1 = \omega_0\sqrt{1-\zeta^2} = \left(1 - \frac{1}{4}\frac{\theta}{\eta}\right)\sqrt{\eta\theta}$. To examine whether the angular frequency of the flow dynamics is really related to ω_1, we first choose some combinations of θ and ω_1, as listed in Table 3, and work out the values of η by

$$\eta = \left(\frac{\omega_1\sqrt{\theta} + \sqrt{\omega_1^2\theta + \theta^2}}{2\theta}\right)^2$$

then the θ and η listed in Table 3 are employed to simulate the evolution process. Trajectories of flow on Path 1 are displayed in Figure 7. The oscillating period decreases when ω_1 increases, but almost remains the same when θ changes. When ω_1 is large enough (e.g., 1.0 or above), doubling ω_1 will reduce the oscillating period by almost half. Nonetheless, the oscillating period is not consistent with the corresponding angular frequency, i.e., their product is not precisely equal to 2π but a little bit larger than that.

Table 3. Values of θ, ω_1 and η

η		ω_1			
		0.5	1.0	2.0	4.0
θ	0.2	1.71	5.49	20.50	80.50
	0.5	0.93	2.47	8.49	32.50
	1.0	0.65	1.46	4.49	16.50
	2.0	0.50	0.93	2.47	8.49

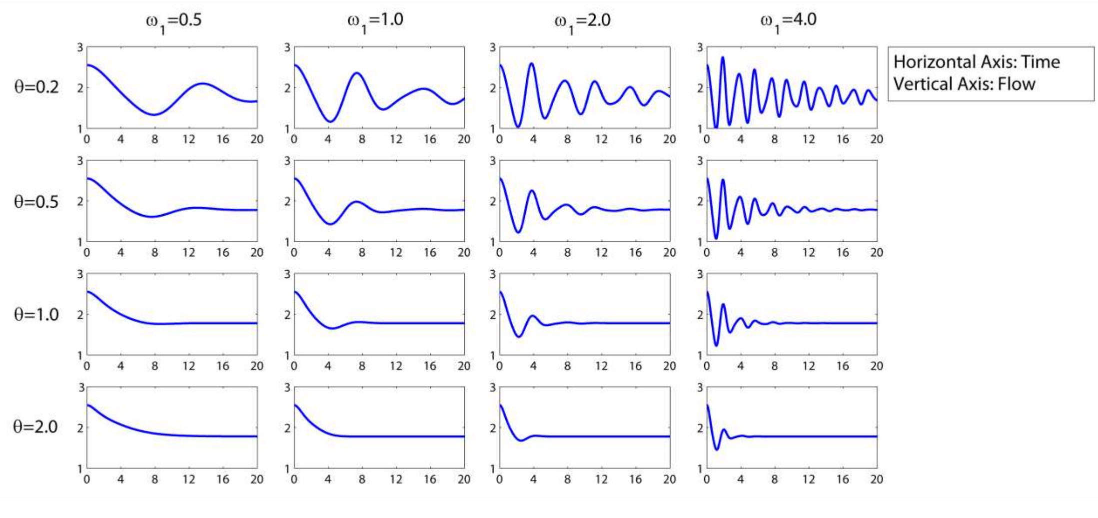

Figure 7. Trajectories of flow on Path 1 with different θ and ω_1

The experimental results in Sections 5.5 and 5.6 indicate that, although the dynamical traffic network is much more complicated than the harmonic oscillator, they do possess a lot of similarity, including the angular frequency and damping ratio.

5.7. Flow evolution under different demand levels

In this part, we investigate the influence of different demand levels on the behavior of the dynamical network. In Figure 8, we look at the trajectories of flow on Path 1 with fixed θ and η under two different demand levels, which are 9 and 18, respectively.

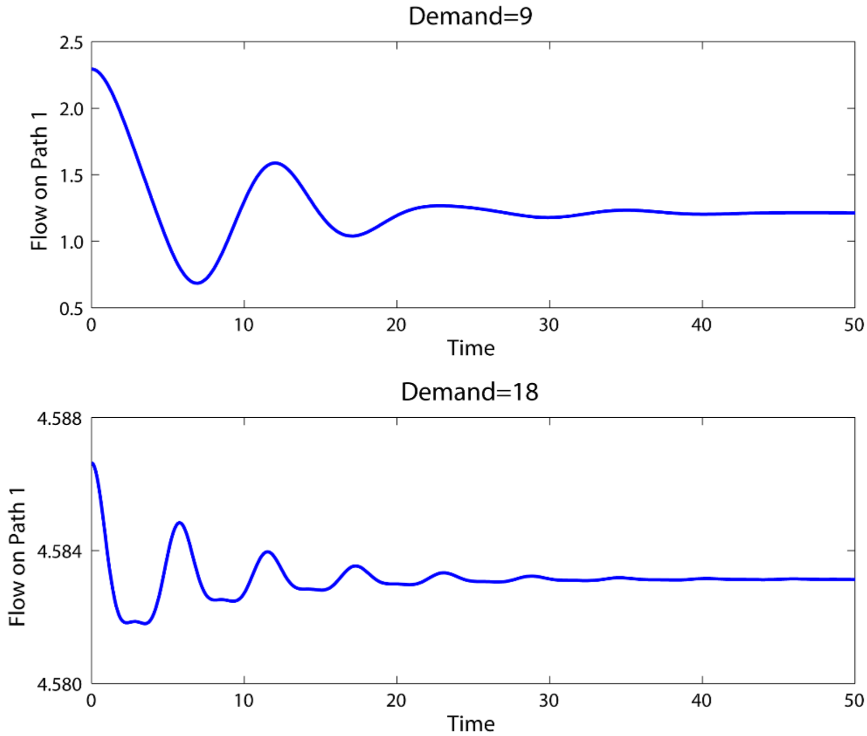

Figure 8. Flow evolution on Path 1 under different demands.

We have two observations in this case. First, the demand would not affect the damping ratio (reflected by the shape and convergence speed of the flow trajectories). Second, increasing demand will increase the damping frequency. In this example, doubling the demand (from 9 to 18) approximately doubles the frequency.

The reason why the demand is related to frequency can be found by looking at the “restoring force”. Since we assume that the link performance function is convex, higher demand leads to higher congestion level, and under this circumstance, a certain amount of flow change will cause higher “restoring force”. By analogy to the harmonic oscillatory system, the damping frequency should increase.

5.8. Evolution of system energies

Given different combinations of θ and η, Figure 9 shows the change of system energies with time, including the kinetic energy, the relative potential energy and the relative total energy. The relative potential energy is defined as the potential energy deducted by the minimum potential energy of the system (i.e., the potential energy of the system at UE). The relative total energy is the sum of kinetic energy and relative potential energy.

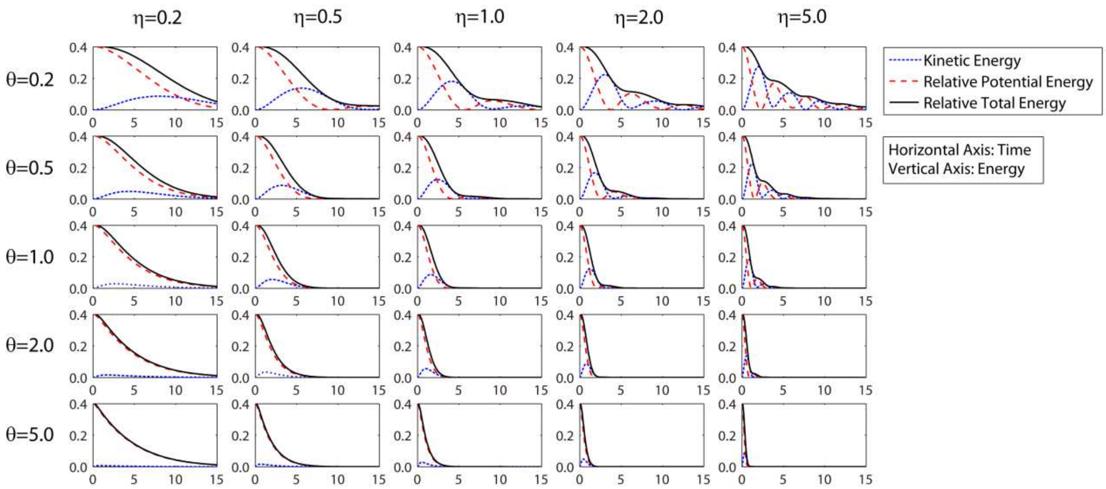

Figure 9. Change of the system energies with different θ and η.

As clearly shown in Figure 9, kinetic energy and potential energy fluctuate and convert to each other during the system evolution. However, the total energy is always decreasing with time: the kinetic energy finally reaches zero and the potential energy finally approaches minimum, which are consistent with the analytical results in Section 4.

6. Conclusions and future research

In this paper, we established the second-order day-to-day dynamics of network traffic flows by considering travelers' learning and route swapping behavior. Travelers' learning behavior is described as their updating behavior regarding the perceived path travel costs, based on the prior perception and real travel costs. Route swapping rate on a certain path is assumed to be related to the perceived travel costs on both this path and all other paths in the corresponding OD pair. Stationary states of our day-to-day model coincide with the UE path flow patterns, which were proven to be stable by LaSalle's theorem. The numerical experiment uncovered some interesting findings, by characterizing our day-to-day model from the perspective of damped oscillatory systems. When travelers are more sensitive to the travel costs, the system converges more slowly. When travelers rely too much on the past information, the system would fall into a long-time oscillation and converge slowly as well. The system is similar to the harmonic oscillator in some important features such as damping ratio, angular frequency and the obedience to the Minimum total potential energy principle.

Two critical parameters θ and η of the day-to-day model determine its oscillation pattern. However, calibrating these parameters in reality might be difficult, since travelers' perception usually cannot be observed or measured directly. Under this circumstance, the concept of potential energy and kinetic energy proposed in this study could be helpful. As we can see from Figure 9 and Figure 10, the combination of θ and η uniquely determines the oscillation pattern of the energies. This property might be useful for calibrating the day-to-day model by investigating the energy evolution process in a practical traffic network, as long as the energy is defined properly.

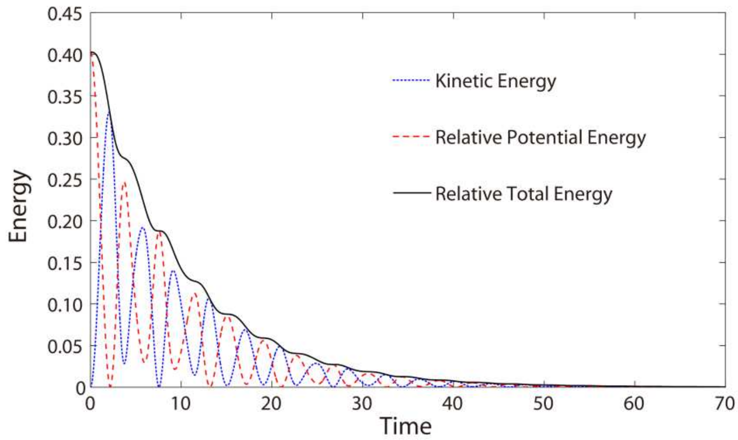

Figure 10. The unique oscillation pattern of energies with $\theta = 0.1$ and $\eta = 10$.

This study builds a bridge from the transportation system to the physical system in day-to-day traffic analysis. It shows that those physical laws that govern the physical world may also apply for human's aggregate social behavior. Through this bridge are a number of potentially valuable avenues that can be explored for further study, such as the empirical analysis of the proposed model, as well as the investigation of system behavior under traffic control, tolling and information provision. In addition, the implication of "mass" in the traffic system requires further discussion.

Acknowledgements

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Appendix A. Equivalence between swapping rule in Eq. (9) and the network tatonnement process (NTP) in Friesz et al. (1994)

By Eq. (39), a discrete NTP based on perceived costs can be constructed as follows,

$$\mathbf{f}^{n+1} - \mathbf{f}^{n} = \alpha[P_{\mathcal{F}}(\mathbf{f}^{n} - \beta\mathbf{C}^{n}) - \mathbf{f}^{n}],\ \alpha > 0,\ \beta > 0 \tag{40}$$

where $\mathbf{C} = (C_{rw}, r \in R_w, w \in W)^{\mathrm{T}}$, and x^n denotes the value of variable x on the discrete time step n. Define $\mathbf{h} = P_{\mathcal{F}}(\mathbf{f}^{n} - \beta\mathbf{C}^{n})$. To solve

$$\min_{\boldsymbol{h}\in\mathcal{F}} \|\mathbf{f}^{n} - \beta\mathbf{C}^{n} - \mathbf{h}\|^2 \tag{41}$$

we write the Lagrangian

$$L = \sum_{w\in W}\sum_{r\in R_w}(f_{rw}^{n} - \beta C_{rw}^{n} - h_{rw})^2 + \sum_{w\in W}\mu_w\left(d_w - \sum_{r\in R_w} h_{rw}\right) \tag{42}$$

and then have the following first-order condition,

$$h_{rw}\frac{\partial L}{\partial h_{rw}} = h_{rw}[2(h_{rw} - f_{rw}^{n} + \beta C_{rw}^{n}) - \mu_w] = 0 \tag{43}$$

Assume that $\mathbf{h} > \mathbf{0}$ always hold, then we have $2(h_{rw} - f_{rw}^{n} + \beta C_{rw}^{n}) - \mu_w = 0$ for all $r \in R_w,\ w \in W$. So $\mu_w = \frac{2\beta}{|R_w|}\sum_{s\in R_w} C_{sw}^{n}$ and $h_{rw} = \frac{\beta}{|R_w|}\sum_{s\in R_w} C_{sw}^{n} + f_{rw}^{n} - \beta C_{rw}^{n}$. Thus if β is small enough, then $\mathbf{h} > \mathbf{0}$ can always hold. Further we have

$$f_{rw}^{n+1} - f_{rw}^{n} = \alpha(h_{rw} - f_{rw}^{n}) = \frac{\alpha\beta}{|R_w|}\sum_{s\in R_w}(C_{sw}^{n} - C_{rw}^{n}) \tag{44}$$

So the discrete NTP degenerates to the swapping rule in Eq. (9) when the projection always yields positive path flows.

Appendix B. Equivalence between swapping rule in Eq. (9) and the projected dynamical system (PDS) in Nagurney and Zhang (1997)

A discrete-time PDS with strictly monotone travel cost functions was presented in Nagurney and Zhang (1997) as follows,

$$\mathbf{f}_w^{n+1} = \arg\min_{\mathbf{f}_w \in K_w} \frac{1}{2} \sum_{r \in R_w} (f_{rw})^2 + \sum_{r \in R_w} (aC_{rw}^n - f_{rw}^n) f_{rw} \tag{45}$$

where $\mathbf{f}_w = (f_{rw}, r \in R_w)^{\mathrm{T}}$, $K_w = \{\mathbf{f}_w | \sum_{r \in R_w} f_{rw} = d_w\}$ and a is a fixed step size. Lagrangian of the minimization problem in Eq. (45) can be written as

$$L(\mathbf{f}_w, u_w) = \frac{1}{2} \sum_{r \in R_w} (f_{rw})^2 + \sum_{r \in R_w} (aC_{rw}^n - f_{rw}^n) f_{rw} + u_w \left(d_w - \sum_{r \in R_w} f_{rw} \right) \tag{46}$$

from which we have

$$\frac{\partial L(\mathbf{f}_w^{n+1}, u_w)}{\partial f_{rw}^{n+1}} = aC_{rw}^n - f_{rw}^n + f_{rw}^{n+1} - u_w$$

With Assumption 1, when $f_{rw}^{n+1} > 0$ for all $r \in R_w$, $w \in W$, we have

$$aC_{rw}^n - f_{rw}^n + f_{rw}^{n+1} - u_w = 0$$

which leads to

$$u_w = \frac{1}{|R_w|} \sum_{i \in R_w} (aC_{iw}^n - f_{iw}^n + f_{iw}^{n+1}) = \frac{a}{|R_w|} \sum_{i \in R_w} C_{iw}^n$$

and thus

$$f_{rw}^{n+1} - f_{rw}^n = u_w - aC_{rw}^n = \frac{a}{|R_w|} \sum_{i \in R_w} (C_{iw}^n - C_{rw}^n) \tag{47}$$

Clearly Eq. (47) follows the same form as that of Eq. (9), thus the two adjustment processes are equivalent. ■

Appendix C. Derivation of the second-order ODE form of the proposed day-to-day dynamics

Substituting Eq. (5) into Eq. (9) yields

$$\begin{aligned} &\frac{f_{rw}(t) - f_{rw}(t - \delta t)}{\delta t} \\ &= -\eta_w \sum_{i \in R_w} \big(C_{rw}(t) - C_{iw}(t) \big) \\ &= -\eta_w \sum_{i \in R_w} \Big(\big(\varphi_w(\delta t) c_{rw}(t - \delta t) + \big(1 - \varphi_w(\delta t)\big) C_{rw}(t - \delta t) \big) \\ &\qquad - \big(\varphi_w(\delta t) c_{iw}(t - \delta t) + \big(1 - \varphi_w(\delta t)\big) C_{iw}(t - \delta t) \big) \Big) \end{aligned} \tag{48}$$

$$= -\eta_w \varphi_w(\delta t) \sum_{i \in R_w} \left(c_{rw}(t - \delta t) - c_{iw}(t - \delta t) \right)$$

$$-\eta_w \left(1 - \varphi_w(\delta t)\right) \sum_{i \in R_w} \left(C_{rw}(t - \delta t) - C_{iw}(t - \delta t) \right)$$

$$= -\eta_w \varphi_w(\delta t) \sum_{i \in R_w} \left(c_{rw}(t - \delta t) - c_{iw}(t - \delta t) \right)$$

$$+\left(1 - \varphi_w(\delta t)\right) \frac{f_{rw}(t - \delta t) - f_{rw}(t - 2\delta t)}{\delta t}$$

Subtracting $\left(1 - \varphi_w(\delta t)\right) \frac{f_{rw}(t) - f_{rw}(t - \delta t)}{\delta t}$ from Eq. (48) and then dividing it by $\varphi_w(\delta t)$, we have

$$\frac{f_{rw}(t) - f_{rw}(t - \delta t)}{\delta t}$$

$$= -\eta_w \sum_{i \in R_w} \left(c_{rw}(t - \delta t) - c_{iw}(t - \delta t) \right) \tag{49}$$

$$-\frac{\delta t\left(1 - \varphi_w(\delta t)\right)}{\varphi_w(\delta t)} \frac{\left(f_{rw}(t) - f_{rw}(t - \delta t)\right) - \left(f_{rw}(t - \delta t) - f_{rw}(t - 2\delta t)\right)}{(\delta t)^2}$$

If $\varphi_w(\delta t)$ is differentiable at $\delta t = 0$, and $\dot{\varphi}_w(0) > 0$, then by l'Hospital's rule, we have

$$\lim_{\delta t \to 0} \frac{\delta t\left(1 - \varphi_w(\delta t)\right)}{\varphi_w(\delta t)} = \lim_{\delta t \to 0} \frac{1 - \varphi_w(\delta t) - \delta t \dot{\varphi}_w(\delta t)}{\dot{\varphi}_w(\delta t)} = \frac{1}{\dot{\varphi}_w(0)} \tag{50}$$

Denote $\theta_w = \dot{\varphi}_w(0)$, then let $\delta t \to 0$ in Eq. (49) and we can obtain the following continuous form,

$$\frac{1}{\eta_w \theta_w} \ddot{f}_{rw} + \frac{1}{\eta_w} \dot{f}_{rw} - \sum_{i \in R_w} (c_{iw} - c_{rw}) = 0, \theta_w > 0, \eta_w > 0 \tag{51}$$

or equivalently,

$$\frac{1}{\eta_w \theta_w |R_w|} \ddot{f}_{rw} + \frac{1}{\eta_w |R_w|} \dot{f}_{rw} - \frac{1}{|R_w|} \sum_{i \in R_w} (c_{iw} - c_{rw}) = 0, \theta_w > 0, \eta_w > 0 \tag{52}$$