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# THE DUFFIN-SCHAEFFER CONJECTURE WITH EXTRA DIVERGENCE II

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ABSTRACT. In [7] the authors set out a programme to prove the Duffin-Schaeffer Conjecture for measures arbitrarily close to Lebesgue measure. In this paper we take a new step in this direction. Given a nonnegative function  $\psi : \mathbb{N} \rightarrow \mathbb{R}$ , let  $W(\psi)$  denote the set of real numbers  $x$  such that  $|nx - a| < \psi(n)$  for infinitely many reduced rationals  $a/n$  ( $n > 0$ ). Our main result is that  $W(\psi)$  is of full Lebesgue measure if there exists a  $c > 0$  such that

$$\sum_{n \geq 16} \frac{\varphi(n)\psi(n)}{n \exp(c(\log \log n)(\log \log \log n))} = \infty.$$

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## 1. INTRODUCTION

We use the following notation:  $p$  denotes a prime number,  $\varphi(n)$  is the Euler phi function,  $\lambda$  denotes Lebesgue measure on  $\mathbb{R}/\mathbb{Z}$ , and  $f \ll g$  means that the absolute value of  $f$  is bounded above by a constant times the absolute value of  $g$ .

Let  $\psi : \mathbb{N} \rightarrow \mathbb{R}$  be a non-negative arithmetical function and for each positive integer  $n$  define  $\mathcal{E}_n \subseteq \mathbb{R}/\mathbb{Z}$  by

$$\mathcal{E}_n := \bigcup_{\substack{a=1 \\ \gcd(a,n)=1}}^n \left( \frac{a - \psi(n)}{n}, \frac{a + \psi(n)}{n} \right).$$

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Denote the collection of points  $x \in \mathbb{R}/\mathbb{Z}$  which fall in infinitely many of the sets  $\mathcal{E}_n$  by  $W(\psi)$ . In other words,

$$W(\psi) := \limsup_{n \rightarrow \infty} \mathcal{E}_n := \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} \mathcal{E}_n .$$

The question we address is:

**Question 1.** *Let  $\psi$  be any non-negative arithmetical function. Under what circumstances is it true that  $\lambda(W(\psi)) = 1$ ?*

It is very easy to give a *necessary* condition for this to happen, namely the divergence of the series:

$$(1) \quad \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \psi(n).$$

This follows from the Borel-Cantelli Lemma, since

$$\lambda(\mathcal{E}_n) \leq 2 \frac{\varphi(n)}{n} \psi(n)$$

and so the convergence of (1) implies that  $\lambda(W(\psi)) = 0$ . It is a central open problem in metric number theory to show that the divergence of (1) is actually *sufficient* to conclude that  $\lambda(W(\psi)) = 1$ .

**Conjecture** (Duffin-Schaeffer 1941). *We have that  $\lambda(W(\psi)) = 1$  if and only if (1) diverges.*

There are several significant partial results towards this conjecture, most notably those due to Khintchine, Duffin & Schaeffer, Erdős, Vaaler, and Pollington & Vaughan [8, 2, 3, 11, 9]. The proofs of these results and others are all given in [6, Chps 2 & 3]. Recently Pollington and two of this paper's authors [7] have considered the effect on the problem of assuming “extra divergence”. They have posed the following question.

**Question 2.** *For what functions  $f$  does the divergence of*

$$(2) \quad \sum_{n=1}^{\infty} f\left(\frac{\psi(n)}{n}\right) \varphi(n)$$

*guarantee that  $\lambda(W(\psi)) = 1$ ?*

In view of the Mass Transference Principle [1] this question is equivalent to investigating the Duffin-Schaeffer Conjecture for (Hausdorff) measures “arbitrarily” close to Lebesgue measure – see [7, §5] for the details. Regarding Question 2 itself, the following result is established in [7].

**Theorem HPV.** *Let  $\psi$  be any non-negative arithmetical function and define the function  $f$  by*

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x \exp\left(\frac{\log x}{\log(-\log x)}\right) & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

*Then the divergence of (2) is sufficient to conclude that  $\lambda(W(\psi)) = 1$ .*

The authors of [7] set as an explicit unsolved problem the task of replacing  $f(x)$  above by  $f(x) = x(-\log x)^{-1}$ . The hope is that as one approaches the situation of the Duffin-Schaeffer Conjecture (that is,  $f(x) = x$ ) one can see more clearly the outstanding problems. In addition any subsequent attack on the conjecture can assume that the series (1) is diverging “slowly” in certain senses. In this paper we make progress towards this goal by establishing the following result.

**Theorem 1.** *Let  $\psi$  be any non-negative arithmetical function and for any  $c > 0$ , define the function  $f_c$  by*

$$f_c(x) = \begin{cases} 0 & \text{if } x = 0, \\ x \exp(-c(\log(-\log x))(\log \log(-\log x))) & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

*Then the divergence of (2) with  $f = f_c$  is sufficient to conclude that  $\lambda(W(\psi)) = 1$ .*

It is worth pointing out that this strengthening of Theorem HPV is not a consequence of simply tweaking the approach taken in [7] – see also the remark at the end of §3 in [7]. By appealing to the Erdős-Vaaler Theorem [11] and to [9, Theorem 2] we can assume without loss of generality throughout the proof that  $1/n \leq \psi(n) \leq 1/2$  whenever  $\psi(n) \neq 0$ . In view of this it suffices to prove the following theorem.

**Theorem 2.** *The Duffin-Schaeffer Conjecture is true for any non-negative arithmetical function  $\psi$  such that the series*

$$(3) \quad \sum_{n=16}^{\infty} \frac{\varphi(n)\psi(n)}{n \exp(c(\log \log n)(\log \log \log n))}$$

*diverges for some  $c > 0$ .*

## 2. THE BASIC FRAMEWORK

Gallagher [4] (see also [6, §2.2]) proved that there is a “zero-one” law for Question 1. That is, for any given function  $\psi$  we have that  $\lambda(W(\psi)) = 0$  or 1. We therefore only need to prove that under our extra divergence hypothesis  $\lambda(W(\psi)) > 0$ . To do this we need the following result [6, Lemma 2.3] whose proof involves little more than the correct application of the Cauchy-Schwartz inequality.

**Lemma 1.** *Let  $\mathcal{A}_n$  be a sequence of Lebesgue measurable subsets of  $\mathbb{R}/\mathbb{Z}$ . Let  $\mathcal{A}$  be the set of  $\alpha$  belonging to infinitely many  $\mathcal{A}_n$ . Then*

$$(4) \quad \lambda(\mathcal{A}) \geq \limsup_{N \rightarrow \infty} \left( \sum_{n=1}^N \lambda(\mathcal{A}_n) \right)^2 \left( \sum_{m,n=1}^N \lambda(\mathcal{A}_m \cap \mathcal{A}_n) \right)^{-1}.$$

The well known Duffin-Schaeffer result [6, Theorem 2.5] toward the Duffin-Schaeffer Conjecture follows from this lemma together with the elementary bound

$$(5) \quad \lambda(\mathcal{E}_m \cap \mathcal{E}_n) \leq 8\psi(n)\psi(m) \quad \text{for } m \neq n.$$

However if the sets in the collection  $\{\mathcal{E}_n\}$  were quasi-pairwise independent, i.e. if

$$(6) \quad \lambda(\mathcal{E}_m \cap \mathcal{E}_n) \ll \psi(n)\psi(m) \frac{\varphi(m)}{m} \frac{\varphi(n)}{n} \quad \text{for } m \neq n,$$

then the Duffin-Schaeffer Conjecture would follow at once from (4) together with Gallagher's zero-one law. Unfortunately (6) does not hold uniformly for all  $m \neq n$ . The best known upper bound for  $\lambda(\mathcal{E}_n \cap \mathcal{E}_m)$  is the following result.

**Lemma 2.** *For  $m \neq n$  we have*

$$(7) \quad \lambda(\mathcal{E}_m \cap \mathcal{E}_n) \ll \lambda(\mathcal{E}_m)\lambda(\mathcal{E}_n)P(m, n),$$

where

$$(8) \quad P(m, n) = \prod_{\substack{p|mn/\gcd(m,n)^2 \\ p > D(m,n)}} \left(1 - \frac{1}{p}\right)^{-1},$$

with

$$(9) \quad D(m, n) = \frac{\max(n\psi(m), m\psi(n))}{\gcd(m, n)}.$$

This was first stated by Strauch [10], but was also given independently by Pollington & Vaughan [9]. The proof is still essentially elementary, but fairly complicated, and needing a simple sieve upper bound. Effectively the same result was given earlier by Erdős [3]. Clearly what needs to be done in applying Lemma 1 is to show that the factor  $P(m, n)$  in Lemma 2 is bounded on average.

It is worth pausing here to see what the real difficulties are in estimating  $\lambda(\mathcal{E}_m \cap \mathcal{E}_n)$ . Two intervals from  $\mathcal{E}_m$  and  $\mathcal{E}_n$  overlap if

$$\left| \frac{a}{m} - \frac{b}{n} \right| < \frac{\psi(m)}{m} + \frac{\psi(n)}{n}.$$

We lose nothing in terms of the order of magnitude of the bound in replacing this with

$$(10) \quad |an - bm| < A(m, n) := 2 \max(m\psi(n), n\psi(m)).$$

The length of the intersection is no more than the smallest length of the two intervals (again nothing is lost in order of magnitude in making this assumption). We thus have

$$\lambda(\mathcal{E}_m \cap \mathcal{E}_n) \leq 2 \min \left( \frac{\psi(m)}{m}, \frac{\psi(n)}{n} \right) \Sigma(m, n),$$

where  $\Sigma(m, n)$  denotes the number of integer solutions to (10) with

$$1 \leq a < m, \quad 1 \leq b < n, \quad \gcd(a, m) = 1, \quad \gcd(b, n) = 1.$$

We thus need to show that, at least on average over  $m, n$ , we have

$$\Sigma(m, n) \ll A(m, n) \frac{\varphi(m)}{m} \frac{\varphi(n)}{n}.$$

Now if  $\gcd(m, n) = 1$  and  $A(m, n)$  is not too small there is no problem with this. The trouble essentially comes when

$$1 \leq \frac{A(m, n)}{\gcd(m, n)} < \log(mn).$$

In that case we are not averaging over enough values of  $h$  in the equation  $an - bm = h$  to get the required bound. This is a *real* problem, not just a deficiency in our knowledge. Our hope would be that the values of  $m$  and  $n$  concerned do not make the major contribution to

$$\sum_{1 \leq m, n \leq N} \lambda(\mathcal{E}_m \cap \mathcal{E}_n).$$

### 3. PROOF OF THEOREM 2

Our proof is in the same spirit as that of [7, Theorem 1], however our approach here is more direct. The proof of [7, Theorem 1] was divided into two steps: (i) dealing with the case when the function  $\psi$  is constant on large intervals in its support, and (ii) utilizing the extra divergence hypothesis to reduce to this case. Our proof here avoids this technical digression and furthermore makes use of a new idea that was not used before and allows us to obtain the stronger result announced in the introduction.

The crucial new ingredient can be described as follows. When we have a collection of integers with a large number of pairs with gcd's falling into the bad intervals (i.e. the intervals which make  $P(m, n)$  large), we can try dividing all of the corresponding values of  $\psi(n)$  by some constant amount in order to shift the intervals. The extra divergence condition gives us enough room to do this in a way that we control the contribution from the  $P(m, n)$  terms, while maintaining the hypothesis that the sum of the measures of our sets diverges.

We proceed to the details of the proof. Without loss of generality, assume that  $\psi(n) \geq n^{-1}$  whenever  $\psi(n) \neq 0$  – see the discussion just before the statement of the theorem. We divide the integers  $n$  into blocks

$$2^{4^h} \leq n < 2^{4^{h+1}}, \quad h \in \mathbb{Z}.$$

It then follows that series (3) diverges with  $n$  restricted to blocks with either all the  $h$  even, or all the  $h$  odd. Without loss of generality we suppose the series diverges over blocks with  $h$  even, and that  $\psi(n) = 0$  for all integers  $n$  which lie in blocks with  $h$  odd. We then note that if  $m < n$ , if  $\psi(m), \psi(n) > 0$ , and if  $m$  and  $n$  are in different blocks then

$$A(m, n) \geq 2n\psi(m) \geq 2nm^{-1} \geq 2n \gcd(m, n)m^{-2} \gg \gcd(m, n)(\log nm).$$

Hence  $P(m, n) \ll 1$  if  $m$  and  $n$  belong to different blocks.

Now we consider a block  $2^{4h} = X \leq m, n < X^4$ . Write  $R = \log \log X$  and

$$\Psi(X) = \sum_{X \leq m, n < X^4} \psi(n)\psi(m) \frac{\varphi(m)}{m} \frac{\varphi(n)}{n}.$$

By one of Mertens' theorems [5, Theorem 328] we have for  $D(m, n) \geq 1$  that

$$P(m, n) \ll \exp \left( \sum_{D(m, n) < p < \log X} \frac{1}{p} \right) \ll \frac{R}{1 + \log D(m, n)}.$$

We let  $\mathcal{D}_j$  be the collection of pairs  $(m, n)$  such that  $e^j \leq D(m, n) < e^{j+1}$ . The idea is going to be to divide each  $\psi(n)$  by a suitable factor (say  $e^k$ ) so that the contribution from  $R$  consecutive ranges for which  $P(m, n) \gg 1$  is not of a larger magnitude than the expected overall contribution. Specifically, we have

$$\sum_{k \leq K} \sum_{k \leq j \leq k+R} \frac{R}{j+1-k} \sum_{(m, n) \in \mathcal{D}_j} \psi(n)\psi(m) \frac{\varphi(m)}{m} \frac{\varphi(n)}{n} \ll R\Psi(X) \log K,$$

since each set  $\mathcal{D}_j$  is counted with weight

$$\leq \sum_{k \leq \min(j, K)} \frac{R}{j+1-k} \ll R \log K.$$

We can therefore choose an integer  $k \leq cR \log R$  such that

$$\sum_{k \leq j \leq k+R} \frac{R}{j+1-k} \sum_{(m, n) \in \mathcal{D}_j} \psi(n)\psi(m) \frac{\varphi(m)}{m} \frac{\varphi(n)}{n} \ll \Psi(X).$$

With this choice of  $k$  write

$$\mathcal{E} = \bigcup_{k \leq j \leq k+R} \mathcal{D}_j.$$

Note that

$$e^k \leq \exp(c(\log \log X)(\log \log \log X)).$$

Now for positive integers  $n$  put

$$\rho(n) = \begin{cases} \psi(n)e^{-k} & \text{if } X \leq n < X^4 \text{ with } X = 2^{4h} \text{ for some even } h, \\ 0 & \text{otherwise,} \end{cases}$$

where  $k = k(X)$  is as above. We now consider the sets  $\mathcal{E}_n(\rho)$ . By construction, we have that

$$\sum_{n=1}^{\infty} \frac{\rho(n)\varphi(n)}{n} = \infty,$$

and

$$\lambda(\mathcal{E}_m(\rho) \cap \mathcal{E}_n(\rho)) \ll \lambda(\mathcal{E}_m(\rho))\lambda(\mathcal{E}_n(\rho))$$

unless  $(m, n) \in \mathcal{E}$ . But now we also have that

$$\sum_{(m,n) \in \mathcal{E}} \rho(m)\rho(n) \frac{\varphi(n)}{n} \frac{\varphi(m)}{m} P(m, n) \ll \sum_{X \leq m, n < X^4} \rho(m)\rho(n) \frac{\varphi(n)}{n} \frac{\varphi(m)}{m}$$

(we note that  $P(m, n)$  here does depend on  $\rho$ ) and so

$$\sum_{m,n=1}^N \lambda(\mathcal{E}_m(\rho) \cap \mathcal{E}_n(\rho)) \ll \left( \sum_{n=1}^N \lambda(\mathcal{E}_n(\rho)) \right)^2$$

for  $N$  taking the values  $2^{4^{h+1}}$ . By Lemma 1 and Gallagher's zero-one law we have that  $\lambda(W(\rho)) = 1$ , and since  $W(\rho) \subseteq W(\psi)$  the proof is completed.

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