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On the charge density and asymptotic tail of a monopole

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Abstract

We propose a new definition for the abelian magnetic charge density of a non-abelian monopole, based on zero-modes of an associated Dirac operator. Unlike the standard definition of the charge density, this density is smooth in the core of the monopole. We show that this charge density induces a magnetic field whose expansion in powers of $1/r$ agrees with that of the conventional asymptotic magnetic field to all orders. We also show that the asymptotic field can be easily calculated from the spectral curve. Explicit examples are given for known monopole solutions.

1 Introduction

Non-abelian monopoles are smooth, static, finite-energy solutions to the Yang-Mills-Higgs equations with non-abelian gauge group. It was first noticed by 't Hooft and Polyakov that to a distant observer they resemble Dirac monopoles in an abelian gauge theory [1, 2]. Thus the singularity of the Dirac monopole can be smoothed out in non-abelian gauge theory.

't Hooft defined [1] an asymptotic abelian magnetic field of a non-abelian monopole with gauge group SU(2). The flux of this magnetic field through the two-sphere at infinity is topologically quantised and non-zero. However, in the core of the monopole this magnetic field is singular [3], and the magnetic charge distribution which induces it typically has delta-function singularities. Thus, while a non-abelian monopole is smooth, the magnetic charge distribution associated to 't Hooft's magnetic field is far from being smooth.

As has been argued by Coleman [4], there is no reason to expect the abelian magnetic field of a non-abelian monopole to be uniquely defined: any magnetic field which agrees with 't Hooft's asymptotically is an equally viable candidate. However, to date no definition of a magnetic field has been proposed which smoothes out the singularities in 't Hooft's field. In this article we remedy this situation: we propose a novel definition of the magnetic charge density of a non-abelian monopole which, unlike 't Hooft's charge density, is smooth. Moreover, we show that the magnetic field induced by this charge density agrees with 't Hooft's asymptotically, at least in the case of BPS monopoles. Our charge density is evaluated by summing the squared norms of zero-modes of a Dirac operator. In this way it resembles the trace of the Bergman kernel used in Kähler geometry, which is a sum of squared norms of zero-modes of a Cauchy-Riemann operator.

The proof that our charge density induces the correct asymptotic magnetic field is based on much of the mathematical formalism that has been developed to study BPS monopoles, including the Nahm transform and spectral curves. A prominent role is played by a function which we call the *tail* of the monopole. This function describes the asymptotics of the Higgs field and was first studied by Hurtubise [5]. The tail function seems not to have since been studied in any great detail, but we feel that it merits more attention; in particular we will show below that it is in many cases relatively easy to evaluate in explicit form, and seems to capture a lot of the structure of the monopole. Another interesting consequence of our work is a proof of a conjecture [6, 7] relating conserved charges of the Nahm equation to asymptotics of an associated Greens' function.

This paper is structured as follows. In section 2 we review standard results relating moments of electric (or magnetic) charge distributions to the asymptotics of the electric (or magnetic) fields that they induce. In section 3 we introduce our charge density and show that its integral agrees with the magnetic flux through the two-sphere at infinity. The proof that the magnetic field induced by this charge density agrees asymptotically with 't Hooft's proceeds in two parts: in section 4 we show that the moments of the charge density equal certain conserved quantities of the Nahm equation, and in section 5 we show using Hurtubise' work on the tail function that these conserved quantities also prescribe the asymptotic expansion of 't Hooft's magnetic field. Section 6 is devoted to the study not of charge densities but instead of monopole asymptotics: in it we present some explicit formulae for the asymptotic fields of monopoles, calculated using the tail function. We discuss promising extensions of this work in section 7.

2 Moments and charge distributions

It is well-known that the moments of a distribution of electric charge determine the asymptotic expansion of the induced electric field [8]. More precisely, let $\rho : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function that decays exponentially as $r \to \infty$ and let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a potential of the induced electric field $e_i = -\partial_i \phi$. The functions ρ and ϕ are related by

$$\rho = \partial_i e_i = -\triangle \phi.$$

Suppose that ϕ has an expansion in powers of $1/r$ of the form

$$\phi = \sum_{\ell=0}^{\infty} \frac{\phi_\ell(\theta, \varphi)}{r^{l+1}}. \tag{1}$$

The functions ϕ_ℓ must then be spherical harmonics of weight ℓ, as $\triangle\phi$ decays exponentially. The expansion (1) is called the multipole expansion of ϕ.

Write

$$x(\zeta) = \frac{1}{2}(x_2 + \mathrm{i}x_3) + \zeta x_1 - \frac{1}{2}(x_2 - \mathrm{i}x_3)\zeta^2,$$

and for each $\ell = 0, 1, 2, \ldots$ let $Q_\ell(\zeta)$ be the polynomial

$$Q_\ell(\zeta) = \int_{\mathbb{R}^3} \rho(\mathbf{x}) x(\zeta)^\ell \mathrm{d}^3 x. \tag{2}$$

The $2l+1$ coefficients of $Q_\ell(\zeta)$ are moments of the distribution ρ. They determine, and are determined by, the spherical harmonics ϕ_ℓ. More precisely:

Proposition 1. *Let ρ be an exponentially decaying function and let ϕ solve $\triangle\phi = -\rho$, such that ϕ has an expansion in powers of $1/r$ of the form* (1)*. Then the coefficients of this expansion and the moments* (2) *of ρ satisfy the identities*

$$Q_\ell(\zeta) = (2\ell+1) \int_{S^2} \phi_\ell(\theta, \phi) n(\zeta)^\ell \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \quad \textit{and} \tag{3}$$

$$\phi_\ell(\theta, \varphi) = \frac{1}{8\pi^2 \mathrm{i}} \oint_\Gamma \frac{Q_\ell(\zeta)}{n(\zeta)^{\ell+1}} \mathrm{d}\zeta, \tag{4}$$

in which

$$n(\zeta) := \frac{x(\zeta)}{r} = \frac{1}{2} e^{\mathrm{i}\varphi} \sin\theta + \zeta \cos\theta - \frac{1}{2}\zeta^2 e^{-\mathrm{i}\varphi} \sin\theta$$

is a polynomial in ζ whose coefficients are spherical functions, and Γ is a small contour which circles the point

$$\zeta = -\frac{x_2 + \mathrm{i}x_3}{x_1 + r} = -\frac{e^{\mathrm{i}\varphi}\sin\theta}{\cos\theta + 1}.$$

Note that, despite this result, the spherical harmonics ϕ_ℓ do not determine the distribution ρ uniquely. This is because given any set of polynomials $Q_\ell(\zeta)$ there are infinitely many distributions ρ that have these as moments.

Although the equivalence of the moments and the spherical harmonics ϕ_ℓ is a standard result, we present a brief proof of equations (3) and (4), as our notation (in particular our choice of parameterising the moments using a polynomial) is non-standard.

Proof. The proof of (3) rests on the fact that the coefficents of $x(\zeta)^l$ solve the Laplace equation. This fact follows by induction from the following two identities, which are easily verified:

$$\begin{aligned}\partial_i\partial_i x(\zeta) &= 0\\ \partial_i x(\zeta)\partial_i x(\zeta) &= 0.\end{aligned}$$

Integrating the right hand side of (2) by parts twice and substituting the series expansion for ϕ then yields

$$\begin{aligned}Q_\ell(\zeta) &= -\lim_{R\to\infty}\int_{\|\mathbf{x}\|\le R} \triangle\phi\, x(\zeta)^\ell\, \mathrm{d}^3x\\ &= \lim_{R\to\infty}\int_{S^2_R}\left(\phi\frac{\partial(x(\zeta)^\ell)}{\partial r} - x(\zeta)^\ell\frac{\partial\phi}{\partial r}\right) r^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\\ &= \lim_{R\to\infty}\sum_{m=0}^{\infty} R^{\ell-m}(\ell+m+1)\int_{S^2}\phi_m\, n(\zeta)^\ell\, \sin\theta\mathrm{d}\theta\mathrm{d}\varphi.\end{aligned}$$

Since $\triangle(x(\zeta)^\ell) = 0$, the coefficients of $n(\zeta)^\ell$ are spherical harmonics of weight ℓ. Now spherical harmonics of different weights are L^2-orthogonal, so the integral over S^2 appearing in the preceding expression vanishes unless $\ell = m$. Thus the expression reduces to the stated result (3).

The second identity (4) follows from the first via representation theory. The space of spherical harmonics of weight ℓ and the space of degree 2ℓ polynomials both carry representations of $\mathfrak{su}(2)$: it is easily checked that the operators

$$\begin{aligned}L_\pm &= (n_2 \pm \mathrm{i}n_3)\frac{\partial}{\partial n_1} - n_1\left(\frac{\partial}{\partial n_2} \pm \mathrm{i}\frac{\partial}{\partial n_3}\right),\\ L_0 &= n_2\frac{\partial}{\partial n_3} - n_3\frac{\partial}{\partial n_2}\\ J_+^{(\ell)} &= \zeta^2\frac{\partial}{\partial\zeta} - 2\ell\zeta\\ J_-^{(\ell)} &= \frac{\partial}{\partial\zeta}\\ J_0^{(\ell)} &= \mathrm{i}\left(\zeta\frac{\partial}{\partial\zeta} - \ell\right)\end{aligned}$$

obey the $\mathfrak{su}(2)$ commutation relations,

$$\begin{aligned}[L_0, L_\pm] &= \pm\mathrm{i}L_\pm & [L_+, L_-] &= 2\mathrm{i}L_0\\ [J_0^{(\ell)}, J_\pm^{(\ell)}] &= \pm\mathrm{i}J_\pm^{(\ell)} & [J_+^{(\ell)}, J_-^{(\ell)}] &= 2\mathrm{i}J_0^{(\ell)}.\end{aligned}$$

These representations are all irreducible.

Equations (3) and (4) define maps between the spaces of spherical harmonics of weight ℓ and the space of degree 2ℓ polynomials, and we aim to show that these maps are inverse to each other. The maps respect the action of $\mathfrak{su}(2)$, in the sense that

$$J_\mu^{(\ell)} \int_{S^2} \phi_\ell(\theta,\phi) n(\zeta)^\ell \sin\theta \mathrm{d}\theta\mathrm{d}\varphi = \int_{S^2} L_\mu \phi_\ell(\theta,\phi) n(\zeta)^\ell \sin\theta \mathrm{d}\theta\mathrm{d}\varphi \tag{5}$$

$$L_\mu \frac{Q_\ell(\zeta)}{n(\zeta)^{\ell+1}}\mathrm{d}\zeta = \oint_\Gamma \frac{J_\mu^{(\ell)} Q_\ell(\zeta)}{n(\zeta)^{\ell+1}}\mathrm{d}\zeta \qquad \text{for } \mu = 0, \pm. \tag{6}$$

Their composition is a linear map from the space of spherical harmonics of weight ℓ to itself that commutes with the action of $\mathfrak{su}(2)$. By Schur's lemma, this map is equivalent to multiplication by a constant, so we only need to show that this constant is 1. We can do so by showing that the map fixes just one element.

We choose the element $\phi_\ell = (n_2 - \mathrm{i}n_3)^\ell$. According to eq. (3) the associated polynomial is

$$\begin{aligned} Q_\ell(\zeta) &= (2\ell+1) \int_{S^2} (n_2 - \mathrm{i}n_3)^\ell n(\zeta)^\ell \sin\theta \mathrm{d}\theta\mathrm{d}\phi \\ &= (2\ell+1) \int_{S^2} (n_2 - \mathrm{i}n_3)^\ell \left(\frac{n_2 + \mathrm{i}n_3}{2}\right)^\ell \sin\theta \mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{(2\ell+1)(2\pi)}{2^\ell} \int_0^\pi \sin^{2\ell+1}\theta \, \mathrm{d}\theta \\ &= \frac{2^\ell(\ell!)^2(4\pi)}{(2\ell)!}. \end{aligned}$$

We now evaluate the right hand side of eq. (4) with this particular Q_ℓ. In order to evaluate the contour integral we factorise the denominator: we find that

$$n(\zeta) = \frac{(\zeta-\zeta_-)(\zeta-\zeta_+)}{\zeta_+ - \zeta_-}, \text{ where } \zeta_\pm = -\frac{n_2 + \mathrm{i}n_3}{n_1 \pm 1},$$

and thus that

$$\begin{aligned} (n(\zeta))^{-\ell-1} &= \frac{1}{(\zeta-\zeta_+)^{\ell+1}} \left(1 - \frac{\zeta-\zeta_+}{\zeta_- - \zeta_+}\right)^{-\ell-1} \\ &= \sum_{m=0}^\infty \begin{pmatrix} \ell+m \\ \ell \end{pmatrix} \frac{(\zeta-\zeta_+)^{m-\ell-1}}{(\zeta_- - \zeta_+)^m}. \end{aligned}$$

Only the $m = \ell$ term in this Laurent series contributes to the integral (4), so

$$\begin{aligned} \frac{1}{8\pi^2\mathrm{i}} \oint_\Gamma \frac{Q_\ell(\zeta)}{n(\zeta)^{\ell+1}}\mathrm{d}\zeta &= \frac{2^\ell(\ell!)^2}{(2\ell)!}\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \begin{pmatrix} 2\ell \\ \ell \end{pmatrix} \frac{1}{(\zeta_- - \zeta_+)^\ell} \frac{1}{\zeta - \zeta_+}\mathrm{d}\zeta \\ &= 2^\ell(\zeta_- - \zeta_+)^{-\ell} \\ &= (n_2 - \mathrm{i}n_3)^\ell. \end{aligned} \tag{7}$$

This equals the original function ϕ_ℓ, so eq. (4) follows from eq. (3) as claimed. □

3 The charge density of a monopole

The Yang-Mills-Higgs energy for an $\mathfrak{su}(2)$ gauge field A_i and adjoint scalar Φ on Euclidean $\mathbb{R}^3$ is

$$E = \int_{\mathbb{R}^3} \left[-\frac{1}{4}\mathrm{Tr}\,(D_i\Phi D_i\Phi) - \frac{1}{8}\mathrm{Tr}\,(F_{ij}F_{ij}) + \lambda(1-\|\Phi\|^2)^2 \right] \mathrm{d}^3x,$$

in which $\lambda \geq 0$ is a parameter and $\|\Phi\|^2 := -\frac{1}{2}\mathrm{Tr}\Phi^2$. A monopole is a finite-energy solution of its Euler-Lagrange equations satisfying the boundary condition

$$\|\Phi\| \to 1 \quad \text{as } r \to \infty.$$

The asymptotic scalar field of a monopole defines a map from the 2-sphere at infinity to the unit sphere in $\mathfrak{su}(2)$, and the topological charge of the monopole is the winding number N of this map.

The non-vanishing asymptotic value for $\|\Phi\|$ breaks the gauge symmetry from SU(2) to U(1). Motivated by this, 't Hooft proposed [1] the following definition of the asymptotic abelian magnetic field of a monopole:

$$b_i' := \frac{1}{4}\epsilon_{ijk}\frac{\mathrm{Tr}(F_{jk}\Phi)}{\|\Phi\|} - \frac{1}{8}\epsilon_{ijk}\frac{\mathrm{Tr}(\Phi D_j\Phi D_k\Phi)}{\|\Phi\|^3}. \tag{8}$$

Another commonly accepted definition for the abelian magnetic field is [9]

$$b_i := \frac{1}{4}\epsilon_{ijk}\frac{\mathrm{Tr}(F_{jk}\Phi)}{\|\Phi\|}. \tag{9}$$

The equations of motion imply that this magnetic field differs from 't Hooft's only by terms which decay exponentially, so these two magnetic fields share the same asymptotic expansion. Both magnetic fields b_i and b_i' have singularities at points where $\Phi = 0$.

The total magnetic charge g of the monopole is defined to be the flux of b_i (or equivalently, of b_i') through the 2-sphere at infinity. It is known in the case of BPS monopoles (defined below) that $g = -2\pi N$, while the same result holds more generally if the fields decay sufficiently fast as $r \to \infty$ [10]. Thus the magnetic charge is topologically quantised.

It is common in the study of monopoles to introduce two twisted Dirac operators with real parameter $s \in (-1, 1)$:

$$\begin{aligned} D_s^\dagger &= \mathrm{i}\sigma_j D_j + \mathrm{i}s + \Phi \\ D_s &= \mathrm{i}\sigma_j D_j - \mathrm{i}s - \Phi. \end{aligned}$$

These act on L^2-normalisable spinors transforming in the fundamental representation of SU(2). Let $\psi_1, \psi_2, \ldots, \psi_n$ be a basis for the space of solutions to $D_s\psi = 0$, let $\chi_1, \chi_2, \ldots \chi_{n'}$ be a basis for the space of solutions to $D_s^\dagger\chi = 0$, and suppose that these bases are both orthonormal:

$$\int \psi_a^\dagger \psi_b \,\mathrm{d}^3\mathbf{x} = \delta_{ab} \quad \text{and} \quad \int \chi_c^\dagger \chi_d \,\mathrm{d}^3\mathbf{x} = \delta_{cd}.$$

We propose

$$\mu_s(\mathbf{x}) = 2\pi \left(-\sum_{a=1}^{n} \psi_a^\dagger \psi_a + \sum_{a=1}^{n'} \chi_a^\dagger \chi_a \right) \tag{10}$$

as a definition of the magnetic charge density of a monopole. Note that this does not depend on the choice of orthonormal bases ψ_a and χ_a. This density is not unique, as it depends on the parameter $s \in (-1, 1)$. It will be demonstrated below that, although the densities μ_s may differ, they induce the same asymptotic magnetic field, with the consequence that all values of $s \in (-1, 1)$ yield equally viable charge densities μ_s. However, if a unique charge density was required then μ_0 seems the most natural choice. Note that all of the densities μ_s decay exponentially in r.

The first requirement of any putative magnetic charge density is that its integral should equal the total magnetic charge $-2\pi N$ as viewed from infinity. Our proposed density meets this requirement: the normalisation conditions above imply that

$$\int_{\mathbb{R}^3} \mu_s \, \mathrm{d}^3 x = 2\pi(n' - n),$$

and an index theorem [11] guarantees that $n - n' = \dim\ker D_s - \dim\ker D_s^\dagger = N$. Thus the topological nature of the magnetic charge is made manifest through the index theorem.

A more sophisticated requirement of a magnetic charge density is that the moments of the density agree with the multipole expansion of the magnetic field (9) in the manner described in the previous section. The main result of this paper is that our proposed density meets this requirement, at least in the case of BPS monopoles. Let us emphasise that this requirement does not determine a unique charge density, as the moments do not determine a unique charge density. However, our charge densities (10) are to date the only known continuous charge densities for monopoles that do meet this requirement.

3.1 Example: the Prasad-Sommerfield 1-monopole

We now present a short calculation of our charge density in a simple case, that of the Prasad-Sommerfield 1-monopole. At the end of section 5 we will present an identity (21) which would allow an efficient calculation of this charge density through the formalism of the Nahm transform. However, for illustrative purposes we present here a calculation starting from the definition (10).

The Prasad-Sommerfield 1-monopole is a spherically symmetric monopole with $N = 1$ which solves the Yang-Mills-Higgs equations in the $\lambda = 0$ limit. It takes the form [9],

$$\begin{aligned} \Phi &= h(r)\frac{x^i}{r} t^i, \\ A_i &= -\frac{1}{2}(1 - k(r))\epsilon_{ijk}\frac{x^j}{r^2} t^k, \\ h(r) &= \coth(2r) - \frac{1}{2r}, \\ k(r) &= 2r \operatorname{csch}(2r), \end{aligned}$$

with t^a generators of SU(2) satisfying $[t^i, t^j] = -2\epsilon_{ijk}t^k$.

By a well-known general argument (to be reviewed below) the equation $D_s^\dagger \chi = 0$ has no non-zero solutions [9]. Therefore we only need to solve the equation $D_s\psi = 0$. By the index theorem, the solution of this equation is unique up to scale, so we may assume it to be spherically symmetric. Thus we make a spherically symmetric ansatz for ψ depending on two complex radial functions $f_1(r), f_2(r)$:

$$\psi_{\alpha a} = \left(f_1(r) + f_2(r)\frac{x^i}{r}t^i \right) \epsilon_{\alpha a}.$$

Note that ψ has two indices: the index $a = 1,2$ transforms in the fundamental representation of the gauge group SU(2), and $\alpha = 1,2$ is a spinor index. Within this ansatz, the equation $D_s\psi = 0$ is equivalent to

$$\begin{aligned} f_1'(r) + \left(\frac{1}{r} + h(r) - \frac{k(r)}{r}\right) f_1(r) - \mathrm{i}s f_2(r) &= 0 \\ f_2'(r) + \left(\frac{1}{r} + h(r) + \frac{k(r)}{r}\right) f_2(r) + \mathrm{i}s f_1(r) &= 0. \end{aligned}$$

In order to simplify this system, we make the substitution $g_a(r) = \sqrt{r \sinh 2r} \coth r\, f_a(r)$. The system becomes

$$\begin{aligned} g_1'(r) - \mathrm{i}s g_2(r) &= 0 \\ g_2'(r) + 4\,\mathrm{csch} 2r\, g_2(r) + \mathrm{i}s g_1(r) &= 0. \end{aligned}$$

The first equation is solved by $g_2 = -\mathrm{i}g_1'/s$, and the second is then equivalent to

$$g_1''(r) + 4\,\mathrm{csch} 2r\, g_1'(r) - s^2 g_1(r) = 0.$$

After making the substitutions $z = \tanh r$, $g_1 = u/z$ one obtains the Legendre equation of degree 2 and order s:

$$(1 - z^2)\frac{\mathrm{d}^2u}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}u}{\mathrm{d}z} + \left(2 - \frac{s^2}{1 - z^2}\right) u = 0.$$

In order that g_1 be finite at $r = 0$, it is necessary that $u(z) = 0$. The solution of the Legendre equation with this boundary condition is

$$u(z) = C(z + s)\left(\frac{1 - z}{1 + z}\right)^{\frac{s}{2}} + C(z - s)\left(\frac{1 + z}{1 - z}\right)^{\frac{s}{2}},$$

with $C \in \mathbb{C}$ an arbitrary constant. Substituting back to our original variables yields

$$\begin{aligned} f_1 &= \frac{2C}{\sqrt{r \sinh 2r}} (\tanh r \cosh sr - s \sinh sr) \\ f_2 &= \frac{-2\mathrm{i}C}{\sqrt{r \sinh 2r}} (\coth r \sinh sr - s \cosh sr). \end{aligned}$$

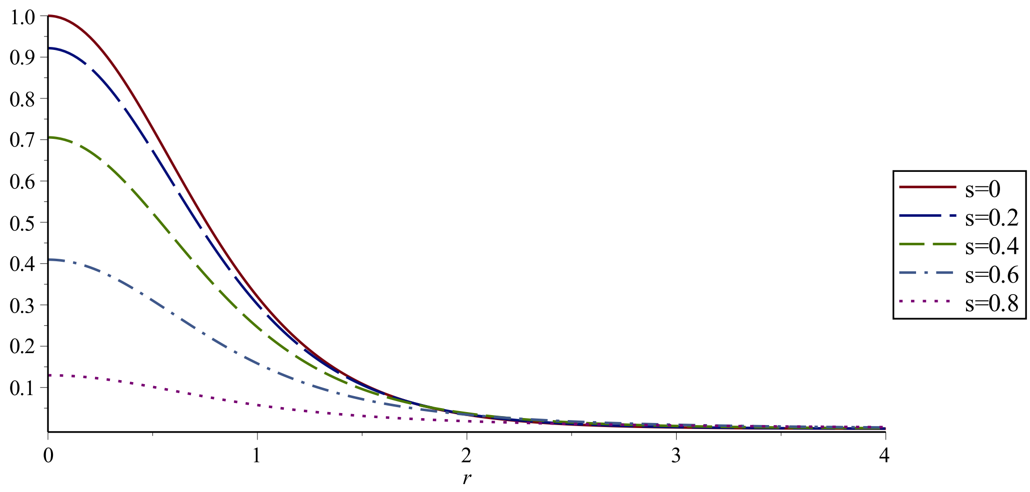

Figure 1: Graphs of the charge density $-\mu_s(r)$ of the Prasad-Sommerfield one-monopole as a function of radius r and for selected values of s.

Hence

$$|\psi|^2 = \frac{8|C|^2}{r}\left(s^2\frac{\cosh 2sr}{\sinh 2r} - 2s\frac{\sinh 2sr\cosh 2r}{\sinh^2 2r} + \frac{\cosh 2sr(\cosh^2 2r+1)}{\sinh^3 2r} - 2\frac{\cosh 2r}{\sinh^3 2r}\right). \tag{11}$$

In order to normalise the solution it will prove useful to note that

$$|\psi|^2 = \frac{2|C|^2}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{\cosh 2sr}{\sinh 2r} - \frac{\cosh 2r}{\sinh 2r} + 1\right).$$

From this identity and two integrations by parts one obtains

$$\begin{aligned}\int_{\mathbb{R}^3}|\psi|^2\mathrm{d}^3x &= 8\pi|C|^2\int_0^\infty r\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{\cosh 2rs}{\sinh 2r} - \frac{\cosh 2r}{\sinh 2r} + 1\right)\mathrm{d}r\\ &= 8\pi|C|^2.\end{aligned}$$

Note that this integral does not depend on s, even though the integrand does. Hence the zero-mode is given the correct normalisation by the choice $C = 1/\sqrt{8\pi}$, which we adopt from now on. The charge density is then $\mu_s = -2\pi|\psi|^2$, with $|\psi|^2$ given in (11).

The charge density is plotted for various values of s in figure 1, from which it can be seen that as $|s|$ increases the distribution of charge becomes less localised. This qualitative observation can be given quantitative meaning by studying the charge radius R_s of the distribution, defined by the formula

$$R_s^2 = \frac{\int_{\mathbb{R}^3} r^2\mu_s\mathrm{d}^3x}{\int_{\mathbb{R}^3}\mu_s\mathrm{d}^3x}.$$

We have obtained the following analytical expression for the charge radius of the Prasad-Sommerfield 1-monopole:

$$R_s^2 = \frac{\pi^2}{4} + \frac{3\pi^2}{8} \tan^2\left(\frac{\pi s}{2}\right).$$

This shows that the charge radius is monotonically increasing in $|s|$ and diverges as $|s| \to 1$. Let us briefly describe how this formula is derived. Integrating by parts twice as above yields

$$\begin{aligned} R_s^2 &= \int_0^\infty r^3 \frac{\mathrm{d}^2}{\mathrm{d}r^2} \left(\frac{\cosh 2rs}{\sinh 2r} - \frac{\cosh 2r}{\sinh 2r} + 1 \right) \mathrm{d}r \\ &= 6 \int_0^\infty r \left(\frac{\cosh 2rs}{\sinh 2r} - \frac{\cosh 2r}{\sinh 2r} + 1 \right) \mathrm{d}r. \end{aligned}$$

This integral will be evaluated by separating it to two terms. The first term can be integrated by parts and expressed in terms of the dilogarithm function Li_2:

$$\begin{aligned} 6 \int_0^\infty r \left(-\frac{\cosh 2r}{\sinh 2r} + 1 \right) \mathrm{d}r &= 3 \int_0^\infty \ln(1 - e^{-4r}) \mathrm{d}r \\ &= \frac{3}{4} \int_0^1 \frac{\ln(1-u)}{u} \mathrm{d}u \\ &= -\frac{3}{4} \mathrm{Li}_2(1). \end{aligned}$$

Since $\mathrm{Li}_2(1) = \pi^2/6$ this contributes $-\pi^2/8$ to the integral. The second term can be expressed in terms of the trigamma function $\psi^{(1)}$:

$$\begin{aligned} 6 \int_0^\infty r \frac{\cosh 2rs}{\sinh 2r} \mathrm{d}r &= \frac{3}{8} \int_0^\infty \frac{e^{-t(1/2-s/2)} + e^{-t(1/2+s/2)}}{1 - e^{-t}} t \, \mathrm{d}t \\ &= \frac{3}{8} \left[\psi^{(1)}\left(\tfrac{1}{2} - \tfrac{s}{2}\right) + \psi^{(1)}\left(\tfrac{1}{2} + \tfrac{s}{2}\right) \right]. \end{aligned}$$

This is further simplified by means of the reflection identity for the trigamma function, which states that

$$\psi^{(1)}(1-z) + \psi^{(1)}(z) = \pi^2 \csc^2(\pi z).$$

Combining the above with elementary trigonometric identities yields the advertised result.

4 Moments and the Nahm transform

BPS monopoles with $N \geq 0$ are solutions of the first order equation

$$D_i \Phi = \frac{1}{2} \epsilon_{ijk} F_{jk}, \tag{12}$$

and the boundary condition

$$\|\Phi\| \sim 1 - \frac{N}{2r} \quad \text{as } r \to \infty.$$

They solve the second order Euler-Lagrange equations for the Yang-Mills-Higgs energy in the limiting case where $\lambda = 0$.

The BPS equation guarantees that

$$D_s D_s^\dagger = D_i D_i + (\mathrm{i}s + \Phi)^2.$$

This operator is negative and therefore $D_s^\dagger$ has no zero-modes χ_a. Therefore our definition of the magnetic charge density reduces in the case of BPS monopoles to

$$\mu_s = -2\pi \sum_{a=1}^{N} \psi_a^\dagger \psi_a. \tag{13}$$

BPS monopoles can be completely constructed through the formalism of the Nahm transform [9, 12, 13]. This transform associates to any monopole the matrix-valued functions

$$\begin{aligned}(T_j(s))_{ab} &= -\mathrm{i} \int x_j \psi_a^\dagger(\mathbf{x}; s) \psi_b(\mathbf{x}; s)\, \mathrm{d}^3\mathbf{x} \\ (T_0(s))_{ab} &= + \int \psi_a^\dagger(\mathbf{x}; s) \frac{\partial}{\partial s} \psi_b(\mathbf{x}; s)\, \mathrm{d}^3\mathbf{x}.\end{aligned}$$

It is a non-trivial but well-known result that these matrices solve the Nahm equation,

$$\frac{\mathrm{d}T_i}{\mathrm{d}s} + [T_0, T_i] = \frac{1}{2}\epsilon_{ijk}[T_j, T_k] \quad \text{for } i = 1, 2, 3,$$

and certain boundary conditions (whose precise form does not concern us here).

The Nahm equation is equivalent to a Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}s} T(\zeta) + [T_+(\zeta), T(\zeta)] = 0,$$

in which

$$\begin{aligned}T(\zeta) &= \frac{1}{2}(T_1 + \mathrm{i}T_2) + \zeta T_3 - \frac{1}{2}(T_1 - \mathrm{i}T_2)\zeta^2 \\ T_+(\zeta) &= T_0 - \mathrm{i}T_3 + \mathrm{i}(T_1 - \mathrm{i}T_2)\zeta.\end{aligned}$$

It follows that the quantities $\mathrm{Tr}\big((\mathrm{i}T(\zeta))^\ell\big)$ are independent of s, for $\ell = 0, 1, 2, \ldots$ [14]. These conserved quantities are in fact equal to the moments of the density (13), as the following proposition shows:

Proposition 2. *Let*

$$M_\ell(\zeta) := \int_{\mathbb{R}^3} \mu_s(\mathbf{x})\, x(\zeta)^\ell \, \mathrm{d}^3 x \tag{14}$$

denote the moments of the charge density (10) *of a BPS monopole and let* $T(\zeta)$ *be the matrix-valued polynomial formed from the associated Nahm data as described above. Then*

$$M_\ell(\zeta) = -2\pi \mathrm{Tr}\big((\mathrm{i}T(\zeta))^\ell\big). \tag{15}$$

Before presenting the proof we note that this implies that the moments of the charge distribution μ_s are independent of $s \in (-1, 1)$. This is why we believe all of the densities μ_s are equally viable candidates for a magnetic charge density. Note that only the moments of the charge density are independent of s, the full density μ_s itself is of course s-dependent.

Proof. The identity (15) will be proved from standard identities for Green's functions. Let $G(\mathbf{x}, \mathbf{x}'; s)$ be the Green's function for $D_s D_s^\dagger$, i.e. the 2×2 matrix-valued function which solves

$$(-D_i D_i - (\mathrm{i}s + \Phi)^2) G(\mathbf{x}, \mathbf{x}'; s) = \delta(\mathbf{x} - \mathbf{x}').$$

For convenience we introduce the notation

$$G(\mathbf{x}, \mathbf{x}'; s)\overleftarrow{D'_s} = \mathrm{i}\sigma_j \left(-\frac{\partial G(\mathbf{x}, \mathbf{x}'; s)}{\partial x'_j} + G(\mathbf{x}, \mathbf{x}'; s) A_j(\mathbf{x}') \right) - G(\mathbf{x}, \mathbf{x}'; s)(\mathrm{i}s + \Phi(\mathbf{x}')).$$

The following identity is well-known (cf. equation (4.35), (4.36) in [13]):

$$\sum_{a=1}^{N} \psi_a(\mathbf{x}; s) \psi_a^\dagger(\mathbf{x}'; s) = \delta_3(\mathbf{x} - \mathbf{x}') - D_s^\dagger G(\mathbf{x}, \mathbf{x}'; z)\overleftarrow{D'_s}. \tag{16}$$

We will use induction and this identity to prove the statement

$$((\mathrm{i}T(\zeta))^\ell)_{ab} = \int \psi_a^\dagger(\mathbf{x}; s) \psi_b(\mathbf{x}; s) x(\zeta)^\ell \, \mathrm{d}^3\mathbf{x} \quad \forall l \in \mathbb{Z},\, l \geq 0, \tag{17}$$

from which our main result (15) follows.

The case $\ell = 0$ of (17) follows directly from the normalisation of the fermion zero modes.

Suppose then that (17) holds in the case $\ell = m$ for some $m \in \mathbb{Z}$. We will show that it must also hold in the case $\ell = m + 1$. Appealing to the Greens' function identity (16) and integrating by parts yields:

$$
\begin{aligned}
&((\mathrm{i}T(\zeta))^{m+1})_{ab} \\
&= \int x(\zeta)^m \psi_a^\dagger(\mathbf{x})\psi_c(\mathbf{x})\psi_c^\dagger(\mathbf{x}')\psi_b(\mathbf{x}')x'(\zeta)\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}' \\
&= \int \psi_a^\dagger(\mathbf{x})\left(\delta_3(\mathbf{x}-\mathbf{x}') - D_s^\dagger G_s(\mathbf{x},\mathbf{x}')\overleftarrow{D_s'}\right)\psi_b(\mathbf{x}')x(\zeta)^m x'(\zeta)\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}' \\
&= \int \psi_a^\dagger(\mathbf{x})\psi_b(\mathbf{x})x(\zeta)^{m+1}\,\mathrm{d}^3\mathbf{x} \\
&\qquad - \int \left(D_s\overline{x(\zeta)}^m\psi_a(\mathbf{x})\right)^\dagger\left(D_s'x'(\zeta)\psi_b(\mathbf{x}')\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'.
\end{aligned}
$$

Observe that $[D_s, x(\zeta)] = \sigma(\zeta) = \left[D_s, \overline{x(\zeta)}\right]^\dagger$, where

$$
\sigma(\zeta) := \frac{1}{2}(\sigma_2 + \mathrm{i}\sigma_3) + \sigma_1\zeta - \frac{1}{2}(\sigma_2 - \mathrm{i}\sigma_3)\zeta^2.
$$

Since in addition $D_s\psi_a = 0$, the unwanted second term on the right equates to

$$
\int \left(D_s\overline{x(\zeta)}^m\psi_a(\mathbf{x})\right)^\dagger\left(D_s'x'(\zeta)\psi_b(\mathbf{x}')\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}' \\
= m\int \psi_a(\mathbf{x})^\dagger\sigma(\zeta)^2 x(\zeta)^{m-1}\psi_b(\mathbf{x}')\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'.
$$

This expression vanishes, because $\sigma(\zeta)^2 = 0$. Therefore the identity (17) holds in the case $\ell = m+1$, and for all $\ell \geq 0$ by the principle of mathematical induction. □

5 Higgs field asymptotics

In the case of BPS monopoles the norm of the scalar field Φ provides a scalar potential for the asymptotic magnetic field (9): it is easily shown using (12) that $b_i = -\partial_i\|\Phi\|$.

Hurturbise has derived [5] an expression for the asymptotic behaviour of $\|\Phi\|$ in terms of spectral curves. We recall that the spectral curve of a BPS monopole is the vanishing set of the polynomial

$$
g(\eta,\zeta) = \det(\eta - \mathrm{i}T(\zeta)).
$$

We note that, like the polynomials $\mathrm{Tr}((\mathrm{i}T(\zeta))^\ell)$, this polynomial g is independent of s.

Definition 3. The *tail* of a BPS monopole is a real function on the complement of a compact subset of $\mathbb{R}^3$, defined by the following contour integral:

$$
\mathscr{V} := -\frac{1}{4\pi\mathrm{i}}\oint_\Gamma \frac{\partial_\eta g(\eta,\zeta)}{g(\eta,\zeta)}\Big|_{\eta=x(\zeta)}\mathrm{d}\zeta. \tag{18}
$$

For sufficiently large r, half of the poles of the integrand cluster near the point $\zeta_+ = -(x_2+\mathrm{i}x_3)/(r+x_1)$ on the Riemann sphere corresponding to $\mathbf{x}/r$, and half near its antipode $\zeta_- = -1/\bar{\zeta}_+$. The contour Γ encloses the former and not the latter. The domain of $\mathscr{V}$ is chosen such that none of the poles move from one cluster to another as $\mathbf{x}$ moves through the domain.

Theorem 4 (Hurtubise [5])**.** *The norm of the Higgs field and tail of a BPS monopole satisfy*

$$\|\Phi\| = 1 + \mathscr{V}$$

up to exponentially decaying terms.

Hurtubise' result can be used to prove:

Theorem 5. *Let (A, Φ) be a BPS monopole, let $s \in (-1, 1)$, and let μ_s be the charge density defined in equation (10). Let ϕ_s be a solution to $\triangle\phi_s = -\mu_s$ and suppose that it admits an asymptotic expansion in powers of $1/r$. Then the asymptotic expansions of ϕ_s and $\|\Phi\| - 1$ agree to all orders.*

The physical interpretation of this theorem is that the multipole expansion of the magnetic field induced by μ_s agrees with that of the magnetic field b_i defined in (9).

Proof. The tail is automatically harmonic almost everywhere, as follows from the formalism of the Penrose transform. Therefore it admits an asymptotic expansion in powers of $1/r$ of the form

$$\mathscr{V} = \sum_{\ell=0}^{\infty} \frac{\mathscr{V}_\ell}{r^{\ell+1}}, \tag{19}$$

in which $\mathscr{V}_\ell$ are spherical harmonics of weight ℓ. By Hurtubise' theorem 4 the function $\|\Phi\| - 1$ admits an expansion in powers of $1/r$ that agrees precisely with this expansion of $\mathscr{V}$.

It is straightforward to derive expressions for these functions $\mathscr{V}_\ell$ from the integral expression (18). First, note that

$$\frac{\partial_\eta g(\eta, \zeta)}{g(\eta, \zeta)} = \mathrm{Tr}\big((\eta - \mathrm{i}T(\zeta))^{-1}\big).$$

Therefore

$$\begin{aligned}\frac{-1}{4\pi\mathrm{i}} \oint_\Gamma \frac{\partial_\eta g(\eta, \zeta)}{g(\eta, \zeta)}\Big|_{\eta = x(\zeta)} \mathrm{d}\zeta &= \frac{-1}{4\pi\mathrm{i}} \oint_\Gamma \mathrm{Tr}\big((x(\zeta) - \mathrm{i}T(\zeta))^{-1}\big)\mathrm{d}\zeta \\ &= \sum_{\ell=0}^{\infty} \frac{-1}{4\pi\mathrm{i}} \oint_\Gamma \frac{\mathrm{Tr}\big((\mathrm{i}T(\zeta))^\ell\big)}{x(\zeta)^{\ell+1}} \mathrm{d}\zeta.\end{aligned}$$

Since $M_\ell(\zeta) = -2\pi \mathrm{Tr}((\mathrm{i}T(\zeta))^\ell)$, we conclude that

$$\mathscr{V}_\ell = \frac{1}{8\pi^2\mathrm{i}} \oint_\Gamma \frac{M_\ell(\zeta)}{n(\zeta)^{\ell+1}} \mathrm{d}\zeta. \tag{20}$$

Suppose now that ϕ_s admits an expansion of the form $\phi_s = \sum_{\ell=0}^{\infty} \phi_\ell / r^{\ell+1}$. By propositions 1 and 2 the coefficients ϕ_ℓ satisfy

$$\phi_\ell = \frac{1}{8\pi^2 \mathrm{i}} \oint_\Gamma \frac{M_\ell(\zeta)}{n(\zeta)^{\ell+1}} \mathrm{d}\zeta.$$

Thus the expansions of ϕ_s and $\|\Phi\| - 1$ coincide. $\square$

In the next section we present some results concerning the explicit evaluation of the tail function $\mathscr{V}$. Before doing so, we pause to point out that the Nahm transform provides a natural definition of the potential function for the magnetic field induced by μ_s and a natural way to evalute it. We recall that the Nahm data Green's function $f(s, s'; \mathbf{x})$ is the $N \times N$ matrix-valued solution to

$$-\left(\left(\frac{\mathrm{d}}{\mathrm{d}s} + T_0\right)^2 + (T_j + \mathrm{i}x_j)^2\right) f(s, s'; \mathbf{x}) = \mathrm{Id}_N \delta(s - s').$$

This Green's function is related to the monopole zero-modes ψ_a by the identity [13],

$$2\pi \psi_a^\dagger(\mathbf{x}; s) \psi_b(\mathbf{x}; s') = ((s - s')^2 - \triangle) f_{ab}(s, s'; \mathbf{x}).$$

It follows that

$$\mu_s = \triangle \mathrm{Tr} f(s, s; \mathbf{x}). \tag{21}$$

Therefore $\phi_s = -\mathrm{Tr} f(s, s; \mathbf{x})$ is a potential for the magnetic field induced by μ_s. This formula could be used to evaluate the charge density of a monopole directly from its Nahm data, even if the monopole fields are themselves not known in explicit form.

In [6, 7] it was conjectured that the asymptotic expansion for $\mathrm{Tr} f(s, s; \mathbf{x})$ is determined to all orders by the conserved quantities $\mathrm{Tr}((\mathrm{i}T(\zeta))^\ell)$ of the Nahm equation. This conjecture provided the initial motivation for our investigations. It can be proved directly from propositions 1 and 2, as in the proof of theorem 5.

6 Evaluating the asymptotic Higgs field

In this section we turn our attention away from the charge density and present some explicit calculations of the asymptotic fields of monopoles, obtained from the tail function discussed above. In general reconstructing a monopole explicitly from its Nahm data is a difficult problem, but we will show below that formulae for the asymptotic fields can be obtained directly from the spectral curve. Formulae will be presented for well-known monopoles with Platonic symmetry.

Our strategy for calculating the tail function is based on the series expansion (19). We expect this series to converge outside of a compact set, and the analytical and numerical results to be presented below are consistent with this expectation. Our strategy is based on equation (20), which expresses the summands $\mathscr{V}_\ell$ in terms of the moments $M_\ell(\zeta)$. We will present below formulae which allow the extraction of the $M_\ell(\zeta)$ from the spectral curve and the efficient evaluation of the contour integral given in equation (20).

The first step in evaluating the tail is to determine the moments $M_\ell(\zeta)$ from the spectral curve. Let $g_m(\zeta)$ be the coefficient of η^{N-m} in the spectral curve, so that

$$g(\eta,\zeta) = \sum_{m=0}^{N} g_m(\zeta)\eta^{N-m}.$$

Note in particular that $g_0 = 1$. Then $g_m(\zeta)$ are elementary symmetric polynomials in the eigenvalues of $T(\zeta)$, while from equation (15) we know that $M_\ell(\zeta)$ are power sums of the same eigenvalues. Newton's identities state that

$$g_\ell(\zeta) = \frac{1}{2\pi\ell}\sum_{m=1}^{\ell} g_{\ell-m}(\zeta)M_m(\zeta).$$

Rearranging this yields the formula

$$M_\ell(\zeta) = -\sum_{m=1}^{\ell-1} g_{\ell-m}(\zeta)M_m(\zeta) + 2\pi\ell g_\ell(\zeta), \tag{22}$$

in which it should be understood that $g_m = 0$ for $m > N$. From this formula the polynomials $M_\ell(\zeta)$ can be calculated recursively.

The second step in evaluating the tail is to determine the spherical harmonics $\mathscr{V}_\ell$ from the polynomials M_ℓ. In principle this can be achieved by evaluating the contour integral (20), but in practice it is useful to have explicit formulae in terms of the basis polynomials $\zeta^0, \zeta^1, \ldots, \zeta^{2\ell}$. The following lemma provides such formulae.

Lemma 6. *Let $M_\ell(\zeta) = \sum_{m=-\ell}^{\ell} M_\ell^m \zeta^{\ell+m}$ and let $\mathscr{V}_\ell$ be the function on S^2 obtained from M_ℓ by the contour integral (20). Let $Y_\ell^m : S^2 \to \mathbb{R}$ be the functions defined by*

$$\left(\frac{-n(\zeta)}{\zeta^2}\right)^\ell = \sum_{m=-\ell}^{\ell} \frac{1}{(-\zeta)^{\ell+m}} Y_\ell^m.$$

Then

$$\mathscr{V}_\ell = \sum_{m=-\ell}^{\ell} M_\ell^m \frac{(\ell-m)!(\ell+m)!}{(4\pi)(\ell!)^2} Y_\ell^m.$$

Note that the functions Y_ℓ^m agree up to normalisation and rotation with the standard spherical harmonics.

Proof. We begin by noting that

$$\left(\frac{-n(\zeta)}{\zeta^2}\right)^\ell = \exp\left(\frac{L_+}{\zeta}\right)\left(\frac{n_1 - \mathrm{i}n_2}{2}\right).$$

The case $\ell = 1$ of this identity can be verified by direct calculation, and the cases with $\ell > 1$ follow because L_+ obeys the Leibniz rule. It follows that

$$Y_\ell^m = \frac{(-L_+)^{\ell+m}}{(\ell+m)!}\left(\frac{n_1 - \mathrm{i}n_2}{2}\right).$$

To evaluate the contour integral (20), we note the following identity (which is easily proved):

$$\zeta^{\ell+m} = \frac{(\ell-m)!}{(2\ell)!}(-J_+^{(\ell)})^{\ell+m}\zeta^0.$$

Here $J_+^{(\ell)}$ is the operator defined in the proof of proposition 2. It follows from this identity and equations (6) and (7) that

$$\begin{aligned}
\frac{1}{8\pi^2\mathrm{i}}\oint_\Gamma \frac{\zeta^{\ell+m}}{n(\zeta)^{\ell+1}}\mathrm{d}\zeta &= \frac{(\ell-m)!}{(2\ell)!}(-L_+)^{\ell+m}\frac{1}{8\pi^2\mathrm{i}}\oint_\Gamma \frac{1}{n(\zeta)^{\ell+1}}\mathrm{d}\zeta \\
&= \frac{(\ell-m)!}{(4\pi)2^\ell(\ell!)^2}(-L_+)^{\ell+m}(n_1 - \mathrm{i}n_2)^\ell \\
&= \frac{(\ell-m)!(\ell+m)!}{4\pi(\ell!)^2}Y_\ell^m.
\end{aligned}$$

The result follows.

□

Lemma 6 provides a means to evaluate $\mathscr{V}_\ell$ from M_ℓ which is easily implemented in standard algebraic software packages. Below we present series expansions of $\mathscr{V}$ for examples of monopoles with Platonic symmetry, along with closed form expressions along certain symmetry axes. In figure 2 we display isosurfaces for $\partial_i\mathscr{V}\partial_i\mathscr{V}$, which approximates the energy density of the monopole. These are created using series expansions for $\mathscr{V}$ up to $\ell = 12$. Removing the last non-zero term from the series expansion did not alter the pictures, so we are confident that the series converges in a neighbourhood of these surfaces. They closely resemble the pictures published in [17, 18], so it seems that much of the structure of a monopole is captured by its abelian tail.

6.1 The tetrahedral 3-monopole

There is a unique monopole with topological charge $N = 3$ and tetrahedral symmetry [15, 16, 17]. Its spectral curve is defined by the polynomial

$$\begin{aligned}
g(\eta,\zeta) &= \eta^3 - \mathrm{i}C_3\zeta(\zeta^4 - 1), \quad \text{where} \\
C_3 &= \frac{2\pi^6}{3^{9/2}\Gamma(2/3)^9}.
\end{aligned}$$

The first few terms in the expansion (19) for $\mathscr{V}$ are easily evaluated. In order to write them down in a concise manner, we make use of the fact that all tetrahedrally-symmetric

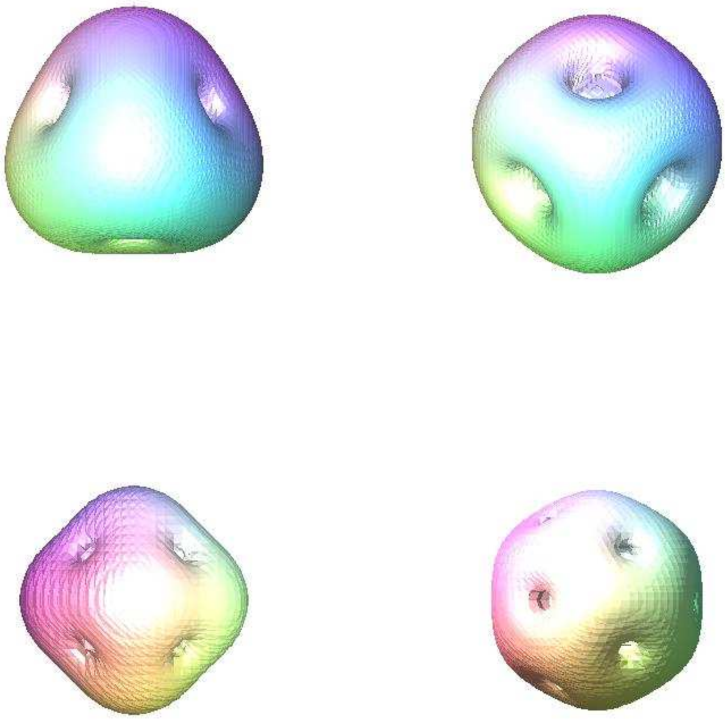

Figure 2: Isosurfaces of $\partial_i \mathscr{V} \partial_i \mathscr{V}$ for the Platonic monopoles with charges 3 (top left), 4 (top right), 5 (bottom left), and 7 (bottom right).

polynomial functions on the two-sphere can be written in terms of the three polynomials $t_2 = n_1^2 + n_2^2 + n_3^2 = 1$, $t_3 = n_1 n_2 n_3$ and $t_4 = n_1^4 + n_2^4 + n_3^4$. In these terms, we find that

$$\mathscr{V} = -\frac{3}{2r} + \frac{15C_3}{r^4} t_3 + \frac{3C_3^2}{4r^7}(17t_2^3 - 21t_2t_4 - 462t_3^2) + \dots$$

Note that in general the terms $\mathscr{V}_\ell$ vanish except when $\ell = 0 \mod 3$. This property does not follow from symmetry considerations alone: for example, there is a tetrahedrally-symmetric spherical harmonic with $\ell = 4$, but this spherical harmonic does not appear in the series expansion for $\mathscr{V}$.

It seems difficult to sum the series expansion for $\mathscr{V}$ in general. However, we have been able to find a closed form expression in some special cases, by working directly from the contour integral expression (18). Restricting eq. (18) to the x_1-axis $(x_1, x_2, x_3) = (t, 0, 0)$

yields the expression

$$\mathscr{V} = -\frac{1}{4\pi\mathrm{i}} \oint_{\Gamma} \frac{3t^2\zeta\mathrm{d}\zeta}{-\mathrm{i}C_3\zeta^4 + t^3\zeta^2 + \mathrm{i}C_3}.$$

By change of variables $w = \zeta^2$, the contour integral becomes

$$\mathscr{V} = -\frac{1}{4\pi\mathrm{i}} \oint_{\Gamma'} \frac{3t^2\mathrm{d}w/2}{-\mathrm{i}C_3w^2 + t^3w + \mathrm{i}C_3}.$$

The denominator of the integrand has two roots, but only the root $w = (t^3 - \sqrt{t^6 - C_3^2})/2\mathrm{i}C_3$ lies inside the contour Γ'. The contour Γ' circles this pole twice because $w = \zeta^2$. Therefore the integral evaluates to

$$\mathscr{V} = -\frac{3t^2}{2\sqrt{t^6 - 4C_3^2}}.$$

It is straightforward to check that the asymptotic expansion of this function agrees with the restriction of the series expansion for $\mathscr{V}$ to the x_1-axis.

The reason why a closed form expression can be obtained on the x_1-axis is that this line has a high degree of symmetry. The symmetry group of the monopole fixes a tetrahedron, and the x_1-axis passes through opposite edges of this tetrahedron. Rotations through π about the x_1-axis fix both the axis and the monopole. This rotational symmetry allowed a simplification of the contour integral, so that the roots of the denominator could be easily found.

There is another line with a high degree of symmetry, namely that passing through a vertex and the centre of the opposing face of the tetrahedron. This line has equation $(x_1, x_2, x_3) = (t, t, t)/\sqrt{3}$. A closed form expression for $\mathscr{V}$ can also be obtained along this line.

To obtain the expression for $\mathscr{V}$ it is convenient to first rotate the monopole so that the desired line is again the x_1-axis. This is accomplished by making a Möbius transform of the spectral curve of the form

$$g(\eta, \zeta) \mapsto (-\bar{b}\zeta + \bar{a})^{2N} g\big(\eta/(-\bar{b}\zeta + a), (a\zeta + b)/(-\bar{b}\zeta + a)\big)$$

with $a = e^{-\pi\mathrm{i}/8}\sqrt{(1 + 1/\sqrt{3})/2}$ and $b = -e^{-\pi\mathrm{i}/8}\sqrt{(1 - 1/\sqrt{3})/2}$. After making this Möbius transformation the spectral curve becomes

$$g(\eta, \zeta) = \eta^3 - \frac{C_3}{\sqrt{27}}(\sqrt{2}\zeta^6 + 10\zeta^3 - \sqrt{2}).$$

With g in this form and $(x_1, x_2, x_3) = (t, 0, 0)$ the contour integral (18) becomes

$$\mathscr{V} = -\frac{1}{4\pi\mathrm{i}} \oint \frac{3t^2\zeta^2\mathrm{d}\zeta}{-\sqrt{2/27}C_3\zeta^6 + (10\sqrt{27}C_3 + t^3)\zeta^3 + \sqrt{2/27}C_3}.$$

This integral is simplified by the substitution $w = \zeta^3$, and evaluates to

$$\mathscr{V} = -\frac{3t^2}{2\sqrt{t^6 + (20/\sqrt{27})C_3t^3 + 4C_3^2}}.$$

We have checked that the expansion of this expression in powers of $1/t$ agrees with the restriction of the series expansion for $\mathscr{V}$ to the line $(x_1, x_2, x_3) = (t, t, t)/\sqrt{3}$.

6.2 The cubic 4-monopole

The unique monopole with topological charge 4 and cubic symmetry has spectral curve [17]

$$\begin{aligned} g(\eta,\zeta) &= \eta^4 + C_4(\zeta^8 + 14\zeta^4 + 1), \quad \text{where} \\ C_4 &= \frac{3\pi^6}{2^8\Gamma(3/4)^8}. \end{aligned}$$

The first few terms in its series expansion for $\mathscr{V}$ are easily calculated, and can be written in terms of the octahedrally-symmetric polynomials $o_2 = t_2$, $o_4 = t_4$ and $o_6 = t_3^2$ as

$$\mathscr{V} = -\frac{4}{2r} + \frac{14C_4}{r^5}(5o_4 - 3o_2^2) + \frac{99C_4^2}{r^9}(208o_6o_2 + 94o_4o_2^2 - 65o_4^2 - 33o_2^4) + \dots.$$

We have been able to evaluate $\mathscr{V}$ in closed form along lines which pass through opposing vertices and opposing faces of the cube fixed by the symmetry group:

$$\begin{aligned} \mathscr{V}(t,0,0) &= -\frac{2t^3}{\sqrt{t^8 + 28C_4t^4 + 192C_4^2}} \\ \mathscr{V}\left(\frac{t}{\sqrt{3}}, \frac{t}{\sqrt{3}}, \frac{t}{\sqrt{3}}\right) &= -\frac{2t^3}{\sqrt{t^8 - 56C_4t^4/3 + 144C_4^2}}. \end{aligned}$$

The series expansions of these functions in powers of $1/t$ agree with the general series expansion quoted above. Both of these expressions are obtained using a similar method to the one used for the 3-monopole. For the second, it is convenient to first make a Möbius transformation with $a = e^{-\pi i/8}\sqrt{(1+1/\sqrt{3})/2}$ and $b = e^{-\pi i/8}\sqrt{(1-1/\sqrt{3})/2}$, after which the spectral curve has the form

$$g(\eta,\zeta) = \eta^4 - \frac{4C_4}{3}(2\sqrt{2}\zeta^6 + 7\zeta^3 - 2\sqrt{2})\zeta$$

and the desired line has moved to $(x_1, x_2, x_3) = (t, 0, 0)$.

6.3 The octahedral 5-monopole

The unique monopole with topological charge 5 and cubic symmetry has spectral curve

$$\begin{aligned} g(\eta,\zeta) &= \eta^5 - C_5(\zeta^8 + 14\zeta^4 + 1)\eta, \quad \text{where} \\ C_5 &= \frac{3\pi^6}{2^6\Gamma(3/4)^8}. \end{aligned}$$

Note that this differs from the result quoted in [18] in that the coefficient of the polynomial in ζ is $-C_5$ rather than C_5 – in attempting to reproduce the calculation of [18] we discovered a sign error.

The first few terms in the series expansion for $\mathscr{V}$ are

$$\mathscr{V} = -\frac{5}{2r} + \frac{14C_5}{r^5}(3o_2^2 - 5o_4) + \frac{99C_5^2}{r^9}(208o_6o_2 + 94o_4o_2^2 - 65o_4^2 - 33o_2^4) + \dots.$$

We have been able to evaluate $\mathscr{V}$ in closed form along lines which pass through opposing vertices and opposing faces of the octahedron fixed by the symmetry group:

$$\mathscr{V}(t,0,0) = -\frac{1}{2t} - \frac{2t^3}{\sqrt{t^8 - 28C_5t^4 + 192C_5^2}}$$
$$\mathscr{V}\left(\frac{t}{\sqrt{3}}, \frac{t}{\sqrt{3}}, \frac{t}{\sqrt{3}}\right) = -\frac{1}{2t} - \frac{2t^3}{\sqrt{t^8 + 56C_5t^4/3 + 144C_5^2}}.$$

The series expansions of these functions in powers of $1/t$ agree with the general series expansion quoted above. Both of these expressions are obtained using a similar method to the one used for the 3-monopole. For the second, it is convenient to first make a Möbius transformation as for the 4-monopole, after which the spectral curve has the form

$$g(\eta,\zeta) = \eta^5 + \frac{4C_5}{3}(2\sqrt{2}\zeta^6 + 7\zeta^3 - 2\sqrt{2})\zeta\eta.$$

6.4 The dodecahedral 7-monopole

The unique charge 7 monopole with dodecahedral symmetry has spectral curve,

$$g(\eta,\zeta) = \eta^7 - C_7(\zeta^{11} - 11\zeta^6 - \zeta)\eta, \quad \text{where}$$
$$C_7 = \frac{16\pi^{12}}{729\Gamma(2/3)^{18}}.$$

The first few terms in the series expansion for $\mathscr{V}$ are

$$\mathscr{V} = -\frac{7}{2r} + \frac{33C_7i_6}{16r^7} + \ldots, \quad \text{where}$$
$$i_6 = 16x_1^6 - 120x_1^4(x_2^2 + x_3^2) + 90x_1^2(x_2^2 + x_3^2)^2$$
$$- 42x_1(x_2^5 - 10x_2^3x_3^2 + 5x_2x_3^4) - 5(x_2^2 + x_3^2)^3.$$

The tail function $\mathscr{V}$ has the following closed form expression along the line $(x_1, x_2, x_3) = (t, 0, 0)$:

$$\mathscr{V} = -\frac{1}{2t} - \frac{3t^6}{\sqrt{t^{12} + 22C_7t^6 + 125C_7^2}}.$$

The series expansion of this function in $1/t$ agrees with the general series expansion quoted above.

The line on which we have evaluated $\mathscr{V}$ passes through the centres of opposite faces of the dodecahedron fixed by the symmetry group. One might ask whether it is also possible to evaluate $\mathscr{V}$ on a line passing through opposite vertices. In order to do so using the method above one would first need to move this line to the x_1-axis by Möbius transformation and then to factor the denominator in the contour integral. On symmetry grounds this denominator is a quartic polynomial in ζ^3, which could in principle be factorised, but we have not attempted to do so.

7 Conclusion

In this paper we have proposed a novel definition (10) for the magnetic charge density of a monopole with gauge group SU(2). This definition differs from standard definitions in that it is smooth and non-singular. We have shown that, in the case of BPS monopoles, the abelian magnetic field which it induces agrees with the standard definitions (8) and (9) to all orders in the multipole expansion.

This result can straightforwardly be extended to the case of SU(n) monopoles. SU(n) monopoles with maximal symmetry-breaking have $n-1$ asymptotic abelian magnetic fields. Hurtubise and Murray have proved in [19] a result which relates the asymptotics of these magnetic fields to the spectral curves and which generalises the result of Hurtubise employed in this paper. The definition (10) of the magnetic charge densities μ_s generalises directly to the case of SU(n), as do our arguments relating these to the Nahm data conserved quantities and the asymptotic magnetic fields. The chief novelty is that different values of s must be chosen to source the different magnetic fields. It may be possible to generalise our result to apply to SU(n) calorons also, however, the analogue of Hurtubise' result for calorons is not currently known.

The tail function, which describes the asymptotics of the norm of the Higgs field, played a key role in our analysis. While the fields of a monopole are in general difficult to construct in explicit form, the tail function is easily evaluated as a series given knowledge of the monopole's spectral curve. Moreover, we have exhibited a number of examples in which this tail function can be evaluated in closed form when restricted to a well-chosen line with a high degree of symmetry. The problem of reconstructing a monopole field explicitly from its Nahm data remains an important problem, and our observation suggests that more progress might be made if attention is restricted to these symmetric lines.

Our results have some consequence for the analysis of magnetic bags. One of us argued in [20] that the Nahm transform for monopoles should converge in a certain large N limit to what was called the $\mathfrak{u}(\infty)$ Nahm transform. It seems that Hurtubise' result, together with our observation relating the tail function to the Nahm data conserved charges, provides a more direct proof of this result.

A final comment concerns other systems that support BPS topological solitons. Analogues to the charge density μ_s could conceivably written down for instantons, vortices, and other solitons. It would be interesting to investigate whether such densities, like the density for monopoles, carry any physical significance.

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