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Marsh, RJ orcid.org/0000-0002-4268-8937 and Reiten, I (2016) Rigid and Schurian modules over cluster-tilted algebras of tame type. Mathematische Zeitschrift, 284 (3). pp. 643-682. ISSN 0025-5874

https://doi.org/10.1007/s00209-016-1668-z

RIGID AND SCHURIAN MODULES OVER CLUSTER-TILTED ALGEBRAS OF TAME TYPE

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Abstract. We give an example of a cluster-tilted algebra Λ with quiver Q, such that the associated cluster algebra $\mathcal{A}(Q)$ has a denominator vector which is not the dimension vector of any indecomposable Λ-module. This answers a question posed by T. Nakanishi. The relevant example is a cluster-tilted algebra associated with a tame hereditary algebra. We show that for such a cluster-tilted algebra Λ, we can write any denominator vector as a sum of the dimension vectors of at most three indecomposable rigid Λ-modules. In order to do this it is necessary, and of independent interest, to first classify the indecomposable rigid Λ-modules in this case.

Introduction

In the theory of cluster algebras initiated by Fomin and Zelevinsky, the authors introduced some important kinds of vectors, amongst them the d-vectors (denominator vectors) [15] and the c-vectors [16]. These vectors have played an important role in the theory. In particular, they have been important for establishing connections with the representation theory of finite dimensional algebras.

Let Q be a finite quiver with n vertices, without loops or two-cycles, and let $\mathcal{A}(Q)$ be the associated cluster algebra with initial cluster $\{x_1, \ldots, x_n\}$. Each non-initial cluster variable is known to be of the form f/m, where $m = x_1^{d_1} \cdots x_n^{d_n}$ for nonnegative integers d_i and f is not divisible by any x_i. Then the associated d-vector is $(d_1, \ldots, d_n)$. For the definition of c-vector we refer to [16]. On the other hand, we have the dimension vectors of the finite dimensional rigid indecomposable KQ-modules.

Assume first that Q is acyclic. Then there are known interesting connections between the d-vectors and the c-vectors on the one hand and the dimension vectors of the indecomposable rigid KQ-modules on the other hand. More specifically, there is a bijection between the non-initial cluster variables and the indecomposable rigid KQ-modules such that the d-vector of a cluster variable coincides with the dimension vector of the corresponding module (see [11, 12, 13]). Furthermore, the (positive) c-vectors of $\mathcal{A}(Q)$ and the dimension vectors of the indecomposable rigid KQ-modules coincide (see [14, 27]).

However, when the initial quiver Q is not acyclic, we do not have such nice connections (see [2, 6, 9] for work in this direction). Answering a question posed to us by Nakanishi, we found an example showing the following:

Date: 26 January 2016.

2010 *Mathematics Subject Classification.* Primary: 13F60, 16G20, 16G70; Secondary: 18E30.

Key words and phrases. Almost split sequences, cluster algebras, cluster categories, cluster-tilted algebras, c-vectors, d-vectors, Q-coloured quivers, tame hereditary algebras.

This work was supported by the Engineering and Physical Sciences Research Council [grant number EP/G007497/1], the Mathematical Sciences Research Institute, Berkeley and NFR FriNat [grant number 231000].

(*) There is a cluster-tilted algebra Λ with quiver Q such that $\mathcal{A}(Q)$ has a denominator vector which is not the dimension vector of any indecomposable Λ-module.

Since we know that there are denominator vectors which are not dimension vectors, it is natural to ask if the denominator vectors can be written as a sum of a small number of dimension vectors of indecomposable rigid Λ-modules. We consider this question for cluster-tilted algebras associated to tame hereditary algebras. Note that by [17, Theorem 3.6], using [5, Theorem 5.2] and [8, Theorem 5.1], such cluster-tilted algebras are exactly the cluster-tilted algebras of tame representation type (noting that the cluster-tilted algebras of finite representation type are those arising from hereditary algebras of finite representation type by [7, Theorem A]). In this case we show that it is possible to use at most 3 summands. We do not know if it is always possible with 2 summands.

In order to prove the results discussed in the previous paragraph we need to locate the indecomposable rigid Λ-modules in the AR-quiver of Λ-mod. This investigation should be interesting in itself. Closely related is the class of indecomposable Schurian modules, which we also describe. If H is a hereditary algebra, then every indecomposable rigid (equivalently, τ-rigid) module is Schurian. So one might ask what the relationships are between the rigid, τ-rigid and Schurian Λ-modules. In general there are τ-rigid (hence rigid) Λ-modules which are not Schurian. However, it turns out that every indecomposable Λ-module which is rigid, but not τ-rigid, is Schurian.

In Section 1, we recall some basic definitions and results relating to cluster categories. In Section 2 we discuss tubes in general. In Section 3 we fix a cluster-tilting object T in a cluster category associated to a tame hereditary algebra and investigate its properties in relation to a tube. Section 4 is devoted to identifying the rigid and Schurian $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{opp}}$-modules. In Section 5, we investigate an example in the wild case which appears to behave in a similar way to the tame case. In Section 6, we give the example providing a negative answer to the question of Nakanishi. Finally, in Section 7, we also show that for cluster-tilted algebras associated to tame hereditary algebras each denominator vector is a sum of at most 3 dimension vectors of indecomposable rigid Λ-modules.

We refer to [3, 4] for standard facts from representation theory. We would like to thank Otto Kerner for helpful conversations about wild hereditary algebras.

1. Setup

In this section we recall some definitions and results related to cluster categories and rigid and τ-rigid objects. We also include some lemmas which are useful for showing that a module is Schurian or rigid.

For a modulus N, we choose representatives $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$, writing $[a]_N$ for the reduction of an integer a mod N. If $N = 0$, we take $\mathbb{Z}_N$ to be the empty set.

We fix an algebraically closed field K; all categories considered will be assumed to be K-additive. For an object X in a category $\mathcal{X}$, we denote by $\mathrm{add}(X)$ the additive subcategory generated by X. Suppose that $\mathcal{X}$ is a module category with AR-translate τ. Then we say that X is *rigid* if $\mathrm{Ext}^1(X, X) = 0$, *τ-rigid* if $\mathrm{Hom}(X, \tau X) = 0$, *Schurian* if $\mathrm{End}(X) \cong K$, or *strongly Schurian* if the multiplicity of each simple module as a composition factor is at most one. Note that any strongly Schurian module is necessarily Schurian.

If $\mathcal{X}$ is a triangulated category with shift [1] and AR-translate τ, we define rigid, τ-rigid and Schurian objects similarly, where we write $\mathrm{Ext}^1(X, Y)$ for $\mathrm{Hom}(X, Y[1])$. For both module categories and triangulated categories, we shall consider objects of the category up to isomorphism.

For modules X, Y in a module category over a finite dimensional algebra, we write $\overline{\mathrm{Hom}}(X, Y)$ for the injectively stable morphisms from X to Y, i.e. the quotient of $\mathrm{Hom}(X, Y)$ by the morphisms from X to Y which factorize through an injective module. We similarly write $\underline{\mathrm{Hom}}(X, Y)$ for the projectively stable morphisms. Then we have the AR-formula:

$$D\overline{\mathrm{Hom}}(X, \tau Y) \cong \mathrm{Ext}^1(X, Y) \cong D\underline{\mathrm{Hom}}(\tau^{-1}X, Y), \tag{1.1}$$

where D denotes the functor $\mathrm{Hom}(-, K)$.

Let $Q = (Q_0, Q_1)$ and $Q' = (Q'_0, Q'_1)$ be quivers with vertices Q_0, Q'_0 and arrows Q_1, Q'_1. Recall that a morphism of quivers from Q to Q' is a pair of maps $f_i : Q_i \to Q'_i$, $i = 0, 1$, such that whenever $\alpha : i \to j$ is an arrow in Q, we have that $f_1(\alpha)$ starts at $f_0(i)$ and ends at $f_0(j)$. In order to describe the modules we are working with, it is convenient to use notation from [25], which we now recall.

Definition 1.1. Let Q be a quiver with vertices Q_0. A Q-*coloured quiver* is a pair (Γ, π), where Γ is a quiver and $\pi : \Gamma \to Q$ is a morphism of quivers. We shall always assume that Γ is a tree.

As Ringel points out, a Q-coloured quiver (Γ, π) can be regarded as a quiver Γ in which each vertex is coloured by a vertex of Q and each arrow is coloured by an arrow of Q. In addition, if an arrow $\gamma : v \to w$ in Γ is coloured by an arrow $\alpha : i \to j$ in Q then v must be coloured with i and w must be coloured with j. We shall draw Q-coloured quivers in this way. Thus each vertex v of Γ will be labelled with its image $\pi(v) \in Q_0$, and each arrow a of Γ will be labelled with its image $\pi(a)$ in Q_1. But note that if Q has no multiple arrows then we can omit the arrow labels, since the label of an arrow in Γ is determined by the labels of its endpoints.

We shall also omit the orientation of the arrows in Γ, adopting the convention that the arrows always point down the page.

As in [25, Remark 4], a Q-coloured quiver (Γ, π) determines a representation $V = V(\Gamma, \pi)$ of Q over K (and hence a KQ-module) in the following way. For each $i \in Q_0$, let V_i be the vector space with basis given by $B_i = \pi^{-1}(i) \subseteq \Gamma_0$. Given an arrow $\alpha : i \to j$ in Q and $b \in \pi^{-1}(i)$, we define

$$\varphi_\alpha(b) = \sum_{b \xrightarrow{\alpha} c \text{ in } \Gamma} c, \tag{1.2}$$

extending linearly.

If $A = KQ/I$, where I is an admissible ideal, and V satisfies the relations coming from the elements of I then it is an A-module. Note that, in general, not every A-module will arise in this way (for example, over the Kronecker algebra). Also, a given module may be definable using more than one Q-coloured quiver (by changing basis).

As an example of a coloured quiver, consider the quiver Q:

$$\begin{tikzcd} 1 \arrow[r] \arrow[rrr, bend left] & 2 \arrow[r] & 3 \arrow[r] & 4. \end{tikzcd} \tag{1.3}$$

Then we have the following Q-coloured quivers and corresponding representations:

$$T_2 = \begin{array}{c} 1 \\ | \\ 4 \end{array}, \qquad \begin{tikzcd} K \arrow[r] \arrow[rrr, bend left, "\mathrm{id}"] & 0 \arrow[r] & 0 \arrow[r] & K. \end{tikzcd} \tag{1.4}$$

$$T_3 = \begin{array}{ccc} & 1 & \\ / & & \backslash \\ 2 & & 4 \end{array}, \qquad \begin{tikzcd} K \arrow[r, "\mathrm{id}"'] \arrow[rrr, bend left, "\mathrm{id}"] & K \arrow[r] & 0 \arrow[r] & K. \end{tikzcd} \tag{1.5}$$

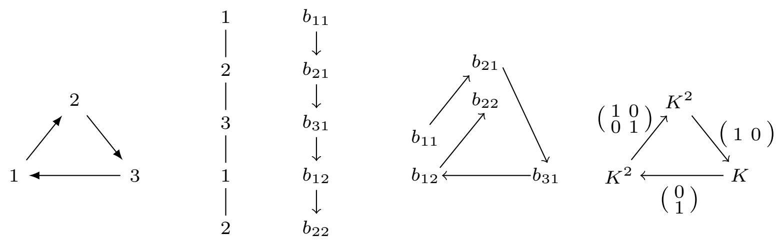

FIGURE 1. A quiver Q, a Q-coloured quiver, together with the redrawing according to Remark 1.2 and the corresponding representation of Q.

Remark 1.2. We will sometimes label the vertices of a Q_Λ-coloured quiver (Γ, π) by writing

$$\pi^{-1}(i) = \{b_{ik} \,:\, k = 1, 2, \ldots\}$$

for i a vertex of Q. Then, if α is an arrow from i to j in Q, (1.2) becomes:

$$\varphi_\alpha(b_{ik}) = \sum_{l,\ b_{ik} \xrightarrow{\alpha} b_{jl} \text{ in } \Gamma} b_{jl}.$$

To aid with calculations, we may also redraw Γ, placing all of the basis elements b_{ij} (for fixed i) close together (according to a fixed embedding of Q in the plane). In this case, we must include the arrowheads on the arrows so that this information is not lost. For an example, see Figure 1.

Definition 1.3. If (Γ, π), (Γ', π') are Q-coloured quivers then we call a map $\varphi : \Gamma \to \Gamma'$ a morphism of Q-coloured quivers if it is a morphism of quivers and $\pi = \pi'\varphi$.

If Γ'' is a full subquiver of Γ and π'' is the restriction of π to Γ then Γ'' is called a Q-coloured subquiver of (Γ, π); note that it is again a Q-coloured quiver.

Remark 1.4. If (Γ', π') is a Q-coloured subquiver of (Γ, π) with the property that every arrow between a vertex in Γ' and a vertex in Γ not in Γ' points towards the vertex in Γ', then it is easy to see that there is a corresponding embedding of modules $V(\Gamma', \pi') \hookrightarrow V(\Gamma, \pi)$. Similarly, if every such arrow points towards Γ', there is a corresponding quotient map $V(\Gamma, \pi) \twoheadrightarrow V(\Gamma', \pi')$.

Let $(\Gamma(1), \pi(1))$ and $(\Gamma(2), \pi(2))$ be Q-coloured quivers. Suppose that there is a Q-coloured quiver (Γ, π) which is isomorphic to a Q-coloured subquiver of $(\Gamma(1), \pi(1))$ with the second property above. Suppose in addition that it is isomorphic to a Q-coloured subquiver of $(\Gamma(2), \pi(2))$ with the first property above. Then there is a KQ-module homomorphism $V(\Gamma(1), \pi(1)) \to V(\Gamma(2), \pi(2))$ given by the composition of the quotient map and the embedding given above.

We fix a quiver Q such that the path algebra KQ has tame representation type. For example, we could take Q to be the quiver (1.3). We denote by KQ-mod the category of finite-dimensional KQ-modules, with AR-translate τ.

We denote by $D^b(KQ)$ the bounded derived category of KQ-mod, with AR-translate also denoted by τ. For objects X and Y in $D^b(KQ)$, we write $\mathrm{Hom}(X, Y)$ for $\mathrm{Hom}_{D^b(KQ)}(X, Y)$ and $\mathrm{Ext}(X, Y)$ for $\mathrm{Ext}_{D^b(KQ)}(X, Y)$. Note that if X, Y are modules, these coincide with $\mathrm{Hom}_{KQ}(X, Y)$ and $\mathrm{Ext}_{KQ}(X, Y)$ respectively.

The category $D^b(KQ)$ is triangulated. Let $\mathcal{C} = \mathcal{C}_Q$ denote the cluster category corresponding to Q, i.e. the orbit category $\mathcal{C}_Q = D^b(KQ)/F$, where F denotes the autoequivalence $\tau^{-1}[1]$ (see [10]). The category $\mathcal{C}$ is triangulated by [20, §4]. Note that an object in $D^b(KQ)$-mod can be regarded as an object in $\mathcal{C}$; in particular this applies to KQ-modules, which can be identified with complexes in $D^b(KQ)$ concentrated in degree zero.

If X, Y are KQ-modules regarded as objects in $\mathcal{C}$, then

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}(X,Y) \oplus \operatorname{Hom}(X,FY)$$

by [10, Prop. 1.5]. We write $\operatorname{Hom}^H_{\mathcal{C}}(X,Y) = \operatorname{Hom}(X,Y)$ and refer to elements of this space as *H-maps* from X to Y, and we write $\operatorname{Hom}(X,FY) = \operatorname{Hom}^F_{\mathcal{C}}(X,Y)$ and refer to elements of this space as *F-maps* from X to Y. So, we have:

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}^H_{\mathcal{C}}(X,Y) \oplus \operatorname{Hom}^F_{\mathcal{C}}(X,Y).$$

Note that

$$\begin{aligned} \operatorname{Hom}^F_{\mathcal{C}}(X,Y) &= \operatorname{Hom}(X,FY) = \operatorname{Hom}(X,\tau^{-1}Y[1]) \\ &\cong \operatorname{Ext}(X,\tau^{-1}Y) \cong D\operatorname{Hom}(\tau^{-1}Y,\tau X) \cong D\operatorname{Hom}(Y,\tau^2 X), \end{aligned} \tag{1.6}$$

where $D = \operatorname{Hom}(-,K)$. If χ is an additive subcategory of $\mathcal{C}$, we write:

$$\operatorname{Hom}^H_{\mathcal{C}/\chi}(X,Y), \quad \operatorname{Hom}^F_{\mathcal{C}/\chi}(X,Y)$$

for the quotients of $\operatorname{Hom}^H_{\mathcal{C}}(X,Y)$ and $\operatorname{Hom}^F_{\mathcal{C}}(X,Y)$ by the morphisms in $\mathcal{C}$ factoring through χ.

A rigid object T in $\mathcal{C}$ is said to be *cluster-tilting* if, for any object X in $\mathcal{C}$, we have $\operatorname{Ext}^1_{\mathcal{C}}(T,X) = 0$ if and only if X lies in $\operatorname{add}(T)$.

We fix a cluster-tilting object T in $\mathcal{C}$. We make the following assumption. As explained in the proof of Theorem 4.10, to find the rigid and Schurian modules for any cluster-tilted algebra arising from $\mathcal{C}$, it is enough to find the rigid and Schurian modules in this case.

Assumption 1.5. The cluster-tilting object T is induced by a KQ-module (which we also denote by T). Furthermore, T is of the form $U \oplus T'$, where U is preprojective and T' is regular. Note that the module T is a tilting module by [10].

Example 1.6. For example, if Q is the quiver in (1.3), we could take T to be the tilting module:

$$T = P_1 \oplus T_2 \oplus T_3 \oplus P_4, \tag{1.7}$$

where T_2 and T_3 are the KQ-modules defined in (1.4), (1.5). Note that T can be obtained from $P_1 \oplus P_2 \oplus P_3 \oplus P_4$ by mutating (in the sense of [18, 24]) first at P_2 and then at P_3. The modules T_2 and T_3 lie in a tube of rank 3 in KQ-mod; see Figure 2.

We define $\Lambda = \Lambda_T = \operatorname{End}_{\mathcal{C}_Q}(T)$ to be the corresponding cluster-tilted algebra. For Example 1.6, Λ is given by the quiver with relations shown in Figure 3 (we indicate how to compute such a quiver with relations explicitly for a similar example in Section 5). Note that this quiver can be obtained from Q by mutating (in the sense of [15]) first at 2 and then at 3.

There is a natural functor $\operatorname{Hom}_{\mathcal{C}}(T,-)$ from $\mathcal{C}$ to Λ-mod. We have:

Theorem 1.7. [7, Thm. A] *The functor* $\operatorname{Hom}_{\mathcal{C}}(T,-)$ *induces an equivalence from the additive quotient* $\mathcal{C}/\operatorname{add}(\tau T)$ *to* Λ*-mod.*

We denote the image of an object X in $\mathcal{C}$ under the functor $\operatorname{Hom}_{\mathcal{C}}(T,-)$ by $\widetilde{X}$. We note the following:

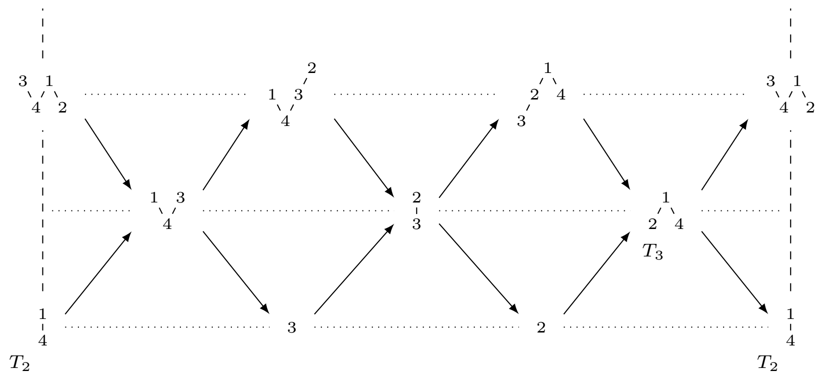

FIGURE 2. Part of the AR-quiver of KQ-mod, where Q is the quiver in (1.3).

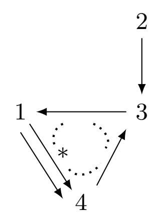

FIGURE 3. The endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{opp}}$ for the tilting module in Example 1.6.

Proposition 1.8. *Let X be an object in $\mathcal{C}$ and $\widetilde{X}$ the corresponding Λ-module. Then*
(a) $\widetilde{X}$ is Schurian if and only if

$$\mathrm{Hom}_{C/\,\mathrm{add}(\tau T)}(X,X) \cong K.$$

(b) $\widetilde{X}$ is rigid if and only if

$$\mathrm{Hom}_{\mathcal{C}/\,\mathrm{add}(\tau T\oplus\tau^2 T)}(X,\tau X) = 0.$$

Proof. Part (a) follows from the equivalence in Theorem 1.7. Part (b) follows from this combined with the AR-formula (1.1), noting that the injective modules in Λ-mod are the objects in the subcategory $\mathrm{add}\,\mathrm{Hom}_{\mathcal{C}}(T,\tau^2 T)$ (see [7], [21, §2]). □

The following statement follows from [1, Thm. 4.1].

Theorem 1.9. [1] *The functor $\mathrm{Hom}_{\mathcal{C}}(T,-)$ induces a bijection between isomorphism classes of indecomposable rigid objects in $\mathcal{C}$ which are not summands of τT and isomorphism classes of indecomposable τ-rigid Λ-modules.*

Since a KQ-module is rigid if and only if the induced object of $\mathcal{C}$ is rigid (by [10, Prop. 1.7]), we have:

Corollary 1.10. *If X is a KQ-module not in $\mathrm{add}(\tau T)$ then X is rigid in KQ-mod if and only if $\widetilde{X}$ is τ-rigid in Λ-mod.*

Since (for modules over any finite-dimensional algebra) every τ-rigid module is rigid, we have that $\widetilde{X}$ is a rigid Λ-module for any rigid KQ-module X.

Remark 1.11. Suppose that X is an indecomposable object of $D^b(KQ)$ which is either a preprojective KQ-module, a preinjective KQ-module or the shift of a projective KQ-module. Assume also that X is not a direct summand of τT. Then X is rigid in $D^b(KQ)$, hence (by [10, Prop. 1.7]) rigid in $\mathcal{C}$. By Theorem 1.9, $\widetilde{X}$ is τ-rigid in Λ-mod. Furthermore, X is Schurian in $D^b(KQ)$. We also have $\mathrm{Hom}^F_{\mathcal{C}}(X,X) \cong D\,\mathrm{Hom}(X,\tau^2 X) = 0$ by (1.6), so X is a Schurian object of $\mathcal{C}$. It follows that $\widetilde{X}$ is a Schurian Λ-module by Proposition 1.8(a). Thus we see that, for any indecomposable transjective object of $\mathcal{C}$ (not a summand of τT), the corresponding Λ-module is Schurian and τ-rigid, hence rigid. Thus the main work in classifying indecomposable Schurian and (τ-)rigid Λ-modules concerns those which arise from tubes in KQ-mod.

Finally, we include some lemmas which will be useful for checking whether a given Λ-module is Schurian or rigid.

Lemma 1.12. *Let X, Y, Z be KQ-modules, regarded as objects in $\mathcal{C}$. Let $f \in \mathrm{Hom}^F_{\mathcal{C}}(X,Y) = \mathrm{Hom}(X, FY)$. Then f factorizes in $\mathcal{C}$ through Z if and only if it factorizes in $D^b(KQ)$ through Z or $F(Z)$.*

Proof. Since f is an F-map, it can only factorize through Z in $\mathcal{C}$ as an H-map followed by an F-map or an F-map followed by an H-map. The former case corresponds to factorizing through Z in $D^b(KQ)$ and the latter case corresponds to factorizing through $F(Z)$ in $D^b(KQ)$. □

Proposition 1.13. *Let A, B, C be objects in $D^b(KQ)$.*

(a) Let $\alpha : A \to C$ and

$$\mathrm{Hom}(B,\tau\alpha) : \mathrm{Hom}(B,\tau A) \to \mathrm{Hom}(B,\tau C)$$

and

$$\mathrm{Hom}(\alpha, B[1]) : \mathrm{Hom}(C, B[1]) \to \mathrm{Hom}(A, B[1])$$

Then $\mathrm{Hom}(B,\tau\alpha)$ is nonzero (respectively, injective, surjective, or an isomorphism) if and only if $\mathrm{Hom}(\alpha, B[1])$ is nonzero (respectively, surjective, injective or an isomorphism). We illustrate the maps $\mathrm{Hom}(B,\tau\alpha)$ and $\mathrm{Hom}(\alpha, B[1])$ below for ease of reference.

$$\begin{tikzcd} B \arrow[rr, "\mathrm{Hom}(B,\tau\alpha)(\gamma)"] \arrow[dr, "\gamma"'] & & \tau C \\ & \tau A \arrow[ur, "\tau\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\mathrm{Hom}(\alpha,B[1])(\delta)"] \arrow[dr, "\alpha"'] & & B[1] \\ & C \arrow[ur, "\delta"'] & \end{tikzcd}$$

(b) Let $\beta : C \to B$ and consider the induced maps:

$$\mathrm{Hom}(\beta,\tau A) : \mathrm{Hom}(B,\tau A) \to \mathrm{Hom}(C,\tau A)$$

and

$$\mathrm{Hom}(A,\beta[1]) : \mathrm{Hom}(A, C[1]) \to \mathrm{Hom}(A, B[1])$$

Then $\mathrm{Hom}(\beta, \tau A)$ *is nonzero (respectively, injective, surjective, or an isomorphism) if and only if* $\mathrm{Hom}(A, \beta[1])$ *is nonzero (respectively, surjective, injective or an isomorphism). We illustrate the maps* $\mathrm{Hom}(\beta, \tau A)$ *and* $\mathrm{Hom}(A, \beta[1])$ *below for ease of reference.*

$$\begin{tikzcd} C \arrow[rr, "\mathrm{Hom}(\beta,\tau A)(\gamma)"] \arrow[dr, "\beta"'] & & \tau A \\ & B \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\mathrm{Hom}(A,\beta[1])(\delta)"] \arrow[dr, "\delta"'] & & B[1] \\ & C[1] \arrow[ur, "\beta[1]"'] & \end{tikzcd}$$

Proof. Part (a) follows from the commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(C, B[1]) \arrow[r, "\sim"'] \arrow[d, "\mathrm{Hom}(\alpha, B[1])"'] & \mathrm{Ext}(C, B) \arrow[r, "\sim"'] & D\,\mathrm{Hom}(B, \tau C) \arrow[d, "D\,\mathrm{Hom}(B,\tau\alpha)"] \\ \mathrm{Hom}(A, B[1]) \arrow[r, "\sim"'] & \mathrm{Ext}(A, B) \arrow[r, "\sim"'] & D\,\mathrm{Hom}(B, \tau A). \end{tikzcd}$$

Part (b) follows from the commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(A, C[1]) \arrow[r, "\sim"'] \arrow[d, "\mathrm{Hom}(A, \beta[1])"'] & \mathrm{Ext}(A, C) \arrow[r, "\sim"'] & D\,\mathrm{Hom}(C, \tau A) \arrow[d, "D\,\mathrm{Hom}(\beta,\tau A)"] \\ \mathrm{Hom}(A, B[1]) \arrow[r, "\sim"'] & \mathrm{Ext}(A, B) \arrow[r, "\sim"'] & D\,\mathrm{Hom}(B, \tau A) \end{tikzcd}$$

□

Proposition 1.14. *Let* A, B *and* C *be indecomposable* KQ*-modules and suppose that* $\mathrm{Hom}(A, B[1]) \cong K$*. Let* $\varepsilon : A \to B[1]$ *be a nonzero map.*

(a) *The map* ε *factors through* C *if and only if there is a map* $\alpha \in \mathrm{Hom}(A, C)$ *such that* $\mathrm{Hom}(B, \tau\alpha) \neq 0$.

(b) *The map* ε *factors through* $C[1]$ *if and only if there is a map* $\beta \in \mathrm{Hom}(C, B)$ *such that* $\mathrm{Hom}(\beta, \tau A) \neq 0$.

Proof. Since $\mathrm{Hom}(A, B[1]) \cong K$, the map ε factors through C if and only if $\mathrm{Hom}(\alpha, B[1]) \neq 0$ for some $\alpha \in \mathrm{Hom}(A, C)$. Part (a) then follows from Proposition 1.13(a). Similarly, ε factors through $C[1]$ if and only if $\mathrm{Hom}(A, \beta[1]) \neq 0$ for some $\beta \in \mathrm{Hom}(C, B)$. Part (b) then follows from Proposition 1.13(b). □

2. Tubes

In this section we recall some facts concerning tubes in KQ-mod. We fix such a tube $\mathcal{T}$, of rank r. Note that $\mathcal{T}$ is standard, i.e. the subcategory of $\mathcal{T}$ consisting of the indecomposable objects is equivalent to the mesh category of the AR-quiver of $\mathcal{T}$.

Let Q_i, for $i \in \mathbb{Z}_r$ be the quasisimple modules in $\mathcal{T}$. Then, for each $i \in \mathbb{Z}_r$ and $l \in \mathbb{N}$, there is an indecomposable module $M_{i,l}$ in $\mathcal{T}$ with socle Q_i and quasilength l; these modules exhaust the indecomposable modules in $\mathcal{T}$. For $i \in \mathbb{Z}$ and $l \in \mathbb{N}$, we define $Q_i = Q_{[i]_r}$ and $M_{i,l} = M_{[i]_r,l}$. Note that the socle of $M_{i,l}$ is $M_{i,1}$. We denote the quasilength l of a module $M = M_{i,l}$ by $\mathrm{ql}(M)$. The AR-quiver of $\mathcal{T}$ is shown in Figure 4 (for the case $r = 3$).

Lemma 2.1. *Let* X *be an object in* $\mathcal{T}$ *of quasilength* ℓ*. Then any path in the AR-quiver of* $\mathcal{T}$ *with at least* ℓ *downward arrows must be zero in* KQ*-mod.*

$$
\begin{tikzcd}[column sep=small]
M_{2,3} \arrow[rr, dotted, no head] \arrow[dr] & & M_{0,3} \arrow[rr, dotted, no head] \arrow[dr] & & M_{1,3} \arrow[rr, dotted, no head] \arrow[dr] & & M_{2,3} \\
& M_{0,2} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & M_{1,2} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & M_{2,2} \arrow[ur] \arrow[dr] & \\
M_{0,1} \arrow[rr, dotted, no head] \arrow[ur] & & M_{1,1} \arrow[rr, dotted, no head] \arrow[ur] & & M_{2,1} \arrow[rr, dotted, no head] \arrow[ur] & & M_{0,1}
\end{tikzcd}
$$

FIGURE 4. The AR-quiver of a tube of rank 3

Proof. By applying mesh relations if necessary, we can rewrite the path as a product of $\ell - 1$ downward arrows (the maximum number possible) followed by an upward arrow and a downward arrow (followed, possibly, by more arrows). Hence the path is zero. $\square$

The following is well-known.

Lemma 2.2. *Let $M_{i,l}$, with $0 \leq l \leq r-1$, and $M_{j,m}$ be objects in $\mathcal{T}$. Then we have the following: (see Figure 5 for an example).*

(a)

$$
\mathrm{Hom}(M_{i,l}, M_{j,m}) \cong \begin{cases} K, & \textit{if } 1 \leq m \leq l-1 \textit{ and } j \textit{ is congruent to a member of} \\ & [i+l-m, i+l-1] \textit{ mod } r; \\ K, & \textit{if } m \geq l \textit{ and } j \textit{ is congruent to a member of } [i, i+l-1] \textit{ mod } r; \\ 0, & \textit{otherwise.} \end{cases}
$$

(b)

$$
\mathrm{Hom}(M_{j,m}, M_{i,l}) \cong \begin{cases} K, & \textit{if } 1 \leq m \leq l-1 \textit{ and } j \textit{ is congruent to a member of } [i-m+1, i] \\ & \textit{mod } r; \\ K, & \textit{if } m \geq l \textit{ and } j \textit{ is congruent to a member of } [i-m+1, i-m+l] \\ & \textit{mod } r; \\ 0, & \textit{otherwise.} \end{cases}
$$

Proof. We first consider part (a). Note that, since the quasilength of $M_{i,l}$ is assumed to be at most r, the rays starting at $M_{i+p,l-p}$ for $0 \leq p \leq l-1$ do not intersect each other. It is then easy to see that, up to mesh relations, there is exactly one path in the AR-quiver of $\mathcal{T}$ from $M_{i,l}$ to the objects in these rays and no path to any other object in $\mathcal{T}$. The result then follows from the fact that $\mathcal{T}$ is standard. A similar proof gives part (b). $\square$

Let $M_{i,l}$ be an indecomposable module in $\mathcal{T}$. The *wing* $\mathcal{W}_{M_{i,l}}$ of $M_{i,l}$ is given by:

$$
\mathcal{W}_{M_{i,l}} = \{M_{j,m} : i \leq j \leq i+l-1,\ 1 \leq m \leq l+i-j\}.
$$

Now fix $M_{i,l} \in \mathcal{T}$ with $l \leq r$. It follows from Lemma 2.2 that if the quasisocle of $X \in \mathcal{T}$ does not lie in $\mathcal{W}_{M_{i,l}}$ then $\mathrm{Hom}(M_{i,l}, X) = 0$. Similarly, if the quasitop of X does not lie in $\mathcal{W}_{M_{i,l}}$ then $\mathrm{Hom}(X, M_{i,l}) = 0$. This implies the following, which we state here as we shall use it often.

Corollary 2.3. *Let M, N, X be indecomposable objects in $\mathcal{T}$, and suppose that M has quasilength at most r, and $M \in \mathcal{W}_N$.*

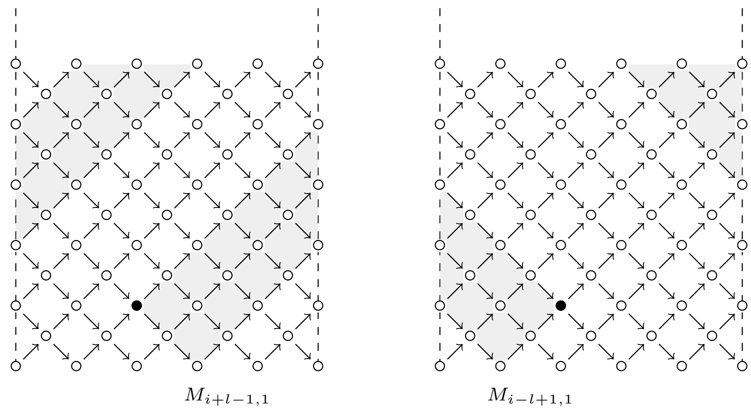

FIGURE 5. The left hand figure shows the modules X in $\mathcal{T}$ for which $\mathrm{Hom}(M_{i,l}, X) \neq 0$ (in the shaded region), for the case $r = 5$. The module $M_{i,l}$ is denoted by a filled-in circle. The right hand figure shows the modules X with $\mathrm{Hom}(X, M_{i,l}) \neq 0$.

(a) *If the quasisocle of X does not lie in $\mathcal{W}_N$ then* $\mathrm{Hom}(M, X) = 0$.
(b) *If the quasitop of X does not lie in $\mathcal{W}_N$ then* $\mathrm{Hom}(X, M) = 0$.

Lemma 2.4. *Let $M_{i,l}$ be an indecomposable module in $\mathcal{T}$. Then we have:*

$$\dim \mathrm{End}(M_{i,l}) = \begin{cases} 1, & 1 \leq l \leq r; \\ 2, & r+1 \leq l \leq 2r; \end{cases} \qquad \dim \mathrm{Hom}(M_{i,l}, \tau M_{i,l}) = \begin{cases} 0, & 1 \leq l \leq r-1; \\ 1, & r \leq l \leq 2r-1; \end{cases}$$

$$\dim \mathrm{Hom}(M_{i,l}, \tau^2 M_{i,l}) = \begin{cases} 0, & 1 \leq l \leq r-2; \\ 1, & r-1 \leq l \leq 2r-2. \end{cases}$$

Proof. The formulas are easily checked using the fact that $\mathcal{T}$ is standard. □

The last lemma in this section also follows from the fact that $\mathcal{T}$ is standard (since the mesh relations are homogeneous).

Lemma 2.5. *Let X, Y be indecomposable objects in $\mathcal{T}$, and let $\pi_1(X,Y), \ldots, \pi_t(X,Y)$ be representatives for the paths in $\mathcal{T}$ from X to Y up to equivalence via the mesh relations. Then the corresponding maps $f_1(X,Y), \ldots, f_t(X,Y)$ form a basis for* $\mathrm{Hom}(X, Y)$.

3. Properties of T with respect to a tube

In this section, we collect together some useful facts that we shall use in Section 4 to determine the rigid and Schurian Λ-modules.

Recall that we have fixed a tube $\mathcal{T}$ in KQ-mod of rank r. Let $T_{\mathcal{T}}$ be the direct sum of the indecomposable summands of T lying in $\mathcal{T}$ (we include the case $T_{\mathcal{T}} = 0$). Let T_k, $k \in \mathbb{Z}_s$ be the indecomposable summands of $T_{\mathcal{T}}$ which are not contained in the wing of any other indecomposable summand of $T_{\mathcal{T}}$, numbered in order cyclically around $\mathcal{T}$. The indecomposable summands of $T_{\mathcal{T}}$ are contained in $\cup_{k \in \mathbb{Z}_s} \mathcal{W}_{T_k}$, where $\mathcal{W}_{T_k}$ denotes the wing of T_k. Note that if $T_{\mathcal{T}} = 0$ then $s = 0$ and $\mathbb{Z}_s$ is the empty set.

A key role is played by the modules τT_k. Let $i_k \in \{0, 1, \dots, r-1\}$ and $l_k \in \mathbb{N}$ be integers such that

$$\tau T_k \cong M_{i_k, l_k}.$$

Note that $l_k \leq r-1$, since T_k is rigid. Then we have the wings

$$\mathcal{W}_{T_k} = \{M_{i,l} \,:\, i_k + 1 \leq i \leq i_k + l_k, 1 \leq l \leq l_k + i_k + 1 - i\}, \tag{3.1}$$

$$\mathcal{W}_{\tau T_k} = \{M_{i,l} \,:\, i_k \leq i \leq i_k + l_k - 1, 1 \leq l \leq l_k + i_k - i\}, \tag{3.2}$$

$$\mathcal{W}_{\tau^2 T_k} = \{M_{i,l} \,:\, i_k - 1 \leq i \leq i_k + l_k - 2, 1 \leq l \leq l_k + i_k - 1 - i\}, \tag{3.3}$$

For $k \in \mathbb{Z}_s$, the quasisimple objects in $\mathcal{W}_{\tau T_k}$ are the Q_i for $i_k \leq i \leq i_k + l_k - 1$. Note that, since $\mathrm{Ext}^1(T, T) = 0$, we have $[i_{k+1} - (i_k + l_k - 1)]_r \neq 0, 1$ (by Lemma 2.2 and the AR-formula). In other words, two successive wings $\mathcal{W}_{\tau T_k}$ and $\mathcal{W}_{\tau T_{k+1}}$ are always separated by at least one quasisimple module.

For $k \in \mathbb{Z}_s$, we define Top_k to be the module $M_{i_k, r+l_k}$. Note that Top_k is the module of smallest quasilength in the intersection of the ray through the injective objects in $\mathcal{W}_{\tau T_k}$ and the coray through the projective objects in $\mathcal{W}_{\tau T_k}$. Let $\mathcal{H}_k$ be the part of $\mathcal{W}_{\mathrm{Top}_k}$ consisting of injective or projective objects in $\mathcal{W}_{\mathrm{Top}_k}$ of quasilength at least r. So

$$\mathcal{H}_k = \{M_{i_k, l} \,:\, r \leq l \leq r + l_k\} \cup \{M_{i_k+p, r+l_k-p} \,:\, 0 \leq p \leq l_k\}. \tag{3.4}$$

The unique object in both of these sets is $\mathrm{Top}_k = M_{i_k, r+l_k}$, the unique projective-injective object in $\mathcal{W}_{\mathrm{Top}_k}$.

Let $\mathcal{R}_k$ (respectively, $\mathcal{S}_k$) be the part of $\mathcal{W}_{\mathrm{Top}_k}$ consisting of non-projective, non-injective objects in $\mathcal{W}_{\mathrm{Top}_k}$ of quasilength at least r (respectively, at least $r-1$). Note that $\mathcal{R}_k \subseteq \mathcal{S}_k$. We have:

$$\mathcal{R}_k = \{M_{i,l} \,:\, i_k + 1 \leq i \leq i_k + l_k - 1,\, r \leq l \leq r + l_k + i_k - i - 1\}; \tag{3.5}$$

$$\mathcal{S}_k = \{M_{i,l} \,:\, i_k + 1 \leq i \leq i_k + l_k,\, r - 1 \leq l \leq r + l_k + i_k - i - 1\}. \tag{3.6}$$

An example is shown in Figure 6.

Lemma 3.1. *The quasisocles of the indecomposable objects in $\mathcal{R}_k$ (respectively, $\mathcal{S}_k$) are the Q_i where $i_k + 1 \leq i \leq i_k + l_k - 1$ (respectively, $i_k + 1 \leq i \leq i_k + l_k$) and the quasitops are the Q_i where $i_k \leq i \leq i_k + l_k - 2$ (respectively, $i_k - 1 \leq i \leq i_k + l_k - 2$). In particular, the quasisocle of an indecomposable object in $\mathcal{R}_k$ lies in $\mathcal{W}_{T_k} \cap \mathcal{W}_{\tau T_k}$ (respectively, in $\mathcal{W}_{T_k}$). The quasitop of an indecomposable object in $\mathcal{R}_k$ (respectively, $\mathcal{S}_k$) lies in $\mathcal{W}_{\tau T_k} \cap \mathcal{W}_{\tau^2 T_k}$ (respectively, $\mathcal{W}_{\tau^2 T_k}$).*

Proof. The first statement follows from (3.5). The quasitop of $M_{i,l}$ is Q_{i+l-1}. Hence, the quasitops of the indecomposable objects in $\mathcal{R}_k$ are the Q_i with

$$(i_k + 1) + r - 1 \leq i \leq (i_k + l_k - 1) + (r + l_k + i_k - (i_k + l_k - 1) - 1) - 1,$$

i.e.

$$i_k + r \leq i \leq r + l_k + i_k - 2.$$

i.e. the Q_i with $i_k \leq i \leq i_k + l_k - 2$, since we are working mod r. Similarly, the quasitops of the indecomposable objects in $\mathcal{S}_k$ are the Q_i with

$$(i_k + 1) + (r - 1) - 1 \leq i \leq (i_k + l_k) + (r + l_k + i_k - (i_k + l_k) - 1) - 1,$$

i.e.

$$i_k + r - 1 \leq i \leq r + l_k + i_k - 2,$$

i.e. the Q_i with $i_k - 1 \leq i \leq i_k + l_k - 2$. The last statements follow from the descriptions of the wings $\mathcal{W}_{T_k}$, $\mathcal{W}_{\tau T_k}$ and $\mathcal{W}_{\tau^2 T_k}$ above (3.1). It is easy to observe the result in this lemma in Figure 6, where the regions $\mathcal{R}_k$ and $\mathcal{S}_k$ are indicated. $\square$

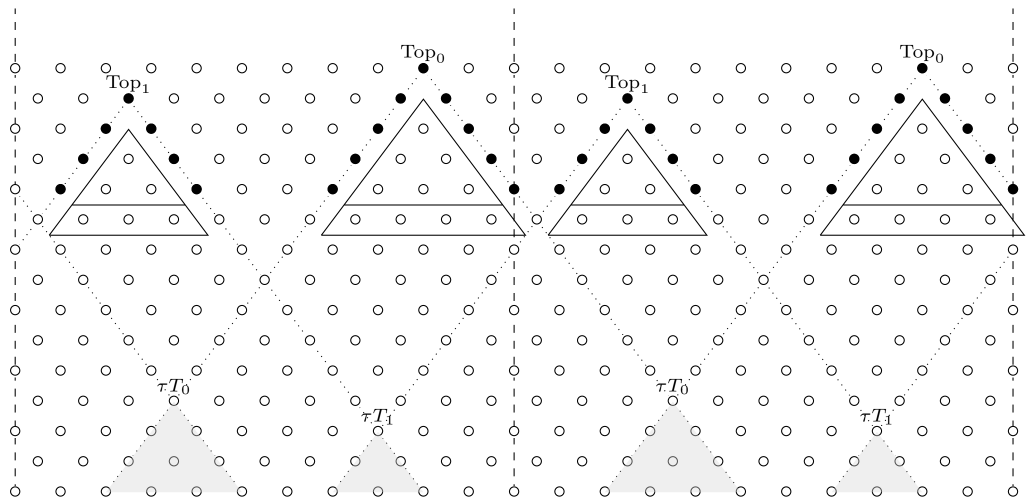

FIGURE 6. The wings $\mathcal{W}_{\tau T_k}$ shown as shaded regions in two copies of $\mathcal{T}$ in the case $r = 11$. The elements in the $\mathcal{H}_k$ are drawn as filled dots. The elements in the regions $\mathcal{R}_k$ and $\mathcal{S}_k$ are enclosed in triangles.

Recall that U denotes the maximal preprojective direct summand of T.

Lemma 3.2. *Let $k \in \mathbb{Z}_s$ and let X be an indecomposable object in $\mathcal{W}_{\tau T_k}$. Then* $\operatorname{Hom}(U, X) = 0$.

Proof. Since U is preprojective, $\operatorname{Hom}(U, -)$ is exact on short exact sequences of modules in $\mathcal{T}$, so $\dim \operatorname{Hom}(U, -)$ is additive on such sequences. This includes, in particular, almost split sequences in $\mathcal{T}$, and it follows that:

$$\dim \operatorname{Hom}(U, X) = \sum_{Y \in \mathcal{W}_X, Y \text{ quasisimple}} \dim \operatorname{Hom}(U, Y). \tag{3.7}$$

Since $\operatorname{Hom}(U, \tau T_k) = 0$, we must have $\operatorname{Hom}(U, Y) = 0$ for all quasisimple modules in $\mathcal{W}_{\tau T_k}$, and the result now follows from (3.7). $\square$

Lemma 3.3. *Suppose that $Y \in \operatorname{ind}(\mathcal{T})$ satisfies* $\operatorname{Hom}_{\mathcal{C}}(T_{\mathcal{T}}, Y) = 0$ *and* $\operatorname{Hom}(U, Y) = 0$. *Then $Y \in \cup_{k \in \mathbb{Z}_s} \mathcal{W}_{\tau T_k}$.*

Proof. Suppose Y satisfies the assumptions above. Then, if V is an indecomposable summand of T in a tube distinct from $\mathcal{T}$, we have $\operatorname{Hom}(V, Y) = 0$ and $\operatorname{Hom}(Y, \tau^2 V) = 0$, so $\operatorname{Hom}_{\mathcal{C}}(V, Y) = 0$. We also have that $\operatorname{Hom}(Y, \tau^2 U) = 0$, since $\tau^2 U$ is preprojective, so $\operatorname{Hom}_{\mathcal{C}}(U, Y) = 0$. Hence, we have $\operatorname{Hom}_{\mathcal{C}}(T, Y) = 0$, so $\operatorname{Ext}_{\mathcal{C}}(Y, \tau T) = 0$. Since T (and hence τT) is a cluster-tilting object in $\mathcal{C}$, this implies that Y lies in $\operatorname{add} \tau T$ and therefore in $\cup_{k \in \mathbb{Z}_s} \mathcal{W}_{\tau T_k}$ as required. $\square$

Proposition 3.4. *Let X be an indecomposable object in $\mathcal{T}$ not lying in $\cup_{k \in \mathbb{Z}_s} \mathcal{W}_{\tau T_k}$. Then* $\operatorname{Hom}(U, X) \neq 0$.

Proof. Since $\dim \operatorname{Hom}(U, -)$ is additive on $\mathcal{T}$, we can assume that X is quasisimple. We assume, for a contradiction, that $\operatorname{Hom}(U, X) = 0$. If we can find a module $Y \in \mathcal{T} \setminus \cup_{k \in \mathbb{Z}_s} \mathcal{W}_{\tau T_k}$ such that $\operatorname{Hom}_{\mathcal{C}}(T_{\mathcal{T}}, Y) = 0$ and $\operatorname{Hom}(U, Y) = 0$ then, by Lemma 3.3, we have a contradiction. We now construct such a module Y, considering various cases for X.

Case 1: Assume that $X \not\cong Q_{i_k-1}$ and $X \not\cong Q_{i_k+l_k}$ for any $k \in \mathbb{Z}_s$, i.e. that X is not immediately adjacent to any of the wings $\mathcal{W}_{\tau T_k}$, $k \in \mathbb{Z}_s$. There is a single module of this kind in the example in Figure 6; this is denoted by X_1 in Figure 7. In this case we take $Y = X$.

If V is an indecomposable summand of $T_{\mathcal{T}}$, then $V \in \mathcal{W}_{T_k}$ for some $k \in \mathbb{Z}_s$. Since the quasisimple module X does not lie in $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{T_k}$, we have $\mathrm{Hom}(V, X) = 0$ by Corollary 2.3. Similarly, $\tau^2 V \in \mathcal{W}_{\tau^2 T_k}$ for some $k \in \mathbb{Z}_s$. Since the quasitop of X (i.e. X) does not lie in $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{T_k}$, we have $\mathrm{Hom}(X, \tau^2 V) = 0$ by Corollary 2.3. Hence $\mathrm{Hom}_{\mathcal{C}}(T_{\mathcal{T}}, X) = 0$, completing this case.

We next suppose that $X \cong M_{i_k+l_k,1}$ for some $k \in \mathbb{Z}_s$ (the case $X \cong M_{i_k-1,1}$ is similar). Recall that there is always at least one quasisimple module between two wings $\mathcal{W}_{\tau T_k}$ and $\mathcal{W}_{\tau T_{k\pm1}}$.

Case 2: Assume first that there are at least two quasisimple modules between the wings $\mathcal{W}_{\tau T_k}$ and $\mathcal{W}_{\tau T_{k+1}}$, so that X is not adjacent to the wing $\mathcal{W}_{\tau T_{k+1}}$. In the example in Figure 7, the object X_2 is an example of this type (with $k = 1$). In this case, we take $Y = M_{i_k,l_k+1}$ (indicated by Y_2 in Figure 7).

By Lemma 3.2, $\mathrm{Hom}(U, M_{i_k+l_k-1,1}) = 0$. By assumption, $\mathrm{Hom}(U, X) = 0$. Since $\dim \mathrm{Hom}(U, -)$ is additive on $\mathcal{T}$, we have $\mathrm{Hom}(U, M_{i_k+l_k-1,2}) = 0$. If $l_k = 1$ then $Y = M_{i_k+l_k-1,2}$ and $\mathrm{Hom}(U, Y) = 0$. If $l_k > 1$ then, since $\dim \mathrm{Hom}(U, -)$ is additive on the short exact sequence:

$$0 \to \tau T_k \to T'_k \oplus M_{i_k+l_k-1,1} \to M_{i_k+l_k-1,2} \to 0,$$

it follows that $\mathrm{Hom}(U, Y) = 0$ in this case also.

Since the quasisocle Q_{i_k} of Y does not lie in $\cup_{k'\in\mathbb{Z}_s}\mathcal{W}_{T_{k'}}$, we have $\mathrm{Hom}(V, Y) = 0$ for all indecomposable summands V of $T_{\mathcal{T}}$ by Corollary 2.3. Since there are at least two quasisimple modules between the wings $\mathcal{W}_{\tau T_k}$ and $\mathcal{W}_{\tau T_{k+1}}$, the quasitop of Y does not lie in $\cup_{k'\in\mathbb{Z}_s}\mathcal{W}_{\tau^2 T_{k'}}$. Hence $\mathrm{Hom}(Y, \tau^2 V) = 0$ for all summands V of $T_{\mathcal{T}}$, by Corollary 2.3. So $\mathrm{Hom}_{\mathcal{C}}(T_{\mathcal{T}}, Y) = 0$, completing this case.

Case 3: We finally consider the case where there is exactly one quasisimple module between the wings $\mathcal{W}_{\tau T_k}$ and $\mathcal{W}_{\tau T_{k+1}}$. In the example in Figure 7, the object X_3 is an example of this type. In this case, we take $Y = M_{i_k,l_k+l_{k+1}+1}$ (indicated by Y_3 in Figure 7).

The quasisimples in $\mathcal{W}_Y$ are the quasisimples in $\mathcal{W}_{\tau T_k}$, the quasisimples in $\mathcal{W}_{\tau T_{k+1}}$ and X. For a quasisimple module Q in one of the first two sets, $\mathrm{Hom}(U, Q) = 0$ by Lemma 3.2. By assumption, $\mathrm{Hom}(U, X) = 0$. Hence, arguing as in Lemma 3.2 and using the additivity of $\dim \mathrm{Hom}(U, -)$ on $\mathcal{T}$, we have $\mathrm{Hom}(U, Y) = 0$.

Since the quasisocle of Y is Q_{i_k}, which does not lie in $\cup_{k'\in\mathbb{Z}_s}\mathcal{W}_{T_{k'}}$, we see that $\mathrm{Hom}(V, Y) = 0$ for any indecomposable summand of $\mathcal{T}_{\mathcal{T}}$ by Corollary 2.3. Similarly, the quasitop of Y is $Q_{i_{k+1}+l_{k+1}-1}$, which does not lie in $\cup_{k'\in\mathbb{Z}_s}\mathcal{W}_{\tau^2 T_{k'}}$. Hence $\mathrm{Hom}(Y, \tau^2 V) = 0$ for any indecomposable summand of $T_{\mathcal{T}}$ by Corollary 2.3. So $\mathrm{Hom}_{\mathcal{C}}(T_{\mathcal{T}}, Y) = 0$, completing this case. □

Lemma 3.5. *Let P be an indecomposable projective KQ-module, and suppose that we have* $\mathrm{Hom}(P, X_0) = 0$ *for some indecomposable module X_0 on the border of $\mathcal{T}$. Then* $\dim \mathrm{Hom}(P, X) \leq 1$ *for all indecomposable modules X on the border of $\mathcal{T}$. Furthermore, if there is some indecomposable module X_1 on the border of $\mathcal{T}$ such that* $\mathrm{Hom}(P, X) = 0$ *for all indecomposable modules $X \not\cong X_1$ on the border of $\mathcal{T}$, then* $\dim \mathrm{Hom}(P, X_1) = 1$.

Proof. This can be checked using the tables in [26, XIII.2]. □

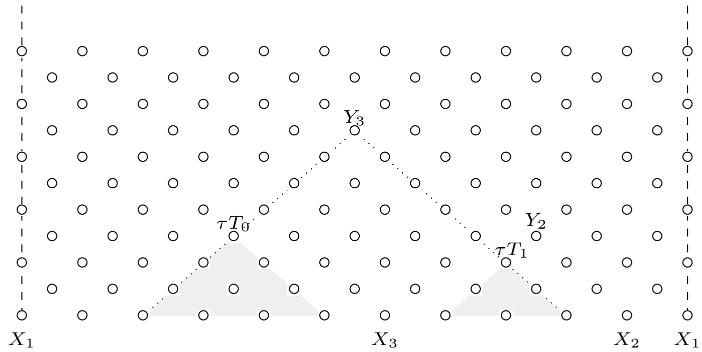

FIGURE 7. Proof of Proposition 3.4. For the quasisimple module X_1, we take $Y = X_1$; for the module X_2, we take $Y = Y_2$, and for the module X_3, we take $Y = Y_3$.

Proposition 3.6. *Suppose that $T_{\mathcal{T}} \neq 0$ and let $X \in \mathcal{T} \setminus \cup_{k\in\mathbb{Z}_s} \mathcal{W}_{\tau T_k}$ be an indecomposable module on the border of $\mathcal{T}$ and V an indecomposable summand of U. Then* $\dim \mathrm{Hom}(V, X) \leq 1$. *Furthermore, if $k = 0$ and T_0 has quasilength $r - 1$, then* $\dim \mathrm{Hom}(V, X) = 1$.

Proof. By applying a power of τ if necessary, we can assume that V is projective. By assumption, $\mathcal{T}$ contains a summand of T, so there is at least one quasisimple module X_0 in $\cup_{k\in\mathbb{Z}_s} \mathcal{W}_{\tau T_k}$. By Lemma 3.2, we have that $\mathrm{Hom}(V, X_0) = 0$. The first part of the lemma then follows from Lemma 3.5. If $k = 0$ and T_0 has quasilength $r - 1$, then $\mathrm{Hom}(V, X_0) = 0$ for every quasisimple in $\mathcal{W}_{\tau T_0}$. Since X is the unique quasisimple in $\mathcal{T}$ not in $\mathcal{W}_{\tau T_0}$, the second part now follows from Lemma 3.5 also. □

Abusing notation, we denote the down-arrows in $\mathcal{T}$ by x and the up-arrows by y. So, for example, x^r means the composition of r down-arrows from a given vertex.

Proposition 3.7. *Let $X = M_{i,l}$ be an indecomposable module in $\mathcal{T}$.*

(a) *Suppose that $r+1 \leq l$. Let $u_X = y^r x^r : X \to X$. Then u_X factors through* $\mathrm{add}(\tau T_{\mathcal{T}})$ *if and only if $X \in \cup_{k\in\mathbb{Z}_s} \mathcal{H}_k \cup \mathcal{R}_k$.*

(b) *Suppose that $r \leq l$. Let $v_X = y^{r-1}x^{r-1} : X \to \tau X$ be the unique nonzero map (up to a scalar), as in Lemma 2.4. Then v_X factors through* $\mathrm{add}(\tau T_{\mathcal{T}} \oplus \tau^2 T_{\mathcal{T}})$ *if and only if $X \in \cup_{k\in\mathbb{Z}_s} (\mathcal{H}_k \cup \mathcal{R}_k \setminus \{\mathrm{Top}_k\})$. Furthermore, v_X factors through both* $\mathrm{add}(\tau T_{\mathcal{T}})$ *and* $\mathrm{add}(\tau^2 T_{\mathcal{T}})$ *if and only if $X \in \cup_{k\in\mathbb{Z}_s} \mathcal{R}_k$.*

(c) *Suppose that $r \leq l$. Let $w_X = y^{r-2}x^{r-2} : \tau^{-1}X \to \tau X$. Then w_X factors through* $\mathrm{add}(\tau T_{\mathcal{T}})$ *if and only if $X \in \cup_{k\in\mathbb{Z}_s} \mathcal{S}_k$.*

Proof. We start with part (a). Note that u_X lies in the basis for $\mathrm{Hom}(X, X)$ given in Lemma 2.5. Also, by the mesh relations, $u_X = x^r y^r$. Let D_X be the diamond-shaped region in $\mathcal{T}$ bounded by the paths $x^r y^r$ and $y^r x^r$ starting at X. It is clear that u_X factors through any indecomposable module in D_X. For an example, see Figure 8, where part of one copy of D_X has been drawn.

If Y lies outside D_X, then any path from X to X in $\mathcal{T}$ via Y must contain more than r downward arrows. By Lemma 2.5 it is a linear combination of basis elements distinct from u. So u cannot factor through the direct sum of any collection of objects outside this region.

Hence u_X factors through $\mathrm{add}(\tau T_{\mathcal{T}})$ if and only if some indecomposable summand of $\tau T_{\mathcal{T}}$ lies in D_X. Since the indecomposable summands of $\tau T_{\mathcal{T}}$ lie in $\cup_{k\in\mathbb{Z}_s} \mathcal{W}_{\tau T_k}$, we see that u_X

factors through $\tau T_{\mathcal{T}}$ if and only if $M_{i+r,l-r}$ (the module in D_X with minimal quasilength) lies in $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$.

The corners of the triangular region $\mathcal{H}_k \cup \mathcal{R}_k$ are $M_{i_k,r}$, $\mathrm{Top}_k = M_{i_k,r+l_k}$ and $M_{i_k+l_k,r}$. The part of $\mathcal{H}_k \cup \mathcal{R}_k$ consisting of modules with quasilength at least $r+1$ is the triangle with corners $M_{i_k,r+1}$, $M_{i_k,r+l_k}$ and $M_{i_k+l_k-1,r+1}$. Hence, $X = M_{i,l}$ lies in $\mathcal{H}_k \cup \mathcal{R}_k$ if and only if $M_{i+r,l-r}$ lies in the triangular region of $\mathcal{T}$ with corners $M_{i_k,1}$, M_{i_k,l_k} and $M_{i_k+l_k-1,1}$, i.e. $\mathcal{W}_{\tau T_k}$. The result follows.

For part (b), we consider the diamond-shaped region E_X bounded by the paths $y^{r-1}x^{r-1}$ and $x^{r-1}y^{r-1}$ starting at X. We have, using an argument similar to the above, that v_X factors through $\mathrm{add}(\tau T_{\mathcal{T}} \oplus \tau^2 T_{\mathcal{T}})$ if and only if some indecomposable direct summand of $\tau T_{\mathcal{T}} \oplus \tau^2 T_{\mathcal{T}}$ lies in E_X. Hence, v_X factors through $\mathrm{add}(\tau T_{\mathcal{T}} \oplus \tau^2 T_{\mathcal{T}})$ if and only if $M_{i+r-1,l-r+1}$ lies in $\cup_{k\in\mathbb{Z}_s}(\mathcal{W}_{\tau T_k} \cup \mathcal{W}_{\tau^2 T_k})$. The corners of the trapezoidal region $\mathcal{H}_k \cup \mathcal{R}_k \setminus \{\mathrm{Top}_k\}$ are $M_{i_k,r}$, $M_{i_k,r+l_k-1}$, $M_{i_k+1,r+l_k-1}$, $M_{i_k+l_k,r}$. Hence $X \in \mathcal{H}_k \cup \mathcal{R}_k \setminus \{\mathrm{Top}_k\}$ if and only if $M_{i+r-1,l-r+1}$ lies in the trapezoidal region with corners $M_{i_k+r-1,1}$, M_{i_k+r-1,l_k}, M_{i_k+r,l_k}, $M_{i_k+l_k+r-1,1}$, i.e. $M_{i_k-1,1}$, M_{i_k-1,l_k}, M_{i_k,l_k}, $M_{i_k+l_k-1,1}$ which is the union $\mathcal{W}_{\tau T_k} \cup \mathcal{W}_{\tau^2 T_k}$. This gives the first part of (b).

We have that v_X factors through $\mathrm{add}(\tau T_{\mathcal{T}})$ (respectively, $\mathrm{add}(\tau^2 T_{\mathcal{T}})$) if and only if $M_{i+r-1,l-r+1}$ lies in $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$ (respectively, $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau^2 T_k}$). Hence v_X factors through both $\mathrm{add}(\tau T_{\mathcal{T}})$ and $\mathrm{add}(\tau^2 T_{\mathcal{T}})$ if and only if $M_{i+r-1,l-r+1}$ lies in $\cup_{k\in\mathbb{Z}_s}(\mathcal{W}_{\tau T_k} \cap \mathcal{W}_{\tau^2 T_k})$. The corners of the triangular region $\mathcal{R}_k$ are $M_{i_k+1,r}$, $M_{i_k+1,r+l_k-2}$ and $M_{i_k+l_k-1,r}$. Hence $X \in \mathcal{R}_k$ if and only if $M_{i+r-1,l-r+1}$ lies in the triangular region with corners $M_{i_k+r,1}$, M_{i_k+r,l_k-1} and $M_{i_k+l_k+r-2,1}$, i.e. $M_{i_k,1}$, M_{i_k,l_k-1} and $M_{i_k+l_k-2,1}$, which is the intersection $\mathcal{W}_{\tau T_k} \cap \mathcal{W}_{\tau^2 T_k}$. This gives the second part of (b).

For part (c), we consider the diamond-shaped region $F_{\tau^{-1}X}$ bounded by the paths $y^{r-2}x^{r-2}$ and $x^{r-2}y^{r-2}$ starting at $\tau^{-1}X$. We have, using an argument similar to the above, that w_X factors through $\mathrm{add}(\tau T_{\mathcal{T}})$ if and only if some indecomposable direct summand of $\tau T_{\mathcal{T}}$ lies in $F_{\tau^{-1}X}$. Hence, w_X factors through $\mathrm{add}(\tau T_{\mathcal{T}})$ if and only if $M_{i+1+r-2,l-r+2} = M_{i+r-1,l-r+2}$ lies in $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$. The corners of the triangular region $\mathcal{S}_k$ are $M_{i_k+1,r-1}$, $M_{i_k+1,r+l_k-2}$ and $M_{i_k+l_k,r-1}$. Hence $X \in \mathcal{S}_k$ if and only if $M_{i+r-1,l-r+2}$ lies in the the triangular region with corners $M_{i_k+1+r-1,1}$, $M_{i_k+1+r-1,l_k}$ and $M_{i_k+l_k+r-1,1}$, i.e. $M_{i_k,1}$, M_{i_k,l_k} and $M_{i_k+l_k-1,1}$, which is the wing $\mathcal{W}_{\tau T_k}$. Part (c) follows. □

4. Rigid and Schurian Λ-modules

We determine which objects X in $\mathcal{T}$ give rise to Schurian and rigid Λ-modules $\widetilde{X}$.

Lemma 4.1. *Let $X = M_{i,l}$ be an indecomposable module in $\mathcal{T}$. Then:*

(a) *If $r+1 \leq l$ and $X \notin \cup_{k\in\mathbb{Z}_s}\mathcal{H}_k \cup \mathcal{R}_k$ then $\widetilde{X}$ is not Schurian.*
(b) *If $r \leq l$ and $X \notin \cup_{k\in\mathbb{Z}_s}\mathcal{H}_k \cup \mathcal{R}_k \setminus \{\mathrm{Top}_k\}$ then $\widetilde{X}$ is not rigid.*

Proof. Firstly note that in both (a) and (b), X cannot be a summand of τT. For part (a), let $u_X = y^r x^r : X \to X$. Since U is preprojective, any composition of maps in $\mathcal{C}$ from X to X factoring through U is zero. By Proposition 3.7(a) and Lemma 1.12, u_X does not factor through $\mathrm{add}(\tau T_{\mathcal{T}})$. It follows that u_X does not factor through $\mathrm{add}(\tau T)$ and hence $\mathrm{Hom}^H_{\mathcal{C}/\mathrm{add}(\tau T)}(X,X) \ncong K$, so $\widetilde{X}$ is not Schurian. A similar argument, using Proposition 3.7(b), gives part (b). □

Lemma 4.2. *Let X be an indecomposable object in $\mathcal{T}$ which is not a summand of τT. Then:*

(a) *$\widetilde{X}$ is a τ-rigid Λ-module if and only if the quasilength of X is at most $r-1$;*

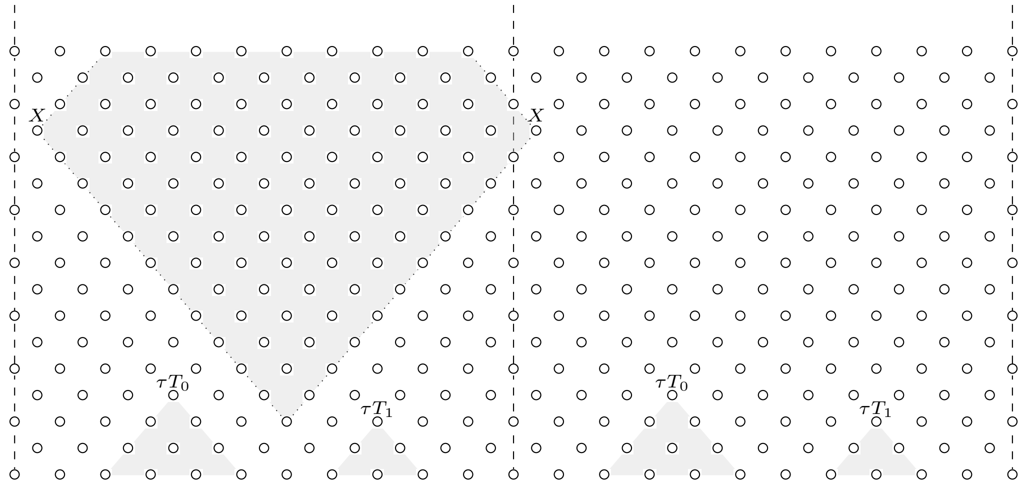

FIGURE 8. Proof of Proposition 3.7: the shaded region indicates part of (one copy of) the diamond-shaped region D_X. In this case, u_X does not factor through $\mathrm{add}(\tau T_{\mathcal{T}})$.

(b) *If the quasilength of X is at most $r-2$, then $\widetilde{X}$ is Schurian.*

Proof. It is well-known (and follows from the fact that $\mathcal{T}$ is standard) that X is a rigid KQ-module if and only if its quasilength is at most $r-1$, so part (a) follows from Corollary 1.10. If the quasilength of X is at most $r-2$, then $\mathrm{Hom}(X,X) \cong K$ and $\mathrm{Hom}(X, \tau^2 X) = 0$ by Lemma 2.4, so

$$\mathrm{Hom}_{\mathcal{C}}(X,X) \cong \mathrm{Hom}(X,X) \oplus D\,\mathrm{Hom}(X,\tau^2 X) \cong K,$$

giving part (b). □

We need the following.

Lemma 4.3. *Let A, B be indecomposable KQ-modules, and assume that* $\mathrm{Hom}(A, B[1]) \cong K$. *Let $\varepsilon \in \mathrm{Hom}(A, B[1])$ be nonzero.*

(a) *If there is a map $\varphi \in \mathrm{Hom}(B, \tau A)$ such that $\mathrm{im}(\varphi)$ has an indecomposable direct summand which does not lie in $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$ then ε factors in $D^b(KQ)$ through $U[1]$.*

(b) *If there is a map $\varphi \in \mathrm{Hom}(B, \tau A)$ such that $\mathrm{im}(\varphi)$ has an indecomposable direct summand which does not lie in $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau^2 T_k}$ then ε factors in $D^b(KQ)$ through $\tau U[1]$.*

Proof. We write φ as $\varphi_2\varphi_1$ where $\varphi_1 : B \to \mathrm{im}(\varphi)$ and $\varphi_2 : \mathrm{im}(\varphi) \to \tau A$. We have the short exact sequence:

$$0 \longrightarrow \ker(\varphi) \longrightarrow B \xrightarrow{\varphi_1} \mathrm{im}(\varphi) \longrightarrow 0 \tag{4.1}$$

For part (a), we apply $\mathrm{Hom}(U, -)$ to this sequence (noting that, since U is preprojective, it is exact on $\mathcal{T}$), to obtain the exact sequence:

$$0 \longrightarrow \mathrm{Hom}(U, \ker(\varphi)) \longrightarrow \mathrm{Hom}(U, B) \xrightarrow{\mathrm{Hom}(U,\varphi_1)} \mathrm{Hom}(U, \mathrm{im}(\varphi)) \longrightarrow 0$$

Since $\mathrm{im}\,\varphi$ has an indecomposable direct summand which does not lie in $\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$, it follows from Proposition 3.4 that $\mathrm{Hom}(U, \mathrm{im}\,\varphi) \neq 0$. Hence, the epimorphism $\mathrm{Hom}(U, \varphi_1)$

is nonzero. Since φ_2 is a monomorphism, $\mathrm{Hom}(U,\varphi)\neq 0$, so there is a map $\beta\in\mathrm{Hom}(U,B)$ such that $\mathrm{Hom}(U,\varphi)(\beta)=\varphi\beta\neq 0$. Hence $\mathrm{Hom}(\beta,\tau A)(\varphi)=\varphi\beta\neq 0$, so $\mathrm{Hom}(\beta,\tau A)\neq 0$. Part (a) now follows from Proposition 1.14(b), taking $C=U$.

For part (b), we apply $\mathrm{Hom}(\tau U,-)$ to the sequence (4.1). Note that $\mathrm{Hom}(\tau U,\mathrm{im}(\varphi))\cong\mathrm{Hom}(U,\tau^{-1}\mathrm{im}(\varphi))\neq 0$ by Proposition 3.4, and the argument goes through as in part (a). □

Lemma 4.4. *Fix $k\in\mathbb{Z}_s$ and let $X\in\mathcal{H}_k\setminus\{\mathrm{Top}_k\}$. Then $\widetilde{X}$ is rigid.*

Proof. Firstly note that X cannot be a direct summand of τT. By the assumption, the quasilength of X lies in the interval $[r,2r-1]$, so, by Lemma 2.4, $\mathrm{Hom}(X,\tau X)\cong K$. Let $u=y^{r-1}x^{r-1}$ be a nonzero element of $\mathrm{Hom}(X,\tau X)$. Then by Proposition 3.7(b), u factors through $\mathrm{add}(\tau T_{\mathcal{T}}\oplus\tau^2T_{\mathcal{T}})$, so $\mathrm{Hom}^H_{\mathcal{C}/\mathrm{add}(\tau T\oplus\tau^2T)}(X,\tau X)=0$.

Suppose that $X\cong M_{i_k,l}$ where $r\leq l\leq r+l_k-1$. We have

$$\mathrm{Hom}^F_{\mathcal{C}}(X,\tau X)=\mathrm{Hom}(X,X[1])\cong D\,\mathrm{Hom}(X,\tau X)\cong K.$$

We apply Lemma 4.3(a) in the case $A=X$, $B=X$. We take $\varphi=u$ and ε to be a nonzero element of $\mathrm{Hom}(X,X[1])$. Then $\mathrm{im}(\varphi)\cong M_{i_k-1,l-r+1}\notin\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$. By Lemma 4.3(a), we have that ε factors through $U[1]$. Hence, regarded as an F-map in $\mathcal{C}$, ε factors through τU. It follows that

$$\mathrm{Hom}^F_{\mathcal{C}/\mathrm{add}(\tau T\oplus\tau^2T)}(X,\tau X)=0.$$

Suppose that $X\cong M_{i_k+p,r+l_k-p}$ where $1\leq p\leq l_k$. We have

$$\mathrm{Hom}^F_{\mathcal{C}}(X,\tau X)=\mathrm{Hom}(X,X[1])\cong D\,\mathrm{Hom}(X,\tau X)\cong K.$$

We apply Lemma 4.3(b) in the case $A=X$, $B=X$. We take $\varphi=u$ and ε to be a nonzero element of $\mathrm{Hom}(X,X[1])$. Then $\mathrm{im}(\varphi)=M_{i_k+p-1,l_k-p-1}\notin\cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$. By Lemma 4.3(b), ε factors through $\tau U[1]$. Hence, regarded as an F-map in $\mathcal{C}$, ε factors through $\tau^2 U$. It follows that

$$\mathrm{Hom}^F_{\mathcal{C}/\mathrm{add}(\tau T\oplus\tau^2T)}(X,\tau X)=0.$$

In either case, we have shown that $\mathrm{Hom}_{\mathcal{C}/\mathrm{add}(\tau T\oplus\tau^2T)}(X,\tau X)=0$, and it follows that $\widetilde{X}$ is rigid by Proposition 1.8(b). □

If $T_{\mathcal{T}}$ contains an indecomposable direct summand of quasilength $r-1$ then $s=1$, $l_0=r-1$ and, by (3.4),

$$\mathcal{H}_0=\{M_{i_0,l}\,:\,r\leq l\leq 2r-1\}\cup\{M_{i_0+p,2r-1-p}\,:,0\leq p\leq r-1\}. \tag{4.2}$$

In particular, $\mathrm{Top}_k=M_{i_0,2r-1}$ has quasilength $2r-1$. In all other cases, Top_k has smaller quasilength.

Lemma 4.5. *Fix $k\in\mathbb{Z}_s$. Suppose that X is an indecomposable object of $\mathcal{T}$ which is not a summand of τT and satisfies either*

(a) *$X\in\mathcal{H}_k$ and $\mathrm{ql}(X)\leq 2r-2$, or*
(b) *$\mathrm{ql}(X)\in\{r-1,r\}$ and $X\notin\cup_{k\in\mathbb{Z}_s}\mathcal{H}_k\cup\mathcal{S}_k$.*

Then $\widetilde{X}$ is Schurian.

Proof. In case (a), $r\leq\mathrm{ql}(X)\leq 2r-2$, and in case (b), $r-1\leq\mathrm{ql}(X)\leq r$. If $\mathrm{ql}(X)\leq r$ then $\mathrm{Hom}(X,X)\cong K$ by Lemma 2.4. If $\mathrm{ql}(X)>r$ then $\mathrm{Hom}(X,X)\cong K^2$. A basis is given by the identity map and the map u_X in Proposition 3.7(a). By Proposition 3.7(a), u_X factors through $\mathrm{add}(\tau T)$. Hence, in either case, $\mathrm{Hom}^H_{\mathcal{C}/\mathrm{add}(\tau T)}(X,X)\cong K$.

Since the quasilength of X lies in $[r-1, 2r-2]$, we have, by Lemma 2.4, that

$$\mathrm{Hom}^F_{\mathcal{C}}(X,X) = \mathrm{Hom}(X, \tau^{-1}X[1]) \cong \mathrm{Ext}(X, \tau^{-1}X) \cong D\,\mathrm{Hom}(\tau^{-1}X, \tau X) \cong K.$$

We apply Lemma 4.3(a) in the case $A = X$, $B = \tau^{-1}X$. We take φ to be the map $w_{\tau^{-1}X}$ in Proposition 3.7(c), the unique nonzero element of $\mathrm{Hom}(\tau^{-1}X, \tau X)$ up to a scalar, and ε to be a nonzero element of $\mathrm{Hom}(X, \tau^{-1}X[1])$.

In case (a), there are two possibilities. If $X \cong M_{i_k,l}$ where $r \leq l \leq r + l_k - 1$, then $\mathrm{im}(\varphi) \cong M_{i_k+r-1,l-r+2} \notin \cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$. If $X \cong M_{i_k+p,r+l_k-p}$ where $1 \leq p \leq l_k$, then $\mathrm{im}(\varphi) \cong M_{i_k+p+r-1,2+l_k-p} \notin \cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}$. In case (b), there are also two possibilities. If $\mathrm{ql}(X) = r-1$, then $X \cong M_{i,r-1}$ where $i \notin \cup_{k\in\mathbb{Z}_s}[i_k+1, i_k+l_k]$. Then

$$\mathrm{im}(\varphi) \cong M_{i+1+(r-2),r-1-(r-2)} = M_{i-1,1} \notin \cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}.$$

If $\mathrm{ql}(X) = r$, then $X \cong M_{i,r}$ where $i \notin \cup_{k\in\mathbb{Z}_s}[i_k, i_k+l_k]$. Then

$$\mathrm{im}(\varphi) \cong M_{i+1+(r-2),r-(r-2)} = M_{i-1,2} \notin \cup_{k\in\mathbb{Z}_s}\mathcal{W}_{\tau T_k}.$$

Applying Lemma 4.3(a), we see that ε factors through $U[1]$. Hence, regarded as an F-map in $\mathcal{C}$, ε factors through τU. It follows that

$$\mathrm{Hom}^F_{\mathcal{C}/\,\mathrm{add}\,\tau T}(X,X) = 0.$$

We have shown that $\mathrm{Hom}_{\mathcal{C}/\,\mathrm{add}(\tau T)}(X,X) \cong K$, and it follows that $\widetilde{X}$ is Schurian by Proposition 1.8(a). $\square$

Lemma 4.6. *Fix $k \in \mathbb{Z}_s$, and let $X \in \mathcal{R}_k$. Then $\widetilde{X}$ is not rigid.*

Proof. Since $X \in \mathcal{R}_k$, we have $r \leq \mathrm{ql}(X) \leq r + l_k - 2 \leq 2r-3$. In particular, this implies that X is not a direct summand of τT. By Lemma 2.4, we have $\mathrm{Hom}(X, \tau X) \cong K$. Let u be a nonzero map in $\mathrm{Hom}(X, \tau X)$, unique up to a nonzero scalar. We have

$$\mathrm{Hom}^F_{\mathcal{C}}(X, \tau X) = \mathrm{Hom}(X, X[1]) \cong D\,\mathrm{Hom}(X, \tau X) \cong K.$$

Let $v \in \mathrm{Hom}(X, X[1])$ be a nonzero map, unique up to a nonzero scalar.

We show first that v cannot factor through V for any indecomposable summand V of τT or $\tau^2 T$. If $\mathrm{Hom}(X, V) = 0$ then we are done, so we may assume that $\mathrm{Hom}(X, V) \neq 0$. Hence, V lies in $\mathcal{T}$ and $\mathrm{ql}(V) \leq r-1$.

By Lemma 2.2, we have that $\mathrm{Hom}(X, V) \cong K$. Let $f \in \mathrm{Hom}(X, V)$ be any nonzero map. Then the number of downward arrows in a path for f (and hence for τf) is at least $\mathrm{ql}(X) - \mathrm{ql}(V) \geq \mathrm{ql}(X) - r + 1$. The number of downward arrows in a path for u is $r-1$, so the number of downward arrows in a path for $\tau f \circ u$ is at least $\mathrm{ql}(X)$, so $\tau f \circ u = 0$ by Lemma 2.1. Since $\{u\}$ is a basis for $\mathrm{Hom}(X, \tau X)$, it follows that $\mathrm{Hom}(X, \tau f) = 0$. Therefore, by Proposition 1.14(a), v cannot factor through V.

We next show that v cannot factor through $\tau^{-1}V[1]$ for any indecomposable summand V of τT. If $\mathrm{Hom}(\tau^{-1}V, X) = 0$ then $\mathrm{Hom}(\tau^{-1}V[1], X[1]) = 0$ and we are done. Therefore, we may assume that $\mathrm{Hom}(\tau^{-1}V, X) \neq 0$.

Suppose first that $V \in \mathcal{T}$, so $\mathrm{ql}(V) \leq r-1$. By Lemma 2.2, we have that $\mathrm{Hom}(\tau^{-1}V, X) \cong K$. Let $g \in \mathrm{Hom}(\tau^{-1}V, X)$ be any nonzero map. The number of downward arrows in a path for u is $r-1$, hence the number of downward arrows in a path for ug is at least $r-1$. As $\mathrm{ql}(\tau^{-1}V) \leq r-1$, it follows from Lemma 2.1 that $ug = 0$. Since $\{u\}$ is a basis for $\mathrm{Hom}(X, \tau X)$, it follows that $\mathrm{Hom}(g, \tau X) = 0$.

Secondly, suppose that V is an indecomposable direct summand of τU or $\tau^2 U$. Let $h \in \mathrm{Hom}(\tau^{-1}V, X)$. By Proposition 3.7(b), we have that u factors through both $\mathrm{add}(\tau T_{\mathcal{T}})$

and $\operatorname{add}(\tau^2 T_{\mathcal{T}})$, so $uh = 0$ as $\tau^{-1}V$ is a direct summand of $T \oplus \tau T$. Since $\{u\}$ is a basis for $\operatorname{Hom}(X, \tau X)$, it follows that $\operatorname{Hom}(h, \tau X) = 0$.

Applying Proposition 1.14(b) to the triple $A = B = X$, $C = \tau^{-1}V$ and $\beta = g$ or h, we obtain that v does not factor through $\tau^{-1}V[1]$.

We have shown that v does not factor in $D^b(KQ)$ through V or $\tau^{-1}V[1]$ for any indecomposable summand V of $\tau T \oplus \tau^2 T$. Since $\operatorname{Hom}(X, X[1]) \cong K$, it follows that v does not factor in $D^b(KQ)$ through $\operatorname{add}(\tau T \oplus \tau^2 T)$ or $\operatorname{add}(\tau^{-1}(\tau T \oplus \tau^2 T)[1])$. By Lemma 1.12, the morphism v, regarded as a morphism in $\operatorname{Hom}_{\mathcal{C}}(X, \tau X)$, does not factor in $\mathcal{C}$ through $\operatorname{add}(\tau T \oplus \tau^2 T)$. Hence:

$$\operatorname{Hom}_{\mathcal{C}/\operatorname{add}(\tau T \oplus \tau^2 T)}(X, \tau X) \neq 0.$$

Therefore $\widetilde{X}$ is not rigid by Proposition 1.8. □

Note that the objects in $\mathcal{S}_k$ (see (3.5)) have quasilength at least $r-1$, so if T has no indecomposable direct summand in $\mathcal{T}$ of quasilength $r-1$, the objects in $\mathcal{S}_k$ are not summands of τT. It is easy to check directly that this holds in the case where T has an indecomposable direct summand T_0 in $\mathcal{T}$ of quasilength $r-1$, since all the indecomposable direct summands of τT in $\mathcal{T}$ lie in $\mathcal{W}_{\tau T_0}$ (see Figure 22).

Lemma 4.7. *Fix $k \in \mathbb{Z}_s$, and let $X \in \mathcal{S}_k$. Then $\widetilde{X}$ is not Schurian.*

Proof. Firstly note that, by the above, X is not an indecomposable direct summand of τT. Since $X \in \mathcal{S}_k$, we have $r - 1 \leq \operatorname{ql}(X) \leq r + l_k - 2 \leq 2r - 3$, so by Lemma 2.4, we have $\operatorname{Hom}(\tau^{-1}X, \tau X) \cong K$. Let u be a nonzero map in $\operatorname{Hom}(\tau^{-1}X, \tau X)$, unique up to a nonzero scalar. We have

$$\operatorname{Hom}^F_{\mathcal{C}}(X, X) = \operatorname{Hom}(X, \tau^{-1}X[1]) \cong D\operatorname{Hom}(\tau^{-1}X, \tau X) \cong K.$$

Let $v \in \operatorname{Hom}(X, \tau^{-1}X[1])$ be a nonzero map, unique up to a nonzero scalar.

We will first show that v cannot factor through V for any indecomposable summand V of τT. If $\operatorname{Hom}(X, V) = 0$ then we are done, so we may assume that $\operatorname{Hom}(X, V) \neq 0$. In particular, we may assume that V lies in $\mathcal{T}$. By Lemma 2.2, $\operatorname{Hom}(X, V) \cong K$. Let $f \in \operatorname{Hom}(X, V)$ be a nonzero map, unique up to a nonzero scalar.

If $\operatorname{ql}(V) \leq r-2$ then the number of downward arrows in a path for f (and hence for τf) is at least $\operatorname{ql}(X) - \operatorname{ql}(V) \geq \operatorname{ql}(X) - r + 2$. If $\operatorname{ql}(V) = r - 1$ then $s = k = 1$ and $V = \tau T_1$. Then, since no object in $\mathcal{S}_1$ is in the coray through τT_1, the number of downward arrows in a path for f (and hence for τf) is at least $\operatorname{ql}(X) - \operatorname{ql}(V) + 1 \geq \operatorname{ql}(X) - r + 2$. The number of downward arrows in a path for u is $r - 2$. Hence in either case the number of downward arrows in a path for $\tau f \circ u$ is at least $\operatorname{ql}(X)$, so $\tau f \circ u = 0$ by Lemma 2.1.

Since $\{u\}$ is a basis for $\operatorname{Hom}(X, \tau X)$, it follows that $\operatorname{Hom}(X, \tau f) = 0$. Therefore, by Proposition 1.14(a), v cannot factor through V.

We next show that v cannot factor through $\tau^{-1}V[1]$ for any indecomposable summand V of τT. If $\operatorname{Hom}(\tau^{-1}V, \tau^{-1}X) = 0$ then $\operatorname{Hom}(\tau^{-1}V[1], \tau^{-1}X[1]) = 0$ and we are done, so we may assume that $\operatorname{Hom}(\tau^{-1}V, \tau^{-1}X) \neq 0$.

Suppose first that $V \in \mathcal{T}$. By Lemma 2.2, $\operatorname{Hom}(\tau^{-1}V, \tau^{-1}X) \cong K$. Let g be a non-zero map in $\operatorname{Hom}(\tau^{-1}V, \tau^{-1}X)$, unique up to a nonzero scalar.

If $\operatorname{ql}(V) \leq r-2$, then the number of downward arrows in a path for g is at least $\operatorname{ql}(X) - \operatorname{ql}(V) \geq \operatorname{ql}(X) - r + 2$. Since the number of downward arrows in a path for u is $r-2$, the number of downward arrows in a path for ug is at least $\operatorname{ql}(X) \geq r - 1 > \operatorname{ql}(\tau^{-1}V)$, so $ug = 0$ by Lemma 2.1.

If $\operatorname{ql}(V) = r - 1$ then $s = k = 1$ and $V = \tau T_1$. Since no element of $\tau^{-1}\mathcal{S}_1$ lies in the ray through $\tau^{-1}V \cong T_1$, a path for g has at least one downward arrow. It follows that a path for ug has at least $r-1 = \mathrm{ql}(\tau^{-1}V)$ downward arrows, so $ug = 0$ in this case also. Since $\{u\}$ is a basis for $\mathrm{Hom}(X, \tau X)$, it follows that, in either case, $\mathrm{Hom}(g, \tau X) = 0$.

Secondly, suppose that V is an indecomposable direct summand of τU, and let $h \in \mathrm{Hom}(\tau^{-1}V, \tau^{-1}X)$. By Proposition 3.7(c), u factors through τT_k, since $X \in \mathcal{S}_k$. Hence, $uh = 0$ as $\tau^{-1}V$ is a direct summand of T. Since $\{u\}$ is a basis for $\mathrm{Hom}(X, \tau X)$, it follows that $\mathrm{Hom}(h, \tau X) = 0$.

Applying Proposition 1.14(b) to the triple $A = B = X$, $C = \tau^{-1}V$, we obtain that v does not factor through $\tau^{-1}V[1]$.

We have shown that v does not factor in $D^b(KQ)$ through V or $\tau^{-1}V[1]$ for any indecomposable summand V of τT. Since $\mathrm{Hom}(X, \tau^{-1}X[1]) \cong K$, it follows that v does not factor in $D^b(KQ)$ through $\mathrm{add}(\tau T)$ or $\mathrm{add}(T[1])$. By Lemma 1.12, the morphism v, regarded as a morphism in $\mathrm{Hom}^F_{\mathcal{C}}(X, X)$, does not factor through $\mathrm{add}(\tau T)$. Hence $\mathrm{Hom}_{\mathcal{C}/\,\mathrm{add}(\tau T)}(X, X) \not\cong K$. Therefore $\widetilde{X}$ is not Schurian by Proposition 1.8. □

Recall (equation 4.2) that if $T_{\mathcal{T}}$ contains an indecomposable direct summand of quasilength $r-1$ then

$$\mathcal{H}_0 = \{M_{i_0,l} \,:\, r \leq l \leq 2r-1\} \cup \{M_{i_0+p, 2r-1-p} \,:, 0 \leq p \leq r-1\}.$$

and $\mathrm{Top}_k = M_{i_0, 2r-1}$. The following lemma shows, in particular, that $\widetilde{\mathrm{Top}_k}$ is Schurian.

Lemma 4.8. *Suppose that $T_{\mathcal{T}}$ contains an indecomposable direct summand T_0 of quasilength $r-1$. Let $X \in \mathcal{H}_0$. Then $\widetilde{X}$ is a strongly Schurian, and hence Schurian, Λ-module.*

Proof. Firstly note that $\mathrm{ql}(X) \geq r$, so X is not a summand of τT. Let V be an indecomposable direct summand of T. Note that the entry in the dimension vector of $\widetilde{X}$ corresponding to V is equal to $\dim \mathrm{Hom}_{\mathcal{C}}(V, X)$.

Suppose first that V is an indecomposable summand of U. Then by Lemma 3.2, we have that $\mathrm{Hom}(V, Y) = 0$ for all objects Y in $\mathcal{W}_{\tau T_0}$. By Proposition 3.6, $\dim \mathrm{Hom}(V, Y) \leq 1$ if $Y = M_{i_0-1,1}$ is the unique object on the border of $\mathcal{T}$ not in $\mathcal{W}_{\tau T_0}$. Using the additivity of $\dim \mathrm{Hom}(V, -)$ on $\mathcal{T}$, we see that $\dim \mathrm{Hom}(V, X) \leq 1$. Since V is preprojective, $\dim \mathrm{Hom}(X, \tau^2 V) = 0$, so, since

$$\mathrm{Hom}_{\mathcal{C}}(V, X) \cong \mathrm{Hom}(V, X) \oplus D\,\mathrm{Hom}(X, \tau^2 V),$$

we have $\dim \mathrm{Hom}_{\mathcal{C}}(V, X) \leq 1$. If V lies in a tube other than $\mathcal{T}$ then $\mathrm{Hom}_{\mathcal{C}}(V, X) = 0$. So we are left with the case where V lies in $\mathcal{T}$.

If $X \cong M_{i_0,l}$ for some l with $r \leq l \leq 2r-1$ then the quasisocle of X is Q_{i_0}, which does not lie in $\mathcal{W}_T$. So, by Corollary 2.3, $\mathrm{Hom}(V, X) = 0$. Since $\mathrm{ql}(V) \leq r-1$, it follows from Lemma 2.2 that $\dim \mathrm{Hom}(X, \tau^2 V) \leq 1$. Hence $\dim \mathrm{Hom}_{\mathcal{C}}(V, X) \leq 1$.

If $X \cong M_{i_0+p, 2r-1-p}$ for some p with $0 \leq p \leq r-1$ then the quasitop of X is $Q_{i_0+p+2r-1-p-1} = Q_{i_0-2}$, which does not lie in $\mathcal{W}_{\tau^2 T}$. So, by Corollary 2.3, we have that $\mathrm{Hom}(X, \tau^2 V) = 0$. Since $\mathrm{ql}(V) \leq r-1$, it follows from Lemma 2.2 that $\dim \mathrm{Hom}(V, X) \leq 1$. Hence $\dim \mathrm{Hom}_{\mathcal{C}}(V, X) \leq 1$.

We have shown that $\widetilde{X}$ is strongly Schurian as required. Since any strongly Schurian module is Schurian, we are done. □

Corollary 4.9. *Let $X \in \cup_{k \in \mathbb{Z}_s} \mathcal{H}_k$. Then $\widetilde{X}$ is Schurian.*

Proof. Firstly note that, since $\mathrm{ql}(X) \geq r$, X is not a direct summand of τT. Suppose $k \in \mathbb{Z}_s$ and $X \in \mathcal{H}_k$. If $\mathrm{ql}(X) \leq 2r-2$ then this follows from Lemma 4.5. The maximal quasilength of an object in $\mathcal{H}_k$ is $\mathrm{ql}(\mathrm{Top}_k) = \mathrm{ql}(M_{i_k, r+l_k}) = r + l_k$. This is only greater than $2r-2$ when l_k is maximal, i.e. equal to $r-1$. Then $s = 1$ (i.e. there is only one indecomposable direct summand of $T_{\mathcal{T}}$ not contained in the wing of another indecomposable direct summand of $T_{\mathcal{T}}$). We must have $k = 0$ and the result follows from Lemma 4.8. □

We have now determined whether $\widetilde{X}$ is rigid or Schurian for all indecomposable modules X in $\mathcal{T}$ which are not direct summands of τT. We summarize this with the following theorem. Note that, by Theorem 1.7, every indecomposable Λ-module is of the form $\widetilde{X}$ for X an indecomposable object in $\mathcal{C}$ which is not a direct summand of τT. Note also that part (a) of the following is a consequence of Lemma 4.2(a), which was shown using [1].

Theorem 4.10. *Let Q be a quiver of tame representation type, and $\mathcal{C}$ the corresponding cluster category. Let T be an arbitrary cluster-tilting object in $\mathcal{C}$. Let X be an indecomposable object of $\mathcal{C}$ which is not a summand of τT and let $\widetilde{X}$ the corresponding Λ-module.*

- (a) *The Λ-module $\widetilde{X}$ is τ-rigid if and only if X is transjective or X is regular and* $\mathrm{ql}(X) \leq r - 1$.
- (b) *The Λ-module $\widetilde{X}$ is rigid if and only if either*
  - (i) *X is transjective, or*
  - (ii) *X is regular and* $\mathrm{ql}(X) \leq r - 1$ *or*
  - (iii) *X is regular and* $X \in \cup_{k \in \mathbb{Z}_s} \mathcal{H}_k \setminus \{\mathrm{Top}_k\}$.
- (c) *The Λ-module $\widetilde{X}$ is Schurian if and only if either*
  - (i) *X is transjective, or*
  - (ii) *X is regular and* $\mathrm{ql}(X) \leq r - 2$, *or*
  - (iii) *X is regular,* $\mathrm{ql}(X) \in \{r-1, r\}$ *and* $X \notin \cup_{k \in \mathbb{Z}_s} \mathcal{S}_k$, *or*
  - (iv) *X is regular,* $\mathrm{ql}(X) \geq r + 1$ *and* $X \in \cup_{k \in \mathbb{Z}_s} \mathcal{H}_k$.

Proof. If X is transjective, the result follows from Remark 1.11, so we may assume that X lies in a tube $\mathcal{T}$. Let r be the rank of $\mathcal{T}$. Replacing T with $\tau^{mr} T$ for some $m \in \mathbb{Z}$ if necessary, we may assume that T is of the form $U \oplus T'$ where U is a preprojective module and T' is regular, i.e. that Assumption 1.5 holds (note that τ is an autoequivalence of $\mathcal{C}$). For part (b), note that if $\mathrm{ql}(X) \leq r - 1$, then $\widetilde{X}$ is τ-rigid by (a), hence rigid. If $\mathrm{ql}(X) \geq r$ and $X \notin \cup_{k \in \mathbb{Z}_s} \mathcal{H}_k \cup \mathcal{R}_k \setminus \{\mathrm{Top}_k\}$ then $\widetilde{X}$ is not rigid by Lemma 4.1. If $\mathrm{ql}(X) \geq r$ and $X \in \mathcal{R}_k$ then $\widetilde{X}$ is not rigid by Lemma 4.6. And if $\mathrm{ql}(X) \geq r$ and $X \in \cup_{k \in \mathbb{Z}_s} \mathcal{H}_k \setminus \{\mathrm{Top}_k\}$ then $\widetilde{X}$ is rigid by Lemma 4.4.

For part (c), note that if $\mathrm{ql}(X) \leq r-2$ then $\widetilde{X}$ is Schurian by Lemma 4.2. If $\mathrm{ql}(X) \geq r+1$ and $X \notin \cup_{k \in \mathbb{Z}_s} \mathcal{H}_k \cup R_k$ then $\widetilde{X}$ is not Schurian by Lemma 4.1. If $\mathrm{ql}(X) \geq r+1$ and $X \in R_k$ then $X \in \mathcal{S}_k$ so $\widetilde{X}$ is not Schurian by Lemma 4.7. If $\mathrm{ql}(X) \geq r + 1$ and $X \in \mathcal{H}_k$ then $\widetilde{X}$ is Schurian by Corollary 4.9.

If $\mathrm{ql}(X) \in \{r-1, r\}$ and $X \notin \cup_{k \in \mathbb{Z}_s} \mathcal{H}_k \cup \mathcal{S}_k$ then $\widetilde{X}$ is Schurian by Lemma 4.5. If $\mathrm{ql}(X) \in \{r-1, r\}$ and $X \in \mathcal{H}_k$ then $\widetilde{X}$ is Schurian by Corollary 4.9. If $\mathrm{ql}(X) \in \{r-1, r\}$ and $X \in \mathcal{S}_k$ then $\widetilde{X}$ is not Schurian by Lemma 4.7. □

Corollary 4.11. *Let Q be a quiver of finite or tame representation type and Λ a cluster-tilted algebra arising from the cluster category of Q. Then every indecomposable Λ-module which is rigid, but not τ-rigid, is Schurian.*

Proof. If Q is of finite representation type, then it is known that every indecomposable object in $D^b(KQ)$ is rigid. Hence, by Theorem 1.10, every indecomposable Λ-module is τ-rigid and the statement is vacuous in this case.

Suppose that Q is of tame representation type. Let $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{opp}}$, where T is a cluster-tilting object in the cluster category $\mathcal{C}$ of Q. Let X be an indecomposable object

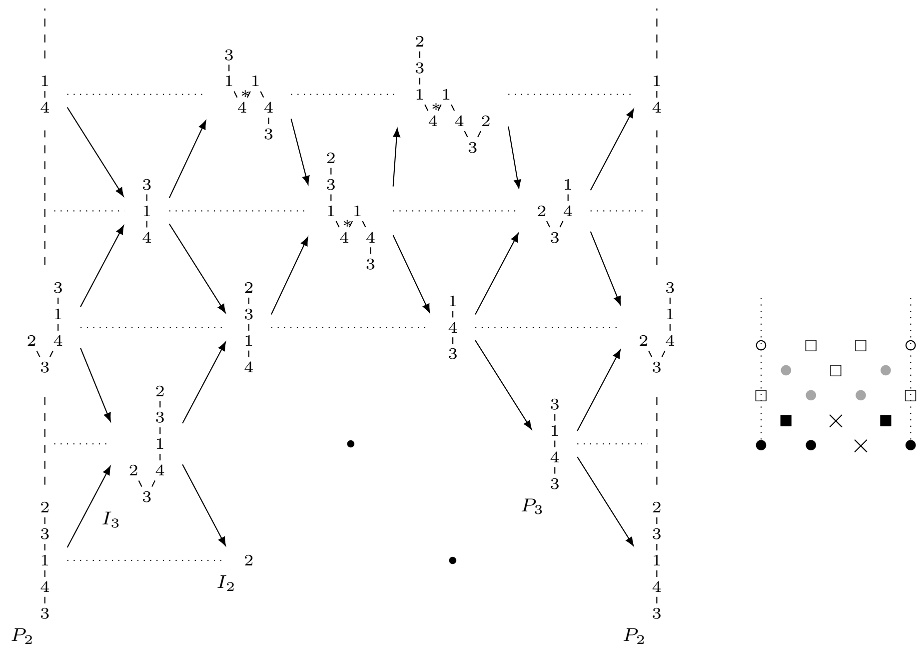

Figure 9. The left hand diagram shows part of the AR-quiver of Λ-mod. The right hand diagram shows the same objects. The symbol for a module is circular if it is Schurian, filled-in with gray if it is rigid but not τ-rigid, and filled-in with black if it is τ-rigid. The symbol $\times$ represents a gap in the AR-quiver (corresponding to an indecomposable direct summand of τT)
.

in $\mathcal{C}$ which is not a summand of τT. If $\widetilde{X}$ is rigid, but not τ-rigid, then by Theorem 4.10, we have that X is regular and $X \in \cup_{k\in\mathbb{Z}_s}\mathcal{H}_k \setminus \{\mathrm{Top}_k\}$. If $\mathrm{ql}(X) = r$, then $\widetilde{X}$ is Schurian by Theorem 4.10(c)(iii), since $\cup_{k\in\mathbb{Z}_s}\mathcal{H}_k \cap \cup_{k\in\mathbb{Z}_s}\mathcal{S}_k$ is empty. If $\mathrm{ql}(X) \geq r+1$, then $\widetilde{X}$ is Schurian by Theorem 4.10(c)(iv). $\square$

In Figure 9, we show part of the AR-quiver of Λ-mod for Example 1.6. The part shown consists of modules coming from the tube in KQ-mod shown in Figure 2. We give a Q_Λ-coloured quiver for each module, where Q_Λ is the quiver of Λ. Note that we need to distinguish between the two arrows between vertices 1 and 4. We do this by decorating the arrow which is involved in the relations with an asterisk. Recall that this then has the following interpretation (see the text after Definition 1.1). Suppose that φ is the linear map corresponding to the decorated (respectively, undecorated) arrow in Q_Λ. Then the image of a basis element $b \in B_1$ (the basis of the vector space at the vertex 1) under φ is the sum of the basis elements $c \in B_4$ which are at the end of an arrow starting at b labelled with (respectively, without) an asterisk. The diagram on the right shows which of these modules are τ-rigid, rigid and Schurian.

In Figure 10, we illustrate the τ-rigid, rigid and Schurian Λ-modules given by Theorem 4.10 for the example in Figure 6 (choosing specific indecomposable summands of T in the wings of the T_i).

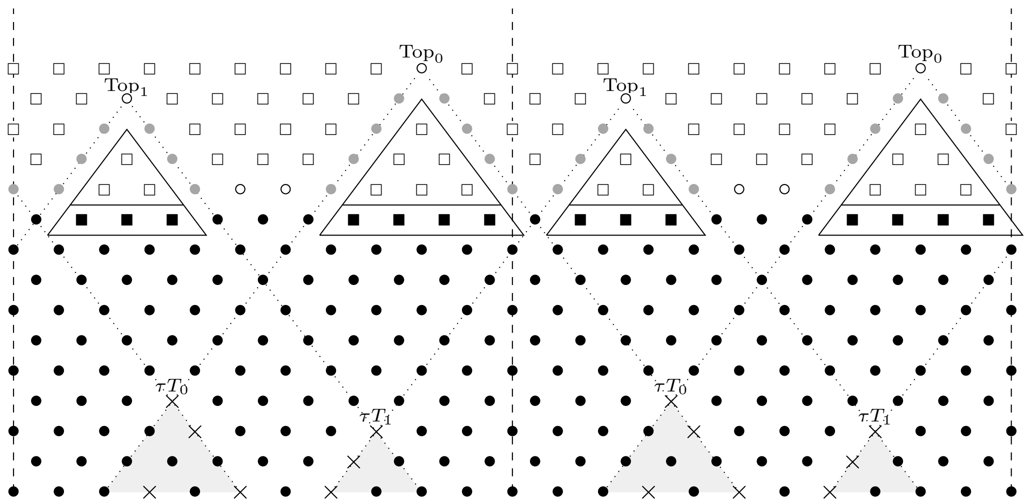

FIGURE 10. Schurian and rigid Λ-modules for a particular choice of tilting module T. The notation is as in Figure 9.

5. Wild case

In this section we determine whether some modules are rigid or Schurian for a specific quiver of wild representation type. We will see that there are some similarities with the tame case.

Let Q be the quiver:

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] \arrow[rrr, bend left] & 2 \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd}$$

and KQ the corresponding path algebra, of wild representation type. Let $P_0, \ldots, P_4$ be the indecomposable projective KQ-modules (with Q-coloured quivers as in (5.1)), and $S_0, \ldots, S_4$ their simple tops. The simple module S_2 is a quasisimple object in a regular component $\mathcal{R}$ of type $\mathbb{Z}A_\infty$ in the AR-quiver of KQ-mod. Figure 11 depicts part of this component.

$$P_0 = \begin{array}{ccc} & 0 & \\ & | & \\ & 1 & \\ / & & \backslash \\ 4 & & 2 \\ & & | \\ & & 3 \\ & & | \\ & & 4 \end{array}, \quad P_1 = \begin{array}{ccc} & 1 & \\ / & & \backslash \\ 4 & & 2 \\ & & | \\ & & 3 \\ & & | \\ & & 4 \end{array}, \quad P_2 = \begin{array}{c} 2 \\ | \\ 3 \\ | \\ 4 \end{array}, \quad P_3 = \begin{array}{c} 3 \\ | \\ 4 \end{array}, \quad P_4 = 4 . \tag{5.1}$$

Lemma 5.1. *Let X be an indecomposable module in $\mathcal{R}$. Then X is rigid if and only if it has quasilength less than or equal to 2.*

Proof. By [19, Thm. 2.6], every rigid module in a regular AR-component of a hereditary algebra has quasilength at most $n-2$, where n is the number of simple modules. In this case, there are 5 simple modules, so no indecomposable module in $\mathcal{R}$ with quasilength at least 4 is rigid.

Since KQ is hereditary and no module in $\mathcal{R}$ is projective, we have

$$\mathrm{Ext}(M, N) \cong \mathrm{Ext}(\tau M, \tau N)$$

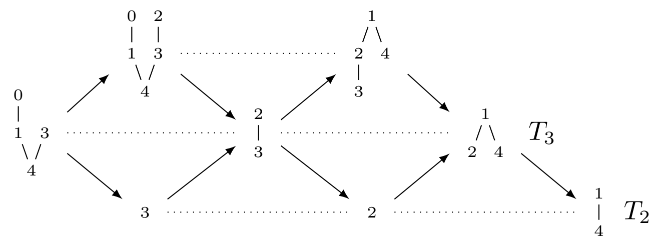

FIGURE 11. Part of the AR-quiver of KQ-mod.

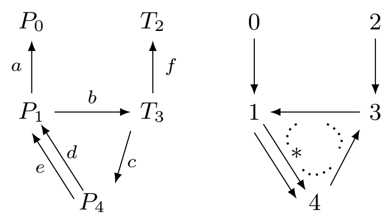

FIGURE 12. Maps between indecomposable projective KQ-modules and the quiver with relations of $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{opp}}$.

for all $M, N \in \mathcal{R}$. Hence an indecomposable module in $\mathcal{R}$ is rigid if and only if every module in its τ-orbit is rigid.

It is easy to check using the AR-formula that the modules S_2, of quasilength 1, and $\begin{smallmatrix}2\\ \mid\\ 3\end{smallmatrix}$, of quasilength 2, are rigid, while the module $\begin{smallmatrix} & 1 & \\ / & & \backslash \\ 2 & & 4 \\ \mid & & \\ 3 & & \end{smallmatrix}$, of quasilength 3, is not rigid. Hence every module in $\mathcal{R}$ of quasilength 1 or 2 is rigid, and no module in $\mathcal{R}$ of quasilength 3 is rigid, and we are done. $\square$

We mutate (in the sense of [18, 24]) the tilting module KQ at P_2, via the short exact sequence:

$$0 \to P_2 \to P_1 \to T_2 \to 0,$$

where $T_2 = \begin{smallmatrix}1\\ \mid\\ 4\end{smallmatrix}$. We obtain the tilting module

$$P_0 \oplus P_1 \oplus T_2 \oplus P_3 \oplus P_4.$$

We mutate this tilting module at P_3, via the short exact sequence

$$0 \to P_3 \to P_1 \to T_3 \to 0,$$

where $T_3 = \begin{smallmatrix} & 1 & \\ / & & \backslash \\ 2 & & 4 \end{smallmatrix}$. This gives the tilting module

$$T = P_0 \oplus P_1 \oplus T_2 \oplus T_3 \oplus P_4,$$

which induces a cluster-tilting object in $\mathcal{C}$.

We define maps a, b, c, d, e, f in $\mathcal{C}$ as follows (see Figure 12). Let a be the embedding of P_1 into P_0, b a surjection of P_1 onto T_3. We have $\mathrm{Hom}(T_3, P_4) = 0$, while

$$\mathrm{Hom}^F_{\mathcal{C}}(T_3, P_4) \cong D\,\mathrm{Hom}(P_4, \tau^2 T_3) \cong K,$$

since $\tau^2 T_3 = \begin{smallmatrix} 0 & & \\ | & & \\ 1 & & 3 \\ & \backslash\, / & \\ & 4 & \end{smallmatrix}$ (see Figure 11). We take c to be a non-zero element of $\mathrm{Hom}^F_{\mathcal{C}}(T_3, P_4)$. There are two embeddings of the simple module $P_4 = S_4$ into P_1 (see (5.1)). We choose d to be the map whose image is the lower 4 in the Q-coloured quiver for P_1 in (5.2), and e to be the map whose image is the upper 4. We take f to be the map from T_3 to T_2 factoring out the simple S_2 in the socle of T_3.

Let $g : P_4 \to P_1$ be equal to d or e. Applying Proposition 1.13(b) with $A = T_3$, $B = \tau^{-1}P_1$, $C = \tau^{-1}P_4$, and $\beta = \tau^{-1}g$ we see that $\mathrm{Hom}(T_3, \tau^{-1}g[1]) = 0$ if and only if $\mathrm{Hom}(\tau^{-1}g, \tau T_3) = 0$, which holds if and only if the map

$$\mathrm{Hom}(g, \tau^2 T_3) : \mathrm{Hom}(P_1, \tau^2 T_3) \to \mathrm{Hom}(P_4, \tau^2 T_3)$$

is zero. We have $\dim \mathrm{Hom}(P_1, \tau^2 T_3) = 1$ (see Figure 11), so let $h : P_1 \to \tau^2 T_3$ be a nonzero map.

From the explicit definition of the maps d and e, we see that $hd = 0$, while $he \neq 0$. Hence $\mathrm{Hom}(d, \tau^2 T_3) = 0$ and $\mathrm{Hom}(e, \tau^2 T_3) \neq 0$. Therefore, $\mathrm{Hom}(T_3, \tau^{-1}d[1]) = 0$ and $\mathrm{Hom}(T_3, \tau^{-1}e[1]) \neq 0$. Hence, $\mathrm{Hom}(T_3, \tau^{-1}d[1])(c) = (\tau^{-1}d[1]) \circ c = 0$, so $dc = 0$ in $\mathcal{C}$. Since the domain of $\mathrm{Hom}(T_3, \tau^{-1}e[1])$ is $\mathrm{Hom}^F_{\mathcal{C}}(T_3, P_4) = \mathrm{Hom}(T_3, \tau^{-1}P_4[1])$, which is spanned by c, we have $\mathrm{Hom}(T_3, \tau^{-1}e[1])(c) \neq 0$, so $ec \neq 0$ in $\mathcal{C}$. Similarly, we can show that $cb = 0$ and $dc = 0$ and that the maps $fb, ad, ae, be, bec, fbec$ and aec are all nonzero in $\mathcal{C}$.

It follows that $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{opp}}$ is given by the quiver Q_Λ with the relations shown in Figure 12 (where we have labelled the arrows with the corresponding maps between indecomposable projectives in KQ-mod — note that these go in the opposite direction).

As in the tame case (see Figure 9), we shall draw modules for Λ as Q_Λ-coloured quivers, decorating the arrow between vertices 1 and 4 which is involved in the relations (corresponding to the map d) with an asterisk.

Note that the AR-quiver of Λ-mod is the image of the AR-quiver of $\mathcal{C}$ under $\mathrm{Hom}_{\mathcal{C}}(T, -)$ by [7, Prop. 3.2], with the indecomposable summands of τT deleted; we will denote them by filled-in vertices.

Let $P_0^\Lambda, \ldots, P_4^\Lambda$ denote the indecomposable projective modules over Λ, $S_0^\Lambda, \ldots, S_4^\Lambda$ their simple tops and $I_0^\Lambda, \ldots, I_4^\Lambda$ the corresponding indecomposable injective modules. We have:

(5.2) $$P_0^\Lambda = \begin{smallmatrix} & 0 & \\ & | & \\ & 1 & \\ / & & \overset{*}{\backslash} \\ 4 & & 4 \\ | & & \\ 3 & & \end{smallmatrix}, \quad P_1^\Lambda = \begin{smallmatrix} & 1 & \\ / & & \overset{*}{\backslash} \\ 4 & & 4 \\ | & & \\ 3 & & \end{smallmatrix}, \quad P_2^\Lambda = \begin{smallmatrix} 2 \\ | \\ 3 \\ | \\ 1 \\ | \\ 4 \\ | \\ 3 \end{smallmatrix}, \quad P_3^\Lambda = \begin{smallmatrix} 3 \\ | \\ 1 \\ | \\ 4 \\ | \\ 3 \end{smallmatrix}, \quad P_4^\Lambda = \begin{smallmatrix} 4 \\ | \\ 3 \end{smallmatrix}.$$

(5.3) $$I_0^\Lambda = 0, \quad I_1^\Lambda = \begin{smallmatrix} 2 & & \\ | & & \\ 3 & & 0 \\ & \backslash\, / & \\ & 1 & \end{smallmatrix}, \quad I_2^\Lambda = 2, \quad I_3^\Lambda = \begin{smallmatrix} 2 & & & \\ | & & & \\ 3 & & 0 & \\ & \backslash\, / & & \\ & 1 & & \\ & | & & \\ & 4 & & 2 \\ & & \backslash\, / & \\ & & 3 & \end{smallmatrix}, \quad I_4^\Lambda = \begin{smallmatrix} & & 2 & \\ & & | & \\ 0 & & 3 & 0 \\ \backslash & & \backslash\, / & \\ 1 & & 1 & \\ & \overset{*}{\backslash}\ / & & \\ & 4 & & \end{smallmatrix}.$$

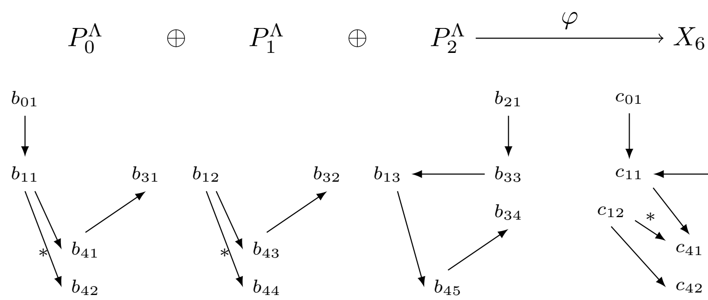

FIGURE 13. The projective cover of E

Lemma 5.2. *Figure 16 illustrates part of the AR-quiver of* Λ*-mod, including the image of the part of the AR-quiver of* $\mathcal{C}$ *shown in Figure 11.*

Proof. Firstly, note that $\mathrm{Hom}_{\mathcal{C}}(T, T_i) \cong P_i^\Lambda$ and $\mathrm{Hom}_{\mathcal{C}}(T, \tau^2 T_i) \cong I_i^\Lambda$, so applying the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$ to the first two rows in Figure 11 gives the first two rows in Figure 16 except for X_1.

If $\alpha : P \to P'$ is a map between projective Λ-modules, we denote by α^* the corresponding map between injective modules, $\alpha^* : D\,\mathrm{Hom}_\Lambda(P, \Lambda) \to D\,\mathrm{Hom}_\Lambda(P', \Lambda)$. A projective presentation of S_2^Λ is:

$$P_3^\Lambda \xrightarrow{\alpha} P_2^\Lambda \longrightarrow S_2^\Lambda \longrightarrow 0,$$

where α is the embedding. So τS_2^Λ is the kernel of $\alpha^* : I_3^\Lambda \to I_2^\Lambda$. Let β be the nonzero map from P_1^Λ to P_3^Λ. Since $\alpha\beta \neq 0$, we have $\alpha^*\beta^* \neq 0$, so α^* must be the map factoring out the lower 2. It follows that $\tau S_2^\Lambda = X_1$, completing the verification of the first two rows in Figure 16.

The irreducible maps from I_3^Λ have targets given by the indecomposable direct summands of $I_3^\Lambda / S_3^\Lambda$, i.e. I_2^Λ and X_3. The irreducible map with target P_3^Λ must be the inclusion of its (indecomposable) radical X_3. We have:

$$\mathrm{Ext}(X_3, \tau X_3) \cong D\,\mathrm{Hom}(\tau X_3, \tau X_3) \cong K,$$

so there is a unique non-split short exact sequence ending in X_3, which must be as shown.

Next, we compute τX_6. From its Q_Λ-coloured quiver, we see that the projective cover of X_6 is given by $\varphi : P_0^\Lambda \oplus P_1^\Lambda \oplus P_2^\Lambda \to X_6$. We need to compute the kernel L of φ. Let $B = \cup_{i \in \{0,1,2,3,4\}} B_i$ be the basis of $P_0^\Lambda \oplus P_1^\Lambda \oplus P_2^\Lambda$ coming from the Q_Λ-coloured quiver given by the disjoint union of the Q_Λ-coloured quivers of P_1^Λ, P_2^Λ and P_3^Λ in (5.2). As in Remark 1.2, we will write the basis elements in B_i as $b_{i1}, b_{i2}, \ldots$ (in an order taking first the basis elements for P_0^Λ, then P_1^Λ and P_2^Λ). We shall also redraw each connected component of this Q_Λ-coloured quiver as in Remark 1.2. We do the same for X_6, using the notation c_{ij}. The result is shown in Figure 13.

Let $L = \ker \varphi$, regarded as a representation with the vector space L_i at vertex i of Q_Λ. We describe a basis for each L_i in the table in Figure 14. This basis is carefully chosen to allow us to give an explicit description of L as a direct sum of indecomposable modules.

Using Figure 13, we can compute the restriction of the linear maps defining P to the submodule L to get the description of L in Figure 15. We obtain a Q_Λ-coloured quiver for this module, and we obtain that $L = \ker \varphi \cong P_1^\Lambda \oplus P_4^\Lambda$.

Vertex i	Action of φ	basis for L_i
0	$b_{01} \mapsto c_{01}$	empty
1	$b_{11} \mapsto c_{11}$ $b_{12} \mapsto c_{12}$ $b_{13} \mapsto c_{11}$	$b_{11} - b_{13}$
2	$b_{21} \mapsto c_{21}$	empty
3	$b_{31} \mapsto 0$ $b_{32} \mapsto c_{32}$ $b_{33} \mapsto c_{31}$ $b_{34} \mapsto 0$	$b_{31} - b_{34}$, b_{34}
5	$b_{41} \mapsto c_{41}$ $b_{42} \mapsto 0$ $b_{43} \mapsto c_{42}$ $b_{44} \mapsto c_{41}$ $b_{45} \mapsto c_{41}$	$b_{41} - b_{45}$, b_{42}, $b_{44} - b_{45}$

FIGURE 14. Computation of a basis for L_i, i a vertex of Q_Λ.

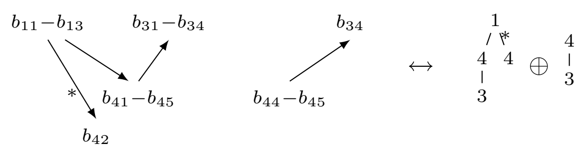

FIGURE 15. The kernel of the projective cover of X_6

Let $\psi : P_1^\Lambda \oplus P_4^\Lambda \to P_0^\Lambda \oplus P_1^\Lambda \oplus P_2^\Lambda$ be the embedding of $\ker\varphi$ into $P_0^\Lambda \oplus P_1^\Lambda \oplus P_2^\Lambda$. We can write ψ as a 3×2 matrix $\psi = (\psi_{ij})$, and the components ψ_{ij} can be read off from the above explicit description of $\ker\varphi$. We have $\psi^* = (\psi_{ij}^*) : I_1^\Lambda \oplus I_4^\Lambda \to I_0^\Lambda \oplus I_1^\Lambda \oplus I_2^\Lambda$. Since $\psi_{11} : P_1^\Lambda \to P_0^\Lambda$ is nonzero, ψ_{11}^* is a surjection onto $I_0^\Lambda \cong S_0^\Lambda$. Since $\psi_{21} : P_1^\Lambda \to P_1^\Lambda$ is the zero map, so is ψ_{21}^*. Since $\psi_{31} : P_1^\Lambda \to P_2^\Lambda$ is nonzero, ψ_{31}^* is a surjection onto $I_2^\Lambda \cong S_2^\Lambda$. Since $\psi_{12} : P_4^\Lambda \to P_0^\Lambda$ is the zero map, so is ψ_{12}^*.

Let $\gamma : P_1^\Lambda \to P_2^\Lambda$ be a nonzero map (unique up to a scalar). Then $\gamma\psi_{22} = 0$, so $\gamma^*\psi_{22}^* = 0$. Hence ψ_{22}^* is the map from I_4^Λ to I_1^Λ whose image is the submodule $\begin{smallmatrix} 0 \\ | \\ 1 \end{smallmatrix}$. Since $\psi_{32} \neq 0$, so is $\psi_{32}^* : I_4^\Lambda \to I_2^\Lambda$, so it must be a surjection onto $I_2^\Lambda \cong S_2^\Lambda$. We thus have an explicit description of the map

$$\psi^* : I_1^\Lambda \oplus I_4^\Lambda \to I_0^\Lambda \oplus I_1^\Lambda \oplus I_2^\Lambda.$$

Using a technique similar to the above, we can compute the kernel τX_6 of ψ^* and verify that it is X_5.

A similar technique can be used to show that $\tau^{-1}X_3 \cong X_4$. We have

$$\operatorname{Ext}(X_4, X_3) \cong D\overline{\operatorname{Hom}}(X_3, \tau X_4) \cong D\overline{\operatorname{Hom}}(X_3, X_3) \cong K.$$

Let φ be the embedding of X_3 into X_7, mapping it to the submodule of this form appearing on the right hand side of the displayed Q_Λ-coloured quiver of this module. Then a computation similar to the above can be done to show that $\operatorname{coker}\begin{pmatrix} \varphi \\ i \end{pmatrix} \cong X_4$, where i is the

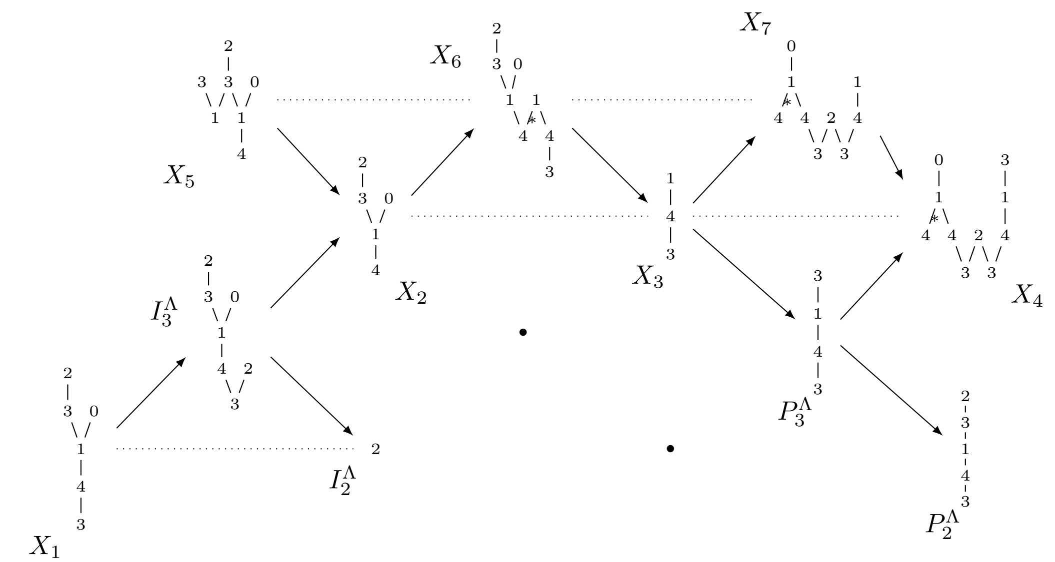

Figure 16. Part of the AR-quiver of Λ-mod

embedding of X_3 into P_3^Λ. This gives a non-split short exact sequence

$$0 \to X_3 \to X_7 \oplus P_3^\Lambda \to X_4 \to 0$$

which must be almost split. This completes the proof. $\square$

Note that the modules in the $\tau^{\pm 1}$-orbits of $I_2^\Lambda, I_3^\Lambda, P_2^\Lambda, P_3^\Lambda$ are all τ-rigid (and hence rigid) by Lemma 5.1 and Corollary 1.10.

Proposition 5.3. *The Λ-modules X_2, X_3, X_5 and X_7 are all rigid, while X_6 is not rigid.*

Proof. We will use Remark 1.4 throughout. We have

$$\mathrm{Ext}(X_2, X_2) \cong D\underline{\mathrm{Hom}}(\tau^{-1}X_2, X_2) \cong D\underline{\mathrm{Hom}}(X_3, X_2).$$

We have $\mathrm{Hom}(X_3, X_2) \cong K$, and any nonzero map from X_3 to X_2 has image $\begin{smallmatrix}1\\|\\4\end{smallmatrix}$ and so factors through the embedding of X_3 into P_2^Λ. See Figure 17, where we highlight in bold the images of the map from X_3 to X_2 and the map from X_3 to P_2^Λ. It follows that X_2 is rigid.

We have

$$\mathrm{Ext}(X_3, X_3) \cong D\overline{\mathrm{Hom}}(X_3, \tau X_3) \cong D\overline{\mathrm{Hom}}(X_3, X_2).$$

In this case, any nonzero map from X_3 to X_2 factors through the embedding of X_3 into I_3^Λ (see Figure 17). It follows that X_3 is rigid.

We have:

$$\mathrm{Ext}(X_5, X_5) \cong D\underline{\mathrm{Hom}}(\tau^{-1}X_5, X_5) \cong D\underline{\mathrm{Hom}}(X_6, X_5).$$

From the Q_Λ-coloured quivers of X_5 and X_6 in Figure 16, we see that S_1^Λ is a quotient of X_6 and is embedded into X_5. Let $f_1 : X_6 \to X_5$ be the composition of these two maps. From the Q_Λ-coloured quiver of X_6, we see that the module $\begin{smallmatrix}1\\|\\4\end{smallmatrix}$ is a quotient of X_6, and is

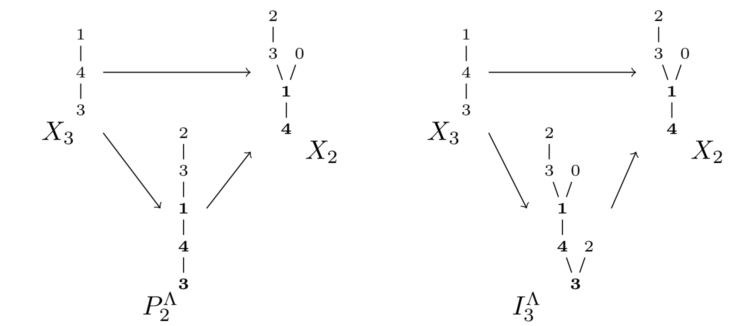

FIGURE 17. Rigidity of X_2 and X_3.

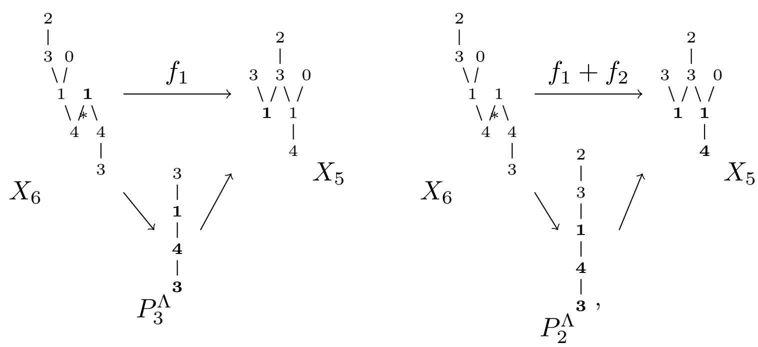

FIGURE 18. Rigidity of X_5.

embedded into X_5; let f_2 be the composition of the two maps. Then it is easy to check that $\{f_1, f_2\}$ is a basis of $\mathrm{Hom}(X_6, X_5)$.

Furthermore, f_1 factors through P_3^Λ: we take the composition of the map from X_6 to P_3^Λ with image isomorphic to X_3 and the map from P_3^Λ to X_5 whose image is the submodule $\begin{smallmatrix}3\\|\\1\end{smallmatrix}$; see Figure 18.

Note that the image of the map $f_1 + f_2$ has basis given by the sum of the basis elements of X_5 corresponding to the two copies of 1 in the Q_Λ-coloured quiver of X_5 and the basis element corresponding to the 4; we indicate the basis elements involved in the right hand diagram in Figure 18. The map $f_1 + f_2$ factors through P_2^Λ: we take the composition of the map from X_6 to P_2^Λ with image isomorphic to X_3 and the map from P_2^Λ to X_5 taking the basis element corresponding to the 2 in P_2^Λ to the basis element corresponding to the 2 in X_5. See Figure 18. Since $\{f_1, f_1 + f_2\}$ is a basis for $\mathrm{Hom}(X_6, X_5)$, it follows that X_5 is rigid.

We have:

$$\mathrm{Ext}(X_7, X_7) \cong D\overline{\mathrm{Hom}}(X_7, \tau X_7) \cong D\overline{\mathrm{Hom}}(X_7, X_6).$$

From the Q_Λ-coloured quivers of X_6 and X_7 in Figure 16, we see that each of the modules $\begin{smallmatrix}1\\|\\4\end{smallmatrix}$ and $\begin{smallmatrix}0\\|\\1\\|\\4\end{smallmatrix}$ is a quotient of X_7 and a submodule of X_6; we set g_1, g_2 to be the maps from

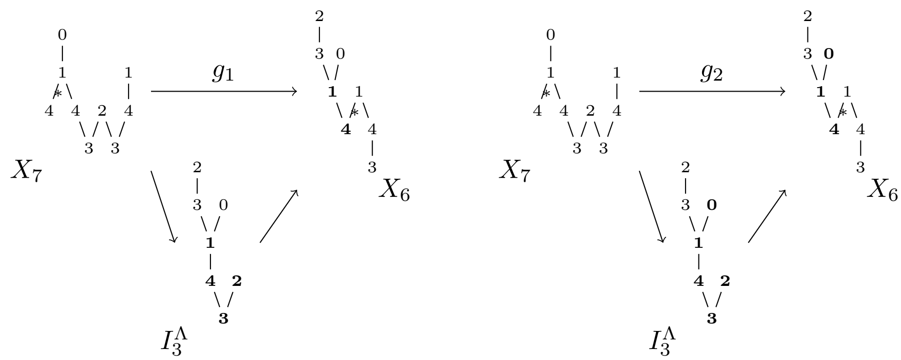

FIGURE 19. Rigidity of X_6.

X_7 to X_6 given by the composition of the quotient map and the embedding in the first and second case respectively. Then it is easy to check that $\{g_1, g_2\}$ is a basis of $\mathrm{Hom}(X_7, X_6)$.

Furthermore, g_1 factors through I_3^Λ: we take the composition of the map from X_7 to I_3^Λ with image $\begin{smallmatrix} & & 1 \\ & & | \\ 2 & & 4 \\ & \backslash\ / & \\ & 3 & \end{smallmatrix}$ and the map from I_3^Λ to X_6 with image isomorphic to X_2 (the composition of the irreducible maps from I_3^Λ to X_2 and from X_2 to X_6); see Figure 19.

The map g_2 also factors through I_3^Λ: we take the composition of the map from X_7 to I_3^Λ with image $\begin{smallmatrix} & & 0 \\ & & | \\ & & 1 \\ & & | \\ 2 & & 4 \\ & \backslash\ / & \\ & 3 & \end{smallmatrix}$ and the map from I_3^Λ to X_6 with image isomorphic to X_2 considered above. See Figure 19. Since $\{g_1, g_2\}$ is a basis for $\mathrm{Hom}(X_6, X_5)$, it follows that X_6 is rigid.

Finally, we have:

$$\mathrm{Ext}(X_6, X_6) \cong D\underline{\mathrm{Hom}}(\tau^{-1}X_6, X_6) \cong D\underline{\mathrm{Hom}}(X_7, X_6).$$

Consider the nonzero map $g_1 : X_7 \to X_6$. The projective cover of X_6 is $P(X_6) \cong P_0^\Lambda \oplus P_1^\Lambda \oplus P_2^\Lambda$, so if g_1 factors through a projective, it must factor through $P(X_6)$. It is easy to check directly that $\mathrm{Hom}(X_7, P_0^\Lambda) = 0$, $\mathrm{Hom}(X_7, P_1^\Lambda) = 0$ and $\mathrm{Hom}(X_7, P_2^\Lambda) = 0$, so $\mathrm{Hom}(X_7, P(X_6)) = 0$. Hence g_1 does not factor through a projective and $\underline{\mathrm{Hom}}(X_7, X_6) \neq 0$. It follows that X_6 is not rigid. $\square$

It is easy to check that X_i is Schurian for $i \in \{1, 2, 3, 5, 7\}$ and not Schurian for $i \in \{4, 6\}$, and that I_2^Λ and P_2^Λ are Schurian, while I_3^Λ and P_3^Λ are not. This gives the picture of Schurian and rigid modules shown on the left hand side of Figure 20 (using the same notation as in Figure 10), corresponding to the modules in Figure 16 (with X_4 omitted, as we have not checked if it is rigid).

In a tube of rank 3, a module is rigid if and only if it has quasilength at most 2, which is also the case in the regular component $\mathcal{R}$. On the right hand side of Figure 20, we show the pattern of τ-rigid, rigid and Schurian modules corresponding to the indecomposable objects in a tube of rank 3. This is from the tame case in Example 1.6, which was shown in Figure 9.

It is interesting to note the similarity of the pattern of τ-rigid, rigid and Schurian Λ-modules in these two cases, and to ask what the pattern is for the whole of $\mathcal{R}$.

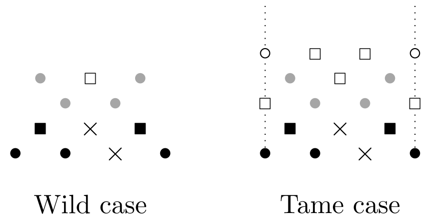

FIGURE 20. τ-rigid, rigid and Schurian Λ-modules in part of a wild example (left hand diagram). In the right hand diagram we recall the τ-rigid, rigid and Schurian modules from the tame case in Example 1.6 shown in Figure 9.

6. A COUNTER-EXAMPLE

In this section, we give the counter-example promised in the introduction. This concerns the relationship with cluster algebras. For background on cluster algebras, we refer to [15, 16]. We fix a finite quiver Q with no loops or 2-cycles and label its vertices $1, 2, \ldots, n$. Let $\mathbb{F} = \mathbb{Q}(x_1, \ldots, x_n)$ be the field of rational functions in n indeterminates over $\mathbb{Q}$. Then the associated cluster algebra $\mathcal{A}(Q)$ is a subalgebra of $\mathbb{F}$. Here, cluster variables and clusters play a key role. The initial cluster variables are $x_1, \ldots, x_n$. The non-initial cluster variables can be written in reduced form f/m, where m is a monomial in the variables $x_1, \ldots, x_n$, $f \in \mathbb{Q}[x_1, \ldots, x_n]$ and $x_i \nmid f$ for all i. Writing $m = x_1^{d_1} \cdots x_n^{d_n}$, where $d_i \geq 0$, we obtain a vector $(d_1, \ldots, d_n)$, which is called the *d-vector* associated with the cluster variable f/m.

On the other hand, let M be an indecomposable finite dimensional KQ-module, and let $S_1, \ldots, S_n$ be the nonisomorphic simple KQ-modules. Then we have an associated *dimension vector* $(d'_1, \ldots, d'_n)$, where d'_i denotes the multiplicity of the simple module S_i as a composition factor of M.

It was then of interest to investigate a possible relationship between the denominator vectors and the dimension vectors of the indecomposable rigid KQ-modules. In the case where Q is acyclic, the two sets coincide (see [11, 12, 13]). When Q is not acyclic, we do not have such a nice correspondence in general, but there are results in this direction in [2, 6, 9]. We have found the following example of a d-vector which is not the dimension vector of an indecomposable KQ-module.

Example 6.1. Let Q be the acyclic quiver from Example 1.6:

$$\begin{tikzcd} 1 \arrow[r] \arrow[rrr, bend left] & 2 \arrow[r] & 3 \arrow[r] & 4. \end{tikzcd} \tag{6.1}$$

and let Λ be the cluster-tilted algebra from this example. The quiver Q_Λ of Λ is shown in Figure 3, and can be obtained from Q by mutating at 2 and then at 3. Recall that the AR-quiver for the largest tube in KQ-mod (which has rank 3) is shown in Figure 2 and the corresponding part of the AR-quiver for Λ-mod is shown in Figure 9. Let M be the KQ-module $\begin{smallmatrix} 1 & & 3 \\ & 4 & \end{smallmatrix}$, which is of quasilength $2 = 3 - 1$ in the tube in Figure 2. The corresponding Λ-module, $\widetilde{M} = I_3^\Lambda = \begin{smallmatrix} & & 2 \\ & & 3 \\ & & 1 \\ 2 & & 4 \\ & 3 & \end{smallmatrix}$, has dimension vector $(1, 2, 2, 1)$. Then we know from [6, Thm. A] that the denominator vector of the corresponding cluster variable in the cluster algebra $\mathcal{A}(Q_\Lambda)$ is $(1,2,1,1) = (1,2,2,1) - (0,0,1,0)$. It is then easy to see that $(1,2,1,1)$ cannot occur as the dimension vector of any indecomposable KQ_Λ-module, by looking at an arbitrary representation with this dimension vector:

$$\begin{tikzcd} & & K^2 \arrow[d] \\ K \arrow[dr, shift left] \arrow[dr, shift right] & & K \arrow[ll] \\ & K \arrow[ur] & \end{tikzcd}$$

Here, a nonzero summand of K^2 has to split off, so that M cannot be indecomposable. Hence we have found a d-vector which is not the dimension vector of any indecomposable KQ_Λ-module. Note that it cannot be the dimension vector of any indecomposable Λ-module either, by the same argument.

There is another interesting class of vectors occurring in the theory of cluster algebras, known as the c-vectors. They were introduced in [16] (see [16] for the definition). In the case of an acyclic quiver Q it is known that the set of (positive) c-vectors coincides with the set of real Schur roots (see [14, 27]), that is, the dimension vectors of the indecomposable rigid KQ-modules.

But the relationship between c-vectors and d-vectors is not so nice in the general case. It is known for any finite quiver Q without loops or two-cycles that each positive c-vector of Q is the dimension vector of a finite dimensional Schurian rigid module over an appropriate Jacobian algebra with quiver Q ([22]; see [14, Thm. 14]). As pointed out in [23], this implies that every positive c-vector of Q is a Schur root of Q, hence a root of Q. Then we get the following:

Proposition 6.2. *There is a finite quiver Q without loops or 2-cycles for which the set of d-vectors associated to $\mathcal{A}(Q)$ is not contained in the set of positive c-vectors of $\mathcal{A}(Q)$.*

Proof. We consider the quiver Q_Λ in Example 6.1. In this case, the set of d-vectors is not contained in the set of dimension vectors of the indecomposable KQ_Λ-modules. If the set of d-vectors of Q_Λ was contained in the set of positive c-vectors of Q_Λ, then we would have a contradiction, since, as we mentioned above, every positive c-vector of Q_Λ is the dimension vector of an indecomposable KQ_Λ-module. □

7. Three dimension vectors

We have seen in Section 6 that there is a cluster-tilted algebra Λ associated to a quiver of tame representation type with the property that not every d-vector of $\mathcal{A}(Q_\Lambda)$ is the dimension vector of an indecomposable Λ-module. So we can ask if it is possible to express each such d-vector as a sum of a small number of such dimension vectors. Our final result shows that, for a cluster-tilted algebra Λ associated to a quiver of tame representation type, it is always possible to write a d-vector for $\mathcal{A}(Q_\Lambda)$ as the sum of at most three dimension vectors of indecomposable rigid Λ-modules.

We do not know whether it is possible to write every d-vector for $\mathcal{A}(Q_\Lambda)$ as a sum of at most two dimension vectors of indecomposable rigid Λ-modules. It would also be interesting to know whether analogous results hold in the wild case.

As before, we fix a quiver Q of tame representation type. We fix an arbitrary cluster-tilting object T in the corresponding cluster category, $\mathcal{C}$. Suppose M is an object in $\mathcal{C}$, with

$$
\begin{tikzcd}[column sep=small]
 & M_U \arrow[dr] & \\
\tau M \arrow[ur] \arrow[rr, dotted, no head] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & M_U \arrow[dr] & \\
\tau M \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & M \\
 & M_L \arrow[ur] &
\end{tikzcd}
$$

FIGURE 21. The mesh ending at an indecomposable object in $\mathcal{T}$. The diagram on the left indicates the case where M is on the border of $\mathcal{T}$.

corresponding Λ-module $\widetilde{M}$. The vertices of the quiver of $\Lambda = \operatorname{End}_{\mathcal{C}}(T)$ are indexed by the indecomposable direct summands $\operatorname{ind}(T)$ of T. The dimension vector of $\widetilde{M}$ is given by the tuple $(d'_V(M))_V$, where V varies over the indecomposable direct summands of T. We have:

$$d'_V(M) = \dim \operatorname{Hom}_{\mathcal{C}}(V, M) = \dim \operatorname{Hom}(V, M) + \dim \operatorname{Hom}(M, \tau^2 V).$$

We shall also write $d'_V(\widetilde{M})$ for $d'_V(M)$. Note that if M lies in $\operatorname{add}(\tau T)$ then $\widetilde{M} = 0$ and $d'_V(M) = 0$ for all $V \in \operatorname{ind}(T)$.

If M is (induced by) an indecomposable module in $\mathcal{T}$, then there is a mesh $\mathcal{M}_M$ in the AR-quiver of $\mathcal{T}$ corresponding to the almost split sequence with last term M. This is displayed in Figure 21, with the diagram on the left indicating the case when M is on the border of $\mathcal{T}$. We denote the middle term whose quasilength is greater (respectively, smaller) than that of M by M_U (respectively, M_L). Note that if M is on the border of $\mathcal{T}$ then M_L does not exist.

For objects X, Y of $\mathcal{C}$ we shall write

$$\delta_{X,Y} = \begin{cases} 1, & \text{if } X \cong Y; \\ 0, & \textit{otherwise}. \end{cases}$$

Lemma 7.1. *Let M be an indecomposable object in $\mathcal{T}$ with mesh $\mathcal{M}_M$ as above. Then:*

$$d'_V(\widetilde{M}) = d'_V(\widetilde{M_U}) + d'_V(\widetilde{M_L}) - d'_V(\tau\widetilde{M}) + \delta_{V,M},$$

where the terms involving M_L do not appear if M is on the border of $\mathcal{T}$.

Proof. If $V \not\cong M$ then the mesh ending at $\widetilde{M}$ in Λ-mod is the image under $\operatorname{Hom}_{\mathcal{C}}(T, -)$ of the mesh ending at M in $\mathcal{C}$ (deleting zero modules corresponding to summands of τT). If $V \cong M$ then $\widetilde{M}$ is an indecomposable projective module, so $\operatorname{rad}(\widetilde{M}) \cong \widetilde{M_L} \oplus \widetilde{M_U}$. $\square$

We assume for the rest of this section that there is an indecomposable direct summand T_0 of $T_{\mathcal{T}}$ with the property that every indecomposable direct summand of $T_{\mathcal{T}}$ lies in the wing $\mathcal{W}_{T_0}$. (In the notation at the beginning of Section 2, we have $s = 1$).

We assume further that the quasilength of T_0 (i.e. l_0) is equal to $r - 1$. We arrange the labelling, for simplicity, so that the quasisimple modules in $\mathcal{W}_{\tau T}$ are the Q_i with $i \in [0, r-2]$, so in the notation from Section 2, $i_0 = 0$. Let

$$D = \{M_{i,l} \,:\, 1 \leq i \leq r-1, r-i \leq l \leq 2r-2-i\}. \tag{7.1}$$

Note that D can be formed from $\mathcal{S}_0$ and its reflection in the line L through the modules of quasilength $r - 1$. It is a diamond-shaped region, with leftmost corner $T_0 \cong M_{1,r-1}$ and rightmost corner $\tau^2 T_0 \cong M_{r-1,r-1}$. The lowest point is the unique quasisimple module Q_{r-1} not in $\mathcal{W}_{\tau T_0}$ and the highest point is the same as the highest point $M_{1,2r-3}$ of $\mathcal{S}_0$, immediately below Top_0; see Figure 22.

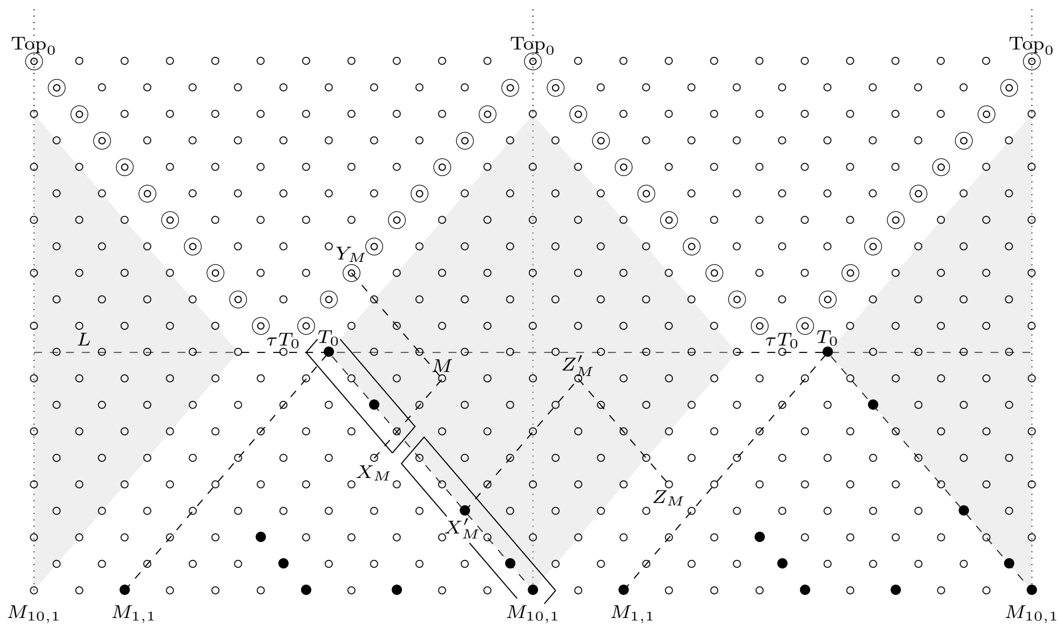

FIGURE 22. Here the rank of the tube is 11. The region D is indicated by the shaded area. The filled-in circles indicate the indecomposable direct summands of T, and the double circles indicate the elements of $\mathcal{H}_0$. The line L divides the region D into two and $\mathcal{S}_0$ consists of the vertices in D on and above the line. The upper boxed region is $\mathcal{I}_M$ (see (7.2)) and the lower boxed region is $\mathcal{I}'_M$ (see (7.8)).

Given an indecomposable module $M = M_{i,l} \in D$, we define:

$$\mathcal{I}_M = \{M_{j,r-j} \,:\, 1 \leq j \leq i\}, \tag{7.2}$$

i.e. the set of indecomposable modules which are injective in $\mathcal{W}_{T_0}$ and lie above or on the (lowest) intersection point, $M_{i,r-i}$, between the ray through M and the coray through T_0. We also set

$$X_M = M_{i,r-i-1}, \qquad Y_M = M_{0,i+l}.$$

Note that X_M is the object in the part of the ray through M below M which is of maximal quasilength subject to not lying in D. Similarly, Y_M is the nearest object to M in the part of the coray through M above M, which is of minimal quasilength subject to not lying in D. See Figure 22.

Lemma 7.2. *Let $M \in D$ and let V be an indecomposable summand of T. Then we have*

$$d'_V(M) = \begin{cases} d'_V(X_M) + d'_V(Y_M) + 1, & V \in \mathcal{I}_M; \\ d'_V(X_M) + d'_V(Y_M), & V \notin \mathcal{I}_M. \end{cases}$$

Proof. Suppose first that V is preprojective, i.e. V is an indecomposable direct summand of U. Since $X_M \in \mathcal{W}_{\tau T_0}$ we have $\mathrm{Hom}_{\mathcal{C}}(V, X_M) = 0$. Note that by Lemmas 3.2 and 3.5,

$$\dim \mathrm{Hom}_{\mathcal{C}}(V, Q_i) = \begin{cases} 0, & 0 \leq i \leq r-2; \\ 1, & i = r-1. \end{cases}$$

It follows from Lemma 3.2 that $d'_V(X) = 0$ for any module $X \in \mathcal{W}_{\tau T_k}$. By Proposition 3.6, $d'_V(Q_{r-1}) = 1$, noting that Q_{r-1} is the unique quasisimple in $\mathcal{T}$ not in $\mathcal{W}_{\tau T_k}$. Using additivity as in the proof of Lemma 3.2, we see that $d'_V(X) = 1$ if $X \in D \cup \mathcal{H}_0$. Since $X_M \in \mathcal{W}_{\tau T_0}$, $Y_M \in \mathcal{H}_0$ and $M \in D$, we have $d'_V(M) = 1$, $d'_V(X_M) = 0$ and $d'_V(Y_M) = 1$, giving the result in this case.

So we may assume that V is an indecomposable direct summand of $T_{\mathcal{T}}$. We prove the result in this case by induction on the minimal length of a path in $\mathcal{T}$ from T_0 to M. The base case is $M \cong T_0$. Then $\mathcal{I}_M = \{T_0\}$. Since $d'_V(\tau M) = 0$, the result in this case follows directly from Lemma 7.1.

We assume that $M \ncong T_0$ and that the result is proved in the case where the minimal length of a path in $\mathcal{T}$ from T_0 to M is smaller. In particular, the result is assumed to be true for all modules in $\mathcal{M}_M \cap D$ other than M.

Case I: If $M = M_{i,r-i}$, with $1 \leq i \leq r-1$ lies on the lower left boundary of D then $\mathcal{M}_M \cap D = \{M_U, M\}$. Applying the inductive hypothesis to M_U and noting that $Y_{M_U} = Y_M$, we have:

$$d'_V(M_U) = \begin{cases} d'_V(X_{M_U}) + d'_V(Y_M) + 1, & V \in \mathcal{I}_{M_U}; \\ d'_V(X_{M_U}) + d'_V(Y_M), & V \notin \mathcal{I}_{M_U}; \end{cases} \tag{7.3}$$

Note that $M_L = X_M$, $\tau M = X_{M_U}$, $Y_{M_U} = Y_M$ and $\mathcal{I}_M = \mathcal{I}_{M_U} \cup \{M\}$ (see Figure 23). By Lemma 7.1 and (7.3), we have:

$$\begin{aligned} d'_V(M) &= d'_V(M_U) + d'_V(M_L) - d'_V(\tau M) + \delta_{V,M} \\ &= d'_V(M_U) + d'_V(X_M) - d'_V(X_{M_U}) + \delta_{V,M} \\ &= \begin{cases} d'_V(X_{M_U}) + d'_V(Y_M) + d'_V(X_M) - d'_V(X_{M_U}) + \delta_{V,M} + 1, & V \in \mathcal{I}_{M_U}; \\ d'_V(X_{M_U}) + d'_V(Y_M) + d'_V(X_M) - d'_V(X_{M_U}) + \delta_{V,M}, & V \notin \mathcal{I}_{M_U}; \end{cases} \\ &= \begin{cases} d'_V(X_M) + d'_V(Y_M) + 1, & V \in \mathcal{I}_M; \\ d'_V(X_M) + d'_V(Y_M), & V \notin \mathcal{I}_M. \end{cases} \end{aligned}$$

Case II: If $M = M_{1,l}$ where $r \leq l \leq 2r-3$ lies on the upper left boundary of D then $\mathcal{M}_M \cap D = \{M_L, M\}$. Applying the inductive hypothesis to M_L and noting that $X_{M_L} = X_M$ (see Figure 23), we have:

$$d'_V(M_L) = \begin{cases} d'_V(X_M) + d'_V(Y_{M_L}) + 1, & V \in \mathcal{I}_{M_L}; \\ d'_V(X_M) + d'_V(Y_{M_L}), & V \notin \mathcal{I}_{M_L}. \end{cases} \tag{7.4}$$

Note that $M_U = Y_M$, $\tau M = Y_{M_L}$, $\mathcal{I}_M = \mathcal{I}_{M_L} = \{T_0\}$ and $\delta_{V,M} = 0$ (see Figure 23). By Lemma 7.1 and (7.4), we have:

$$\begin{aligned} d'_V(M) &= d'_V(M_U) + d'_V(M_L) - d'_V(\tau M) + \delta_{V,M} \\ &= d'_V(Y_M) + d'_V(M_L) - d'_V(Y_{M_L}) + \delta_{V,M} \\ &= \begin{cases} d'_V(Y_M) + d'_V(X_M) + d'_V(Y_{M_L}) - d'_V(Y_{M_L}) + 1, & V \in \mathcal{I}_{M_L}; \\ d'_V(Y_M) + d'_V(X_M) + d'_V(Y_{M_L}) - d'_V(Y_{M_L}), & V \notin \mathcal{I}_{M_L}; \end{cases} \\ &= \begin{cases} d'_V(X_M) + d'_V(Y_M) + 1, & V \in \mathcal{I}_M; \\ d'_V(X_M) + d'_V(Y_M), & V \notin \mathcal{I}_M. \end{cases} \end{aligned}$$

Case III: If $M = M_{i,l}$ with $1 \leq i \leq r-1, r-i \leq l \leq 2r-2-i\}$, but is not in one of the cases above, then $\mathcal{M}_M \cap D = \{M_L, M_U, \tau M, M\}$. Note that $X_{M_U} = X_{\tau M}$, $Y_{M_U} = Y_M$,

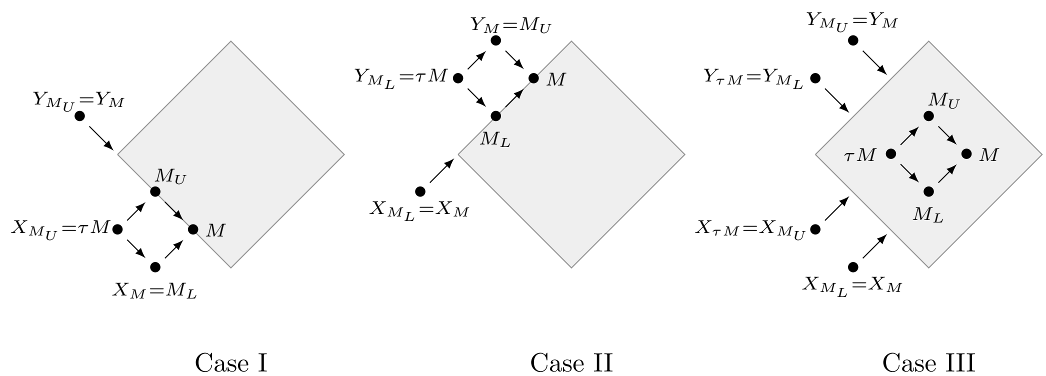

FIGURE 23. Proof of Lemma 7.2. The shaded region is the region D.

$X_{M_L} = X_M$, $Y_{M_L} = Y_{\tau M}$ and $\delta_{V,M} = 0$. We also have that $\mathcal{I}_{M_U} = \mathcal{I}_{\tau M}$ and $\mathcal{I}_{M_L} = \mathcal{I}_M$. Applying the inductive hypothesis to M_L, M_U and τM, we have:

$$d'_V(M_U) = \begin{cases} d'_V(X_{\tau M}) + d'_V(Y_M) + 1, & V \in \mathcal{I}_{\tau M}; \\ d'_V(X_{\tau M}) + d'_V(Y_M), & V \notin \mathcal{I}_{\tau M}; \end{cases} \tag{7.5}$$

$$d'_V(M_L) = \begin{cases} d'_V(X_M) + d'_V(Y_{\tau M}) + 1, & V \in \mathcal{I}_M; \\ d'_V(X_M) + d'_V(Y_{\tau M}), & V \notin \mathcal{I}_M; \end{cases} \tag{7.6}$$

$$d'_V(\tau M) = \begin{cases} d'_V(X_{\tau M}) + d'_V(Y_{\tau M}) + 1, & V \in \mathcal{I}_{\tau M}; \\ d'_V(X_{\tau M}) + d'_V(Y_{\tau M}), & V \notin \mathcal{I}_{\tau M}. \end{cases} \tag{7.7}$$

By Lemma 7.1 and (7.5)–(7.7), we obtain:

$$\begin{aligned} d'_V(M) &= d'_V(M_U) + d'_V(M_L) - d'_V(\tau M) \\ &= \begin{cases} d'_V(X_M) + d'_V(Y_M) + 1, & V \in \mathcal{I}_M; \\ d'_V(X_M) + d'_V(Y_M), & V \notin \mathcal{I}_M. \end{cases} \end{aligned}$$

The result now follows by induction. □

Let $\mathcal{I}$ denote the set of all injective objects in $\mathcal{W}_{T_0}$ and set

$$\mathcal{I}'_M = \mathcal{I} \setminus \mathcal{I}_M, \tag{7.8}$$

i.e. the set of objects in the coray through T_0 which are on or below the lowest intersection point with the ray through M. Suppose that there is an indecomposable direct summand of T in $\mathcal{I}'_M$. Let $X'_M \cong M_{j,r-j}$ be such a summand with maximal quasilength and set $Z_M = M_{0,j-2}$. Otherwise, we set $Z_M = M_{0,r-2}$.

Remark 7.3. In the first case above, the object Z_M can be constructed geometrically as follows. Let $Z'_M = M_{j,r-2}$ be the unique object in the ray through X'_M of quasilength $r-2$. Then $Z_M = M_{0,j-2}$ is the unique object in the coray through Z'_M which is a projective in $\mathcal{W}_{\tau T_0}$. See Figure 22.

Lemma 7.4. *Let V be an indecomposable direct summand of T. Then*

$$d'_V(Z_M) = \begin{cases} 1, & V \in \mathcal{I}_M \setminus T_0; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. If V is preprojective (i.e. an indecomposable direct summand of U) then, since $Z_M \in \mathcal{W}_{\tau T_0}$, we have $d'_V(Z_M) = 0$ by Lemma 3.2.

Suppose that V is an indecomposable direct summand of $T_{\mathcal{T}}$. The quasisocle of Z_M is Q_0, which does not lie in $\mathcal{W}_{T_0}$. Since $V \in \mathcal{W}_{T_0}$, it follows from Corollary 2.3 that $\mathrm{Hom}(V, Z_M) = 0$. Hence (using (1.6)), we have:

$$d'_V(Z_M) = \dim \mathrm{Hom}^F_{\mathcal{C}}(V, Z_M) = \dim \mathrm{Hom}(Z_M, \tau^2 V).$$

Consider first the case where there is an indecomposable direct summand of T in $\mathcal{I}'_M$, so X'_M is defined. We have $\mathrm{Hom}(Z_M, \tau^2 V) \neq 0$ if and only if $\mathrm{Hom}(\tau^{-2} Z_M, V) \neq 0$. By Lemma 2.2 and the fact that $V \in \mathcal{W}_{T_0}$, this holds if and only if V lies in the rectangle with corners $\tau^{-2} Z_M = M_{2,j-2}$, $M_{2,r-2}$, $M_{j-1,1}$ and $M_{j-1,r-j+1}$ In this case, $\dim \mathrm{Hom}(Z_M, \tau^2 V) = 1$.

As V and X'_M are indecomposable direct summands of T, we have that $\mathrm{Hom}(V, \tau X'_M) = 0$. So, again using Lemma 2.2, V cannot lie in the rectangle with corners $M_{1,j-1}$, $M_{j-1,1}$, $M_{j-1,r-j}$ and $M_{1,r-2}$. Combining this fact with the statement in the previous paragraph, we see that $\mathrm{Hom}(Z_M, \tau^2 V) \neq 0$ if and only if $V \in \mathcal{I}$, $V \not\cong T_0$ and V has quasilength greater than $\mathrm{ql}(X'_M) = r - j$.

However, by the definition of X'_M, there are no indecomposable direct summands of V in $\mathcal{I}'_M$ with quasilength greater than $\mathrm{ql}(X'_M)$. Hence $\mathrm{Hom}(Z_M, \tau^2 V) \neq 0$ if and only if $V \in \mathcal{I}_M \setminus \{T_0\}$.

If there is no indecomposable direct summand of T in $\mathcal{I}'_M$, then $Z_M = M_{0,r-2}$. By Lemma 2.2 and the fact that $V \in \mathcal{W}_{T_0}$, we have that

$$\dim \mathrm{Hom}(Z_M, \tau^2 V) = \dim \mathrm{Hom}(\tau^{-2} Z_M, V)$$

is 1 if and only if V lies in the coray through $\tau^{-2} Z_M = M_{2,r-2}$, i.e. if and only if $V \in \mathcal{I} \setminus \{T_0\}$. Since there is no indecomposable summand of T in $\mathcal{I}'_M$, this holds if and only if $V \in \mathcal{I}_M \setminus \{T_0\}$, and the proof is complete. □

Proposition 7.5. *Let $M \in D$ and let V be an indecomposable summand of T. Then we have:*

$$d'_V(M) = d'_V(X_M) + d'_V(Y_M) + d'_V(Z_M) + \delta_{V,T_0}.$$

Proof. This follows from Lemmas 7.2 and 7.4. □

Theorem 7.6. *Let Q be a quiver of tame representation type and let $\mathcal{C}$ be the corresponding cluster category. Let T be a cluster-tilting object in $\mathcal{C}$ and $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{opp}}$ the corresponding cluster-tilted algebra. Let Q_Λ be the quiver of Λ and $\mathcal{A}(Q_\Lambda)$ the corresponding cluster algebra. Then any d-vector of $\mathcal{A}(Q_\Lambda)$ can be written as a sum of at most three dimension vectors of indecomposable rigid Λ-modules.*

Proof. Let M be a rigid indecomposable object in $\mathcal{C}$ which is not an indecomposable direct summand of τT and x_M the corresponding non-initial cluster variable of $\mathcal{A}(Q_\Lambda)$. By [6, Thm. A], if M is transjective or in a tube of rank r containing no indecomposable direct summand of T of quasilength $r-1$ then the d-vector of x_M coincides with the dimension vector of the Λ-module $\widetilde{M}$.

Suppose that M lies in a tube which contains an indecomposable direct summand T_0 of T of quasilength $r-1$. If M is contained in the wing $\mathcal{W}_{\tau T_0}$ then the d-vector of x_M again coincides with the dimension vector of $\widetilde{M}$. If not, then M must lie in the region D defined in (7.1) (after Lemma 7.1) (note that in addition it must have quasilength at most $r-1$, but we don't need that here). By construction, the quasilengths of X_M and Z_M are both

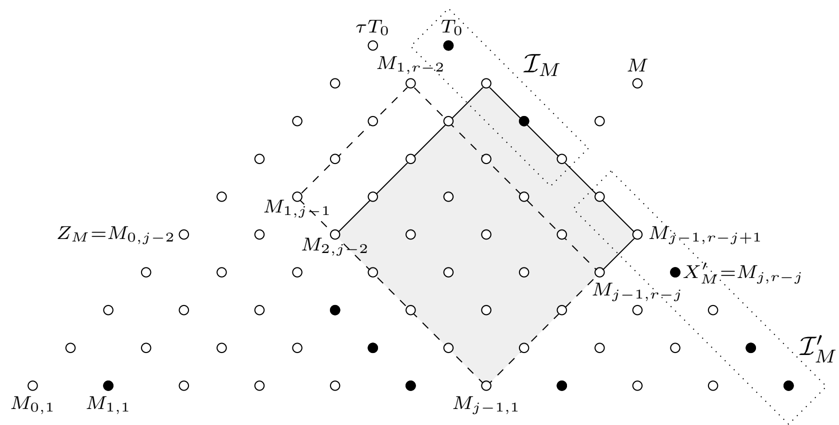

FIGURE 24. Part of the proof of Lemma 7.4 in the case $r = 11$, $j = 7$. The dotted rectangles show $\mathcal{I}_M$ and $\mathcal{I}'_M$. Since $V \in \mathcal{W}_{T_0}$, we have $\mathrm{Hom}(\tau^{-2}Z_M, V) \neq 0$ if and only if V lies in the shaded rectangle. Since $\mathrm{Ext}(V, X'_M) = 0$, V cannot lie in the dashed rectangle. By the definition of X'_M, V cannot lie in the part of $\mathcal{I}'_M$ above X'_M. Hence $\mathrm{Hom}(Z_M, \tau^2 V) \neq 0$ if and only if $V \in \mathcal{I}_M \setminus \{T_0\}$.

less than or equal to $r-1$, so they are τ-rigid Λ-modules by Lemma 4.2. Since Y_M lies in $\mathcal{H}_0$, it follows from Theorem 4.10 that Y_M is a rigid Λ-module. By [6, Thm. A], the d-vector $(d_V)_{V\in \mathrm{ind}(T)}$ of x_M satisfies

$$d_V(x_M) = d'_V(M) - \delta_{V,T_0}.$$

The result now follows from Proposition 7.5. □

Acknowledgements Both authors would like to thank the referee for very helpful comments and would like to thank the MSRI, Berkeley for kind hospitality during a semester on cluster algebras in Autumn 2012. RJM was Guest Professor at the Department of Mathematical Sciences, NTNU, Trondheim, Norway, during the autumn semester of 2014 and would like to thank the Department for their kind hospitality.

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