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Itchy Feet vs Cool Heads: Flow of Funds in an Agent-based Financial Market[*]

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Abstract

Investors tend to move funds when they are unhappy with their current portfolio managers' performance. We study the effect of the size of this flow of funds in an agent-based model of the financial market. The model combines the discrete choice approach from agent-based modelling, where all capital is mobile, with the evolutionary finance framework where all growth is endogenous. Our results show that, if investors exhibit recency bias in evaluating portfolio managers' performance, even a small amount of freely flowing capital has a huge impact on the market dynamics and the survival of noise traders. We also find that investors' intensity of choice is a driving force for excess volatility and extreme price movements when the size of the flow of funds is large.

Keywords: Portfolio management; agent-based financial market; evolutionary finance; flow of funds.
JEL classification: D53; G18; C63.

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1 Introduction

Providers of portfolio management services chase excess returns in the asset market as well as new money from investors. These are two closely intertwined goals: A portfolio manager who outperforms many of their peers tends to see exogenous growth through the inflow of money from new and existing clients as well as endogenous growth through returns on the capital employed.[1]

The exogenous growth of investment funds through the inflow (or outflow) of money is at the heart of much of the agent-based literature on financial markets, see, e.g., the textbook Hommes [26] and the surveys Hommes [25], Chiarella et al. [14], Hommes and Wagener [28], and Lettau [35]. These models are populated by a small number of different investment styles and an infinite number of clients who move money between the available styles based on differences in performance, measured, e.g., as a weighted average of realised excess returns. To capture the impact of portfolio managers' performance on the reallocation of investors' money, this literature generally employs a discrete choice model. In the absence of borrowing/lending constraints, strategies can lever their positions without limit and, as a result, have a disproportional short-term price impact. As money under management does not matter for a fund's asset allocation, asset prices are driven by the dynamic of expectations about future excess returns which can result in excess volatility with consistent deviation of asset prices from fundamental values.

Endogenous growth through investment returns and its consequences for asset prices in evolutionary, agent-type finance models have been studied in Amir et al. [1] and Evstigneev et al. [20, 21]. These models contain a small number of portfolio managers who aim to grow funds under management but do not face client attrition. The price impact of investors is proportional to their funds, and there is no leveraging. A main result in that literature is that there is only one asset price system that is stable (in the long term)

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[1]See, e.g., the survey papers by Constantinides, Harris and Stulz [16, Chapters 14, 15, 21 and 22] and Anderson and Ahmed [2].

against the entry of new investment styles. This benchmark price system is given by the expected value of the discounted sum of relative asset payoffs (a generalisation of the Kelly investment rule).

This paper combines exogenous and endogenous growth of funds in one model. Investors can move their funds between portfolio mangers with different styles, but the total amount of freely flowing capital is a model parameter. There is no leveraging: the more funds a portfolio manager holds, the stronger its price impact. By varying the size of the flow of funds in this model, we can explore the relative importance of the two different sources of growth for asset price dynamics.

The exogenous amount of freely flowing capital in each time period can be interpreted as the average client's degree of patience. If the proportion is small, most investors keep cool heads and tend to stick with their portfolio manager even during long periods of poor performance. On the other hand, when this amount is large, clients have itchy feet and tend to desert an under-performing portfolio manager quickly. There is a substantial difference between this approach to modelling the flow of funds and the usual discrete choice formula in agent-based models of financial markets: we can control the amount of freely flowing capital and thus the general degree of impatience in the market by varying the level of client attrition. The discrete choice formula is used however to model the destination of the free capital. The idea of modelling non-switching and switching investors is similar to the one of Dieci et al. [18] with the same motivation. However, their model is based on the framework of Brock and Hommes [8, 9] where the budget effect and the interdependence between wealth and prices are left in the background. An exception is Bottazzi and Dindo [5] who study agents with decision rules that can be driven by past prices.

The agent-based part of the model presented here is most closely related to that part of this literature that forbids short-selling: Anufriev and Dindo [3], LeBaron [30, 31, 32, 33] and Levy, Levy and Solomon [36, 37, 38, 39]. In these papers the budget constraint limits the potential market impact of the different investment styles. This is in contrast to the models where unlimited positions are possible (e.g. Chiarella, Dieci and Gardini [13] and Brianzoni,

Mammana and Michetti [7]) and those where asset prices are driven only by funds' expectations about future returns (e.g. Brock and Hommes [8, 9], Gaunersdorfer and Hommes [23]).

The evolutionary finance part of the model extends Evstigneev et al. [22] by adding an explicit mechanism that reallocates a certain proportion of funds between the different portfolio managers. In these models leverage is excluded and therefore the available budget constrains the positions that a fund manager can take on as well as their market impact. The hybrid model presented here bridges the gap between the agent-based and the evolutionary approach. It can be used as a powerful tool to obtain insightful results regarding the feedback loop between the exogenous flow of funds with budget effect, the endogenous growth of wealth, and the price dynamics.

Our paper is also closely related to inquires into the interaction of *passive* and *active* learning dynamics, as defined in LeBaron [34]. Passive learning refers to the market force by which wealth accumulates on investment strategies which have done well (in relative terms). Active learning refers to the switching behaviour by which investors reallocate wealth into strategies which have performed well in the past. As LeBaron points out, although both learning types and their consequences on the price dynamics have been extensively studied in isolation, the interaction between the two remains largely unexplored.

We are particularly interested in the impact of the size of the flow of funds on systematic deviations of prices from fundamental values as well as on excess volatility. This inquiry has both theoretical as well as practical aspects. Under the discrete choice model, all capital is ready to move at any time. In evolutionary finance models, all funds stay with the same portfolio manager. In reality however clients' behaviour fits neither description. Investors do not continuously monitor the performance of all portfolio managers and move funds at all times, nor do they ignore performance and never switch to managers with superior performance. As stressed by Dieci et al. ([18], p. 520): *"Empirical evidence has suggested that, facing different trading strategies and complicated decision, the proportions of agents relying on particular strategies may stay at constant level or vary over time."*.

Since our model separates the clients' allocation decision from the amount of freely flowing capital, we look more closely into the relation between behavioural aspects, such as differences of opinions, recency bias in performance evaluation, conservatism bias (e.g., Edwards [19]) and rational herding, and the model parameters. We also explore the impact of some of these behavioural phenomena on the asset price dynamics.

The next section introduces the general framework of the hybrid model and provides a specification with three investment styles. The detailed numerical study of the model is provided in Section 3. Section 4 concludes. All proofs are collected in an appendix. The software and data are available at www.schenk-hoppe.net/software/flow-of-funds/.

2 Model

2.1 General framework

We consider a financial market in which $K \geq 1$ risky assets and one risk-free asset are traded at discrete points in time $t = 0, 1, \dots$. Risky assets $k = 1, \dots, K$ pay dividends $D_{t,k} \geq 0$. $(D_{t,1}, \dots, D_{t,K})$ is a stationary stochastic process with $\sum_{k=1}^{K} D_{t,k} > 0$ and $ED_{t,k} \equiv \bar{D}_k < \infty$. Risky assets are in constant positive supply, normalised to 1, and their prices $P_{t,k}$ will be determined through short-run equilibrium of supply and demand. The risk-free asset $k = 0$ has a constant price $P_{t,0} = 1$ and pays a constant interest $r > 0$ per period.

There are $I \geq 1$ portfolio managers (funds) in the market which manage wealth on behalf of their clients. The portfolio held by fund i at time t is denoted by a vector $\theta_t^i = (\theta_{t,0}^i, \theta_{t,1}^i, \dots, \theta_{t,K}^i)$ representing the number of units of each asset. The quantity $\theta_{t,k}^i$ is given by

$$\theta_{t,k}^i = (1 - c) \frac{\lambda_{t,k}^i W_t^i}{P_{t,k}}, \quad k = 0, 1, \dots, K, \tag{1}$$

where W_t^i is the wealth managed by fund i at time t, $1 - c \in (0, 1)$ is the fraction of wealth reinvested in every period (the remainder is, e.g., used

for management fees or clients' consumption), and $\lambda_t^i = (\lambda_{t,0}^i, ..., \lambda_{t,K}^i)$ is a vector of investment proportions with $\lambda_{t,k}^i \geq 0$ and $\sum_{k=0}^{K} \lambda_{t,k}^i = 1$. We assume that λ_t^i can depend on past observations of dividends, asset prices and investment strategies up to time $t-1$. These strategies can also exhibit additional, inherent randomness as long as it is independent from dividends and other strategies at times $t, t+1, ...$

The value of fund i's holdings at the end of investment period $[t, t+1)$ is equal to

$$V_{t+1}^i = \sum_{k=1}^{K} [P_{t+1,k} + D_{t+1,k}] \theta_{t,k}^i + (1+r)\theta_{t,0}^i. \tag{2}$$

Equation (2) describes the *endogenous* change in an investment fund's wealth, i.e., the gains or losses of a fund's wealth due to the asset returns.

The *exogenous* change of wealth under management is caused by clients moving investments between the different funds at the end of each period. The decision to reallocate investments is driven by observed performance of funds. We assume that each time period a fraction $\beta \in [0, 1]$ of all investments is allocated according to some performance measure. The remaining fraction $1 - \beta$ stays with the fund where it is currently invested. Formally,

$$W_{t+1}^i = (1 - \beta)V_{t+1}^i + q_t^i \beta \bar{V}_{t+1}, \tag{3}$$

where $\bar{V}_{t+1} = \sum_{i=1}^{I} V_{t+1}^i$ is the aggregate wealth under management of all funds and q_t^i, $i = 1, ..., I$, are proportions ($q_t^i \geq 0$, $\sum_i q_t^i = 1$) that depend on the funds' performance up to time t. Equation (3) says that, after clients' completed their reallocation of funds, the actual wealth W_{t+1}^i managed by an investment fund i at the beginning of the period $[t+1, t+2)$, consists of two parts: the wealth that stays with this fund, $(1 - \beta)V_{t+1}^i$, and the (new) wealth received by this fund, $q_t^i \beta \bar{V}_{t+1}$. The value of W_{t+1}^i is the budget of fund i which is available for investment at time $t + 1$. The parameter β allows to control the maximum amount of capital that can flow between the funds. In contrast to other agent-based models, for each \$1 of wealth, only \$$\beta$ will be reallocated.

If clients cannot move investments between funds ($\beta = 0$), equation (3)

implies that each fund's budget W_{t+1}^i is equal to the value V_{t+1}^i of its time $t+1$ holdings. Then wealth under management can only grow endogenously. If clients can move investments between funds ($\beta > 0$), the actual budget W_{t+1}^i is affected by the exogenous flows of wealth. Depending on the sign of $W_{t+1}^i - V_{t+1}^i$, the net inflow or outflow of wealth into a fund i is given by

$$\beta \left(q_t^i \bar{V}_{t+1} - V_{t+1}^i \right).$$

Market clearing for each risky asset requires $\sum_{i=1}^I \theta_{t,k}^i = 1$, $k = 1, ... K$, which is equivalent to

$$P_{t,k} = (1 - c) \sum_{i=1}^I \lambda_{t,k}^i W_t^i. \tag{4}$$

Since the price of the risk-free asset is 1, $\theta_{t,0}^i = (1 - c)\lambda_{t,0}^i W_t^i$ is equal to the amount invested in that asset. With specification (3), we can write the dynamic of wealth under management as

$$
\begin{aligned}
V_{t+1}^i &= \sum_{k=1}^K \frac{\lambda_{t,k}^i[(1 - \beta)V_t^i + \beta q_{t-1}^i \bar{V}_t]}{\langle \lambda_{t,k}, (1 - \beta)V_t + \beta q_{t-1}\bar{V}_t \rangle} \times \\
&\quad \left((1 - c)\left[(1 - \beta)\langle \lambda_{t+1,k}, V_{t+1} \rangle + \beta \langle \lambda_{t+1,k}, q_t \rangle \bar{V}_{t+1} \right] + D_{t+1,k} \right) \\
&\quad + (1 + r)(1 - c)\lambda_{t,0}^i\left[(1 - \beta)V_t^i + \beta q_{t-1}^i \bar{V}_t \right],
\end{aligned}
$$

with $i = 1, ..., I$, where $\langle x, y \rangle = \sum_i x_i y_i$ denotes the scalar product, and (dropping the time index) $V = (V^1, ..., V^I)^{\mathrm{T}} \in \mathbb{R}_+^I$, $\lambda_k = (\lambda_k^1, ..., \lambda_k^I)^{\mathrm{T}} \in \mathbb{R}_+^I$, and $q = (q^1, ..., q^I)^{\mathrm{T}} \in \mathbb{R}_+^I$ with $\mathbb{R}_+$ denotes the set of non-negative real numbers. Equivalently, using vector notation,

$$
\begin{aligned}
V_{t+1} &= \Theta_t \left[(1 - c)\left[(1 - \beta)\Lambda_{t+1} + \beta \Lambda_{t+1} q_t \mathbf{1} \right] V_{t+1} + D_{t+1} \right] \\
&\quad + (1 + r)(1 - c)\Delta\lambda_{t,0}\left[(1 - \beta)V_t + \beta q_{t-1}\mathbf{1}V_t \right], \tag{5}
\end{aligned}
$$

where $\Lambda = (\lambda_k^i) \in \mathbb{R}_+^{K \times I}$, $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^I$, $D = (D_1, ..., D_K)^{\mathrm{T}} \in \mathbb{R}_+^K$, and $\Delta\lambda_0 \in \mathbb{R}_+^{I \times I}$ has entries λ_0^i on the diagonal and zero otherwise. Denoting

$W_t = (W_t^1, ..., W_t^I)^T \in \mathbb{R}_+^I$, the portfolio matrix $\Theta \in \mathbb{R}_+^{I \times K}$ is given by

$$\Theta_{ik} = \frac{\Lambda_{ki} W^i}{(\Lambda W)_k} = \frac{\Lambda_{ki}[(1-\beta)V^i + \beta q^i \bar{V}]}{\left[\Lambda\big((1-\beta)V + \beta q \mathbf{1} V\big)\right]_k}.$$

Conditions ensuring that the dynamic (5) is well-defined are provided in the following result which is proved in Appendix A. Here $\mathbb{R}_{++}^I = \{a \in [0, \infty)^I : \sum_{i=1}^I a_i > 0\}$ denotes the set of non-zero vectors with non-negative coordinates.

Proposition 2.1. *For any $V_t \in \mathbb{R}_{++}^I$, there is a unique $V_{t+1} \in \mathbb{R}_{++}^I$ solving (5) provided there is at least one fund i with $V_t^i > 0$, which is fully diversified in risky assets, i.e., $\lambda_{t,k}^i > 0$ and $\lambda_{t+1,k}^i > 0$ for all $k \geq 1$. If $\beta = 1$ it is further required that $q_t^i > 0$.*

The proof of this result also yields an explicit expression of the dynamic:

$$V_{t+1} = \left[\mathrm{Id} - (1-c)\Theta_t\big((1-\beta)\Lambda_{t+1} + \beta\Lambda_{t+1}q_t\mathbf{1}\big)\right]^{-1} \times$$
$$\left[\Theta_t D_{t+1} + (1+r)(1-c)\Delta\lambda_{t,0}\big[(1-\beta)V_t + \beta q_{t-1}\mathbf{1}V_t\big]\right] \quad (6)$$

with Id the I-dimensional identity matrix.[2] Using this result, each fund's budget W_{t+1}^i can be computed by inserting (6) into the specification (3).

2.2 Benchmark

Assume there is one risky and one risk-free asset, and a single portfolio manager. Then the dynamic (6) reduces to

$$V_{t+1} = \frac{D_{t+1} + (1+r)(1-c)\lambda_{t,0}V_t}{1 - (1-c)\lambda_{t+1,1}} \quad (7)$$

and the risky asset's price is

$$P_{t+1} = (1-c)\lambda_{t+1,1}V_{t+1}. \quad (8)$$

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[2] Setting $\beta = 0$ in (6), one obtains the dynamic studied in Evstigneev et al. [22].

One therefore finds the recursive equation

$$P_{t+1} = \frac{(1-c)\lambda_{t+1,1}}{1-(1-c)\lambda_{t+1,1}}\left[D_{t+1} + (1+r)\frac{1-\lambda_{t,1}}{\lambda_{t,1}}P_t\right]. \tag{9}$$

Let the dividend (D_t) be a stationary process with a finite variance. Then we have the following result (its proof is given in Appendix A):

Proposition 2.2. *Let $c \geq r/(1+r)$ and assume that $\lambda_{t,1} \equiv \lambda \in (0,1]$ is a constant and $\sigma^2 = Var(D_0) < \infty$.*

(i) The process

$$P_t^* = b \sum_{n=-\infty}^{0} a^{-n} D_{t+n} \tag{10}$$

with

$$a = (1+r)\frac{(1-c)(1-\lambda)}{1-(1-c)\lambda}, \quad b = \frac{(1-c)\lambda}{1-(1-c)\lambda} \tag{11}$$

is a stationary solution to the dynamics (9).

(ii) For each $P_0 \geq 0$, the price process P_t converges a.s. to the path (P_t^*) in the following sense

$$\lim_{t\to\infty} |P_t - P_t^*| = 0 \quad a.s.$$

(iii) The expectation and variance of the stationary solution are as follows

$$EP_t^* = \frac{b}{1-a}ED_0 = \frac{b}{1-a}\bar{D}, \quad Var(P_t^*) = \frac{b^2\sigma^2}{1-a^2}.$$

Interpretation. Suppose $c = r/(1+r)$. Then the above result shows that the price of the risky asset converges to a stationary process (P_t^*). Moreover the expectation of this process is equal to the fundamental value:

$$E(P_t^*) = \frac{\bar{D}}{r}. \tag{12}$$

This result holds regardless of the portfolio manager's investment strategy λ as long as it is a constant. The volatility of the price process (P_t^*) however

depends on the specific strategy. For $c = r/(1+r)$, one finds that

$$Var(P_t^*) = \frac{\sigma^2 \lambda/r}{r + (2+r)(1-\lambda)}.$$

In the particular case $\sigma = 0$, the price process becomes deterministic. Then the condition $c = r/(1+r)$ implies that the fundamental value is a unique fixed point of the price process.

To have the fundamental value as an equilibrium benchmark, we set $c = r/(1+r)$ throughout the remainder of the paper.

2.3 Model specification

We focus on the model with one risky and one risk-free asset. The investment proportion for the risky asset can then be expressed by a single number $\lambda_t \in [0,1]$. Given λ_t^i, fund i invests according to $(1 - \lambda_t^i, \lambda_t^i)$. We assume that dividends are i.i.d. with truncated normal distribution $\mathcal{N}(\mu, \sigma)^+$, see Appendix B for details on the implementation.

Investment strategies of portfolio managers. We consider three investment strategies, each followed by one portfolio manager: fundamental, trend-following and noise trading. These investment styles are the most commonly studied in the heterogeneous agent-based literature, see, e.g., Hommes [25, 26]. All three strategies are based on subjective forecasts of expected *cum-dividend* excess returns. For each investment style, the forecast $\hat{F}_t$ of the excess return of risky over the risk-free asset between the current and the next period is computed as follows.

The *fundamental fund* forecasts price reversal to the fundamental value:

$$\hat{F}_t^F = \frac{\bar{D}/r + \bar{D} - P_{t-1}}{P_{t-1}} - r = \left[\frac{\bar{D}/r}{P_{t-1}} - 1\right](1+r).$$

Here $\bar{D}$ is the expected dividend payment, and $\bar{D}/r$ the risky asset's fundamental value.

The *trend-chasing fund* interpolates the trend observed in the last L

periods to forecast future excess return:

$$\hat{F}_t^T = (1/L)\sum_{j=1}^{L}\frac{P_{t-j} + D_{t-j} - P_{t-1-j}}{P_{t-1-j}} - r.$$

The *noise-trader fund* makes randomly revised forecasts according to an AR(1) process:

$$\hat{F}_t^N = \delta\hat{F}_{t-1}^N + \epsilon\xi_t,\ \xi_t \sim N(0,1) \tag{13}$$

with constants $\delta \in (0,1)$ and $\epsilon > 0$.

Given a forecast $\hat{F}_t$, the demand of each fund at time t is determined by a demand function that maps $\hat{F}_t$ into an investment proportion λ_t. To ensure the dynamics of funds' wealth is well-defined (Proposition 2.1), the value of λ_t has to be bounded by zero and one, i.e., there is no leverage or short-selling. On the other hand, we want agents' portfolio decision to be proportional to changes in their forecasts.

A natural candidate for the demand function is a symmetric S-shaped smooth function whose values are in the interval $(0,1)$ and where the slope at zero is controlled by a parameter. Similar to Lebaron [29, 30, 31] and Chiarella, Dieci and Gardini [12, 13], we adopt a sigmoid demand function to determine the investment proportions of each fund type with respect to their forecasts of excess returns.[3] The value of λ_t is given by[4]

$$\lambda_t = (1/\pi)\arctan(\alpha\hat{F}_t) + 1/2. \tag{14}$$

The investment proportions in the risk-free and the risky asset are given by $(1 - \lambda_t, \lambda_t)$. The parameter $\alpha \in (0,1)$ describes how strongly the fund reacts to perceived future excess return. Formula (14) guarantees that for

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[3]LeBaron argues that, in a model with heterogenous agents, deriving optimal demand from utility maximisation requires an agent to be informed about the states of other agents which is unrealistic. Agents' demand is therefore modeled with a simple rule based on a sigmoid function. Chiarella et al. use sigmoid functions to capture agents' demand to capture expectations of an increase in market risk when the absolute value of expected excess return rises, e.g., during periods of booms or crashes.

[4]The trigonometric function arctan takes values in $(-\pi/2, +\pi/2)$. Hence dividing by π and adding $1/2$ gives values between 0 and 1.

any forecast the fund takes long positions in both risk-free and risky asset because $\lambda_t \in (0,1)$. The S-shaped demand curve captures a simple but general heuristic in investment — investors tend to increase (or decrease) diminishingly their investments in the risky asset along with the increase (or decrease) in perceived future excess return. Different to models where investment proportions are derived from utility maximisation (e.g. Levy, Levy, and Solomon [39] and Chiarella and He [15]) with a purpose to characterise agents' optimal behaviour, the demand function (14) used here is to facilitate behaviourally the modelling of the three typical investment styles mentioned above[5].

Flow of clients' money. We follow Brock and Hommes [9] in assuming that investors reallocate the wealth that is withdrawn from funds in proportions

$$q_t^i = \frac{\exp(\gamma f_t^i)}{\sum\limits_{j=1}^{I} \exp(\gamma f_t^j)}, \tag{15}$$

where f_t^i is the sum of discounted realised log returns of fund i:

$$f_t^i = \log(\phi_t^i) + \rho f_{t-1}^i \tag{16}$$

with

$$\phi_t^i = \frac{V_t^i}{(1-c)W_{t-1}^i}. \tag{17}$$

Here ϕ_t^i is fund i's realised return between period $t-1$ and t, and $\rho \in [0,1]$ is the discounting factor. The parameter $\gamma \geq 0$ measures clients' sensitivity

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[5]Levy, Levy, and Solomon [39] and Chiarella and He [15] may represent two typical approaches for solving investment proportions via maximising CRRA type utility functions in agent-based models. Both approaches require ad-hoc assumptions to restrict investment proportions between zero and one. The former solves investment proportions and the market clearing price simultaneously by a numerical search procedure. This method is equivalently to assume that each agent in the market knows others' investment strategies so that all agents are able to compute the current market clearing price. The latter derives investment proportions based on a closed-form solution to utility maximisation with particular assumptions on the wealth dynamics. The resulting investment proportions depend linearly on expected excess returns. A negative expected excess return will cause agents to withdraw immediately all investments from the risky asset regardless of their previous positions.

to differences in observed performance of portfolio managers.

A key feature of (15) is the heterogeneity of clients' choices. For every parameter value $\gamma \in [0, \infty)$, some positive (gross) amount of wealth flows to each investment fund. But the actual net flow is dependent on all funds' performances. A crucial issue for the model dynamics is therefore which condition implies an increase or decrease in the proportion q^i of a fund i between time t and $t+1$. A short discussion of this neglected issue follows.

Let us look at the sign function $sgn(q_{t+1}^i - q_t^i)$ where a positive (negative) sign of $q_{t+1}^i - q_t^i$ refers to a net increase (decrease) of wealth. Define the improvement of an investment fund's strategy $i = 1, ..., I$ during a time interval $[t, t+1]$ as:

$$\Delta_{t+1}^i = f_{t+1}^i - f_t^i,$$

and the average improvement of all investment funds' strategies during the time interval $[t, t+1]$ as:

$$\bar{\Delta}_{t+1} = \frac{1}{I} \sum_{i=1}^{I} \Delta_{t+1}^i.$$

The condition which triggers the increase or decrease in q^i can be characterised in the following proposition.

Proposition 2.3. *For each investment fund's strategy $i = 1, ..., I$, the sign of $q_{t+1}^i - q_t^i$ is determined by the sign of $\Delta_{t+1}^i - \bar{\Delta}_{t+1}$:*

$$sgn(q_{t+1}^i - q_t^i) = sgn(\Delta_{t+1}^i - \bar{\Delta}_{t+1}).$$

This result shows that clients have a tendency to choose funds whose improvements are higher than the average level. Note that a fund j which has the highest performance measure f_{t+1}^j at time $t+1$ does not necessarily have a higher improvement Δ_{t+1}^j than the average level. The 'best performed' fund type j may lose clients at time $t + 1$ if its improvement Δ_{t+1}^j falls below the average. Such a property reveals that clients with itchy feet hold a different interpretation for the performance measure: the improvement of

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an investment fund. Itchy-feet clients are sensitive to the improvements of funds and thus liable to overreact to changes in performance when choosing between funds.

Proposition 2.3 further provides insights into the difference between the two widely studied type-switching mechanisms: The discrete choice approach as described by equation (15) and the replicator dynamics used in evolutionary game theory. Branch and McGough [6] apply the later in the framework of Brock and Hommes [8]. They find that the replicator dynamics implies that the proportion q_t^i for agents who choose a predictor i at time t will increase (decrease) at time $t+1$ if the value of predictor i's performance measure is higher (lower) than the value of the average performance measure across all available predictors. Therefore, under the replicator dynamics, the use of some predictors will cease over time. But according to Proposition 2.3 under the discrete choice approach, agents' choice of predictors is sensitive to the improvement in predictors' performance measures rather than the difference between predictors' performance measures. Hence even predictors that are dominated can survive if their performance is volatile enough.

2.4   Discussion of behavioural aspects

Clients' psychology may play an important role in the choice of investment funds, especially when clients are boundedly rational. By linking to the behavioural finance literature, we explain and discuss some psychological elements and behavioural phenomena which are covered by the model.

Differences of opinion. The foundation for the switching mechanism (15) is the randomized discrete choice framework of McFadden [41], whereas Brock and Hommes [8, 9] utilised it in dynamic equilibrium models of financial markets to study the adaptation of investors. In discrete choice studies, not all agents necessarily choose the option, here the investment fund, which is indicated (by the model) to have the highest performance measure. Such a phenomenon corresponds to the case of $\gamma \in [0, \infty)$ in (15). The finite value of γ implies that agents are heterogeneous in making choices of available options. The reason for such heterogeneity of agents is explained by McFad-

den [41] as unmodeled idiosyncratic components in agents' utility function or randomness in agents' preferences, while Brock and Hommes [8, 9] attribute this heterogeneity to agents' bounded rationality.

Our model characterises various types of differences of opinion among investors. First, the differences in prior beliefs is modelled by the set of different fund types. The presence of cool-head investors may reinforce this type of differences of opinion. Second, the switching mechanism (15) with $\gamma \in [0, \infty)$ is able to capture the differences of opinion in fund selection and the phenomenon that clients hold different interpretations of the public performance measure (16) of each fund type. When $\gamma \in [0, \infty)$, each itchy-feet investor can be thought as having a private measurement or interpretation for the performance of each strategy. As revealed by Proposition 2.3, the improvement of strategy $\Delta_t^i = f_t^i - f_{t-1}^i$ can be regarded as an example of this kind of private measurement or interpretation for the performance of each strategy. This may lead to the phenomenon that some itchy-feet investors choose fund types according to the value of public performance measure, i.e., the value of f_t^i, while some may make their choices based on the value of improvement of each fund type, i.e., the value of Δ_t^i.

In this switching mechanism, the distribution of clients' investments among different fund types becomes more (or less) diversified when the value of γ becomes low (or high). For this reason, the degree of differences of opinion among investors in strategy selection can be measured by the value of $\gamma \in [0, \infty)$. A lower (or higher) value of γ corresponds to a higher (or lower) degree of differences of opinion.

Conservatism bias and rational herding. Different degrees of differences of opinion in strategy-switching may represent different behavioural phenomena, such as conservatism bias and rational herding. Edwards [19] identified the phenomenon of conservatism which describes that people react conservatively to new information, and they are slow to change an established view. In the context of performance-driven fund flows, a micro level foundation for conservatism is that switching investors tend to be less sensitive to the evidence of the performance of each strategy. A resulting manifestation on the macro level is that the average amount of the net flows

of wealth is low. Such a phenomenon can be captured when the value of γ is low.[6]

Rational herding refers to the tendency that investors to react to information about the behaviour of other investors. According to Bruce [11], rational herding happens because some investors believe others can perform better than themselves, therefore they follow or mimic others' behaviour. Such a phenomenon can be captured when the value of γ is high, since a high value of γ implies that investors are less conservative and a large proportion of investors will switch fast to the best performing investment strategy.

In our model, the market impact of the differences of opinion in strategy selection and its related behavioural phenomena can be studied by exploring how different values of γ in (15) affects the market dynamics.

Recency bias. This cognitive bias is related to the way that individuals digest information. Recency bias refers to the tendency of individuals to assign more importance to more recent information compared to those farther in the past. In the behavioural finance literature, recency bias have been widely studied in relation to asset valuation and evaluation of funds' performance. For example, as stated in Pompian ([43], p. 216): "*one of the most obvious and most pernicious manifestation of recency bias among investors pertains to their misuse of investment performance records for mutual funds and other types of funds. Investors track managers who produce temporary outsized returns during a one-, two-, or three-year period and make investment decisions based on such recent experience*".

In our model, recency bias in performance evaluation is captured through the parameter $\rho \in [0, 1)$ in (16). Decreasing the value of ρ represents an increase in the degree of recency bias. In the extreme case $\rho = 0$, the

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[6]In each time period, the intensity of the actual flow of wealth between investment strategies is controlled by the value of γ, while the value of β governs the proportion of the total amount of wealth that is potentially to flow. We refer to conservatism as a behavioural attribute of itchy-feet investors, as in our model cool-head investors do not look at the performance of each fund type and their wealth does not participate in the flow-of-funds. The presence of cool-head investors can be regarded as a form of rational inattention (e.g. Sims [45]), sticky-information (e.g. Mankiw and Reis [40]), or status quo bias (e.g. Samuelson and Zeckhauser [44]) in agents' decision-making.

performance of each fund is assessed by its most recent realised return. In contrast, the case $\rho = 1$ describes clients who have an infinite memory and are unbiased in performance evaluation. This setting allows us to explore how investors' recency bias in performance evaluation affects the market dynamics.

3   Simulation results

Our numerical analysis of the model focuses on the impact of the flow of clients' funds. We explore the effect of the proportion (β) of the total wealth that is allocated according to observed performance of funds. We also study the impact of the degree of the recency bias (ρ) in measuring performance and the role of the intensity of choice (γ).

Table 1 collects the parameter values used in the numerical simulations. Each time period in the simulation corresponds to a week in real time.

Parameter	Value	Explanation
r	0.001	Interest rate per week (5.3% p.a.)
c	$r/(1+r)$	Consumption rate
D_t	$\mathcal{N}(\mu, \sigma)^+$	Dividend i.i.d. random variable
$\bar{D}$	1	Mean of dividend process
σ	0.2	Standard deviation of dividend process
L	30	Observation horizon trend-chasing fund
α^L	L	Scaling parameter trend-chasing fund
α^F	0.25	Scaling parameter fundamental fund
δ	0.97	Discounting for noise trader fund
ϵ	0.2	Standard deviation for noise trader fund
β	$\{0, 10^{-8}, 10^{-7}, ..., 1\}$	Proportion of freely flowing wealth
γ	$\{1, 2, ..., 10\}$	Intensity of choice
ρ	$\{0.99, 1\}$	Discount rate of observations

Table 1: Parameters and their values

Initial conditions. To ensure a level playing field, the value of each fund's performance measure is set to 0 at the initial time $t = 0$. Aggregate initial wealth is set to 2,000 and is equally distributed across the three

funds. To be able to determine the three fund's investment strategies at the initial time, we define L observations of the price and dividend for the time periods $t = -L, ..., -1$. These data provide an upward trend in the price with a constant growth rate 0.0015 per period in order to initialise the trend following behaviour.[7] The L dividends are set to their expected value. The first period where the flow of funds can occur is from time $t = 1$ to $t = 2$, i.e., after the first actual realisation of the performance measure.

3.1 Size of flow of funds

We first explore the effects of the size of the flow of clients' funds. This proportion is given by the value of the parameter β which determines the proportion of the total wealth that is allocated according to funds' performance in any given period of time. In the extreme situation $\beta = 0$ all funds' growth is purely driven by returns on their investments (the evolutionary finance case where all clients keep a cool head). As the value of β increases, more capital is ready to move in any period which entail higher growth rates of wealth under management of better performing funds. Underperforming funds on the other hand will lose investment wealth faster. The other extreme is $\beta = 1$ where all capital is ready to move and funds' superior performance attracts an inflow of new capital (the agent-based case where all clients have itchy feet).

3.1.1 No recency bias

Table 2 collects data on the long-run averages of wealth under management by the three different funds, trading volume, excess return and standard deviation of the price and excess return. Clients have no recency bias and weigh all observations equally, i.e., the discount rate applied is $\rho = 1$.

All quantities in the table are calculated from 10 independent runs of the model by averaging over $N = 1$ million time periods after an initial (discarded) 900,000 time periods. Trading volume per time step is calculated

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[7]These data will not affect the results as the predefined pattern in the price will be fully cleared by the randomness of the model after L periods from $t = 0$.

as $\text{Vol}_t = \sum_{i=1}^{I} |\theta_t^i - \theta_{t-1}^i|/2$. Excess return per time period is given by $\ln(\frac{P_t + D_t}{P_{t-1}}) - \ln(1 + r)$. Table 2 reports annualised values.

β	Funds' wealth (in %)			Std. dev Price	Std. dev Excess ret.	Volume (in %)	Excess ret. (in %)
	Noise	Fund.	Trend				
0	0.0505	75.09	24.86	4.4815	0.003137	0.402	0.000418
1E-8	.0199	73.96	26.02	4.4959	0.003157	0.415	-0.000711
1E-7	.0193	77.16	22.82	4.3256	0.003087	0.374	0.000178
1E-6	.0122	84.04	15.95	4.3520	0.003046	0.300	-0.000271
1E-5	.0006	89.52	10.48	4.3583	0.002913	0.206	-0.000051
1‰₀	0.0151	86.28	13.71	4.3137	0.002956	0.250	0.000478
1‰	0.0237	79.41	20.57	4.4111	0.003106	0.362	-0.000332
1%	0.0256	75.82	24.15	4.4101	0.003083	0.348	0.000108
10%	0.0251	74.73	25.24	4.4924	0.003087	0.347	-0.000488
100%	0.0246	75.70	24.28	4.3697	0.003082	0.350	0.000308

Table 2: Market characteristics for different proportions of free flow of funds (β). Intensity of choice is set to $\gamma = 2$. Clients have no recency bias, i.e., $\rho = 1$.

The results in Table 2, columns 1-4, show the average amounts of wealth under management by the three funds for different values of β. The noise trader fund holds less than 0.0256% of the total wealth under management, the fundamental fund has about three-quarters, and the trend-chasing fund about one-quarter. The noise trader fund therefore has a negligible impact on the price. Among the two large funds, fundamental investment dominates. Since the trading volume per period is extremely low (column 7), it follows that both funds essentially hold identical portfolios.

As a consequence of the fact that almost all wealth is held in portfolios invested according to fundamental values, the price of the risky asset is very close to the fundamental value (columns 5-6). The excess returns are almost zero and the volatility of the price is close to its benchmark value $\sigma^* = 4.4688$ (see Proposition 2.2).

These findings hold true for all values of β. These scenarios range from no freely flowing capital ($\beta = 0$) to all wealth being allocated according to performance in each period ($\beta = 1$). In all of these markets, pricing is in

line with fundamentals, there is almost no excess volatility, and the noise trader fund plays a negligible role. We can draw the following conclusion. If clients have no recency bias ($\rho = 1$) then the market prices assets according to fundamental values for all proportions β of freely flowing capital.

3.1.2 Recency bias

We repeat the above exercise but assume that clients exhibit a mild recency bias by setting $\rho = 0.99$. Clients discount the realised performance of all funds, hence place greater weight on more recent observations. In the current case, the last observation ($L = 30$) is weighted by a factor of about 74%. We therefore refer to this scenario as mild recency bias.

Table 3 collects the simulation results. The first observation is that recency bias has a pronounced effect on the market dynamics. Compared to Table 2, all but the first row (where the value of ρ is irrelevant as no capital moves) contain different numbers in Table 3. The fundamental value fund is the largest throughout but both the trend chasing fund and, in particular, the noise trader fund increase their wealth under management when β is larger.

β	Funds' wealth (in %)			Std. dev Price	Std. dev Excess ret.	Volume (in %)	Excess ret. (in %)
	Noise	Fund.	Trend				
0	.0505	75.09	24.86	4.4815	0.003137	0.402	0.000418
1‰₀	7.42	63.89	28.69	62.163	0.113713	32.20	0.020348
1‰	18.05	51.89	30.06	161.88	0.283485	71.03	0.130028
1%	26.62	41.88	31.50	253.23	0.429098	95.70	0.327618
10%	28.37	39.80	31.86	277.10	0.457822	99.56	0.382398
100%	28.84	39.51	31.65	285.83	0.459898	102.79	0.417128

Table 3: Market characteristics for different proportions of free flow of funds (β). Intensity of choice is set to $\gamma = 2$. Clients exhibit a mild recency bias with $\rho = 0.99$.

There is excess return in all scenarios where some capital can move ($\beta > 0$) and the risky asset's price is much more volatile than the fundamental value. The market adds price risk and a risk premium. We also observe a

considerable trading volume.

Even with a small amount of wealth being allocated according to fund performance, the noise trader fund plays a substantial role. For instance, for $\beta = 1\text{‰}$ the fund on average manages more than 7.4% of all wealth (up from 0.0151%), which turns out to have a strong impact on price volatility of the risky asset and trading volume. The standard deviation of the price increases by an order of magnitude and trading volume increases by a factor of more than 100.

The larger the value of β, the more pronounced these effects. However, the increase in the noise trader fund's wealth, price volatility and trading volume is strongest for values of β up to 1%. A further increase from 1% up to 100% of capital being reallocated according to performance has a comparatively minor impact.

At the maximum level $\beta = 100\%$, the fundamental value fund manages less than 40% while the two other funds are roughly of equal size with each having about 30% of wealth under management. The risk premium (0.417) is about a thousand times higher than in the case $\beta = 0$.

While in the rational market observed in the absence of recency bias ($\rho = 1$) was accompanied by a negligible amount of noise trading, the recency bias creates a tremendous room for the noise trader fund. With a wealth of 28.84% of the total, the noise trading fund has substantial price impact. Analogously to the noise trader literature (De Long et al. [17]), one can conclude that the noise trader fund 'creates its own space'.

We can draw the following conclusion. Clients' recency bias, even a mild one, when combined with small fraction of freely moving capital, can have a tremendous impact on the price and market dynamics. Noise trading and trend chasing both are investment styles that are viable over the long term.

The agent-based literature mostly focuses on the case where individual investors use only the most recent realised return or profit to evaluate the performance of each investment strategy (i.e., $\rho = 0$). How investors' memory biases (such as the recency bias and short memory) in evaluating and selecting investment strategies can impact the market dynamics has received less attention. LeBaron [31, 33] studies the role of investors' memory

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lengths in strategy evaluation using an agent-based model where the strategy selection mechanism is mainly driven by the discrete choice model as described by (15). Similar to the model presented here, that model forbids short-selling and allows the impact of wealth on the price.

LeBaron shows that in a market where investors have heterogeneous memory lengths in evaluating the performance of each strategy the presence of investors who exhibit "small sample bias" (short memory length) increases the market volatility in the long term. The volatility becomes smaller if more investors with long memory lengths are present in the market. Furthermore, it has been reported that if all investors have sufficient long memory lengths (greater than 28 years) when come to evaluate the performance of each strategy, the price may converge to a rational benchmark (e.g. the fundamental value).

By comparing our results with those reported by LeBaron [31, 33], the following important consensus can be reached. First and foremost, the cognitive bias in investors' memory plays a crucial role in affecting strategy evaluation and selection, which may yield a substantial impact on the long-term market dynamics. Second, the "long memory" of investors in evaluating and selecting investment strategies tends to stabilise the market in the long term. Third, recency and small sample bias both point to the conjecture that investors with relatively short memory in strategy selection may destabilise the market. These "short memory" investors create an evolutionary space where different investment strategies are able to thrive. Not just those strategies with good cumulative performance, even those with relatively bad cumulative performance are viable. As stressed by LeBaron ([31], p. 7206): "*these results contrast sharply with the commonly held wisdom in finance that "bad" strategies will eventually be driven out of the market*".

However, the situation and result can be quite different in models where unlimited positions are possible and asset prices are driven only by investors' expectations about future returns. Hommes [10], for instance, shows that an increase in memory length, i.e., a larger value of the parameter ρ, of all investors in strategy evaluation and selection tends to destabilise the market. Hommes et al. [27] further report that whether memory stabilises

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or destabilises may critically depend on how past performances are weighted. Changing the weights may reverse the result, that is, an increase in memory may also stabilise the market.[8] Although different conclusions are drawn in different market contexts, these results reveal that investors' memory in evaluating and selecting strategies is a critical behavioural element in shaping the long term dynamics.

3.2 Intensity of choice

The above results show that all three funds can co-exist in the long run, provided clients exhibit at least some degree of recency bias. We now turn to the question whether a higher intensity of choice is to the advantage or disadvantage of either the noise trading or the trend chasing fund.

The intensity of choice, which is set by the parameter γ, determines how strongly clients react to differences in fund performance. The higher the value of γ, the more of the freely flowing capital will go to the better performing funds. If γ is low, the time average of a fund's performance matters more than its volatility. If γ is high, variance in performance carries a higher penalty in competition for clients' wealth.

A key feature of the switching mechanism is that, if the value of the intensity of choice parameter γ is finite, not all clients necessarily choose the fund type which is indicated (by the model) to have the highest performance measure. Clients may hold different opinions in selecting fund types. Furthermore, clients under this switching mechanism may have different interpretations of the public performance measure of each fund type (see Proposition 2.3). The degree of differences of opinion in type-switching can be measured by the value of parameter γ. Clients' conservatism bias and herding type of behaviour are associated with the degree of differences of opinion in type-switching. Clients' conservative or herding type of behaviour can be observed when the value of γ is relatively low or high. We

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[8]Hommes et al. [27] show that an increase in memory stabilises the market if the normalisation of performance measure changed to: $(1 - \rho)X_t^i + \rho Y_{t-1}^i$ where $\rho \in [0, 1)$, X_t^i is the profit strategy i at time t, and Y_{t-1}^i is the performance measure of strategy i at time $t - 1$.

investigate how different values of γ impact the market dynamics.

To demonstrate the effect of a higher intensity of choice when clients exhibit recency bias ($\rho = 0.99$) we consider 10 scenarios with parameter values $\gamma = 1, 2, ..., 10$. Table 4 summarises the market characteristics of these scenarios.

γ	Funds' wealth (in %)			Std. dev Price	Std. dev Excess ret.	Volume (in %)	Excess ret. (in %)
	Noise	Fund.	Trend				
1	30.75	36.75	32.50	303.061	0.492027	106.19	0.519018
2	28.83	39.44	31.73	285.309	0.455402	102.04	0.410298
3	27.18	41.91	30.91	269.53	0.434343	100.95	0.345498
4	25.77	43.68	30.55	277.981	0.408359	99.31	0.368258
5	24.82	45.73	29.45	281.61	0.399342	100.77	0.332078
6	23.68	47.76	28.56	302.576	0.392855	101.57	0.408968
7	22.84	50.46	26.70	353.622	0.402047	104.03	0.513918
8	22.02	53.38	24.61	387.082	0.422043	105.50	0.560178
9	21.01	56.46	22.52	435.641	0.443613	105.23	0.675598
10	21.02	58.66	20.32	455.214	0.454744	108.04	0.699088

Table 4: Market characteristics for different values of intensity of choice γ. Clients exhibit a mild recency bias, $\rho = 0.99$. $\beta = 1$.

The results on the long-run average of wealth under management for the three funds in Table 4 are interesting. More performance-sensitive clients benefit investment in fundamentals. When the intensity of choice is very low, the three funds are of roughly equal size. With an increasing intensity of choice, the fundamental value fund increases in size at the expense of the two other funds which are affected almost equally. However, these are average proportions of wealth under management. Columns 5 and 6 in Table 4 show that the short-term dynamics gets more volatile when the intensity of choice γ is high. Indeed average price volatility, trading volume and asset return all exhibit U-shaped patterns with respect to γ.[9]

Both a lower and higher values of γ can lead to higher levels of price

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[9]The U-shaped dependence on γ could cause issues in empirical estimations. Here the knowledge of the funds' strategies and the size of their wealth under management is needed to decide whether one is in a low γ or a high γ regime.

volatility and trading volume. The causes for these observed high price volatility and trading volume can be quite different. To illustrate the drivers behind the high price volatility and trading volume, Figure 1 depicts the time series of the price of the risky asset and the funds' wealth under management when $\gamma = 2$, and Figure 2 does the same for $\gamma = 8$. When the value of γ is relatively low ($\gamma = 2$), itchy-feet clients are less sensitive to the performance of each fund type leading to a well diversified wealth distribution among the three fund types. The low level of net flows of wealth which is implied by the low value of γ serves as a macro level manifestation of the conservatism bias of the itchy-feet clients. The scenario corresponds to that in the bottom row of Table 3.

Figure 1 illustrates that for low intensities of choice ($\gamma = 2$) the wealth managed by each of the three funds is almost equal. This can be interpreted as a high degree of differences of opinion in the population of clients which, in turn, entails high price volatility and trading volume.

When the intensity is high ($\gamma = 8$) clients' are prone to herding which can be viewed as less differences of opinion. Herding causes fluctuations of funds' wealth under management and thereby drives booms and crashes. These findings reveal that not only a high degree of differences of opinion but also herding with a lower degree of differences of opinion can cause high price volatility and trading volume. This finding points to an alternative explanation to the differences-of-opinion literature on explaining the high trading volume observed in real markets.

Moreover, the differences-of-opinion literature usually ignores the evolutionary perspectives of financial markets such as adaptation and market selection. We have shown that, in a evolutionary context, excess fluctuations of the price which are caused solely by differences of opinion (in terms of different investment strategies or prior beliefs, different views in strategy selections and different interpretations of the public performance measure) are only a *temporary* market phenomenon. These differences of opinion are not sufficient to explain the *persistence* of high trading volume, whereas additional insights can be obtained by analysing investors' heuristics and biases in strategy-switching. Our simulation results show that the high trading
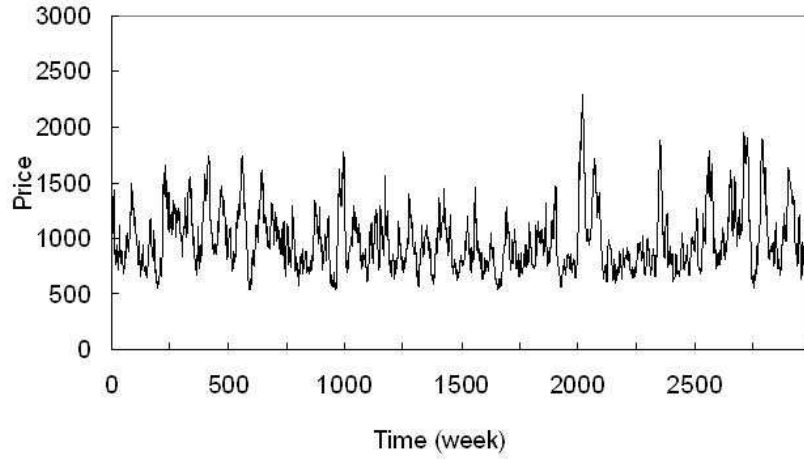
(a) Price of risky asset
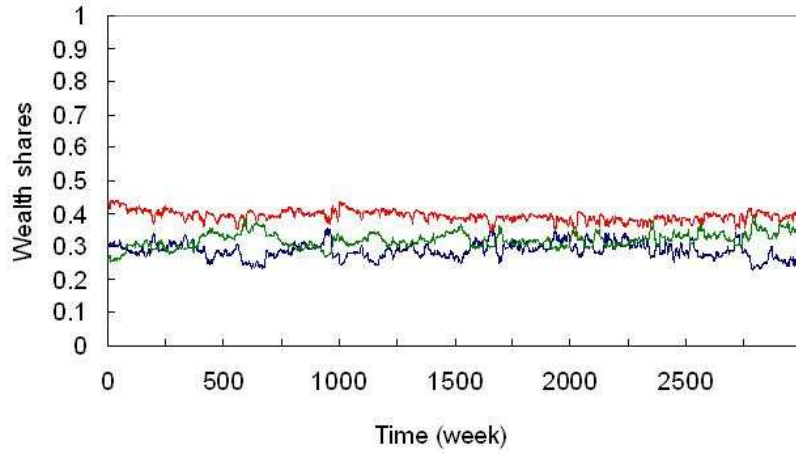
(b) Proportion of wealth under management: fundamental value fund (red, top), trend-chaser fund (green, middle), noise trader fund (blue, bottom)

Figure 1: Time series of funds' wealth under management (in proportions of total wealth) for recency bias ($\rho = 0.99$), all wealth freely flowing ($\beta = 1$) and low intensity of choice ($\gamma = 2$).
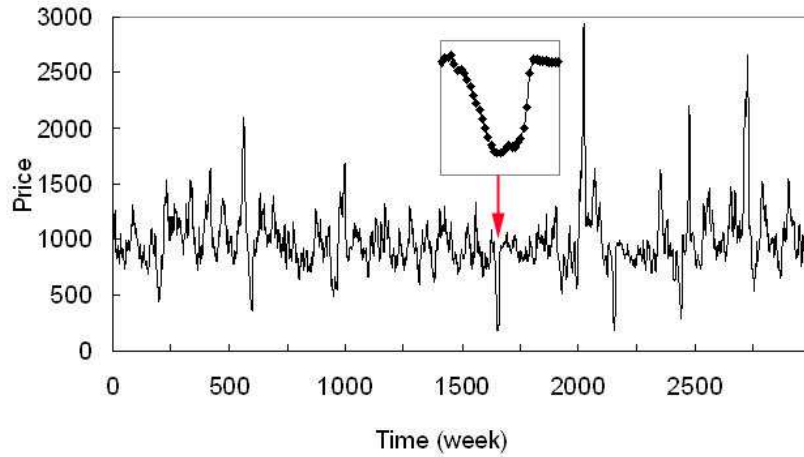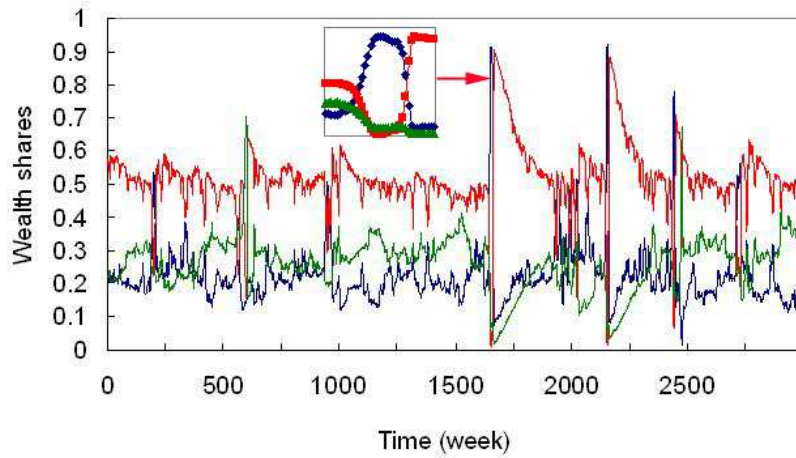
(a) Price of risky asset

(b) Proportion of wealth under management: fundamental value fund (red, top), trend-chaser fund (green, middle), noise trader fund (blue, bottom)

Figure 2: High intensity of choice ($\gamma = 8$) in the scenario of Figure 1.

volume is triggered by differences of opinion and amplified by conservatism bias or herding behaviour, while it is investors' recency bias in performance evaluation which maintains the persistence of differences of opinion and high

trading volume.

To further illustrate how the values of the intensity of choice impact the time series of log-returns of the risky asset, Table 5 collects the summary statistics for 10 different scenarios with γ ranging from 1 to 10. For comparison purpose, the path for the noise fund's investment proportions and the path for the dividends are fixed across all the scenarios. For each run, the time series of log-returns are sampled over 10,000 periods after an initial 1 million periods.

γ	Mean	Max.	Min.	Std. dev	p-quantile		Skewness	Kurtosis
					$p = 1\%$	$p = 99\%$		
1	0.000020	0.3023	-0.2545	0.0683	-0.1675	0.1693	0.0453	3.4240
2	0.000019	0.2779	-0.2501	0.0636	-0.1561	0.1555	0.0434	3.3740
3	0.000018	0.2620	-0.2453	0.0600	-0.1480	0.1453	0.0392	3.3221
4	0.000016	0.2539	-0.2410	0.0573	-0.1389	0.1374	0.0323	3.2717
5	0.000015	0.2467	-0.2384	0.0554	-0.1349	0.1324	0.0267	3.2822
6	0.000014	0.4605	-0.2653	0.0545	-0.1331	0.1295	0.0895	4.2057
7	0.000012	0.6818	-0.3128	0.0557	-0.1388	0.1368	0.3969	8.7877
8	0.000011	1.1125	-0.3949	0.0594	-0.1459	0.1396	1.4351	30.8357
9	0.000014	1.2360	-0.4941	0.0630	-0.1665	0.1551	1.7602	36.0831
10	0.000008	0.9861	-1.2813	0.0643	-0.1718	0.1698	0.1880	40.4167

Table 5: Summary statistics for the time series of log-returns of the risky asset (excluding dividend) under different values of intensity of choice γ. Clients exhibit a mild recency bias, $\rho = 0.99$. $\beta = 1$.

Table 5 shows that increasing γ decreases the mean of log-returns (which eventually goes to zero). In contrast, the magnitude of maximum and minimum log-returns, standard deviations, the length between 1%-quantile and 99%-quantile, and kurtosis exhibit a U-shaped pattern with respect to γ.

The values of the 1%-quantile and the 99%-quantile show that 98% of returns are bounded in mid ranges with a maximum interval between -17.18% and 16.98%. These 98% of returns as well as the standard deviation of returns are quite stable over the whole range of γ. However, huge booms and crashes occur for higher values of γ as evidenced by the large maximum and minimum returns. The extremely high kurtosis of returns in these cases im-

plies that increasing γ increases the frequency of small returns (and reduces that of high returns) but at the same time makes large price movements more extreme.

The reason for this behaviour is as follows. If the intensity of choice is relatively low, individual investors are not sensitive to the performance of each fund type. The aggregate wealth is well diversified across the three funds most of the time, which leaves some space for a volatile market since the fundamental investment fund does not have a comparative advantage in terms of relative wealth. If the intensity of choice is high, investors are sensitive to the performance of each fund. From time to time, large price changes trigger large shifts of wealth between the funds, and thereby induce further price changes. These in turn feed back to more changes in funds' wealth shares. (This is illustrated in the insets in Figure 2 (a) and (b)). During these periods, a highly volatile market with booms and crashes ensues. On the other hand, the high intensity of choice also increases the average wealth held in the fundamental investment fund (see Table 4 and Figure 2). This leads to lone periods of time where the market is much less volatile.

Another important question is whether the effect of the intensity of choice is proportional to the value of β (the proportion of the freely flowing capital). To this end, we perform the same exercise as in Table 5 but with β set to 1% (down from 100%). Table 6 depicts the summary statistics for scenarios with relatively low and high values of γ.

γ	Mean	Max.	Min.	Std. dev	p-quantile		Skewness	Kurtosis
					$p = 1\%$	$p = 99\%$		
1	0.000021	0.2792	-0.2372	0.0643	-0.1559	0.1563	0.0420	3.3428
2	0.000021	0.2609	-0.2501	0.0597	-0.1443	0.1456	0.0437	3.3596
9	0.000019	0.2059	-0.2453	0.0419	-0.1028	0.1034	0.0484	3.4906
10	0.000019	0.2024	-0.1602	0.0404	-0.0997	0.0991	0.0489	3.5129

Table 6: Summary statistics for the time series of log-returns of the risky asset (excluding dividend) under different values of intensity of choice γ. Clients exhibit a mild recency bias, $\rho = 0.99$. $\beta = 0.01$.

Comparing the results in Tables 5 and 6 we find the following. When

the value of the intensity of choice is low, reducing the value of β by a factor of one-hundred only has a minor impact on the return series. The standard deviation in both cases $\gamma = 1$ and $\gamma = 2$ drops only by 0.4%. Changes in other the magnitude of the maximum and minimum returns, 1%-quantile and 99%-quantile, skewness and kurtosis are also very small. In contrast, if the intensity of choice is high, reducing the value of β substantially affects the return series. Both skewness and kurtosis are much smaller in Table 6 than in Table 5 for $\gamma = 9$ and $\gamma = 10$. The values for the 1% and 99% quantiles are both closer to zero, and the extremely large booms and crashes disappeared. Based on these results, we can conclude that the market is much more sensitive to changes in the intensity of choice when the parameter β is large.

These findings show that in our model the intensity of choice has a pronounced impact on the market dynamics when many investors have itchy feet. But when the cool heads are much more numerous, the intensity of choice has very little impact. This observation might be of interest to empirical researchers. For instance, Boswijk et al. [4] use S&P 500 data to estimate the intensity of choice in an agent-based model. They find the existence of two expectation regimes, one fundamentalist and one trend-following. But the intensity of choice is not significantly different from zero. The authors stress that this is a common result in type-switching regression models because large changes in the intensity of choice cause only small variations in the wealth holdings of the different investment styles. Our model, in contrast, shows that the intensity of choice does have a strong impact – but only if enough investors have itchy feet to generate a sufficient amount of freely flowing capital. Moreover, based on the data of mutual fund flows, statistically significant estimates of the intensity of choice parameter are reported by Goldbaum and Mizrach [24].

4   Conclusion

The paper brings together two strands of literature, agent-based models of financial markets where investment funds grow exogenously and evolutionary finance where all growth is endogenous. By embedding the discrete

choice approach into an evolutionary finance framework, the resulting model allows the coexistence of itchy-feet investors who tend to desert an underperforming portfolio manager quickly and cool-head investors who would stick with their portfolio manager even during long periods of poor performance. The model separates the clients' decision problem where to invest their funds and the amount of capital that flows freely within a period in time. If many investors have itchy feet, more capital will be reallocated but if there are many cool heads, client attrition is low and less capital will move in a period.

Numerical analysis of the model shows that a very small amount of freely flowing capital can have a huge impact on the market dynamics if clients exhibit recency bias in evaluating fund performance. In particular, even with a mild degree of recency bias, the flow of funds is able to create an evolutionary space where the investment strategies of fundamental trading, trend chasing and the noise trading all survive in the long-run. Moreover, in contrast to pure expectations-based type-switching models, the intensity of clients' choice is an important factor in driving excess volatility and extreme price movements if enough investors have itchy feet.

The approach offers several directions for further research. We only look at the standard one-asset-one-bond model, and we restrict our analysis to 3 decision rules. Further we exclude short-selling and long-leveraging. This type of constraint is absent in most agent-based models of financial markets which, in general, use borrowing as a main driver for excess volatility. Empirical issues are not covered in the paper, and it might be interesting to see how well a calibration can fit stock index dynamics in real markets.

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A   Proofs

Proof of Proposition 2.1. Let us first prove an auxiliary result.

Lemma A.1. *Let $A \in \mathbb{R}^{I \times K}$ and $B \in \mathbb{R}^{K \times I}$ be non-negative matrices. Suppose*

(i) $\sum_{i=1}^{I} A_{ik} < 1$ *for all* $k = 1, ..., K$; *and*
(ii) $\sum_{k=1}^{K} B_{ki} \leq 1$ *for all* $i = 1, ..., I$.

Then the matrix $\mathrm{Id} - AB$ *is invertible and the inverse has only strictly positive elements.*

Proof. We show that $C = \mathrm{Id} - AB$ has a strict column-dominant diagonal:

$$\sum_{j=1, j \neq i}^{I} |C_{ji}| < C_{ii} \text{ for all } i = 1, ..., I. \tag{18}$$

Then applying Murata [42, Corollary (p. 22) and Theorem 23 (p. 24)] yields the assertion.

As

$$C_{ji} = \mathbf{1}_{\{i=j\}} - \sum_{k=1}^{K} A_{jk} B_{ki}$$

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and $A_{jk}B_{ki} \geq 0$, (18) is equivalent to:

$$\sum_{j=1}^{I}\sum_{k=1}^{K} A_{jk}B_{ki} < 1 \text{ for all } i = 1, ..., I.$$

Indeed we find that the term on the left-hand side is bounded by

$$\sum_{k=1}^{K}\left(\sum_{j=1}^{I} A_{jk}\right)B_{ki} < \sum_{k=1}^{K} B_{ki} \leq 1$$

where we first use assumption (i) and then (ii). $\qquad\square$

Proof of Proposition 2.1. Consider the system (5). We apply Lemma A.1 to show that the matrix

$$\text{Id} - (1-c)\Theta_t\Big((1-\beta)\Lambda_{t+1} + \beta\Lambda_{t+1}q_t\mathbf{1}\Big)$$

is invertible and that all elements of its inverse are strictly positive. Let $A = (1-c)\Theta_t$ and $B = (1-\beta)\Lambda_{t+1} + \beta\Lambda_{t+1}q_t\mathbf{1}$. Since $c < 1$, one has

$$\sum_{i=1}^{I} A_{ik} = 1 - c < 1.$$

Further

$$\sum_{k=1}^{K} B_{ki} = (1-\beta)\sum_{k=1}^{K}\lambda_k^i + \beta\sum_{k=1}^{K}\sum_{i=1}^{I}\lambda_k^i q^i \leq 1 - \beta + \beta = 1.$$

This gives the result because the non-negative vector

$$\Theta_t D_{t+1} + (1+r)(1-c)\Delta\lambda_{t,0}\big[(1-\beta)V_t + \beta q_{t-1}\mathbf{1}V_t\big]$$

has at least one strictly positive entry. Indeed, by assumption, $W_t^i = (1-\beta)V_t^i + \beta q^i\bar{V}_t > 0$ and $\lambda_{t,k}^i > 0$ for all $k \geq 1$. Therefore $\Theta_{t,k}^i > 0$. Since $D_{t+1,k} \geq 0$ and $\sum_{k=1}^{K} D_{t+1,k} > 0$, we finally find $\sum_{k=1}^{K}\Theta_{t,k}^i D_{t+1,k} > 0$. $\qquad\square$

Proof of Proposition 2.2. Recall that the dividend process (D_t) is stationary. Under the assumption of the proposition, the price dynamics (9)

can be written as

$$P_{t+1} = aP_t + bD_{t+1} \tag{19}$$

with $a = a(c, r, \lambda)$ and $b = b(c, r, \lambda)$ defined in (11). The price process has an autoregressive form with a stationary sequence of innovations bD_{t+1}. Equation (19) has a unique stationary solution if $|a| < 1$ and the variance of the innovation is finite. Since a is strictly decreasing in λ, setting $\lambda = 0$ one finds that $a(c, r, 0) \leq 1$ if and only if $c \geq r/(1+r)$. Also, $a(c, r, 1) = 0$, hence, $0 \leq a < 1$. To verify that (10) is a stationary solution to (19), observe that

$$P_{t+1}^* = b \sum_{n=-\infty}^{0} a^{-n} D_{t+1+n} = b \sum_{n=-\infty}^{-1} a^{-n} D_{t+1+n} + bD_{t+1}$$

$$= ab \sum_{n=-\infty}^{0} a^{-n} D_{t+n} + bD_{t+1} = aP_t^* + bD_{t+1}.$$

One has $EP_t^* = b/(1-a)ED_0 < \infty$ and $P_t^* \geq 0$ (as $D_t \geq 0$) therefore P_t^* is finite a.s. Denoting $v = Var(P_t^*)$ one obtains the relationship $v = a^2 v + b^2 \sigma^2$ with a unique solution $v = b^2 \sigma^2 / (1 - a^2)$.

Consider any price path P_t with $P_0 \geq 0$. Then

$$|P_t - P_t^*| = a|P_{t-1} - P_{t-1}^*| = \cdots = a^t |P_0 - P_0^*|$$

which a.s. converges to zero as $t \to \infty$. $\qquad\square$

Proof of Proposition 2.3. Rearranging equation (15) gives:

$$q_t^i = \frac{\exp(\gamma f_t)}{\sum\limits_{i=1}^{I} \exp(\gamma f_t)} = \frac{1}{1 + \sum_{j \neq i}^{I} \exp[\gamma(f_t^j - f_t^i)]}. \tag{20}$$

Inserting (20) into $\mathrm{sgn}(q_{t+1}^i - q_t^i)$ gives:

$$
\begin{aligned}
\mathrm{sgn}(q_{t+1}^i - q_t^i) &= \mathrm{sgn}\left(\frac{1}{1 + \sum_{j \neq i}^I \exp[\gamma(f_{t+1}^j - f_{t+1}^i)]} - \frac{1}{1 + \sum_{j \neq i}^I \exp[\gamma(f_t^j - f_t^i)]} \right) \\
&= \mathrm{sgn}\left(\sum_{j \neq i}^I \exp[\gamma(f_t^j - f_t^i)] - \sum_{j \neq i}^I \exp[\gamma(f_{t+1}^j - f_{t+1}^i)] \right) \\
&= \mathrm{sgn}\left(\sum_{j \neq i}^I [(f_{t+1}^i - f_t^i) - (f_{t+1}^j - f_t^j)] \right) \\
&= \mathrm{sgn}\left(\sum_{j \neq i}^I (\Delta_{t+1}^i - \Delta_{t+1}^j) \right) \\
&= \mathrm{sgn}\left(I\Delta_{t+1}^i - \sum_{j=1}^I \Delta_{t+1}^j \right).
\end{aligned}
\tag{21}
$$

Since dividing $I > 0$ on the right-hand side of (21) does not change its sign, one has

$$
\mathrm{sgn}(q_{t+1}^i - q_t^i) = \mathrm{sgn}\left(\Delta_{t+1}^i - \frac{\sum_{j=1}^I \Delta_{t+1}^j}{I} \right).
\tag{22}
$$

Finally, using $\bar{\Delta}_{t+1} = \sum_{j=1}^I \Delta_{t+1}^j$ in (22) gives:

$$
\mathrm{sgn}(q_{t+1}^i - q_t^i) = \mathrm{sgn}(\Delta_{t+1}^i - \bar{\Delta}_{t+1}).
$$

$\square$

B   Calibration of dividend distribution

In the numerical analysis the dividends D_t are univariate and i.i.d. with distribution $\mathcal{N}(\mu, \sigma^2)^+$ – the normal distribution truncated to non-negative numbers. We shall show how to choose μ and σ so that the truncated distribution has a given mean and variance.

Denote by ϕ_+ the density of $\mathcal{N}(\mu, \sigma^2)^+$. The probability that $\mathcal{N}(\mu, \sigma^2)$ random variable takes values in the interval $\mathbb{R}_{++}$ is $\Phi(\frac{\infty - \mu}{\sigma}) - \Phi(\frac{0 - \mu}{\sigma})$ with $\Phi(\cdot)$ the cumulative distribution function of the standard normal distribu-

tion. Therefore, using Bayes theorem the conditional density function is given by

$$\phi_+(x) = \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{-\mu}{\sigma}\right)} = \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{\mu}{\sigma}\right)}, \qquad x \geq 0,$$

where $\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is the probability distribution function of the standard normal distribution. The expected value is obtained using the moment generating function $M(\tau)$ and equals

$$ED_t = M'(\tau)|_{\tau=0} = \mu + \sigma\frac{\phi(h)}{\Phi(-h)} \tag{23}$$

with $h = -\frac{\mu}{\sigma}$. The variance is given by:

$$VarD_t = M''(\tau)|_{\tau=0} - (M'(\tau)|_{\tau=0})^2 = \sigma^2\left[1 + \frac{h\phi(h)}{\Phi(-h)} - \left(\frac{\phi(h)}{\Phi(-h)}\right)^2\right].$$
$$\tag{24}$$

Rewriting the system of equations (23) and (24) gives:

$$ED_t = \mu + \sigma\frac{\phi(h)}{\Phi(-h)}, \qquad \sigma^2 = VarD_t + ED_t(ED_t - \mu). \tag{25}$$

Inserting σ from the second equation into the first gives a non-linear equation for μ which we solved numerically. When the mean of D_t equals to 1 and the standard deviation is 20%, the corresponding parameters for the truncated normal distribution are $\mu = 0.9999997026250630$ and $\sigma^2 = 0.0400002973749369$.