Critical Plane Approach to Multiaxial Variable Amplitude Fatigue Loading

Yingyu Wang

Key Laboratory of Fundamental Science for National Defense-Advanced Design Technology of Flight Vehicle, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, China
yywang@nuaa.edu.cn

Luca Susmel

Department of Civil and Structural Engineering, the University of Sheffield, Sheffield S1 3JD, UK
l.susmel@sheffield.ac.uk

ABSTRACT. A new critical plane approach based on the modified Manson-Coffin curve method (MMCCM) is presented in this paper for predicting the fatigue lifetime under variable amplitude (VA) multiaxial fatigue loading. The critical plane is assumed to be the plane experiencing the maximum variance of the resolved shear strain. Fatigue damage depends on both the amplitude of resolved shear strain and the stress ratio, which is determined by the mean value and the variance of the stress perpendicular to the critical plane as well as for the variance of the shear stress resolved along the direction experiencing the maximum variance of the resolved shear strain. Damage load cycles are obtained by the rain flow counting of the resolved shear strain time history. The Palmgren-Miner's linear damage rule is applied to calculate the cumulative fatigue damage. The accuracy and reliability of the proposed approach is checked by using several experimental data taken from the literature. Predicted fatigue lives based on the new approach are seen to be in sound agreement with the experimental fatigue lives.

KEYWORDS. Multiaxial fatigue; Variable amplitude loading; Critical plane; Fatigue life prediction.

INTRODUCTION

Fatigue life prediction approaches based on the concept of the critical plane are generally accepted to be more accurate for multiaxial fatigue life estimation. The critical plane concept is based on the physical observation that cracks initiate and grow on some specific planes. We believe there are three aspects needed to be considered when applying the critical plane approach to estimate fatigue lifetime under multiaxial variable amplitude loading, i.e.: (i) determining the orientation of the critical plane, (ii) counting the fatigue cycles and (iii) calculating the amplitude and mean value of the stress/strain components relative to the critical plane.

As to the available critical plane approaches, Findley [1] determined the critical plane by maximizing a linear combination of shear stress amplitude and maximum value of the normal stress. Brown and Miller [2] defined the critical plane as the plane experiencing the maximum shear strain amplitude. Then, Wang and Brown [3] proposed a modified

version of their criterion by defining the critical plane as the one of experiencing not only the maximum shear strain range, but also the largest value of the normal strain excursion. Fatemi and Socie [4] assumed that the critical plane is the plane of maximum shear strain amplitude when the crack initiation process is Mode II governed. From the brief review presented above, it becomes clear that there are different ways to define the critical plane itself. However, it is not an easy task to determine the direction of the critical plane when the applied multiaxial load history varies randomly. Several investigations [5-9] have been published focusing on the methods for the determination of the critical plane under multiaxial random loading. Susmel [9] have recently formalized a novel technique, the Shear stress-Maximum Variance Method (τ-MVM). The Maximum Variance Method (MVM) postulates that the critical plane can be defined as that plane containing the direction (passing through the assumed critical point) that experiences the maximum variance of the resolved shear stress. The peculiarity of the MVM is that, it is not necessary to calculate the stress/strain components relative to any plane passing through the assumed critical point to determine the critical plane. Therefore, from a computational point of view, it is more efficient than the other existing methods.

In order to take into account of the cyclic hardening or softening in the fatigue failure criteria, both stress and strain components are recommended be used to estimate the fatigue damage extent. As to the in-field usage of the available VA multiaxial fatigue life estimation techniques, correctly performing the cycle counting under such loading complex conditions is one of the trickiest aspects [10]. Even though there are a few methods suitable for counting the cycles under uniaxial loading, among these the rainflow cycle counting method [11] has been most widely and successfully used. The number of published papers dealing with VA multiaxial loading histories is small [12]. Bannantine and Socie [13] proposed a method based on the critical plane concept and rainflow cycle counting. Wang and Brown [14] proposed a cycle counting method based on rainflow and a modified von Mises equivalent strain.

The accuracy of the results predicted by using the critical plane approach to a great extent relies on the determination of the amplitude and mean value of the normal and shear stresses acting on a particular plane [15]. For a particular plane under complex loading, the direction of the normal stress does not change over time, since just the magnitude of the normal stress vector varies. However, the shear stress vector on this particular plane changes in both magnitude and direction. Therefore, the tip of the shear stress vector draws an imaginary curve on that plane. In this situation, it is not an easy task to calculate the amplitude and mean value of the shear stress on a given plane. There are several techniques proposed to address this problem, namely the Longest Chord Method, the Longest Projection Method, the Minimum Circumscribed Circle concept, and the Minimum Circumscribed Ellipse Method. These methods have been discussed in the Refs [9, 15].

The present paper summarizes an attempt of extending the use of the Modified Manson-Coffin Curve Method (MMCCM) to those situations where complex variable amplitude loadings are involved. By making the most of our previous experience [10, 16-18], now we intend to use the stain-based MVM in conjunction with MMCCM to predict fatigue lifetime under VA multiaxial fatigue loading. In more detail, the MMCCM is suggested here as being applied in conjunction with the maximum variance method (MVM) to estimate fatigue lifetime by directly post-processing the strain state relative to that material plane containing the direction along which the variance of the resolved shear strain is maximized. Since, by definition, the resolved shear strain is a monodimensional quantity, the rainflow method can directly be used to count the loading cycles. After counting the cycles and estimating the fatigue damage for each counted cycle, fatigue lifetime can be predicted directly by using the Palmgren-Miner's linear damage rule. Finally, the validation of the new approach by experimental data is presented.

THE SHEAR STRAIN_MAXIMUM VARIANCE METHOD (γ_MVM)

The shear strain_Maximum Variance Method (γ_MVM) assumes that the critical plane can be defined as that plane containing the direction that experiences the maximum variance of the resolved shear strain.

In order to calculate the shear strain relative to a generic material plane, Δ, and resolved along a generic direction, *q*, consider a body subjected to an external system of forces resulting in a triaxial strain state at point O (Fig. 1a). Point O is taken as the center of the absolute system of coordinates, *Oxyz*. The time-variable strain state at point O is defined through the following strain tensor:

$$[\varepsilon(t)] = \begin{bmatrix} \varepsilon_x(t) & \varepsilon_{xy}(t) & \varepsilon_{xz}(t) \\ \varepsilon_{yx}(t) & \varepsilon_y(t) & \varepsilon_{yz}(t) \\ \varepsilon_{zx}(t) & \varepsilon_{zy}(t) & \varepsilon_z(t) \end{bmatrix} = \begin{bmatrix} \varepsilon_x(t) & \frac{1}{2}\gamma_{xy}(t) & \frac{1}{2}\gamma_{xz}(t) \\ \frac{1}{2}\gamma_{xy}(t) & \varepsilon_y(t) & \frac{1}{2}\gamma_{yz}(t) \\ \frac{1}{2}\gamma_{xz}(t) & \frac{1}{2}\gamma_{yz}(t) & \varepsilon_z(t) \end{bmatrix} \qquad (1)$$

where ε_x, ε_y and ε_z are the three normal strains, and γ_{xy}, γ_{xz} and γ_{yz} are the total shear strains.

The orientation of a generic material plane, Δ, having normal unit vector $\mathbf{n}$ can be defined through angles φ and θ (Fig. 1b). φ is the angle between axis x and the projection of unit vector $\mathbf{n}$ on plane x-y. θ is the angle between $\mathbf{n}$ and axis z. A new system of coordinates, Onab, can now be defined. The unit vectors defining the orientation of axes n, a and b can be expressed, respectively, as follows:

$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{bmatrix}; \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ -\sin(\theta) \end{bmatrix} \tag{2}$$
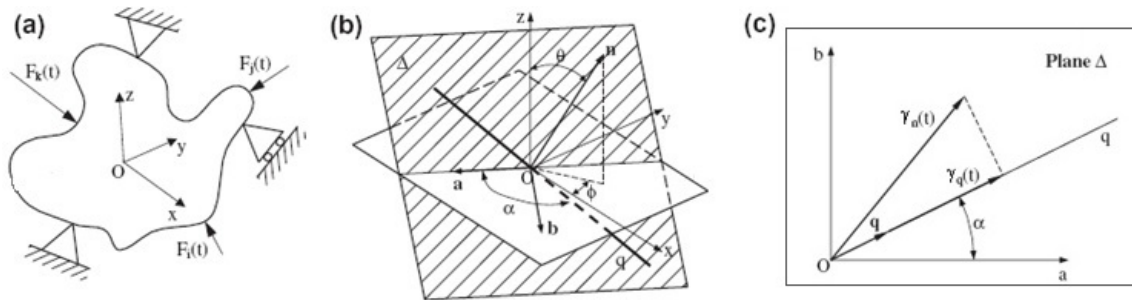

Figure 1: Generic plane and shear strain resolved along one generic direction in a body subjected to an external system of forces

Consider now a generic direction q lying on plane Δ and passing through point O. α is the angle between direction q and axis a. The unit vector defining the orientation of q can be calculated as follows:

$$\mathbf{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\sin(\varphi) + \sin(\alpha)\cos(\theta)\cos(\varphi) \\ -\cos(\alpha)\cos(\varphi) + \sin(\alpha)\cos(\theta)\sin(\varphi) \\ -\sin(\alpha)\sin(\theta) \end{bmatrix} \tag{3}$$

According to the definition reported above, the instantaneous value of the shear strain resolved along direction q, $\gamma_q(t)$, can then be calculated as:

$$\frac{\gamma_q(t)}{2} = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix} \begin{bmatrix} \varepsilon_x(t) & \frac{1}{2}\gamma_{xy}(t) & \frac{1}{2}\gamma_{xz}(t) \\ \frac{1}{2}\gamma_{xy}(t) & \varepsilon_y(t) & \frac{1}{2}\gamma_{yz}(t) \\ \frac{1}{2}\gamma_{xz}(t) & \frac{1}{2}\gamma_{yz}(t) & \varepsilon_z(t) \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{4}$$

In order to make the calculation easier, it is useful to express $\gamma_q(t)$ through the following scalar product:

$$\frac{\gamma_q(t)}{2} = \mathbf{d} \cdot \mathbf{s}(t) \tag{5}$$

where $\mathbf{d}$ is the vector of direction cosines, that is:

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} n_x q_x \\ n_y q_y \\ n_z q_z \\ n_x q_y + n_y q_x \\ n_x q_z + n_z q_x \\ n_y q_z + n_z q_y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[\sin(\theta) \cdot \sin(2\varphi) \cdot \cos(\alpha) + \sin(2\theta) \cdot \cos(\varphi)^2 \cdot \sin(\alpha)] \\ \frac{1}{2}[-\sin(\theta) \cdot \sin(2\varphi) \cdot \cos(\alpha) + \sin(2\theta) \cdot \sin(\varphi)^2 \cdot \sin(\alpha)] \\ -\frac{1}{2}\sin(2\theta) \cdot \sin(\alpha) \\ \frac{1}{2}\sin(2\theta) \cdot \sin(2\varphi) \cdot \sin(\alpha) - \sin(\theta) \cdot \cos(2\varphi) \cdot \cos(\alpha) \\ \cos(2\theta) \cdot \cos(\varphi) \cdot \sin(\alpha) - \cos(\theta) \cdot \sin(\varphi) \cdot \cos(\alpha) \\ \cos(2\theta) \cdot \sin(\varphi) \cdot \sin(\alpha) - \cos(\theta) \cdot \cos(\varphi) \cdot \cos(\alpha) \end{bmatrix} \tag{6}$$

and $s(t)$ is a six-dimensional vector process depending on $[\varepsilon(t)]$ and defined as:

$$s(t) = \begin{bmatrix} \varepsilon_x(t) & \varepsilon_y(t) & \varepsilon_z(t) & \frac{1}{2}\gamma_{xy}(t) & \frac{1}{2}\gamma_{xz}(t) & \frac{1}{2}\gamma_{yz}(t) \end{bmatrix} \tag{7}$$

According to the quantities defined above, the variance of the shear strain, $\gamma_q(t)$, resolved along direction q can then be calculated directly as:

$$\mathrm{Var}\left[\frac{\gamma_q(t)}{2}\right] = \mathrm{Var}[\textstyle\sum_k d_k s_k(t)] = \sum_i \sum_j d_i d_j \mathrm{Cov}[s_i(t), s_j(t)] = \mathbf{d}^T[C]\mathbf{d} \tag{8}$$

where $[C]$ is a symmetric square matrix of order six, and the terms of the covariace matrix are defined as

$$C_{ij} = \mathrm{Cov}[s_i(t), s_j(t)] \tag{9}$$

where, when $i = j$ then $\mathrm{Cov}[s_i(t), s_j(t)] = \mathrm{Var}[s_i(t)]$, whereas when $i \neq j$ then $\mathrm{Cov}[s_i(t), s_j(t)] = \mathrm{Cov}[s_j(t), s_i(t)]$.

Now $[C]$ can be rewritten in explicit form by using both the variance and covariance terms:

$$[C] = \begin{bmatrix} V_x & C_{x,y} & C_{x,z} & C_{x,xy} & C_{x,xz} & C_{x,yz} \\ C_{x,y} & V_y & C_{y,z} & C_{y,xy} & C_{y,xz} & C_{y,yz} \\ C_{x,z} & C_{y,z} & V_z & C_{z,xy} & C_{z,xz} & C_{z,yz} \\ C_{x,xy} & C_{y,xy} & C_{z,xy} & V_{xy} & C_{xy,xz} & C_{xy,yz} \\ C_{x,xz} & C_{y,xz} & C_{z,xz} & C_{xy,xz} & V_{xz} & C_{xz,yz} \\ C_{x,yz} & C_{y,yz} & C_{z,yz} & C_{xy,yz} & C_{xz,yz} & V_{yz} \end{bmatrix} \tag{10}$$

where

$$V_i = \mathrm{Var}[\varepsilon_i(t)] \quad \text{for } i = x, y, z, xy, xz, yz \tag{11}$$

$$C_{i,j} = \mathrm{CoVar}[\varepsilon_i(t), \varepsilon_j(t)] \quad \text{for } i = x, y, z, xy, xz, yz \tag{12}$$

Then Eq. (8) can be rewritten in the following simple form:

$$\mathrm{Var}[\gamma_q(t)] = \mathbf{d}^T[C]\mathbf{d} \tag{13}$$

Eq. (13) makes it evident that the determination of the direction experiencing the maximum variance of the resolved shear strain is actually a conventional multi-variable optimization problem. It can be satisfactorily solved by simply using the so-called Gradient Ascent Method [19]. Figure 2 reports the flowchart summarising the algorithm which was proposed in the Ref. [9] to be used to determine the orientation of the critical plane.

After determining the direction of the maximum variance, the shear strain resolved along this direction, $\gamma_{MV}(t)$, can directly be evaluated, at any instant, t, of the load history, through the following relationship:
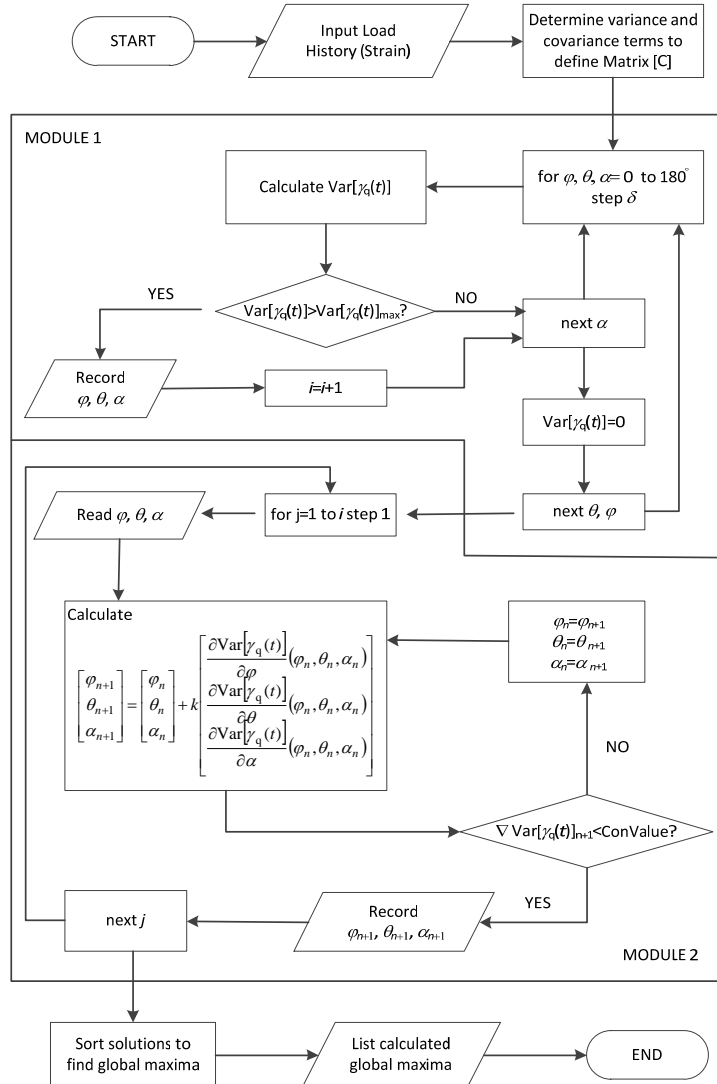
Figure 2: Flowchart summarizing the algorithm for the determination of the critical plane orientation.

$$\frac{\gamma_{MV}(t)}{2} = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix} \begin{bmatrix} \varepsilon_x(t) & \frac{1}{2}\gamma_{xy}(t) & \frac{1}{2}\gamma_{xz}(t) \\ \frac{1}{2}\gamma_{xy}(t) & \varepsilon_y(t) & \frac{1}{2}\gamma_{yz}(t) \\ \frac{1}{2}\gamma_{xz}(t) & \frac{1}{2}\gamma_{yz}(t) & \varepsilon_z(t) \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{14}$$

And the shear stress resolved along the maximum variance direction, $\tau_{MV}(t)$, and the stress normal to the plane experiencing the maximum variance of resolved shear strain, $\sigma_n(t)$, can be evaluated, at any instant, t, of the load history, through the following relationship:

$$\tau_{MV}(t) = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix} \begin{bmatrix} \sigma_x(t) & \tau_{xy}(t) & \tau_{xz}(t) \\ \tau_{xy}(t) & \sigma_y(t) & \tau_{yz}(t) \\ \tau_{xz}(t) & \tau_{yz}(t) & \sigma_z(t) \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{15}$$

$$\sigma_n(t) = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} \sigma_x(t) & \tau_{xy}(t) & \tau_{xz}(t) \\ \tau_{xy}(t) & \sigma_y(t) & \tau_{yz}(t) \\ \tau_{xz}(t) & \tau_{yz}(t) & \sigma_z(t) \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{16}$$

EVALUATION OF STRAIN AND STRESS COMPONENTS RELATIVE TO THE CRITICAL PLANE

Consider the body subjected to a complex system of time variable forces shown in Fig.3. The applied load history is defined in the time interval $[0, T]$ and it is assumed to result in a VA multiaxial strain state at point O. By using the shear strain_MVM, the orientation of the potential critical plane can be determined through that direction which experiences the maximum variance of the resolved shear strain. After determining the direction of the maximum variance (MV), the shear strain resolved along this direction, $\gamma_{MV}(t)$, the shear stress resolved along the maximum variance direction, $\tau_{MV}(t)$, and the stress normal to the plane experiencing the maximum variance of the resolved shear strain, $\sigma_n(t)$, can be evaluated, at any instant, t, of the load history, through Eq.(14)-Eq.(16).

Now, attention can initially be focused on stress component, $\sigma_n(t)$, which is defined in the time interval $[0, T]$. The mean value of such a stress component is equal to

$$\overline{\sigma}_{n,m} = \frac{1}{T} \int_0^T \sigma_n(t) dt \tag{17}$$

The equivalent amplitude of $\sigma_n(t)$ is suggested here as being calculated as follows:

$$\overline{\sigma}_{n,a} = \sqrt{2 \cdot \mathrm{Var}[\sigma_n(t)]} \tag{18}$$

where $\mathrm{Var}[\sigma_n(t)]$ is the variance of stress component $\sigma_n(t)$, i.e.

$$\mathrm{Var}[\sigma_n(t)] = \frac{1}{T} \int_0^T [\sigma_n(t) - \overline{\sigma}_{n,m}]^2 dt \tag{19}$$

Consider now the shear stress, $\tau_{MV}(t)$, resolved along direction MV. Owing to the fact that also $\tau_{MV}(t)$ is a monodimensional stress quantity defined in the time interval $[0, T]$, its mean value and its equivalent amplitude can directly be determined by following a strategy similar to the one adopted above to calculate $\overline{\sigma}_{n,m}$ and $\overline{\sigma}_{n,a}$. In particular, $\overline{\tau}_m$ is equal to

$$\overline{\tau}_m = \frac{1}{T} \int_0^T \tau_{MV}(t) dt \tag{20}$$

whereas the equivalent amplitude of the shear stress resolved along direction MV takes on the following value:

$$\overline{\tau}_a = \sqrt{2 \cdot \mathrm{Var}[\tau_{MV}(t)]} \tag{21}$$

where

$$\mathrm{Var}[\tau_{MV}(t)] = \frac{1}{T} \int_0^T [\tau_{MV}(t) - \overline{\tau}_m]^2 dt \tag{22}$$

Then a stress ratio $\overline{\rho}$, which will be used to adapt the Manson-Coffin curve to the degree of multi-axiality and non-proportionality of the stress/strain state at the assumed crack initiation site, can be defined as follows:

$$\overline{\rho} = \frac{\overline{\sigma}_{n,m} + \overline{\sigma}_{n,a}}{\overline{\tau}_a} \tag{23}$$

To conclude the present section it is important to highlight that, in general, there exist two or more different directions which experience the maximum variance of the resolved shear strain, so that, it is always possible to locate, at

least, two potential critical planes: amongst all the potential critical planes, the one which has to be used to estimate fatigue lifetime is that experiencing the largest value of $\bar{\rho}$.

THE MMCCM TO ADDRESS THE VARIABLE AMPLITUDE PROBLEM

The use of the MMCCM to situations involving VA multiaxial fatigue loading is based on the following assumptions: (i) initiation and initial propagation of Stage I cracks occur on that material plane containing the direction experiencing the maximum variance of the resolved strain, $\gamma_{MV}(t)$; (ii) the fatigue damage depends also on the stress ratio $\bar{\rho}$ which is defined in the previous Section. Fig. 3 summarizes the approach proposed in the present paper to be followed to estimate fatigue lifetime of engineering materials subjected to in-field VA multiaxial fatigue loading. In more detail, consider a body loaded by a complex system of time variable forces resulting in a VA multiaxial strain state at the assumed crack initiation site (i.e. point O in Fig. 3a). Initially, by making use of the shear strain_MVM, the orientation of the candidate critical planes will be determined through that direction, MV, experiencing the maximum variance of the resolved shear strain (Fig. 3b). After determining the direction of the maximum variance of the shear strain, the shear strain resolved along this direction, $\gamma_{MV}(t)$, the shear stress resolved along the maximum variance direction, $\tau_{MV}(t)$, and the stress normal to the critical planes, $\sigma_n(t)$, can be evaluated, at any instant, t, of the load history, through Eq.(14)-Eq.(16) (Fig. 3c and d). Subsequently, the calculated values for $\bar{\sigma}_{n,m}$ (Eq. 24), $\bar{\sigma}_{n,a}$ (Eq. 25) and $\bar{\tau}_a$ (Eq. 28) have to be used to estimate, according to Eq. 23, critical plane stress ratio $\bar{\rho}$. The critical plane is the plane, which experiences the maximum variance of the resolved shear strain, undergoing the maximum $\bar{\rho}$.

According to the determined value for $\bar{\rho}$, the profile of the corresponding modified Manson-Coffin curve can be described by using the following general relationship:

$$\gamma_a = \frac{\tau'_f(\bar{\rho})}{G}(2N_f)^{b(\bar{\rho})} + \gamma'_f(\bar{\rho}) \cdot (2N_f)^{c(\bar{\rho})} \tag{24}$$

where $\tau'_f(\bar{\rho}), \gamma'_f(\bar{\rho}), b(\bar{\rho}), c(\bar{\rho})$ are material functions, which can be determined by the following equations.

$$b(\bar{\rho}) = \frac{b \cdot b_0}{(b_0 - b)\bar{\rho} + b} \tag{25}$$

$$\tau'_f(\bar{\rho}) = \frac{\tau_{A,ref}(\bar{\rho})}{(2N_A)^{b(\bar{\rho})}} \tag{26}$$

$$\tau_{A,ref}(\bar{\rho}) = \bar{\rho}\frac{\sigma'_f}{2}(2N_A)^b + (1-\bar{\rho})\tau'_f(2N_A)^{b_0} \tag{27}$$

$$\gamma'_f(\bar{\rho}) = \bar{\rho} \cdot (1 + \nu_p)\varepsilon'_f + (1 - \bar{\rho})\gamma'_f \tag{28}$$

$$c(\bar{\rho}) = \frac{c \cdot c_0}{(c_0 - c)\bar{\rho} + c} \tag{29}$$

where ν_p is Poisson's ratio for plastic strain, N_A is the reference number of cycles to failure.

After determining the appropriate reference fatigue curve as above, by post-processing the shear strain resolved along direction MV, the classical rain-flow method allows the corresponding shear strain spectrum to be built directly (Figure 3g and h). Finally, from such spectrum fatigue strength under VA multiaxial fatigue loading can be predicted according to Palmgren-Miner's rule (Figure 3i). Here it is assumed that the damage occur when the damage sum equal to 1.

VALIDATION BY EXPERIMENTAL DATA

In order to check the accuracy of the proposed approach in estimating fatigue lifetime under VA multiaxial fatigue loading, a number of experimental data were selected from the technical literature [12, 20-22]. The summary of the static and fatigue properties of the considered materials is reported in Table 1. When the material constants listed in the above table were not directly available in the original sources, they were calculated as follows.
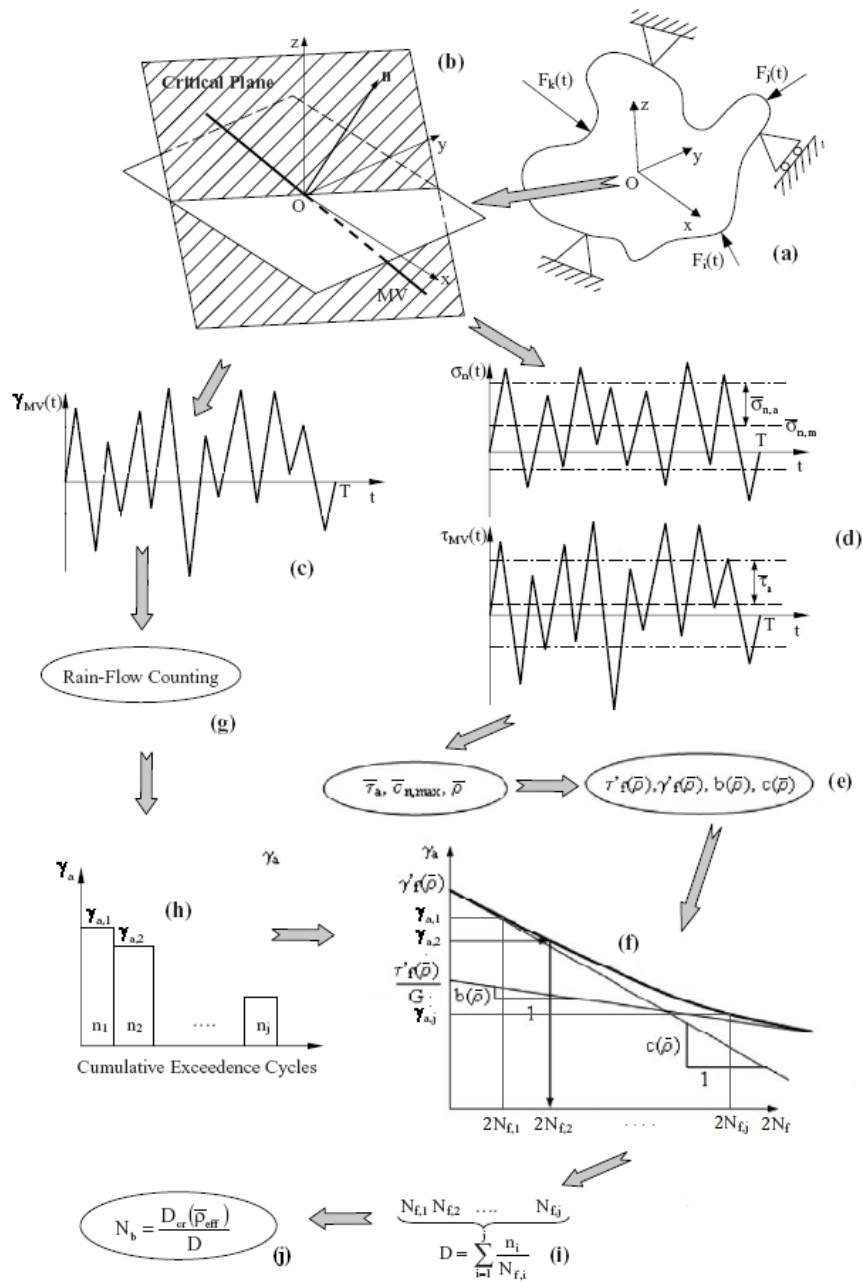
Figure 3: In-field use of the MMCCM to estimate fatigue lifetime under VA multiaxial fatigue loading.

$$G = \frac{E}{2(1+\mu)} \quad \gamma'_f = \sqrt{3}\varepsilon'_f \quad \tau'_f = \frac{\sigma'_f}{\sqrt{3}} \quad b_0 = b \quad c_0 = c$$

Lastly, the investigated materials' nonproportional hardening was taken into account by making the following assumption [23]:

$$K'_{NP} = 1.25 \cdot K' \quad n'_{NP} = n'$$

Both the strain loading history and the stress loading history at the assumed crack initiation site are needed. If the stress loading histories were not given in the original literature, they were calculated by using the model proposed by Jiang and Sehitoglu [24]. Some stress loading histories under several loading paths for one complete loading block, which were

obtained by using Jiang's model, are shown in Table 2. Table 2 also includes the resolved shear strain histories for one loading block along the direction of maximum variance of the resolved shear strain in the critical plane and the spectrums of the resolved shear stain which were obtained by using the Rain-Flow cycle counting method under these loading paths.

Predicted fatigue lives are compared with experimental fatigue lives for S45C steel and SNCM630 steel under VA multiaxial fatigue loading in Figure 4. Figure 4 also shows the loading paths applied to specimens made of S45C steel and SNCM630 steel. As it can be seen from this figure, 81% of the data are within scatter bands of 3. Data points under loading blocks with a considerable portion of axial loading such as AT, TA and AV are on the outside of scatter bands of 3 from predictions. Fortunately, they are conservative.

The predicted versus experimental fatigue lifetime diagram for 1050 QT steel and 304L steel under discriminating axial-torsion stain paths with random and incremental changes in straining direction is reported in Figure 5. As it can be seen from Figure 5, all the data are within scatter bands of 3 from predictions.

Figure 6 shows the predicted versus experimental fatigue lifetime diagram for pure titanium and titanium alloy BT9 under step loading and block loading composed of different combinations of axial, torsion and 90°out-of-phase axial-torsion strain paths. As it can be seen from Figure 6, 90% of the data are within scatter bands of 3 from predictions, while 10% of the data are on the margin of scatter bands of 3. All data drop in scatter bands of 4.
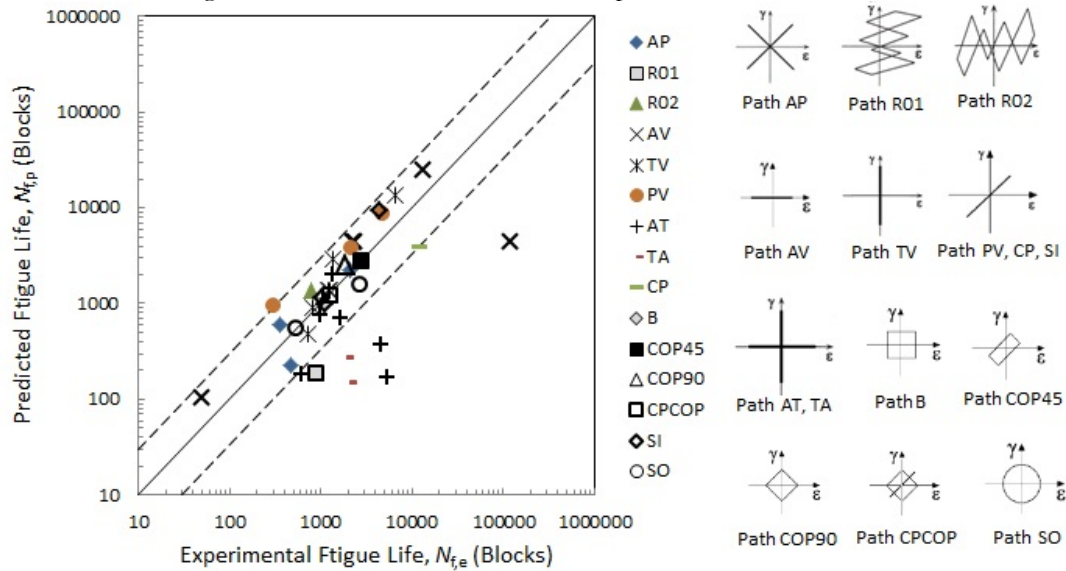

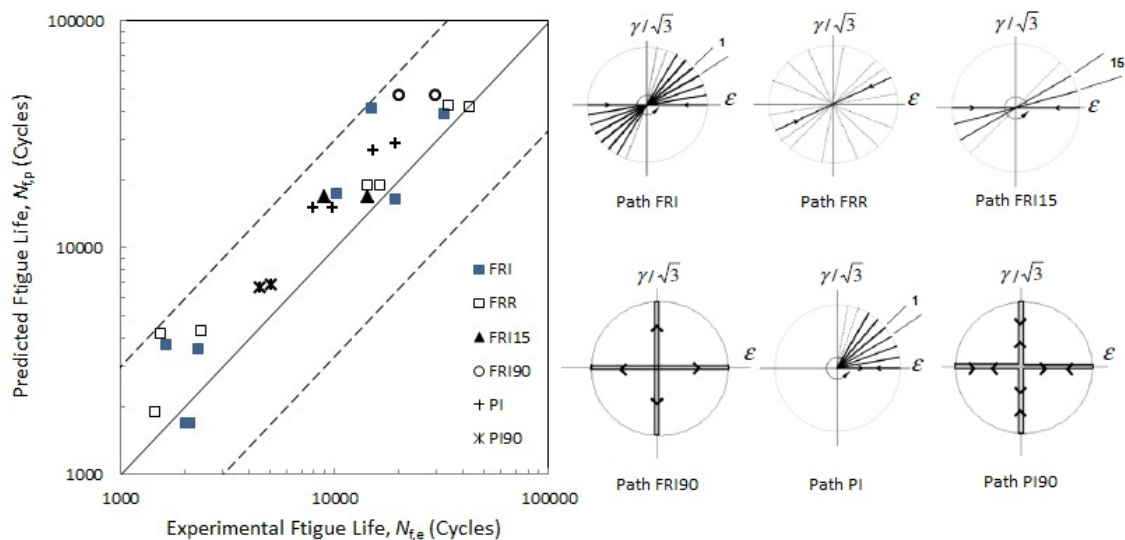Figure 4: Comparison of observed and predicted fatigue lives by the MMCCM for S45C and SNCM630.

Figure 5: Comparison of observed and predicted fatigue lives by the MMCCM for 1045 QT steel and 304L steel.

Material	Ref.	E (MPa)	G (MPa)	v	ε_f'	σ_f' (MPa)	b	c	γ_f'	τ_f' (MPa)	b_0	c_0	K' (MPa)	n'	K_{NP}' (MPa)	n_{NP}'
S45C	[20]	186,000	70,600	0.28	0.359	923	-0.099	-0.519	0.198	685	-0.12	-0.36	1215	0.217	1519	0.217
SNCM630	[21]	196,000	77,000	0.273	1.54	1272	-0.073	-0.823	1.51	858	-0.061	-0.706	1056	0.054	1320	0.054
1050 QT Steel	[12]	203,000	81,000	0.27	2.01	1346	-0.062	-0.725	3.48	777	-0.062	-0.725	1461	0.06	1420	0.113
304L steel	[12]	195,000	77,000	0.27	0.122	1287	-0.145	-0.394	0.211	743	-0.145	-0.394	680	0.214	5056	0.373
Pure titanium	[22]	112,000	40,000	0.4	0.548	647	-0.033	-0.646	0.417	485	-0.069	-0.523	-	-	-	-
Titanium alloy BT9	[22]	118,000	43,000	0.37	0.278	1180	-0.025	-0.665	0.18	881	-0.0082	-0.47	-	-	-	-

Table 1: Static and fatigue properties of the investigated materials.

Material	Path Code	Strain Path	Stress Path	Resolved shear strain history	Spectrum
S45C steel	R01				
	R02				
1050 QT Steel	FRI				

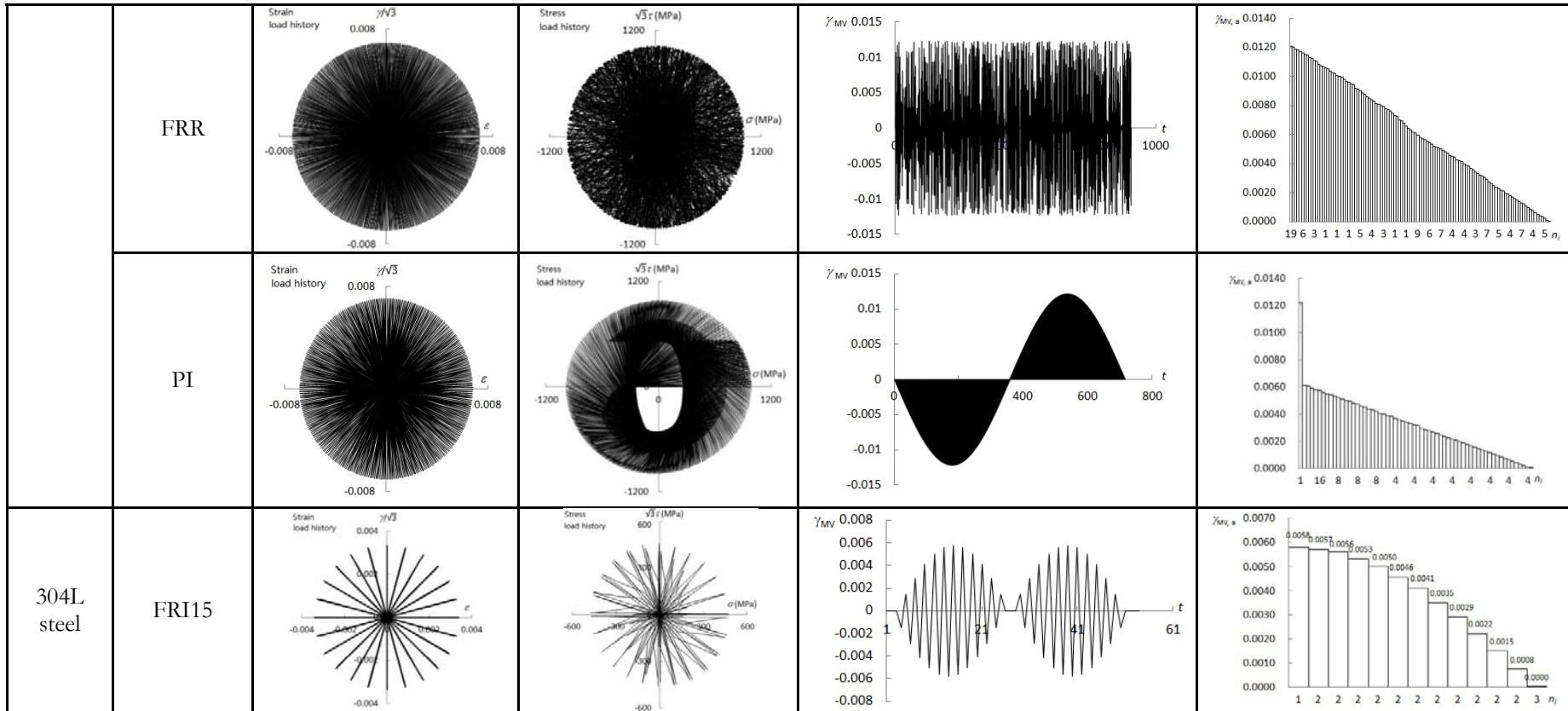|					
	FRR				
304L steel	PI				
	FRI15				

Table 2: Stain paths, stress paths, resolved shear strain histories and spectrums of the resolved shear strain for one complete strain block under several investigated loading paths.
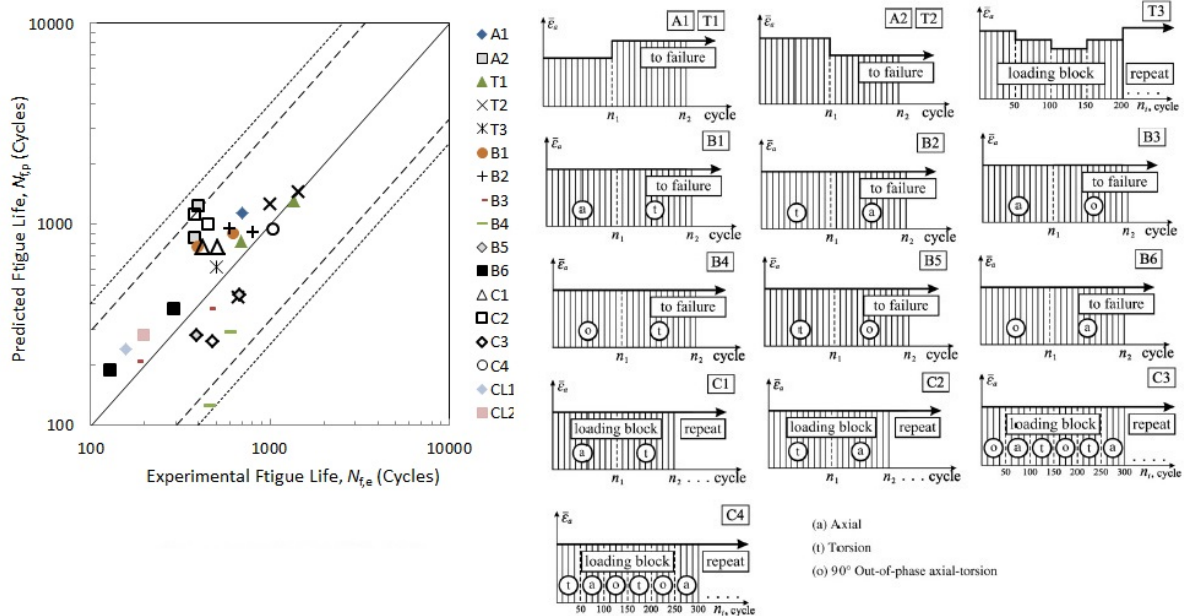
Figure 6: Comparison of observed and predicted fatigue lives by the MMCCM fo pure titanium and titanium alloy BT9.

CONCLUSIONS

(1) The MMCCM originally proposed by Susmel et al. [6] for multiaxial constant amplitude loading was reformulated to estimate fatigue lifetime of structures subjected to VA multiaxial fatigue loading. Satisfactory fatigue life predictions were obtained by using the MMCCM, when coupled with the Rain-Flow cycle counting method and Palmgren-Miner linear cumulative damage rule for several different materials under VA multiaxial fatigue loading. The critical plane was determined by using the γ_MVM.

(2) The MVM is an efficient and valid method to determine the orientation of the critical plane when it is used to evaluate fatigue damage under complex variable amplitude multiaxial fatigue loading.

(3) Because the controlling parameter of the MMCCM is just the shear strain resolved along the direction of maximum variance of the resolved shear strain when the MMCCM is used to estimate fatigue damage under variable amplitude multiaxial fatigue loading, cycles can directly be counted by using rainflow method.

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