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Robust control strategies for multi–inventory systems with average flow constraints

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Abstract

In this paper we consider multi–inventory systems in presence of uncertain demand. We assume that i) demand is unknown but bounded in an assigned compact set and ii) the control inputs (controlled flows) are subject to assigned constraints. Given a long–term average demand, we select a nominal flow that feeds such a demand. In this context, we are interested in a control strategy that meets at each time all possible current demands and achieves the nominal flow in the average. We provide necessary and sufficient conditions for such a strategy to exist and we characterize the set of achievable flows. Such conditions are based on linear programming and thus they are constructive. In the special case of a static flow (i.e. a system with 0–capacity buffers) we show that the strategy must be affine. The dynamic problem can be solved by a linear-saturated control strategy (inspired by the previous one). We provide numerical analysis and illustrating examples.

Key words: Inventory control, Robust control, Bounded disturbances, Manufacturing systems, Linear programming.

1 Introduction

Multi–inventory systems [12,26] are formed by buffers, where raw materials/subassemblies/finished products are stored, connected by processing links, along which items are produced or transported. Such systems are met in several different contexts, such as manufacturing [2,7,8,14,16,20,21], network routing [13], communications [9], water distribution [15], logistics and traffic control [18]. Hence, their control is of relevant economic interest. The control concerns storage and processing operations and aims at meeting the external demand of finished products [10,12].

In the literature, there are many contributions on the design, and possibly the optimization, of the system controls with respect to static criteria in the assumption that the demand is known in advance (see, e.g., [24]). Unfortunately, many real systems work in uncertain and time–varying conditions. Thus, a feedback approach is preferable [1,13,14,19] to assure robustness against uncertain events such as failures or unknown demand rate. In this context, several authors deal with the problem of transient optimality (see, e.g., [2,17,19,22]) and near-optimality (see, e.g., [25]). However, few of them explicitly consider uncertainties in the demand or supply flows (see, e.g., [4,6,7]).

We pursue a deterministic approach by assuming that the external input (we will name it for brevity "the demand") is unknown–but–bounded within given constraint sets. Under this assumption, the basic problem we are investigating is the stability of the multi-retailer system. In a context of fluid models, the stability of the system consists in keeping the buffer levels within assigned constraints or driving them to prescribed levels [4] [6]. In those references it is shown that for continuous–time models there exists a strategy assuring convergence to any target buffer level if a certain "control dominance" necessary and sufficient condition is satisfied. Some optimality criteria for the transient are considered in [3].

In this paper, we are considering, in some sense, a mixed–type problem. We start from the observation that, in a typical production/distribution system, there are different values of the controlled flows, satisfying the constraints, that face a same fixed level of demand (this is a peculiar situation in which the controlled process ma-

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trix is a "large one"). Therefore we have a degree of freedom in choosing, at each time, the workload distribution among controlled links that satisfies at each time the current demands. Beside satisfying at each time the current demand, we are also concerned with the long–term utilization of the system. At a certain time, a generic link may be requested to work harder, than expected in the average, or to be underutilized due to demand fluctuations. However, the average utilization of the links should be adapted to the "average" behavior of the demand and possibly determined by a (steady–state) optimality criterion. This problem is important because in many context, balancing between links is fundamental since over–utilization of some links may cause failures or produce high costs.

In this work we simultaneously consider the two following aspects.

- *Instantaneous fluctuations* — These are assumed unknown due to the large number of unpredictable factors that influence the demand. The control must face all possible variations, within prescribed limits, in order to meet the demand. So, these fluctuations can require a control flow which is, instantaneously, completely unbalanced with respect to the nominal one.
- *Long term information* — Forecasts about long–term average demand values are generally *much more reliable*. Quite accurate statistics over long–time horizons are often available. Besides, long–term values are sometimes fixed (for instance established by contract). The long–term average demand, henceforth also called nominal demand, should be faced, in the average, by the nominal flow, whenever possible.

Therefore we are seeking for a stabilizing strategy capable of balancing the flow in the long run. A basic question is the following. Given the system structure and the controlled flows constraints and assumed that the demand has a known (deterministic) average value, can we find a stabilizing strategy which assures, in the average a prescribed controlled value? As we will see by means of a trivial examples such a strategy does not exists even if the nominal flow is feasible and feeds the nominal demand in steady–state. It will be apparent that this fact is due to the simultaneous presence of flow constraints and demand uncertainties.

We will refer to controlled process matrix that, in general, may not be the incidence matrix of graph. In this sense, a main contribution of the paper is in the generality of the topology of the systems, which are not necessary networks. The main results of the paper are reported next.

- We first consider static strategies (i.e. we assume 0–capacity buffers). We provide necessary and sufficient conditions for the existence of a strategy which is able to meet all the possible demands and assures the desired flow average, whenever the demand meets its nominal average. Such conditions are based on linear programming and are constructive.
- We characterize the set of all flows corresponding to the nominal demand which can be achieved in the average.
- We show that, if the necessary and sufficient conditions are satisfied, then the static strategy is affine. Such an affine function characterizes the actual average flows even in the cases in which the demand average is different from the nominal one.
- We show that the very conditions, valid in the static case, are sufficient for the existence of a dynamic strategy, based on the feedback of the buffer levels. These conditions are also necessary under appropriate, quite general, assumptions.
- We show that the proposed feedback strategy is a linear-saturated dynamic control. The introduced dynamics is, basically, an integrator that gets rid of the load unbalancing. The control synthesis is based on the mentioned linear programming conditions.
- We prove that the problem of establishing whether a nominal flow is achievable or not is an easy problem. Actually, this is done through a polynomial algorithm that selects a candidate strategy and verifies the mentioned conditions at each iteration.

The paper will finally present applications and discussion of the proposed theory.

2 Problem Formulation

Consider the following continuous time system

$$\dot{x}(t) = Bu(t) - w(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a vector whose components are the buffer levels, $u(t) \in \mathbb{R}^m$ is the controlled flow vector, B is the controlled process matrix and $w(t) \in \mathbb{R}^n$ is an exogenous (uncontrolled) input, typically modeling demand, whose value is externally determined. To model backlog $x(t)$ may be less than zero.

We assume that u and w are subject to the next constraints

$$u(t) \in \mathcal{U} = \{u : u^- \leq u \leq u^+\}, \tag{2}$$

where u^- and u^+ are assigned vectors and the expression is to be intended component-wise. We assume that w is constrained as follows

$$w(t) \in \mathcal{W}, \tag{3}$$

where $\mathcal{W}$ is a polytope. We also introduce the following assumptions.

Assumption 1 *Matrix B has full row rank.*

If the above assumption is not satisfied, the system is unreachable. As we will see soon, the problem becomes trivial if B is square therefore we will consider the case in which B is a "fat matrix".

Given a vector function of time $f : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ we introduce the following notation

$$Av[f] = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\, dt. \tag{4}$$

Function $Av[f]$ will be referred to as the deterministic average of f, henceforth the *average*, and we will always assume that such a value exists whenever considered.

Assumption 2 *The set $\mathcal{W}$ includes $\bar{w} = Av[w]$ in its relative interior*[1] *.*

We will consider static and dynamic stabilizing policies for the system according to the following definitions.

Definition 3 *The function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a* static balancing strategy *if for $u(t) = \Phi(w(t))$,*

$$Bu(t) = w(t),$$

and $u(t) \in \mathcal{U}$, for all $w(t) \in \mathcal{W}$, for all $t \geq 0$.

If a static balancing strategy is applied, as a consequence we have $\dot{x}(t) = 0$. Therefore (from a ideal point of view) the buffer level remains bounded since the system meets at each time the current demand. Clearly this is not a feedback strategy and the resulting system is not stabilized[2] .

Our ultimate goal is solving the dynamic problem of steering the system buffer to the neighborhood of a prescribed level.

Definition 4 *Given $\epsilon > 0$ and a reference value $\bar{x}$, an ϵ-*stabilizing strategy *is a feedback control for which there exists a continuous positive function $\phi(t)$, monotonically decreasing and converging to 0 as $t \rightarrow \infty$ such that for all $w(t) \in \mathcal{W}$ and for all $x(0)$, the conditions $u(t) \in \mathcal{U}$ and*

$$\|x(t) - \bar{x}\| \leq \max\{\|x(0)\|\phi(t), \epsilon\}$$

hold true.

[1] we mean that $\bar{w}$ is an interior point of $\mathcal{W}$ with respect to the smallest linear subspace including it, for instance given a vector $v \neq 0$, 0 is in the relative interior of a segment joining v and $-v$

[2] indeed infinitesimal perturbations on w may cause buffer overflow

We introduce the following basic conditions [4] as a preliminary result.

Theorem 5 *For the considered system*

i *there exists a static balancing strategy as in Definition 3 if and only if*

$$\mathcal{W} \subseteq B\mathcal{U}; \tag{5}$$

ii *there exists a feedback stabilizing strategy as in Definition 4 if and only if*

$$\mathcal{W} \subseteq int\{B\mathcal{U}\}. \tag{6}$$

Henceforth, we assume that the appropriate necessary and sufficient condition is met (depending on which kind of strategy we are considering). Assume to apply either a balancing or an ϵ-stabilizing strategy. As a consequence, $x(t)$ remains constant or bounded. Then, by integrating (1) we have that, necessarily,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T [Bu(t) - w(t)]\, dt = \lim_{T\to\infty} \frac{1}{T} [x(t) - x(0)] = 0,$$

which implies that the average value of w is equal to the average value of Bu

$$B\ Av[u(t)] = Av[w(t)]. \tag{7}$$

Given a nominal average flow $\bar{w}$, unless B is square (the problem would be trivial in this case) there are several possible vectors (average controlled flows) $\bar{u} = Av[u(t)]$ such that $B\bar{u} = \bar{w}$. By exploiting this redundancy, we are actually interested in selecting a nominal flow $\bar{u}$ that supports the average of the demand $Av[w]$ whenever $Av[w] = \bar{w} \in \mathcal{W}$.

Formally, the problem is the following.

Problem 6 *Assume that the average $\bar{w} \in \mathcal{W}$ is given. Consider the feasible flow $\bar{u} \in \mathcal{U}$ such that*

$$B\bar{u} = \bar{w}.$$

Provide a yes–no answer to the question: does there exist a static balancing (or dynamic ϵ–stabilizing) strategy such that whenever $Av[w] = \bar{w}$ then $Av[u] = \bar{u}$? In the case of a positive answer we will say that $\bar{u}$ is achievable.

We stress that, given $\bar{w} \in \mathcal{W}$, not all the vectors $\hat{u}$ such that $B\hat{u} = \bar{w}$ can be achieved as average flows, as shown next.

Example 7 *Consider the scalar system*

$$\dot{x}(t) = u_1(t) + u_2(t) - w(t),$$

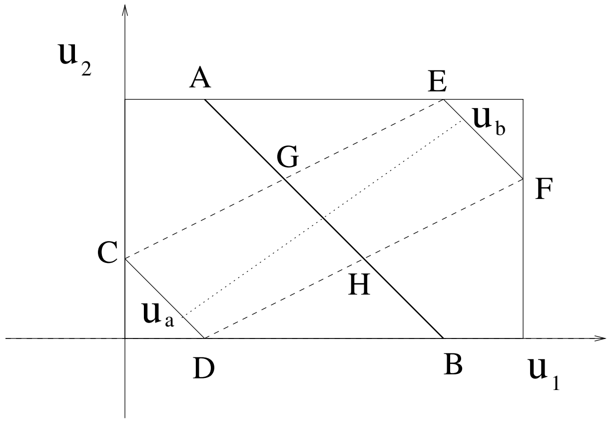

Fig. 1. The achievable averages

where

$$0 \le u_1 \le 5, \quad 0 \le u_2 \le 3, \quad 1 \le w \le 7.$$

Assume that $Av[w] = 4$. The all candidate average flows are those such that

$$\bar{u}_1 + \bar{u}_2 - 4 = 0,$$

precisely those on the central line A–B in Fig. 7. Now the feasible flows to meet the demand $u_1 + u_2 = w = 1$ are those on the line C–D, while the feasible flows to meet the demand $u_1 + u_2 = w = 7$ are those on the line E–F. Now, if the demand periodically jumps from 1 to 7 as follows: $w(t) = 1$ for $kT \le t < kT + T/2$ and $w(t) = 7$ for $kT + T/2 \le t < (k+1)T$ then its average is $\bar{w} = 4$ but it can be faced only by points of the type u_a and u_b respectively. It is therefore clear that the only suitable average values are those on the line A–B which are included between the two dashed lines (segment G–H). Actually it is not difficult to see that, for a generic $\bar{w}$ the achievable average flows $\bar{u}$ for this problem are all the points on the line $\bar{u}_1 + \bar{u}_2 = \bar{w}$ confined between such dashed lines (i.e. such that $-1 \le 2\bar{u}_2 - \bar{u}_1 \le 1$).

In the following sections we will solve constructively the problem for both static and dynamic strategies.

3 Achievable average: the static case

In this section we consider the case in which the controlled flow is a function of the demand w so that $Bu(t) = w(t)$. Note that this control strategy can not stabilize the queue lengths since the time derivative of the queue lengths is made zero. This situation occurs in several problems (for instance in power supply). This section has to be considered as a prelude to the dynamic case in which we will use the necessary and sufficient conditions derived here.

For the simple notations we work under the following assumption.

Assumption 8 *The nominal average "demand" is zero, i.e. $\bar{w} = Av[w] = 0 \in \mathcal{W}$.*

This is not a restriction because under the conditions (5) or (6) there exists u_0 (we will assume equal to $\bar{u}$ for convenience) such that $Bu_0 = \bar{w}$, the nominal average. Then we can translate the problem by writing the new model

$$\dot{x}(t) = B(u(t) - u_0) - [w(t) - \bar{w}] = B\delta u(t) - \delta w(t)$$

and by translating the constraints as

$$u^- - u_0 \le \delta u(t) \le u^+ - u_0, \quad \delta w(t) \in \mathcal{W} - \bar{w}.$$

where $Av[\delta w] = 0$. If we assume $u_0 = \bar{u}$ the question is weather a static balancing strategy exists such that *any* null average demand implies a null average flow. The following theorem, whose proof will be given later, provides an answer.

Theorem 9 *Under Assumption 1 and 2 let condition (5) be satisfied. Then there exists a static balancing strategy that achieves the average $Av[u] = 0$ whenever $Av[w] = 0$ if and only if there exists a "tall" matrix D $m \times n$ such that*

$$BD = I \tag{8}$$

$$u^- \le Dw^{(i)} \le u^+, \quad i = 1, \ldots, s. \tag{9}$$

where $w^{(i)}$ are the vertices of $\mathcal{W}$. Moreover, if such necessary and sufficient conditions are satisfied, then the static strategy is linear

$$u(t) = Dw(t). \tag{10}$$

The previous theorem allows us to check a single candidate $\bar{u}$ we fixed to zero. We can now characterize the *set of achievable average flows*, namely the set of all vectors such that $Av[w] = 0$ implies $Av[u] = \bar{u} \in \mathcal{U}$.

Corollary 10 *The set of all achievable average flows, provided that a suitable static balancing strategy is applied, is made up by all the vectors $\bar{u} \in ker[B]$ such that there exists a matrix D, $m \times n$, with*

$$BD = I \tag{11}$$

$$u^- \le Dw^{(i)} + \bar{u} \le u^+, \quad i = 1, \ldots, s. \tag{12}$$

In this case the static strategy is affine

$$u = Dw + \bar{u}.$$

PROOF. It follows immediately from the theorem by applying the translation $u - \bar{u}$. □

We have seen that as long as a strategy achieving the average exists, this has to be linear (or affine taking into account possible translations on w). As a consequence of the linearity we have the following property.

Corollary 11 *If the necessary and sufficient conditions (8) and (9) are satisfied, then the average constraints are satisfied not only on the infinite horizon, but on every finite horizon as well, in the sense that for all* $T > 0$

$$\frac{1}{T}\int_0^T w(t)dt = 0 \quad \textit{implies} \quad \frac{1}{T}\int_0^T u(t)dt = 0.$$

Remark 12 *It should be noticed that, if the demand average is not the nominal one, but* $Av[w] = \hat{w}$*, then the corresponding average flow is characterized by*

$$Av[u] = DAv[w].$$

This means that D *may be thought of as a "partitioning law" for the workload* $Av[u]$ *and thus chosen via some optimality criterion.*

3.1 Proof of the theorem

To prove the theorem we need the next lemma.

Lemma 13 *Consider a convex cone* $\mathcal{C} \subset \mathbb{R}^n$ *centered in* 0 *with a non–empty interior. Consider two subspaces* $\mathcal{Y}$ *and* $\mathcal{Z} \subset \mathbb{R}^n$ *and define*

$$\hat{\mathcal{Y}} \doteq \mathcal{Y} \bigcap \mathcal{C}, \quad \hat{\mathcal{Z}} \doteq \mathcal{Z} \bigcap \mathcal{C}.$$

Assume that $\hat{\mathcal{Y}}$ *includes an element interior to* $\mathcal{C}$, $y \in int\{\mathcal{C}\}$*. Then* $\hat{\mathcal{Y}} \subseteq \hat{\mathcal{Z}}$ *implies* $\mathcal{Y} \subseteq \mathcal{Z}$.

PROOF. We initially observe that, since $\mathcal{Y}$ and $\mathcal{Z}$ are subspaces, we can prove the lemma by showing that $dim(\mathcal{Y} \bigcap \mathcal{Z}) = dim(\mathcal{Y})$. To this end, note that, as $\mathcal{C}$ is not empty, $dim(\mathcal{C}) = n$; as $\mathcal{C}$ and $\mathcal{Y}$ are polytopes, $y \in int\{\mathcal{C}\}$ implies that there exists $\delta > 0$ such that $y + \delta e^{(i)} \in \hat{\mathcal{Y}}$, for each vector $e^{(i)}$ belonging to a basis of the subspace $\mathcal{Y}$. Hence, $dim(\hat{\mathcal{Y}}) = dim(\mathcal{Y})$. Finally, as $\mathcal{Y} \bigcap \mathcal{Z} \supseteq \mathcal{Y} \bigcap \mathcal{Z} \bigcap \mathcal{C}$, then $\hat{\mathcal{Y}} \subseteq \hat{\mathcal{Z}}$ implies $\mathcal{Y} \bigcap \mathcal{Z} \supseteq \hat{\mathcal{Y}}$. Hence, $dim(\mathcal{Y} \bigcap \mathcal{Z}) \geq dim(\mathcal{Y})$. As $dim(\mathcal{Y} \bigcap \mathcal{Z}) \leq \min\{dim(\mathcal{Y}), dim(\mathcal{Z})\}$ we obtain $dim(\mathcal{Y} \bigcap \mathcal{Z}) = dim(\mathcal{Y})$. □

We can now prove the theorem.

Sufficiency. We assume that (8) and (9) hold and prove that (10) is the desired strategy. Indeed, strategy (10) is static and balancing, since $u(t) = Dw(t)$ implies $Bu(t) = BDw(t) = w(t)$. In addition, for all $w \in \mathcal{W}$, we have that $w = \sum \alpha_i\, w^{(i)}$, $\sum \alpha_i = 1$, $\alpha_i \geq 0$, then

$$u = Dw = \sum \alpha_i\, Dw^{(i)} \in \mathcal{U}.$$

Strategy (10) also achieves the average $Av[u] = 0$ whenever $Av[w] = 0$, since $u(t) = Dw(t)$ implies also $Av[u] = Av[Dw] = DAv[w]$.

Necessity. We assume that there exists a static balancing strategy $u = \Phi(w)$ such that $u \in \mathcal{U}$ for all $w \in \mathcal{W}$ and such that $Av[w] = 0$ implies $Av[u] = 0$ and we prove that (8) and (9) hold. Given the nonnegative unit–sum vector $\alpha = [\alpha_1\ \alpha_2 \ldots \alpha_s]$, $\alpha \geq 0$, $\bar{1}^T\alpha = 1$ consider a periodic demand $w(t)$ of period T defined as follows

$$w(t) = w^{(k)}, \qquad \sum_{i=0}^{k-1} \alpha_i T \leq t \leq \sum_{i=0}^{k} \alpha_i T,$$

for $k = 1, 2, \ldots, s$, with $\alpha_0 = 0$ Namely, $w(t)$ assumes the vertex value $w^{(i)}$ for the portion of period $\alpha_i T$. This demand is feasible and its average is

$$Av[w] = \sum_{i=1}^{s} \alpha_i\, w^{(i)}.$$

Now, no matter how the α_i are chosen, the above static balancing strategy feeds any possible demand $w^{(i)}$ through a controlled flow $u^{(i)} = \Phi(w^{(i)})$ which verifies

$$B\Phi(w^{(i)}) = w^{(i)}.$$

As a consequence the average flow is

$$Av[u] = \sum_{i=1}^{s} \alpha_i\, \Phi(w^{(i)}) = \sum_{i=1}^{s} \alpha_i\, u^{(i)}.$$

Denote by $W = [w^{(1)}\ w^{(2)}\ \ldots\ w^{(s)}]$ the matrix including the vertices of $\mathcal{W}$ and by $U = [u^{(1)}\ u^{(2)}\ \ldots\ u^{(s)}]$ the corresponding input values (note that this means $BU = W$). In view of the assumption, we have that $Av[w] = 0$ implies $Av[u] = 0$, which can be written as

$$W\alpha = 0, \quad \Rightarrow \quad U\alpha = 0, \tag{13}$$

(actually $W\alpha = 0$ iff $U\alpha = 0$). Therefore the positive kernel of W, precisely the intersection of $ker[W]$ with the positive hortant, is included in the positive kernel of U.

We remind now that 0 belongs to the relative interior of $\mathcal{W}$ by Assumption 2.

Then in the α space, there exists a positive vector $\hat{\alpha} > 0$, $\bar{1}^T\hat{\alpha} = 1$ such that $W\hat{\alpha} = 0$. Then we can apply the lemma and claim that

$$ker[W] \subseteq ker[U].$$

On the other hand we have, by construction that $W = BU$ and then $U\alpha = 0$ implies $W\alpha = BU\alpha = 0$, so that $ker[U] = ker[W]$.

This means that the columns of U can be generated as linear combination of the columns of W and vice-versa and therefore the two matrices have the same row rank. Therefore, there exists a matrix $\hat{D}$ $m \times n$ such that

$$U = \hat{D}W.$$

Then

$$W = BU = B\hat{D}W.$$

Now, if W has full row rank, this implies $B\hat{D} = I$ and then (8). Actually, the equation implies that $B\hat{D}$, is the identity within the subspace generated by the column of W. We show now that we can always find a right inverse of B, precisely a matrix D such that $B\hat{D} = I$. Assume that W has not full row rank and take the matrix $Q = P^{-1}$ such that

$$QW = \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix} \quad \text{and let} \quad \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 \end{bmatrix} \doteq \hat{D}\ P$$

with $\tilde{W}_1$ full row rank equal to ρ. Consider the equation $B\hat{D}W = W$ and premultiply both its sides by Q as $QB\hat{D}W = QB\hat{D}PQW = QW$ we achieve by substitution

$$QB \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix},$$

where $\tilde{D}_1$ has necessarily full column rank (equal to ρ). Note that we can replace $\tilde{D}_2$ by a tall matrix Δ (with $n - \rho$ columns)

$$QB \begin{bmatrix} \tilde{D}_1 & \Delta \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix}$$

Take Δ such that $QB\Delta = \begin{bmatrix} 0 & I \end{bmatrix}^T$ (this is possible because QB has full row rank) and augment the previous equation as follows

$$QB \begin{bmatrix} \tilde{D}_1 & \Delta \end{bmatrix} \begin{bmatrix} \tilde{W}_1 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 & 0 \\ 0 & I \end{bmatrix}.$$

Note that the rightmost matrix has full row rank. By multiplying on the left by P we achieve

$$B \underbrace{\begin{bmatrix} \tilde{D}_1 & \Delta \end{bmatrix} Q}_{\doteq D} \underbrace{P \begin{bmatrix} \tilde{W}_1 & 0 \\ 0 & I \end{bmatrix}}_{\doteq [W\ W^*]} = BD[W\ W^*] =$$

$$= P \begin{bmatrix} \tilde{W}_1 & 0 \\ 0 & I \end{bmatrix} = [W\ W^*]$$

with $[W\ W^*]$ of full row rank. As previously observed, this means that $BD = I$. Now we have to show that $U = DW$. This is easy

$$DW = \begin{bmatrix} \tilde{D}_1 & \Delta \end{bmatrix} Q\ P \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{D}_1 & \Delta \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 \end{bmatrix} Q\ P \begin{bmatrix} \tilde{W}_1 \\ 0 \end{bmatrix} = \hat{D}W = U.$$

Then, for each row $w^{(k)}$ of W $Dw^{(k)} = u^{(k)} = \Phi(w^{(k)})$ and thus (9) is automatically satisfied. $\square$

Example 14 *(Example 7 cont'd) Let us briefly consider again the simple system of Example 7. Since the long–term average demand is $\bar{w} = 4$ we can select a nominal flow $\bar{u} = [2.5 \quad 1.5]'$. By translating the axes to the origin $u_0 = \bar{u}$, we have the new model*

$$\begin{aligned} \dot{x}(t) &= (u_1(t) - 2.5) + (u_2(t) - 1.5) - (w(t) - 4) \\ &= \delta u_1(t) + \delta u_2(t) - \delta w(t), \end{aligned}$$

where$-2.5 \leq \delta u_1(t) \leq 2.5$, $-1.5 \leq \delta u_2(t) \leq 1.5$, *and* $-3 \leq \delta w(t) \leq +3$. *Now, there exist a variety of matrices D that verify conditions (8) and (9). As an example, we can choose $D = \left[\frac{2}{3} \quad \frac{1}{3}\right]'$. The resulting static strategy for the translated problem is then*

$$\delta u(t) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \delta w(t),$$

which in the original axes takes on the form

$$u(t) = D\delta w + u_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \delta w(t) + \begin{bmatrix} 2.5 \\ 1.5 \end{bmatrix}.$$

There are counterexamples which prove that Theorem 9 does not hold when $0 \notin rel\ int\{\mathcal{W}\}$, (in general when $\bar{w} \notin rel\ int(\mathcal{W})$) in the sense that the provided conditions become sufficient only.

4 Achievable average flows with dynamic strategies

Here we show how to achieve an average flow by a dynamic stabilizing strategy. The main results of the section is Theorem 19, which basically states that the conditions for the existence of a dynamic strategy which, achieves a certain average are the same of the static case. We will first show, in the next subsection, that conditions (8) and (9) are sufficient for the existence of a dynamic ϵ–stabilizing strategy of the form

$$\begin{aligned} \dot{y}(t) &= f(y(t), x(t), w(t)) \\ u(t) &= g(y(t), x(t), w(t)). \end{aligned} \tag{14}$$

To provide results about necessity of (8) and (9) we need to better characterize the class of dynamic strategies by additional assumptions. This will be done in the subsequent subsection.

4.1 Sufficiency of the conditions

Let assumptions (8) and (9) be satisfied and consider the corresponding matrix D. Equation (8) means that D is a right inverse of B and it is a standard property of linear algebra that this is equivalent to the existence of two matrices C and F which "square" B and D producing two matrices inverse to each other, namely such that

$$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} D & F \end{bmatrix} = I. \tag{15}$$

Consider the following augmented system

$$\begin{aligned} \dot{x}(t) &= Bu(t) - w(t) \\ \dot{y}(t) &= Cu(t). \end{aligned} \tag{16}$$

The additional dynamic variable $\dot{y}(t) = Cu(t)$ has the goal of keeping trace of the load unbalancing with respect to the desired average 0.

The first step is to show that under (8) and (9), the extended system (16) satisfies the stabilizability conditions (6) as well (in the extended state–space), precisely for all $w \in \mathcal{W}$ there exists $u \in \mathcal{U}$ such that

$$\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix} u,$$

or equivalently that, for all $w \in \mathcal{W}$, there exists $u \in \mathcal{U}$ such that

$$u = \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} w \\ 0 \end{bmatrix} = Dw.$$

The existence of such u is an immediate consequence of (9). Indeed, it is easy to verify that, if $\mathcal{W} \in int\{B\mathcal{U}\}$, then the u which corresponds to w is in the interior of the extended set. Then the problem can be solved as follows.

- Determine D such that (8) and (9) are satisfied.
- Determine C and F such that (15) is satisfied.
- Design a control which stabilizes (16).

Observe that Theorem 5 applied to the extended system (16) guarantees the existence of such a stabilizing control.

Here we propose a new strategy based on a variable transformation. In the following we exploit (for the first time) the structure of the set $\mathcal{U}$. Consider the new variable $z(t)$ defined as

$$z(t) = \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix} z(t)$$

This variable satisfies the equation

$$\dot{z}(t) = u(t) - Dw(t). \tag{17}$$

The new system (17) is decoupled in its state variable, precisely it is equivalent to

$$\dot{z}_i(t) = u_i(t) - D_i w(t), \tag{18}$$

where we have denoted by D_i the ith row of D and where $u_i^- \leq u_i \leq u_i^+$. Denote by

$$\rho_i^- = \min_{w \in \mathcal{W}} D_i w,$$
$$\rho_i^+ = \max_{w \in \mathcal{W}} D_i w,$$

The stabilizability conditions are equivalent to the fact that for all $w \in \mathcal{W}$

$$u_i^- < \rho_i^- < \rho_i^+ < u_i^+.$$

Henceforth, without restriction, we consider the single–buffer case, namely the scalar system

$$\dot{z}(t) = u(t) - r(t),$$

with

$$\rho^- \leq r(t) \leq \rho^+, \quad u^- \leq u(t) \leq u^+.$$

Define the saturated control (see Fig. 2)

$$u(t) = sat_{[u^-, u^+]}(-\kappa z(t)) \tag{19}$$

with $\kappa > 0$ and where

$$sat_{[\alpha,\beta]}(\zeta) = \begin{cases} \beta, & \text{if } \zeta > \beta, \\ \zeta, & \text{if } \alpha \le \zeta \le \beta, \\ \alpha, & \text{if } \zeta < \alpha. \end{cases}$$

We will use the same notation (19) for the multi–input control derived applying the formula component–by–component. Note that this control function is Lipschitz

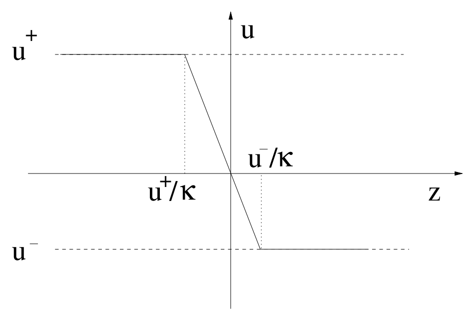

Fig. 2. The function (19)

continuous. For $\kappa \to \infty$, the control (19) converges to the bang bang control

$$bb_{[u^-,u^+]}(\zeta) = \begin{cases} u^+, & \text{if } \zeta > 0, \\ 0, & \text{if } \zeta = 0, \\ u^-, & \text{if } \zeta < 0, \end{cases}$$

which is of the type considered in [4].

Theorem 15 *The variable $z(t)$ with the control (19) converges to the interval $[-u^+/\kappa, -u^-/\kappa]$ (which includes 0 as an interior point). Therefore the global system converges to the corresponding hyper–box (i.e. that delimited by $-u_i^+/\kappa \le z_i \le -u_i^-/\kappa$, $i = 1, 2, \ldots, m$).*

PROOF. The proof derives from the fact that, for $z \ge -u^-/\kappa$, we have that the control is saturated to its lower level $u = u^-$, then

$$\dot{z} = u^- - r \le u^- - \rho^- < 0. \tag{20}$$

Conversely for $z \le -u^+/\kappa$ we have that $u = u^+$, then

$$\dot{z} = u^+ - r \ge u^+ - \rho^+ > 0. \tag{21}$$

Therefore $z(t)$ reaches the interval in finite time and is ultimately confined in it. □

As a consequence of the previous theorem we have that, choosing κ large enough, we can bound z in an arbitrarily small interval. Therefore we achieve ϵ–stability. We have now to show that the controller so obtained satisfies the average requirement. Indeed variable $z(t)$ remains bounded so $\|z(t) - z(0)\| \le \xi$. By integrating (17) we have that

$$\frac{1}{T}\int_0^T u(t)dt - \frac{1}{T}\int_0^T Dw(t)dt = \frac{z(T) - z(0)}{T} \to 0$$

as $T \to \infty$. This yields

$$Av[u] = Av[Dw],$$

that is all we need to claim that sufficiency of (8) and (9) is proved.

We briefly consider now the case in which the control is allowed to be discontinuous. This case is important in all the systems in which several controlled arcs are of the switching (on–off) type.

Corollary 16 *The system equipped with the bang–bang control $u = bb_{[u^-,u^+]}(z)$ is such that $z(t) \to 0$. The origin is reached in finite time which is equal to*

$$\tau_{max} = \max_i \quad \max\{\frac{z_i(0)}{-u_i^- - \rho_i^-}, \frac{-z_i(0)}{u_i^+ + \rho_i^+}\}.$$

PROOF. It is an easy consequence of the fact that the derivative can be bounded for $z > 0$ as in (20) and for $z < 0$ as in(21). □

Remark 17 *The proposed strategy works under measurement errors. Bounded measurement errors on $x(t)$ imply bounded errors on $z(t)$. By reasoning component-wise we achieve systems of the form*

$$\dot{z} = sat_{[u^-,u^+]}(-\kappa(z + \delta_z)) - r$$

whose state remains bounded as long as the error δ_z is such. Precisely via elementary analysis it can be shown that if $|\delta_z| \le \delta_{meas}$ then z will be ultimately confined in the interval $[-(\epsilon + \delta_{meas}), (\epsilon + \delta_{meas})]$.

4.2 Proof of necessity

In this section we show the necessity of conditions (8) and (9) for the existence of ϵ–stabilizing strategy in the general class that satisfy the next assumption.

Assumption 18 *The strategy must assure*

- *Boundedness of compensator variables: (along with those of the plant) there exists $\mu > 0$, $\nu > 0$ and $\bar{t}$ such that for any arbitrary t_0, the conditions $\|x(t_0) - \bar{x}\| \le \nu$ and $\|y(t_0)\| \le \mu$ imply $\|x(t) - \bar{x}\| \le \nu$ and $\|y(t)\| \le \mu$, for all $t > t_0 + \bar{t}$.*

- *Uniqueness at steady state: For each vertex $w^{(k)}$ of $\mathcal{W}$ there exists a corresponding $u^{(k)}$ such that for $w(t) \equiv w^{(k)}$ and for all initial condition in t_0 as above*

$$\left| \frac{1}{\tau} \int_{t_0}^{t_0+\tau} (u(\sigma) - u^{(k)}) d\sigma \right| \leq \psi(\tau),$$

where $\psi(\tau)$ is a positive monotonically decreasing continuous function converging to 0 as $\tau \to \infty$.

The assumption means that i) the additional dynamics represented by variable $y(t)$ must be bounded under bounded disturbances w "at least after some time"; and that ii) for a constant $w^{(k)}$ the strategy replies with a unique vector $u^{(k)}$ in the average. In the case of a continuous control this just means that, at steady state, $w^{(k)}$ is faced by a precise flow vector $u^{(k)}$. The strange formulation is due to the fact that for some discontinuous strategies the value $u^{(k)}$ may be a value not actually achieved at any time. For instance consider the system

$$\dot{x} = u - w$$

with the control $u = -bb_{[-1,1]}(x)$. If w is constant equal to 1/2, then the corresponding $u = 1/2$, but the value is never achieved.

We show now that under Assumptions 1 and 2, if there exists an ϵ–stabilizing strategy which satisfies also Assumption 18 and which meets the average value $Av[u] = 0$ whenever $Av[w] = 0$ for all $w(t) \in \mathcal{W}$, then (8) and (9) must be satisfied.

Consider the matrix W made up by the vertices of $\mathcal{W}$ and fix a positive vector α such that $W\alpha = 0$ and $\sum_{i=1}^{s} \alpha_i = 1$ (the fact that α can be positive is due to Assumption 2). Given $T > 0$ consider again the demand $w(t)$ periodic of period T, defined as follows

$$w(t) = w^{(k)}, \quad T_k \leq t \leq T_{k+1} \quad \text{where} \quad T_k = \sum_{i=0}^{k-1} \alpha_i T,$$

where $\alpha_0 = 0$, having average $Av[w] = W\alpha$. Let $\bar{x} = 0$ and $\|x(t_0)\| \leq \nu$ and $\|y(t_0)\| \leq \mu$ for $t_0 = 0$. For T large enough these bounds are true, by assumption, for any t_0 chosen as any "switching time" of w (just take $T\alpha_i \geq \bar{t}$ for all i). Assume that for $Av[w] = 0$ $Av[u] = 0$. Then the control average on $[0, T]$ is (noticing that $[T_{i+1} - T_i]/T = \alpha_i$) is the limit of the following function

$$h(T) \doteq \frac{1}{T} \int_0^T u(t)dt = \frac{1}{T} \sum_{k=1}^{s} \int_{T_k}^{T_{k+1}} u(t)dt =$$

$$= \frac{1}{T} \sum_{k=1}^{s} \alpha_k u^{(k)} + \frac{1}{T} \sum_{k=1}^{s} \int_{T_k}^{T_{k+1}} u(t)dt - \sum_{k=1}^{s} \frac{T_{k+1} - T_k}{T} u^{(k)} =$$

$$= \frac{1}{T} \sum_{k=1}^{s} \alpha_k u^{(k)} + \frac{1}{T} \sum_{k=1}^{s} \int_{T_k}^{T_{k+1}} \left[u(t) - u^{(k)} \right] dt =$$

$$= \frac{1}{T} \sum_{k=1}^{s} \alpha_k u^{(k)} + \sum_{k=1}^{s} \alpha_k \underbrace{\frac{1}{T_{k+1} - T_k} \int_{T_k}^{T_{k+1}} \left[u(t) - u^{(k)} \right] dt}_{\to 0, \text{ as } T \to \infty}.$$

The fact that the rightmost quantity converges to 0 as $T \to \infty$, follows from Assumption 18. Since also $h(T) \to 0$, we have that

$$\frac{1}{T} \sum_{i=1}^{s} \alpha_i u^{(k)} = 0.$$

Since $\alpha > 0$, we have proved condition (13). The remaining part of the proof proceeds exactly as the proof of Theorem 9 so necessity is proved.

We can then formalize the result as follows.

Theorem 19 *Under Assumptions 1 and 2, let the stabilizability condition (6) be satisfied. Then there exists a control, in the class of strategies satisfying Assumption 18, which achieves the average 0 whenever $Av[w] = 0$ if and only if there exists a "tall" matrix D $m \times n$ such that (8) and (9) are satisfied.*

Remark 20 *The provided theory can be easily applied to systems with production/transportation delays along the lines proposed in [5] for discrete–time systems. The extension to the continuous–time case is simple as we show next. Consider the model*

$$\dot{x}(t) = B_0 u(t) + \sum_{k=0}^{s} B_k u(t - \tau_k) - w(t)$$

where τ_k are known delay (see [5] for details). Consider the variable "inventory position"

$$x_{ip}(t) = x(t) + \sum_{k=1}^{s} \int_{t-\tau_k}^{t} B_k\, u(\sigma) d\sigma$$

By differentiating x_{ip} we derive the following equation

$$\dot{x}_{ip}(t) = Bu(t) - w(t)$$

where we have defined $B \doteq \sum_{k=0}^{s} B_k$. Since $u(t)$ is bounded, $x(t)$ is bounded if and only if $x_{ip}(t)$ is bounded.

Then the exposed theory applies without modifications if we deal with the problem of keeping $x_{ip}(t)$ bounded.

Note that this strategy provides boundedness, but not epsilon-stabilization (except when ε is larger than $\sum_k \tau_k \max_{u\in\mathcal{U}} ||B_k u||$). Achieving regulation would require controller design which takes into account delays. This is a challenging problem, especially when the delays are unknown and/or time-varying (which is usually the case in practice). An attempt to solve this problem in the case of communication networks has been made by [23].

5 Existence of achievable average flows

In this section, we show that verifying whether a given average flow is achievable is an easy problem. This can be accomplished in polynomial time by an algorithm that iteratively selects a candidate control strategy and checks if the necessary and sufficient conditions are satisfied. The algorithm stops when the conditions are satisfied returning a possible strategy, or establishes that no strategy exists such that the average flow $\bar{u}$ is achievable. To study the difficulty of such a problem, henceforth we assume that $\mathcal{W}$ is known through its external representation, i.e., it is described by means of the inequalities defining its facets.

We initially determine whether a given flow $\bar{u} \in U$ is achievable. More formally we face the following problems.

Problem 21 *Assume that a feasible flow $\bar{u} \in \mathcal{U} \cap ker[B]$, and a matrix $D \in \mathbb{R}^{m\times n}$, such that $BD = I$, are given. Provide a yes–no answer to the question whether $\bar{u}$ is achievable with the strategy $u = Dw + \bar{u}$.*

Problem 22 *Assume that a feasible flow $\bar{u} \in \mathcal{U} \cap ker[B]$ is given. Provide a yes–no answer to the question whether there exists a matrix $D \in \mathbb{R}^{m\times n}$ such that $\bar{u}$ is achievable with the strategy $u = Dw + \bar{u}$.*

Observe that, in general, an achievable flow may not exist. This fact is due to the *simultaneous presence of control constraints and input uncertainty.* Indeed it is trivial to see that, when the demand set $\mathcal{W}$ is a singleton, any flow $u \in \mathcal{U}$ such that $Bu = w$ is achievable. On the other hand, if the control is unbounded, i.e., $u^+ = +\infty$ and $u^- = -\infty$, conditions (11) and (12) in Corollary 10 hold trivially. Therefore the following questions are natural: given $\mathcal{W}$, which is the "smallest" box $\mathcal{U}$ for which an achievable value exists? Or, given $\mathcal{U}$, which is the "largest" uncertainty set $\mathcal{W}$ for which an achievable value exists? These questions are stated more formally in the following optimization problems.

Problem 23 *Assume that a set $\mathcal{W} \subset \mathbb{R}^n$, a matrix $B \in \mathbb{R}^{n\times m}$, and a cost vector $c = [c^+|c^-]' \in \mathbb{R}^{2m}$ are given. Find vectors $u^+ \in \mathbb{R}^{m+}$, $u^- \in \mathbb{R}^{m-}$, of minimum cost $c^{+\prime}u^+ + c^{-\prime}u^-$ such that there exist a matrix $D \in \mathbb{R}^{m\times n}$ and a vector $\bar{u} \in ker[B]$ for which conditions (11) and (12) in Corollary 10 hold.*

Problem 24 *Assume that a set $\mathcal{U} \subset \mathbb{R}^m$, a set $\widehat{\mathcal{W}} \subset \mathbb{R}^n$, a matrix $B \in \mathbb{R}^{n\times m}$ are given. Find the maximum scalar α such that there exist a matrix $D \in \mathbb{R}^{m\times n}$ and a vector $\bar{u} \in ker[B]$ for which conditions (11) and (12) in Corollary 10 hold for $\mathcal{W} = \alpha\widehat{\mathcal{W}}$.*

In the following subsections, we show that we can solve all the above problems through linear programming. The resulting linear programming formulations may present an exponential number of constraints, proportional to the number of vertices of $\mathcal{W}$. Nevertheless we show that Problem 21 can be easily solved by a polynomial time procedure and that we can use such a procedure as an oracle for solving the remaining problems by constraint generation [11].

5.1 Solution of Problem 21

We can give a positive answer to Problem 21 if and only if $\bar{u}$ and D satisfy conditions (12) in Corollary 10. Conditions (12) may be exponential in number. However, we can easily answer Problem 21 by solving $2m$ linear programming problems. In particular let x_k denote the component k of the generic vector x. Then, to provide a positive answer to Problem 21, we have to verify whether $\max_{w\in W}(Dw)_k + \bar{u}_k$ (respectively, $\min_{w\in W}(Dw)_k + \bar{u}_k$) is less than or equal to u_k^+ (respectively, greater than or equal to u_k^-) for each $k = 1, \ldots, m$. Note that, when a verification fails, the solution of the corresponding linear programming returns a vertex $w^{(i)}$ of $\mathcal{W}$ for which condition (12) does not hold. In the following we refer to such a vertex as the *violating vertex* for Problem 21.

5.2 Solution of Problem 22

To provide a positive answer to Problem 22 we have to determine whether a feasible solution D to the linear programming problem defined by conditions (11) and (12) exists. We can solve the linear programming problem in polynomial time by constraint generation [11], i.e., generating iteratively only the constraints that are necessary to identify the desired solution. In particular, we can use the following algorithm that has the vector $\bar{u}$ as input:

(1) Let $\mathcal{I}$ be a subset of the indices denoting the vertices of $\mathcal{W}$. Go to step 2.
(2) Solve the linear programming problem defined by the conditions (11) and the conditions (12) corresponding to the vertices with indices in $\mathcal{I}$. If the linear programming problem has no feasible solution, exit and provide a negative answer to Problem 22;

otherwise let $\hat{D}$ be the feasible solution obtained and go to step 3.

(3) Solve Problem 21 for $\bar{u}$ and $\hat{D}$. If Problem 21 has a positive answer, exit and provide a positive answer to Problem 22, $\hat{D}$ is the desired matrix; otherwise let $w^{(i)}$ be the violating vertex, set $\mathcal{I} = \mathcal{I} \cup \{i\}$ and go to step 2.

Problem 22 becomes particularly easy if $\mathcal{W}$ is a box and we introduce the additional requirement that the $u(w) - \bar{u} \geq 0$ when $w \geq 0$, in other words, if we desire that the control strategy reacts with a positive perturbation to a positive perturbation of the demand. This additional assumption implies that matrix D must have all its entries non negative. Assume, by contradiction, that a generic entry d_{pq} of D is negative and the strategy $u = Dw + \bar{u}$ is applied. Is this case, the component p of perturbation $u - \bar{u}$ is negative for any positive demand w such that $w_k > 0$ for $k = p$ and $w_k = 0$ otherwise.

If the above additional hypotheses hold, we can answer Problem 22 by determining whether the following linear programming problem has a feasible solution.

$$u^- \leq Dw^- + \bar{u} \leq u^+, \tag{22}$$
$$u^- \leq Dw^+ + \bar{u} \leq u^+, \tag{23}$$
$$D \geq 0. \tag{24}$$

In particular, note that imposing the feasibility of the control reactions to demands corresponding to the two vertices w^- and w^+ of $\mathcal{W}$ is sufficient to guarantee that the same strategy is feasible for the remaining $2^m - 2$ vertices and hence for all the demands in $\mathcal{W}$. Define as $\Delta w^{(p)} = \begin{cases} w_k^+ - w_k^-, & \text{if } p = k \\ 0, & \text{otherwise} \end{cases}$, for $p = 1, \ldots, n$. It is immediate to verify that $w^+ = w^- + \sum_{p=1}^{m} \Delta w^{(p)}$ and, in general, that each vertex $w^{(i)}$ of $\mathcal{W}$ can be expressed as $w^{(i)} = w^- + \sum_{p \in I} \Delta w^{(p)}$, where I is an appropriate subset of $\{1, \ldots, m\}$. As $\Delta w^{(p)} \geq 0$ and $D\Delta w^{(p)} \geq 0$, the following condition holds for all vertex $w^{(i)}$ of $\mathcal{W}$

$$u^- \leq Dw^- + \bar{u} \leq Dw^{(i)} + \bar{u} = D(w^- + \sum_{p \in I} \Delta w^{(p)}) + \bar{u}$$
$$\leq D(w^- + \sum_{p=1}^{m} \Delta w^{(p)}) + \bar{u} = Dw^+ + \bar{u} \leq u^+.$$

5.3 Solution of Problem 23

We reformulate Problem 23 as the following linear programming problem

$$\min_{u^-, u^+, \bar{u}, D} z = c^{+\prime} u^+ + c^{-\prime} u^- \tag{25}$$
$$B\bar{u} = 0 \tag{26}$$
$$BD = I \tag{27}$$
$$u^- \leq Dw^{(i)} + \bar{u} \leq u^+ \quad i = 1, \ldots, s \tag{28}$$
$$u^+ \geq 0, \; u^- \leq 0. \tag{29}$$

Then, we can use an algorithm similar to the one introduced in Subsection 5.2 to solve the above linear programming problem by iteratively generating constraints (28).

5.4 Solution of Problem 24

We reformulate Problem 24 as following

$$\max_{\alpha, \bar{u}, D} z = \alpha \tag{30}$$
$$B\bar{u} = 0 \tag{31}$$
$$BD = I \tag{32}$$
$$u^- \leq D\alpha w^{(i)} + \bar{u} \leq u^+ \quad i = 1, \ldots, s \tag{33}$$
$$\alpha \geq 0. \tag{34}$$

Although the above problem is not linear, it can be easily linearized by defining $\beta = \frac{1}{\alpha}$ and $\hat{u} = \frac{\bar{u}}{\alpha}$ to obtain

$$\min_{\beta, \bar{u}, D} z = \beta \tag{35}$$
$$B\hat{u} = 0 \tag{36}$$
$$BD = I \tag{37}$$
$$\beta u^- \leq Dw^{(i)} + \hat{u} \leq \beta u^+ \quad i = 1, \ldots, s \tag{38}$$
$$\beta \geq 0. \tag{39}$$

Now, we can use again an algorithm similar to the one introduced in Subsection 5.2 to solve the above linear programming problem by iteratively generating constraints (33).

The value of α, solution of problem (30) - (34), indicates to which extent we can expand the set $\mathcal{W}$ so that it is still contained in $B\mathcal{U}$ and a linear control strategy exists. More general formulations of Problem 24 with weaker constraints on the shape of $\mathcal{W}$ could be proposed, but in general they turn out to be non linearizable. A trivial exception occurs when no shape constraint is imposed. In this case, the largest set $\mathcal{W}$ is obviously $\mathcal{W} = B\mathcal{U}$.

Remark 25 *Through the paper, we have used the translation $\delta u = u - u_0$ and $\delta w = w - w_0$ so that $Av[\delta w] = 0$ so that $\delta u = Av[\delta w]$ is sought in the kernel of B. An interesting problem is to find δu so that the actual average flow $\delta u + u_0$ has the smallest component along the kernel of B, because in this way we minimize useless circulation in the system. By writing $\delta u + u_0 = Mp_1 + M^{\perp} p_2$ where M is a basis of $ker[B]$ and $M^{\perp}$ and orthogonal basis, we can minimize $\|Mp_1\|$ by solving a quadratic problem.*

Example 26 *Let us solve Problem 3 for the system depicted in Fig. 3 (B is then the incidence matrix of the*

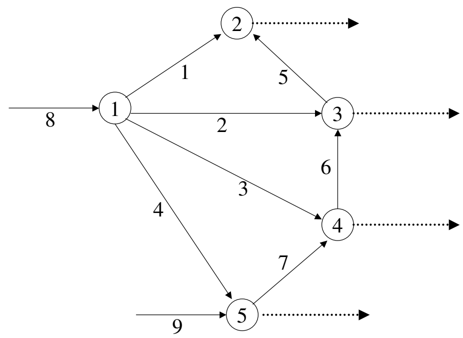

Fig. 3. Example of a system with 5 nodes and 9 arcs.

arcs	1	2	3	4	5	6	7	8	9
upper bounds	3	2	3	3	3	3	3	5	5

Table 1
Controlled flows constraints

nodes	1	2	3	4	5
upper bounds	0	2	3	2	2
averages	0	1	2	1	1

Table 2
Demand bounds

network). Table 1 summarizes the controlled upper flows constraints (the lower constraints are all set to 0*) whereas Table 2 the demand bounds and the long–term average demands. Now, given the nominal demand* $\bar{w} = [0\ 1\ 2\ 1\ 1]$ *and the nominal balancing flow* $\bar{u} = [1\ 1\ 1\ 0\ 0\ 1\ 1\ 3\ 2]' \in \mathcal{U}$ *(which is* $\bar{w} = B\bar{u}$*) we have to determine whether* $\bar{u}$ *is an achievable average flow, namely, it is such that if* $Av[w] = \bar{w}$ *then* $Av[u] = \bar{u}$*. If we translate the variables by setting* $\delta u \doteq u - \bar{u}$ *and* $\delta w = w - \bar{w}$*, according to the exposed theory this is equivalent to the existence of a static strategy which can be expressed as* $\delta u = D\delta w$*, where* D *is a matrix which satisfies conditions (8)(9). To determine such a matrix, we implement the algorithm proposed in Section 5.2. First, we give* $\bar{u}$ *as input and initialize the subset* $I = \{1\}$*. We solve the linear program defined by conditions (11) and (12) corresponding to the only vertex* $w^{(1)} = [0\ 0\ 3\ 0\ 0]'$ *and obtain a first matrix* $\hat{D}$*. Observe that the hypercube* $\mathcal{W}$ *has* 2^4 *vertices. Solving Problem 21 with the given* $\bar{u}$ *and* $\hat{D}$ *we obtain as violating vertex* $w^{(2)} = [0\ 0\ 0\ 0\ 0]'$*. We update* $I = \{1, 2\}$ *and solve the linear program with conditions (11) and (12) corresponding to the vertices* $w^{(1)}$ *and* $w^{(2)}$*. The procedure stops after* 6 *iterations returning as violating vertices*

$$w^{(1)} = [0\ 0\ 3\ 0\ 0]',\ w^{(2)} = [0\ 0\ 0\ 0\ 0]',\ w^{(3)} = [0\ 0\ 0\ 0\ 2]',$$
$$w^{(4)} = [0\ 2\ 0\ 0\ 0]',\ w^{(5)} = [0\ 0\ 0\ 2\ 0]',\ w^{(6)} = [0\ 2\ 3\ 2\ 0]'$$

and matrix $D = \hat{D}$ *defined as*

$$\hat{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ -0.1 & 0 & 0.5 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0 \\ 0.6 & 1 & 1 & 0 & 0 \\ 0.4 & 0 & 0 & 1 & 1 \end{bmatrix}. \qquad (40)$$

Basically, the columns of the above matrix establish that i) the demand at node 2 is satisfied by a flow through arc 8 and 1, ii) the demand at node 3 is satisfied by a flow through arc 8, which splits in two equal parts, the first one going through arc 2 and the second one through arc 3 and 6, iii) the demand at node 4 is entirely satisfied by a flow through arc 9 and 7, iv) finally the demand at node 5 is satisfied by a flow through arc 9. Obviously, the first column has no particular meaning since the demand at node 1 is null.

6 Conclusions and Discussion

The problem faced in this paper consists in satisfying a fluctuating demand while meeting long–term average specifications. We have provided necessary and sufficient conditions for this problem to be solvable via static strategies and we have seen that if this condition is met, then the static strategy is linear. We have then shown that the same necessary and sufficient conditions still hold when we consider a wide class of dynamic strategies. The proposed dynamic stabilizing control is achieved by introducing the auxiliary buffer variable $y(t)$ which has the precise meaning of keeping trace of the load unbalancing.

This fact is particularly interesting in the case in which the controlled arcs may have inactivity periods (for instance in the case of failures). For instance assume that $\|z(0)\| \le \epsilon$ with ϵ arbitrarily small. This last condition can always be assured. Also, assume that for a certain period $[0, t_{fail}]$ the control $u(t)$ is not the desired one due to some failure. Typically this situation can be faced by adopting "emergency strategies" (see, e.g., [4]) to keep the real buffer level $x(t)$ bounded. However $y(t)$ might diverge (this is the case if an arc from which a certain positive average flow is expected undergoes a failure) and then $z(t) = Dx(t) + Fy(t)$ might diverge as well. When the situation is restored (the arc repaired) at time t_{fail} the value $z(t_{fail}) = Dx(t_{fail}) + Fy(t_{fail})$ is of fundamental importance. Indeed the restored system is assured to reach the ϵ–ball again in finite time so that for some $T > t_{fail}$ the condition $\|z(T)\| \leq \epsilon$ is still met. By integrating over the period $[0, T]$ we derive

$$\frac{1}{T} \int_0^T [u(t) - Dw(t)]dt = \frac{z(T) - z(0)}{T}$$

Since ϵ can be made arbitrarily small, we can achieve the finite average relation $\bar{u}_T = D\bar{w}_T$ "approximately" in finite time.

Among the limitation of the paper, we stress that we have not considered buffer constraints. Actually, these can be easily taken into account. For instance one may assume

$$x^- \leq x(t) \leq x^+, \quad y^- \leq y(t) \leq y^+,$$

and use a Lyapunov approach as proposed in [4] for the extended system with these constraints. Note that by assigning the new bounds y^- and y^+ we may limit the "mismatch variable" $y(t)$.

Further developments of this work include the investigation of special categories of systems, for instance those in which B is an incidence matrix for which stronger results could be found. Furthermore, here have considered the average in a deterministic sense. Facing the problem assuming a stochastic demand characterization is certainly of interest. We have seen that, in general, it is not possible to keep the buffers bounded while meeting the average in a worst–case setting. Now, our main question is whether in a stochastic framework the situation is different, namely we can meet the average while assuring stochastic stability. So far, we have only conjectures but not sound answers.

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