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Consensus for switched networks with unknown but bounded disturbances

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Abstract

We consider stationary consensus protocols for networks of dynamic agents with switching topologies. The measure of the neighbors' state is affected by Unknown But Bounded disturbances. Here the main contribution is the formulation and solution of what we call the ϵ -consensus problem, where the states are required to converge in a tube of ray ϵ asymptotically or in finite time.

1 Introduction.

Consensus protocols are distributed control policies based on neighbors' state feedback that allow the coordination of multi-agent systems. According to the usual meaning of consensus, the system state must converge to an equilibrium point with all equal components in finite time or asymptotically [1, 4, 6, 9, 10, 11, 12, 13, 15].

The novelty of our approach is in the presence of Unknown But Bounded (UBB) disturbances [2] in the neighbors' state feedback. Actually, despite the literature on consensus is now becoming extensive, only few approaches have considered a disturbance affecting the measurements. In our approach we have assumed an UBB noise, because it requires the least amount of a-priori knowledge on the disturbance. Only the knowledge of a bound on the realization is assumed, and no statistical properties need to be satisfied. Moreover, we recall that starting from [2], the UBB framework has been used in many different fields and applications, such as, mobile robotics, vision, multi-inventory, data-fusion and UAV's and in estimation, filtering, identification and robust control theory.

Because of the presence of UBB disturbances convergence to equilibria with all equal components is, in general, not possible. The main contribution is then the introduction and solution of the ϵ -consensus problem, where the states converge in a tube of ray ϵ asymptotically or in finite time. In solving the ϵ -consensus problem we focus on linear protocols and present a rule for estimating the average from a compact set of candidate points, say it lazy rule, such that the optimal estimate for the i th agent is the one which minimizes the distance from x_i .

The system under consideration consists of n dynamic agents that reach consensus on a group decision value by implementing *distributed* and *stationary control policies* based on disturbed neighbors' state feedback. Neighborhood

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relations are defined by the existence of communication links between nearby agents. Here, we assume that the set of communication links are bidirectional and define a time-varying connected communication network.

The presentation of the results is organized as follows. We first solve the ϵ -consensus asymptotically and in finite time for networks with fixed topology (we look at them as switched systems with dwell time of length infinite). To be more precise, for a given protocol, we find a tube of minimum radius that the agents reach asymptotically. Trivially, any tube of radius strictly greater than the minimum one can be reached in finite time. We do this by introducing polyhedra of equilibrium points and studying their stability. The above result means that, in general, the value of ϵ cannot be chosen arbitrarily small. We point out its relation with the amplitude ξ of the disturbances. We also consider additional assumptions on the disturbance realization, beyond its inclusion in D , and show that different type of disturbances lead to different values for the minimum radius. For certain disturbance realizations the agents are shown to asymptotically reach 0-consensus. The last part of this paper, extends the above results to the case of switching topology. For a given dwell time, we find a tube that can be reached in finite time. Higher dwell times imply the convergence to tubes of lower radius.

The paper is organized as follows. In Section 2, we set up the new framework of switched networks under UBB disturbances and formulate the ϵ -consensus problem (Problem 1). In Section 3 we introduce the linear protocol and the lazy rule. In Section 4 we study networks with fixed topology. In Section 5, we extend the obtained results to networks with switching topology. Finally, in Section 6, we draw some conclusions.

2 Switched networks.

Consider a system of n dynamic agents $\Gamma = \{1, \dots, n\}$ and let \mathcal{E} be a finite set of possible edgesets connecting the agents in Γ . We model the interaction topology among agents through a network (graph) $G_{\sigma(t)} = (\Gamma, E_{\sigma(t)})$, with time variant edgeset $E_{\sigma(t)} \in \mathcal{E}$, where $\sigma(t)$ is a *switching function* $\sigma : \mathbb{R}^+ \rightarrow \mathcal{I}$ and \mathcal{I} is the index set associated with the elements of \mathcal{E} . Also, let us call *switching time* a time t such that $\sigma(t^-) \neq \sigma(t^+)$ and let us call *switching interval* the time interval between two consecutive switching times. In the rest of this section, to avoid pathological behaviors arising when the switching times have a finite accumulation point (see, e.g., the *zeno behavior* in [7]) and in accordance with [6, 14], we make the following assumption (see, e.g., the notion of *dwell time* in [6, 5]).

Assumption 1 *The switching intervals have a finite minimum length $\tau > 0$.*

Henceforth τ is referred to as the *dwell time*. We also assume that the edgesets in \mathcal{E} induce undirected connected not complete graphs on Γ . For each $k \in \mathcal{I}$, the network $G_k = (\Gamma, E_k)$ is undirected if $(i, j) \in E_k$ then $(j, i) \in E_k$. The network G_k is connected if for any agent $i \in \Gamma$ there exists a path, i.e., a sequence of edges in E_k , $(i, i_1)(i_1, i_2) \dots (i_r, j)$, that connects it with any other agent $j \in \Gamma$. Finally, the network G_k is not complete if each agent i is connected (with one edge) only to a subset of other vertices $N_{ik} = \{j : (i, j) \in E_k\}$ called *neighborhood of i* .

Each edge (i, j) in the edgeset E_k means that there is communication from j to i . As (j, i) is also in the edgeset E_k the communication is bidirectional, namely, if agent i can receive information from agent j then also agent j can receive information from agent i . Also, G_k not complete means that each agent i exchanges information only with its neighbors. Here and in the following, $\mathbf{1}$ stands for the vector $(1, 1, \dots, 1)^T$.

2.1 Unknown But Bounded disturbances.

Let \mathcal{T} be the set of switching times. For all $i \in \Gamma$, consider the family of *first-order* dynamical systems controlled by a *distributed* and *stationary* control policy

$$\begin{aligned}\dot{x}_i &= u_{i\sigma(t)}(x_i, y^{(i)}) \quad \forall t \geq 0, t \notin \mathcal{T} \\ x_i(t^+) &= x_i(t^-) \quad \forall t \in \mathcal{T}\end{aligned}\tag{1}$$

where $y^{(i)}$ is the information vector from the agents in $N_{i\sigma(t)}$ with generic component j defined as follows,

$$y_j^{(i)} = \begin{cases} y_{ij} & \text{if } j \in N_{i\sigma(t)}, \\ 0 & \text{otherwise.} \end{cases}$$

In the above equation, y_{ij} is a disturbed measure of x_j obtained by agent i as

$$y_{ij} = x_j + d_{ij}$$

and d_{ij} is an UBB disturbance, i.e., $-\xi \leq d_{ij} \leq \xi$ with a-priori known $\xi > 0$. Hereafter, we denote by $d = \{d_{ij}, (i, j) \in \Gamma^2\}$ the disturbance vector and by D the hypercube $D = \{d : -\xi \leq d_{ij} \leq \xi, \forall (i, j) \in \Gamma^2\}$ of the possible disturbance vectors. We assume that any disturbance realization $\{d(t) \in D, t \geq 0\}$ is continuous over time. Note that both d and D are independent of the topology of network G (which may change over time) as they are defined on all the possible pairs of agents in Γ and not only on the links between them. The continuity hypothesis on the disturbance realizations can be weakened and most of our results keep holding true. However, we hold the continuity assumption to make the proofs of our results simpler and more readable.

2.2 Problem formulation.

Before stating the problem we need to introduce the notions of equilibrium point for a given disturbance realization $d(t)$, and of ϵ -consensus.

Definition 1 A point x^* is an equilibrium point for a given disturbance realization $\{d(t) \in D, t \geq 0\}$ if there exists $\bar{t} \geq 0$ such that $u_{\sigma(t)}(x_i^*, y^{(i)}) = 0$, for all $i \in \Gamma$, for all $t \geq \bar{t}$.

According to the usual meaning of consensus, the system state must converge to an equilibrium point $x^* \in \{\pi \mathbf{1}\}$ in finite time or asymptotically. Hereafter, when we refer to points of type $\pi \mathbf{1}$, we always understand that π may assume any value in \mathbb{R} and we denote by $\{\pi \mathbf{1}\}$ the set $\{x : \exists \pi \in \mathbb{R} \text{ s.t. } \pi \mathbf{1}\}$.

Because of the presence of UBB disturbances convergence to $\{\pi \mathbf{1}\}$ is, in general, not possible. This motivates the following definition of ϵ -consensus, describing the cases where the system state is driven in finite time within a bounded tube of radius ϵ ,

$$T = \{x \in \mathbb{R}^n : |x_i - x_j| \leq 2\epsilon, \forall i, j \in \Gamma\}.\tag{2}$$

Definition 2 We say that a protocol $u_{\sigma(t)}(\cdot)$ makes the agents reach ϵ -consensus in finite time if there exists a finite time $\bar{t} > 0$ such that the system state $x(t) \in T$ for all $t \geq \bar{t}$. Furthermore, we say that a protocol $u_{\sigma(t)}(\cdot)$ makes the agents reach ϵ -consensus asymptotically, if the system state $x(t) \rightarrow T$ for $t \rightarrow \infty$.

The above definition for $\epsilon = 0$ (say it 0-consensus) coincides with the usual definition of (asymptotical) consensus. However, for a generic $\epsilon > 0$, the ϵ -consensus in finite time does not necessarily implies the convergence of the state x to an equilibrium $x^* \in T$. In other words, x can be driven to T and keep on oscillating within it for the rest of the time.

Problem 1 (ϵ -consensus problem) *Given the switched system (1), determine a (distributed stationary) protocol $u_{\sigma(t)}(\cdot)$ that makes the agents reach ϵ -consensus in finite time or asymptotically for any initial state $x(0)$. Furthermore, study the dependence of the tube radius ϵ on the sets \mathcal{E} and D and on the dwell time τ .*

In the rest of this paper we focus on linear protocols, and present a rule for estimating the average from a compact set of candidate points, say it lazy rule, such that the optimal estimate for the i th agent is the one which minimizes the distance from x_i .

3 Linear protocols and *lazy* rule.

A typical consensus problem is the average consensus one, i.e., the system state converges to the average of the initial state. Its success derives from the fact that, in absence of disturbances, it can be simply solved by linear protocols.

Let the linear protocol be given as

$$u_{i\sigma(t)}(x_i, y^{(i)}) = \sum_{j \in N_{i\sigma(t)}} (\tilde{y}_{ij} - x_i), \quad \text{for all } i \in \Gamma \quad (3)$$

where \tilde{y}_{ij} is the estimate of state x_j on the part of agent i . For a given disturbed measure y_{ij} the state x_j and consequently its estimate \tilde{y}_{ij} must belong to the interval

$$\tilde{y}_{ij} \in [y_{ij} - \xi, y_{ij} + \xi]. \quad (4)$$

The crucial point is how to select \tilde{y}_{ij} from the interval $[y_j - \xi, y_j + \xi]$. The next example shows that there may not exist equilibria if we choose simply $\tilde{y}_{ij} = y_{ij}$.

Example 1 *A three-agent network with a fixed edgeset E_k , $N_{1k} = \{1, 2\}$, $N_{2k} = \{1, 2, 3\}$ and $N_{3k} = \{2, 3\}$. A simple criterion is to let $\tilde{y}_{ij} = y_{ij} = x_i + d_{ij}$*

$$\begin{aligned} \dot{x}_1 &= (x_2 + d_{12}) - x_1 \\ \dot{x}_2 &= [(x_1 + d_{21}) - x_2] + [(x_3 + d_{23}) - x_2] \\ \dot{x}_3 &= (x_2 + d_{32}) - x_3 \end{aligned}$$

Find equilibria by imposing $\dot{x} = 0$ and obtain

$$x_1 = x_2 + d_{12} \quad (5)$$

$$0 = d_{12} + d_{21} + d_{23} + d_{32} \quad (6)$$

$$x_3 = x_2 + d_{32} \quad (7)$$

There exist equilibria only if $d_{12} + d_{21} + d_{23} + d_{32} = 0$, that is, for generic values of d_{12} , d_{21} , d_{23} , and d_{32} we cannot guarantee the convergence of the system.

Let $\tilde{y}^{(i)} = \{\tilde{y}_{ij}, i \in N_i\}$ be defined according to the *lazy* rule

$$\tilde{y}^{(i)} = \arg \min_{\tilde{y}_{ij} \in [y_{ij} - \xi, y_{ij} + \xi], j \in N_{i\sigma(t)}} \left| \sum_{j \in N_{i\sigma(t)}} (\tilde{y}_{ij} - x_i) \right|. \quad (8)$$

Note that as $u(\cdot, \cdot)$ in protocol (3) depends on $\sum_{j \in N_{i\sigma(t)}} \tilde{y}_{ij}$, the existence of multiple solutions $\tilde{y}^{(i)}$ for (8) is not an issue. This is clearer if one observes that multiple solutions induce the same value $\sum_{j \in N_{i\sigma(t)}} \tilde{y}_{ij}$ for $u(\cdot, \cdot)$

in protocol (3). Given the lazy rule (8), protocol (3) turns out to have a feedback structure as, for each $i \in \Gamma$, the quantity $\sum_{j \in N_{i\sigma(t)}} \tilde{y}_{ij}$ can be computed as

$$\sum_{j \in N_{i\sigma(t)}} \tilde{y}_{ij} = \begin{cases} \sum_{j \in N_{i\sigma(t)}} y_{ij} + |N_{i\sigma(t)}| \xi & \text{if } x_i > \frac{\sum_{j \in N_{i\sigma(t)}} y_{ij}}{|N_{i\sigma(t)}|} + \xi \\ |N_{i\sigma(t)}| x_i & \text{if } \frac{\sum_{j \in N_{i\sigma(t)}} y_{ij}}{|N_{i\sigma(t)}|} - \xi \leq x_i \leq \frac{\sum_{j \in N_{i\sigma(t)}} y_{ij}}{|N_{i\sigma(t)}|} + \xi \\ \sum_{j \in N_{i\sigma(t)}} y_{ij} - |N_{i\sigma(t)}| \xi & \text{if } x_i < \frac{\sum_{j \in N_{i\sigma(t)}} y_{ij}}{|N_{i\sigma(t)}|} - \xi \end{cases} . \quad (9)$$

Hereafter, when we refer to the linear protocol (3), we always understand that the agents choose $\tilde{y}^{(i)}$ as in (8).

4 Fixed topology.

In this section we consider a network with fixed topology, i.e., a network $G = (\Gamma, E)$, with edgeset E constant over time. As the edgeset E remains constant, for the easy of notation, we drop the index $\sigma(t)$ from all the notation used throughout this section. Also when we refer to system (1) and to a protocol (3) we always mean that they are associated to the network G .

4.1 Equilibrium points.

For a network with fixed topology, we prove that the equilibrium points exist and belong to polyhedra depending on the type of disturbance realization. In particular, we state a first result in the case of constant disturbance d , and extend such a result to the case where the disturbance d takes on values in specific subsets of D .

Lemma 1 *Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). If the disturbance d is constant over time, then:*

(i) *a point x is an equilibrium point for $u(\cdot)$ if and only if it belongs to the polyhedron*

$$P(d, E) = \left\{ x : -\frac{\sum_{j \in N_i} d_{ij}}{|N_i|} - \xi \leq \frac{\sum_{j \in N_i} x_j}{|N_i|} - x_i \leq -\frac{\sum_{j \in N_i} d_{ij}}{|N_i|} + \xi, \forall i \in \Gamma \right\}; \quad (10)$$

(ii) *$P(d, E)$ includes all the points in $\{\pi \mathbf{1}\}$; in addition, $\{\pi \mathbf{1}\} = \bigcap_{d \in D} P(d, E)$.*

(iii) *$P(d, E)$ has $\mathbf{1}$ as only extreme ray up to multiplication by a non-zero scalar.*

Proof. (i) A point x is an equilibrium point if and only if $u_i(x_i, y^{(i)}) = 0$ for all $i \in \Gamma$. This condition is equivalent to $\min_{\tilde{y}_{ij} \in [y_{ij} - \xi, y_{ij} + \xi], j \in N_i} |\sum_{j \in N_i} (\tilde{y}_{ij} - x_i)| = 0$ that, as $y_{ij} = x_j + d_{ij}$, in turn becomes

$$\sum_{j \in N_i} (\tilde{y}_{ij} - x_i) = 0, \quad \forall i \in \Gamma \quad (11)$$

$$x_j + d_{ij} - \xi \leq \tilde{y}_{ij} \leq x_j + d_{ij} + \xi, \quad \forall i \in \Gamma, \forall j \in N_i. \quad (12)$$

The polyhedron $P(d, E)$ is the projection of the solutions $(x_i, \tilde{y}^{(i)})$, for $i \in \Gamma$, of system (11)-(12) in the space of the x variables.

(ii) For any $x \in \{\pi \mathbf{1}\}$ it holds that $\frac{\sum_{j \in N_i} x_j}{|N_i|} - x_i = 0$. Also, $-\frac{\sum_{j \in N_i} d_{ij}}{|N_i|} - \xi \leq 0 \leq -\frac{\sum_{j \in N_i} d_{ij}}{|N_i|} + \xi$ because $-\xi \leq d_{ij} \leq \xi$ for any $i \in \Gamma, j \in N_i$. Then, $\{\pi \mathbf{1}\} \subseteq P(d, E)$. To prove that $\{\pi \mathbf{1}\} = \bigcap_{d \in D} P(d, E)$ we show that $P(\xi, E) = \{\pi \mathbf{1}\}$. To see this last argument, from (10) with $d_{ij} = \xi$ for all i and j we have that $\frac{\sum_{j \in N_i} x_j}{|N_i|} - x_i \leq 0$

for all $i \in \Gamma$ and for any $x \in P(\xi, E)$. The latter means that the state x_i of each agent i must not be less than the average state of its neighbors in N_i and this situation occurs only if all the agents have the same state.

(iii) The vector $\mathbf{1}$ is an extreme ray as it is immediate to verify that if $x \in P(d, E)$ then $x + \pi \mathbf{1} \in P(d, E)$ for any $\pi \in \mathbb{R}$. To prove that a vector $\mathbf{1}$ is the unique extreme ray, up to multiplication by a non-zero scalar, consider a vector v not parallel to $\mathbf{1}$. We note that $0 \in P(d, E)$ and we prove that for some $\pi \in \mathbb{R}$ the point $0 + \pi v \notin P(d, E)$. As $-\frac{\sum_{j \in N_i} d_{ij}}{|N_i|} - \xi$ and $-\frac{\sum_{j \in N_i} d_{ij}}{|N_i|} + \xi$ are fixed values, we have that $\pi v \in P(d, E)$ for any $\pi \in \mathbb{R}$ if and only if $v_i - \frac{\sum_{j \in N_i} v_j}{|N_i|} = 0$ for all $i \in \Gamma$. The latter conditions define a linear system with $n - 1$ independent conditions (provided that G is connected) and the solutions are of type $v = \eta \mathbf{1}$ for $\eta \in \mathbb{R}$ contradicting the hypothesis that v is not parallel to $\mathbf{1}$. \square

In the proof of the previous theorem, we have observed that $-\frac{\sum_{j \in N_i} d_{ij}}{|N_i|} - \xi \leq 0 \leq -\frac{\sum_{j \in N_i} d_{ij}}{|N_i|} + \xi$ for all $i \in \Gamma$. When such inequalities hold strictly, $P(d, E)$ is a full-dimensional polyhedron. Actually, any x of type $(0, \dots, 0, \delta, 0, \dots, 0)$ belongs to $P(d, E)$ if we choose $\delta > 0$ sufficiently small. However, not all the polyhedra $P(d, E)$ are full-dimensional as it is apparent by reminding that $P(\xi, E) = \{\pi \mathbf{1}\}$.

In the following we generalize the results of Lemma 1 to the case in which the disturbance is not constant over time. In other words, we are concerned with the study of the equilibrium points for generic disturbance realizations $\{d(t) \in D, t \geq 0\}$. First, we can say that only the points in $\{\pi \mathbf{1}\}$ are equilibrium points for all the possible disturbance realizations $\{d(t) \in D, t \geq 0\}$. To see this, observe that i) they are the only equilibrium points if $d(t) = \xi$ for all t and ii) condition (8) implies $u(x_i, y^{(i)}) = 0$, for all $i \in \Gamma$ for any realization $\{d(t) \in D, t \geq 0\}$.

We will show in the next lemma that, under certain assumptions, all equilibrium points belong to $P(Q, E) = \bigcup_{d \in Q} P(d, E)$, for any subset $Q \subseteq D$. Before introducing the lemma, consider, without loss of generality, the box $Q = \{d \in D : d^- \leq d \leq d^+\} \subseteq D$ where d^- and d^+ are in D and $d^- \leq d^+$ componentwise. Then it holds

$$P(Q, E) = \left\{ x \in \mathbb{R}^n : -\frac{\sum_{j \in N_i} d_{ij}^+}{|N_i|} - \xi \leq \frac{\sum_{j \in N_i} x_j}{|N_i|} - x_i \leq -\frac{\sum_{j \in N_i} d_{ij}^-}{|N_i|} + \xi, \forall i \in \Gamma \right\}. \quad (13)$$

To prove (13), denote by Ξ the set on the rhs of (13) and note that it holds either $\Xi \supseteq \bigcup_{d \in Q} P(d, E)$ and $\Xi \subseteq \bigcup_{d \in Q} P(d, E)$. Actually, for any $d \in Q$, it holds $d^- \leq d \leq d^+$ then $P(d, E) \subset \Xi$, hence $\Xi \supseteq \bigcup_{d \in Q} P(d, E)$. Also, to prove $\Xi \subseteq \bigcup_{d \in Q} P(d, E)$, consider a generic point $\hat{x} \in \Xi$. It belongs to $P(\hat{d}, E)$, where for any $i \in \Gamma$ we set

$$\hat{d}_{ij} = \begin{cases} d_{ij}^- & \text{if } j \in N_i, \frac{\sum_{j \in N_i} \hat{x}_j}{|N_i|} - \hat{x}_i \geq 0 \\ d_{ij}^+ & \text{if } j \in N_i, \frac{\sum_{j \in N_i} \hat{x}_j}{|N_i|} - \hat{x}_i < 0 \\ d_{ij}^+ & \text{otherwise} \end{cases}. \quad (14)$$

As $\hat{d} \in Q$ by construction, we have $P(\hat{d}, E) \subseteq \bigcup_{d \in Q} P(d, E)$ which implies $\Xi \subseteq \bigcup_{d \in Q} P(d, E)$. Then, we can conclude that (13) holds true.

In particular, it holds

$$P(D, E) = \left\{ -2\xi \leq \frac{\sum_{j \in N_i} x_j}{|N_i|} - x_i \leq 2\xi, \forall i \in \Gamma \right\}. \quad (15)$$

Let us define, for a given realization $d(t)$ and a subset Q of D , the value

$$\mu(Q, t_1, t_2) = \max\{\Delta : t_1 \leq \tilde{t} \leq \tilde{t} + \Delta \leq t_2 \text{ s.t. } d(t) \in Q \text{ for all } \tilde{t} \leq t \leq \tilde{t} + \Delta\}. \quad (16)$$

In other words, given a time interval $[t_1, t_2]$, the value $\mu(Q, t_1, t_2)$ is the length of the longest subinterval where $d(t)$ remains in Q .

Lemma 2 *Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). Consider a disturbance realization $\{d(t) \in D, t \geq 0\}$ and box $Q = \{d \in D : d^- \leq d \leq d^+\} \subseteq D$. Assume that there exist two nonnegative finite numbers M and δ , such that $\mu(Q, t, t + M) > \delta$, for all $t \geq 0$. Then, equilibrium points x exist and belong to $P(Q, E)$.*

Proof. We first observe that the points $\{\pi\mathbf{1}\}$ are equilibrium points for a given disturbance realization $d(t)$ and also that they belong to $P(Q, E)$. We then prove by contradiction that $x \notin P(Q, E)$ cannot be an equilibrium point. If $x \notin P(Q, E)$, at least for one of its component, say it i , it holds that either $\frac{\sum_{j \in N_i} x_j}{|N_i|} - x_i < -\frac{\sum_{j \in N_i} d_{ij}^+}{|N_i|} - \xi$ or $\frac{\sum_{j \in N_i} x_j}{|N_i|} - x_i > -\frac{\sum_{j \in N_i} d_{ij}^-}{|N_i|} + \xi$. The previous conditions imply that the value of $u(x_i, y(i))$ is either strictly less than zero or strictly greater than zero for all $d \in Q$. Then, for any $\bar{t} \geq 0$, there exists a time interval of length greater than or equal to δ such that $u(x, y^{(i)})$ is always either strictly greater than 0 or less than 0. Hence x is not an equilibrium point. \square

An immediate consequence of the above lemma is the following corollary.

Corollary 1 *Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). Consider a disturbance realization $\{d(t) \in D, t \geq 0\}$ and a finite set $\mathcal{Q} = \{Q_1, Q_2, \dots\}$ of boxes of D . Assume that there exist two nonnegative finite numbers M and δ , such that $\mu(Q_r, t, t + M) > \delta$, for all $Q_r \in \mathcal{Q}$, for all $t \geq 0$. Then, equilibrium points x for $u(\cdot)$ exist and belong to $\bigcap_{Q_r \in \mathcal{Q}} P(Q_r, E)$.*

In addition, Corollary 1 gives us a hope that if the disturbance realization enjoys some general properties the system can reach an equilibrium point close to the set $\{\pi\mathbf{1}\}$. As an example, consider a disturbance realization in Corollary 1 characterized, at least, by $\mathcal{Q} = \{Q_1, Q_2\}$, with $Q_1 = \{d \in D : -\xi \leq d \leq -\hat{d}\}$ and $Q_2 = \{d \in D : \hat{d} \leq d \leq \xi\}$, with $0 < \hat{d} \leq \xi$, we obtain that the only equilibrium points x are in

$$P(Q_1, E) \cap P(Q_2, E) = \left\{ \frac{\sum_{j \in N_i} \hat{d}_{ij}}{|N_i|} - \xi \leq \frac{\sum_{j \in N_i} x_j}{|N_i|} - x_i \leq -\frac{\sum_{j \in N_i} \hat{d}_{ij}}{|N_i|} + \xi, \forall i \in \Gamma \right\}. \quad (17)$$

The above set obviously defines a neighborhood of the set $\{\pi\mathbf{1}\}$, as $\{\pi\mathbf{1}\} \subseteq P(Q, E)$ for any possible subset Q of D . Interesting is that the radius of the neighborhood becomes smaller and smaller as $\hat{d} \rightarrow \xi$ and that $P(Q_1, E) \cap P(Q_2, E) = \{\pi\mathbf{1}\}$ if $\hat{d} = \xi$. The same results hold, for all the situations in which we can guarantee the disturbance realizations characterized by $\mathcal{Q} = \{Q_1, Q_2\}$, such that $P(Q_1, E) \cap P(Q_2, E)$ is equal to a neighborhood of $\{\pi\mathbf{1}\}$ with a small radius.

The following corollary asserts that $P(D, E)$ is the minimal set including all the possible equilibrium points for a policy $u(\cdot)$ given an unknown but bounded disturbance in D .

Corollary 2 *Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). If the disturbance is unknown but bounded in D , then*

- (i) *given any $x \in P(D, E)$ then there exists a disturbance realization $\{d(t) \in D, t \geq 0\}$ that has x as an equilibrium point;*
- (ii) *given any disturbance realization $\{d(t) \in D, t \geq 0\}$ then all its equilibrium points belong to $P(D, E)$.*

Proof. (i) Any generic point $\hat{x} \in P(D, E)$ is trivially an equilibrium point for the corresponding realization $d(t) = \hat{d}$, $t \geq 0$, where \hat{d} is defined as in (14).

(ii) Any disturbance realization $\{d(t) \in D : t \leq 0\}$ has, for any finite $M > 0$, $\mu(D, t, t + M) = M$ then all its equilibrium points for $u(\cdot)$ belong to $P(D, E)$ by Lemma 2. □

The following example shows that the value of the parameter ϵ defining tube T in (31) cannot be chosen arbitrarily small.

Example 2 Consider the network $G = (\Gamma, E)$ with $\Gamma = \{1, \dots, n\}$ and $E = \{(i, i + 1) : i = 1, \dots, n - 1\}$. Let $x_{i+1}(t) = x_i(t) + \xi$ for any arbitrary value of $x_1(t)$. This point is an equilibrium as long as $d_{ij}(t) = 0$ for all $i \in \Gamma$ and $t \geq 0$. In this situation, the value ϵ defining T in (31) is equal to $\frac{n-1}{2}\xi$.

Corollary 2 suggests a way to determine a strict upper bound $\bar{\epsilon}$ for ϵ . We have

$$\bar{\epsilon} = \max_{i, j \in \Gamma} \max_{x \in P(D, E)} \{x_i - x_j\}, \quad (18)$$

whose brute force computation requires the solution of $n(n-1)$ linear programming problems of type $\max_{x \in P(D, E)} \{x_i - x_j\}$. Then, the computation of $\bar{\epsilon}$ becomes polynomial.

4.2 Stability.

In this subsection we prove the asymptotic stability of the equilibrium points. To this end, we have to introduce a basic property of the stationary protocol $u(\cdot)$ whose components have the feedback form (3). In the following, we denote by $\text{sign} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ the function that returns 1 if its argument is positive, -1 if its argument is negative, 0 if its argument is null.

Lemma 3 Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). Either $\text{sign}(u_i(x_i, y^{(i)})) = \text{sign}(\sum_{j \in N_i} (x_j - x_i))$ or $\text{sign}(u_i(x_i, y^{(i)})) = 0$, for each $i \in \Gamma$, for each $t \geq 0$.

Proof. For each $i \in \Gamma$, for each $t \geq 0$, given the protocol $u_i(x_i, y^{(i)}) = \sum_{j \in N_i} (\tilde{y}_{ij} - x_i)$, consider the solution of the linear problem that defines the value of $\tilde{y}^{(i)}$

$$z_i = \min_{\tilde{y}_{ij} \in [y_{ij} - \xi, y_{ij} + \xi], j \in N_i} \left| \sum_{j \in N_i} (\tilde{y}_{ij} - x_i) \right|. \quad (19)$$

If $z_i = 0$ the lemma is proved. If $z_i > 0$ two situations can occur, the value of $\sum_{j \in N_i} (\tilde{y}_{ij} - x_i)$ is either strictly positive or strictly negative, for any $\tilde{y}_{ij} \in [y_{ij} - \xi, y_{ij} + \xi]$, $j \in N_i$. We claim that if $\sum_{j \in N_i} (\tilde{y}_{ij} - x_i) > 0$ then $u_i(x_i, y^{(i)}) > 0$ and $\sum_{j \in N_i} (x_j - x_i) > 0$. If $\sum_{j \in N_i} (\tilde{y}_{ij} - x_i) > 0$ for any $\tilde{y}_{ij} \in [y_{ij} - \xi, y_{ij} + \xi]$, $j \in N_i$ then, by definition, $u_i(x_i, y^{(i)}) > 0$, as the chosen \tilde{y}_{ij} must belong to $[y_{ij} - \xi, y_{ij} + \xi]$. In addition, we have $\sum_{j \in N_i} (x_j + d_{ij} - \xi - x_i) > 0$, hence $\sum_{j \in N_i} (x_j - x_i) > \sum_{j \in N_i} (\xi - d_{ij}) \geq 0$, as $-\xi \leq d_{ij} \leq \xi$.

A symmetric argument holds if $\sum_{j \in N_i} (\tilde{y}_{ij} - x_i) < 0$, for any $\tilde{y}_{ij} \in [y_{ij} - \xi, y_{ij} + \xi]$, $j \in N_i$. □

Theorem 1 Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). Then, the system trajectory converges to equilibrium points in $P(D, E)$.

Proof. We prove the convergence to equilibrium points in $P(D, E)$ by introducing a candidate Lyapunov function $V(x) = \frac{1}{2} \sum_{(i,j) \in E} (x_j - x_i)^2$. Trivially, $V(x) = 0$ if and only if $x \in \{\pi \mathbf{1}\}$; $V(x) > 0$ for all $x \notin \{\pi \mathbf{1}\}$. We now prove that $\dot{V}(x) < 0$ for all $x \notin P(D, E)$. On this purpose, for $\dot{V}(x)$ we can write

$$\begin{aligned} \dot{V}(x) &= \sum_{(i,j) \in E} (x_j - x_i)(u_j - u_i) = - \sum_{i \in \Gamma} u_i \sum_{j \in N_i} (x_j - x_i) = \\ &= - \sum_{i \in \Gamma} \text{sign}(u_i) \text{sign} \left(\sum_{j \in N_i} (x_j - x_i) \right) |u_i| \left| \sum_{j \in N_i} (x_j - x_i) \right| \end{aligned} \quad (20)$$

From Lemma 3, if $\sum_{j \in N_i} (x_j - x_i) = 0$ then $u_i(x, y^{(i)}) = 0$. This in turns implies that $\dot{V}(x)$ is null if and only if $u_k(x) = 0$. The latter observation is sufficient to prove that i) the state trajectory converges to $P(D, E)$ and that ii) the convergence is to an equilibrium point. Indeed, for $t \rightarrow \infty$, we have $\dot{V} \rightarrow 0$. Then, $u \rightarrow 0$ and consequently $\dot{x} \rightarrow 0$. □

Theorem 2 *Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). Consider a disturbance realization $\{d(t) \in D, t \geq 0\}$ and a box $Q = \{d \in D : d^- \leq d \leq d^+\} \subseteq D$. Assume that there exist two nonnegative finite numbers M and δ , such that $\mu(Q, t, t + M) > \delta$, for all $t \geq 0$. Then, the system trajectory converges to equilibrium points in $P(Q, E)$.*

Proof. We prove the convergence to equilibrium points in $P(Q, E)$ following the same argument used in the proof of Theorem 1. We only note that now we have more information on the disturbance. In particular we know that, for every time interval of length M , it spends at least a time δ assuming values in Q . The explicit dependence of the disturbance on time makes the Lyapunov function time-varying.

To prove the system stability we make use of the results in [8]. We define the function $p : \mathbb{R} \rightarrow [0, \infty)$

$$p(t) = \begin{cases} 1 & \text{if } d(t) \in Q \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

It is immediate to verify that $p(\cdot)$ satisfies the conditions in Remark 3 in [8], in particular, there exists three finite values $\bar{p}, M, \delta > 0$ such that $0 \leq p(t) \leq \bar{p}$, $\int_t^{t+M} p(s) ds \geq \delta$ for all $t \geq 0$. We also define the function $W : \mathbb{R}^n \rightarrow [0, \infty)$

$$W(x) = \begin{cases} 0 & \text{if } x \in P(Q, E) \\ \min_{\tilde{y}^{(i)} \in [x+d-\xi, x+d+\xi], d \in Q} \left\{ \left| \sum_{j \in N_i} (\tilde{y}_{ij} - x_i) \right| \left| \sum_{j \in N_i} (x_j - x_i) \right| \right\} & \text{if } x \notin P(Q, E) \end{cases} \quad (22)$$

Observe that $W(x) = 0$ for $x \in P(Q, E)$, whereas $0 < W(x) < |u_i| \left| \sum_{j \in N_i} (x_j - x_i) \right|$ for all $x \notin P(Q, E)$ and for all $t \geq 0$. Hence $\dot{V}(x) \leq -p(t)W(x) \leq 0$ for all x and all $t \geq 0$ and, in particular, $\dot{V}(x) = 0$ for all $t \geq 0$, only for $x \in P(Q, E)$. The system trajectory converges to $P(Q, E)$. Finally, we note that the system trajectory converges to an equilibrium point as $\dot{V}(x)$ is null if and only if $u_k(x) = 0$. □

An immediate consequence of the above theorem is the following corollary.

Corollary 3 *Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). Consider a disturbance realization $\{d(t) \in D, t \geq 0\}$ and a finite set $\mathcal{Q} = \{Q_1, Q_2, \dots\}$ of boxes of D . Assume that there exist two nonnegative finite numbers M and δ , such that $\mu(Q_r, t, t + M) > \delta$, for all $Q_r \in \mathcal{Q}$, for all $t \geq 0$. Then, the system trajectory converges to $\bigcap_{Q_r \in \mathcal{Q}} P(Q_r, E)$.*

Finally, we can conclude that for a disturbance realization that in Corollary 1 and in Corollary 3 is characterized, at least, by $\mathcal{Q} = \{Q_1, Q_2\}$, with $Q_1 = \{d \in D : -\xi \leq d \leq -\hat{d}\}$ and $Q_2 = \{d \in D : -\hat{d} \leq d \leq \xi\}$, with $0 < \hat{d} \leq \xi$, the system trajectory converges to a neighborhood of the set $\{\pi \mathbf{1}\}$ with the ray of the neighborhood that becomes smaller and smaller as $\hat{d} \rightarrow \xi$ and $P(Q_1, E) \cap P(Q_2, E) = \{\pi \mathbf{1}\}$ if $\hat{d} = \xi$.

4.3 Bounds for $x(t)$.

In this subsection, we determine bounds for the minimum and maximum value that the components of $x(t)$ assume over the time depending on the initial state $x(0)$ and for any disturbance realization $\{d(t) \in D : t \geq 0\}$. In particular, we prove that, when we apply the lazy rule (8), we always obtain

$$\alpha(x(0)) \leq \lim_{t \rightarrow \infty} x_i(t) \leq \beta(x(0)), \quad \text{for all } i \in \Gamma. \quad (23)$$

As a further result, we also show that the difference between the maximum and the minimum agent states may not increase over the time.

Given a network G and an initial state $x(0)$, the main idea is to replace G by a much simpler network H composed by only two agents and such that the initial state of H is equal to the two maximal values of the initial state of network G . The result is that the maximal value assumed by the states of G is always bounded by the values assumed by the states of H .

Let us denote by $H = (\{a, b\}, \{(a, b)\})$ a system with only two connected agents. Let $x^H(t)$ be the state of H , namely, $x^H(t) = \{x_a^H(t), x_b^H(t)\}$. Let the components of $x^H(t)$ be subject to a constant disturbance $d^H(t) = \xi$. Let us also define $i_1(t) = \arg \max_{j \in \Gamma} \{x_j(t)\}$ and $i_2(t) = \arg \max_{j \in \Gamma \setminus \{i_1(t)\}} \{x_j(t)\}$, for all $t \geq 0$. Actually, $i_1(t)$ and $i_2(t)$ are the two agents with the first two maximal states. Obviously, $i_1(t)$ and $i_2(t)$ depend on time t . Analogously, define $i_n(t) = \arg \min_{j \in \Gamma} \{x_j(t)\}$ and by $i_{n-1}(t) = \arg \min_{j \in \Gamma \setminus \{i_n(t)\}} \{x_j(t)\}$, for all $t \geq 0$.

Lemma 4 *Given the system (1) on $G = (\Gamma, E)$ with initial state $x(0)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). Consider the system $H = (\{a, b\}, \{(a, b)\})$ with an initial state $x_a^H(0) = x_{i_1(0)}(0)$ and $x_b^H(0) = x_{i_2(0)}(0)$. Then, for all $t \geq 0$, $x_a^H(t) \geq x_i(t)$, for all $i \in \Gamma$ and $x_b^H(t) \geq x_i(t)$, for all $i \in \Gamma \setminus \{i_1(t)\}$.*

Proof. Observe that $x_i(t)$ is a differentiable variable for all $i \in \Gamma$. The same property holds for $x_a^H(t)$ and $x_b^H(t)$. In addition it holds that $x_a^H(t) \geq x_b^H(t)$ for any $t \geq 0$.

At time $t = 0$ the thesis holds by definition of values $x_a^H(0)$ and $x_b^H(0)$. By contradiction, assume that at some time instant $\bar{t} > 0$ the thesis is false, i.e., there exist some $i, j \in \Gamma$ such that either $x_a^H(\bar{t}) < x_i(\bar{t})$ or $x_a^H(\bar{t}) \geq x_i(\bar{t})$ but $x_b^H(\bar{t}) < x_j(\bar{t})$. By continuity, there must also exist $0 \leq t < \bar{t}$ where one of the following conditions holds

- i) $x_a^H(t) = x_i(t)$, $x_b^H(t) = x_j(t)$, $x_b^H(t) \geq x_k(t)$, for all $k \in \Gamma \setminus \{i, j\}$, and either $x_a^H(t + dt) < x_i(t + dt)$ or $x_b^H(t + dt) < x_j(t + dt)$;
- ii) $x_a^H(t) = x_i(t)$, $x_b^H(t) > x_k(t)$, for all $k \in \Gamma \setminus \{i\}$, and $x_a^H(t + dt) < x_i(t + dt)$;
- iii) $x_a^H(t) > x_i(t)$, $x_b^H(t) = x_j(t)$, $x_b^H(t) \geq x_k(t)$, for all $k \in \Gamma \setminus \{i, j\}$, and $x_b^H(t + dt) < x_j(t + dt)$;

Consider case i). It holds $\dot{x}_a^H(t) = \tilde{y}_{ab}^H(t) - x_a^H(t) \leq 0$ and $\dot{x}_i(t) = \sum_{r \in N_i} (\tilde{y}_{ir}(t) - x_i(t)) \leq 0$ and, in particular, $\tilde{y}_{ir}(t) - x_i(t) \leq 0$, for all $r \in N_i$. As by hypothesis $\tilde{y}_{ab}^H(t) - x_a^H(t) \geq \tilde{y}_{ir}(t) - x_i(t)$ for any $r \in \Gamma$, we have $\dot{x}_a^H(t) \geq \dot{x}_i(t)$, hence the inequality $x_a^H(t + dt) < x_i(t + dt)$ is false. It also holds $\dot{x}_b^H(t) = \tilde{y}_{ba}^H(t) - x_b^H(t) \geq 0$ and $\dot{x}_j(t) =$

$\sum_{r \in N_{j\sigma(t)}(\tilde{y}_{jr}(t) - x_j(t)) \leq \sum_{r \in N_{j\sigma(t)} \setminus \{i\}}(\tilde{y}_{jr}(t) - x_j(t)) + (\tilde{y}_{ji}(t) - x_j(t))$. As by hypothesis $\tilde{y}_{ba}^H(t) - x_b^H(t) \geq \tilde{y}_{ji}(t) - x_j(t)$ and $\sum_{r \in N_{j\sigma(t)} \setminus \{i\}}(\tilde{y}_{jr}(t) - x_j(t)) \leq 0$, we have $\dot{x}_a^H(t) \geq \dot{x}_i(t)$, hence the inequality $x_b^H(t+dt) < x_j(t+dt)$ is false. Hence Conditions i) cannot hold.

We can use the first part of the above argument to prove that Conditions ii) cannot hold, and use the second part to prove that Conditions iii) cannot hold. □

The following corollary holds.

Corollary 4 *Given the system (1) on $G = (\Gamma, E)$ with initial state $x(0)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). Then:*

(i) *The values assumed by the state trajectory $x(t)$ for $t \rightarrow \infty$ satisfy the following inequalities for any disturbance realization $d(t)$*

$$x_{i_n}(0) \leq \alpha(x(0)) \leq \lim_{t \rightarrow \infty} x_i(t) \leq \beta(x(0)) \leq x_{i_1}(0), \quad \text{for all } i \in \Gamma, \quad (24)$$

where the bounds $\alpha(x(0))$ and $\beta(x(0))$ depend on the initial state $x(0)$ as follows

$$\alpha(x(0)) = \max \left\{ x_{i_n}(0), \frac{x_{i_n}(0) + x_{i_{n-1}}(0)}{2} - \xi - \frac{\xi}{2} \ln \frac{-x_{i_n}(0) + x_{i_{n-1}}(0) + \xi}{\xi} \right\} \quad (25)$$

$$\beta(x(0)) = \min \left\{ x_a^H(0), \frac{x_a^H(0) + x_b^H(0)}{2} + \xi + \frac{\xi}{2} \ln \frac{x_a^H(0) - x_b^H(0) - \xi}{\xi} \right\}. \quad (26)$$

(ii) *The value $x_{i_1}(t)$ is non increasing on t and the value $x_{i_n}(t)$ is non decreasing on t .*

Proof. To prove (i), consider the network H as defined in the proof of Lemma 4 and let us study the evolution of $x^H(t)$ over the time. If $x_a^H(0) - x_b^H(0) \leq 2\xi$, the system state evolves according to $\dot{x}_a^H(t) = 0$ and $\dot{x}_b^H(t) = x_a^H(0) - x_b^H(t)$. If $x_a^H(0) - x_b^H(0) > 2\xi$, the system state evolves according to $\dot{x}_a^H(t) = x_b^H(t) + 2\xi - x_a^H(t)$ and $\dot{x}_b^H(t) = x_a^H(t) - x_b^H(t)$, as long as $0 \leq t \leq \hat{t}$, where \hat{t} is such that $x_a^H(\hat{t}) - x_b^H(\hat{t}) = 2\xi$, i.e., $\hat{t} = \frac{1}{2} \ln \frac{x_a^H(0) - x_b^H(0) - \xi}{\xi}$. For $t \geq \hat{t}$, the system state evolves according to $\dot{x}_a^H(t) = 0$ and $\dot{x}_b^H(t) = x_a^H(\hat{t}) - x_b^H(t)$. Hence, for $t \geq 0$, we have

$$x_a^H(t) = \begin{cases} x_a^H(0) & \text{if } x_a^H(0) - x_b^H(0) \leq 2\xi \\ \frac{\xi(1+2t-e^{-2t}) + x_a^H(0)(1+e^{-2t}) + x_b^H(0)(1-e^{-2t})}{2} & \text{if } x_a^H(0) - x_b^H(0) > 2\xi \text{ and } t \leq \hat{t} \\ \frac{x_a^H(0) + x_b^H(0)}{2} + \xi + \frac{\xi}{2} \ln \frac{x_a^H(0) - x_b^H(0) - \xi}{\xi} & \text{if } x_a^H(0) - x_b^H(0) > 2\xi \text{ and } t > \hat{t} \end{cases} \quad (27)$$

$$x_b^H(t) = \begin{cases} x_a^H(0) - x_b^H(0)e^{-t} & \text{if } x_a^H(0) - x_b^H(0) \leq 2\xi \\ \frac{\xi(-1+2t+e^{-2t}) + x_a^H(0)(1-e^{-2t}) + x_b^H(0)(1+e^{-2t})}{2} & \text{if } x_a^H(0) - x_b^H(0) > 2\xi \text{ and } t \leq \hat{t} \\ x_a^H(\hat{t}) - x_b^H(\hat{t})e^{-t} & \text{if } x_a^H(0) - x_b^H(0) > 2\xi \text{ and } t > \hat{t} \end{cases}. \quad (28)$$

For $t \rightarrow \infty$, $x_a^H(t)$ and $x_b^H(t)$ converge to $\beta(x(0)) = \min \left\{ x_a^H(0), \frac{x_a^H(0) + x_b^H(0)}{2} + \xi + \frac{\xi}{2} \ln \frac{x_a^H(0) - x_b^H(0) - \xi}{\xi} \right\}$. Hence, $\lim_{t \rightarrow \infty} x_i(t) \leq \beta(x(0))$, for all $i \in \Gamma$. With an analogous argument, we can prove $\lim_{t \rightarrow \infty} x_i(t) \geq \alpha(x(0)) = \max \left\{ x_{i_n}(0), \frac{x_{i_n}(0) + x_{i_{n-1}}(0)}{2} - \xi - \frac{\xi}{2} \ln \frac{-x_{i_n}(0) + x_{i_{n-1}}(0) + \xi}{\xi} \right\}$, for all $i \in \Gamma$.

To prove (ii), observe that we have

$$x_{i_1}(0) = x_a^H(0) \geq x_a^H(t) \geq x_{i_1}(t),$$

where the first equality and the last inequality hold by definition, whereas the inequality $x_a^H(0) \geq x_a^H(t)$ derives straightforwardly from the fact that $\dot{x}_a^H(t) \leq 0$ for all $t \geq 0$.

Corollary 4 (ii) proves that the system trajectory $x(t)$ is bounded as $t \rightarrow \infty$ and also that the difference between the maximum and the minimum agent states may not increase over the time. More formally, denote by $\mathcal{V}(x(t)) = x_{i_1}(t) - x_{i_n}(t)$ then

$$\mathcal{V}(x(t)) \geq \mathcal{V}(x(t + \Delta t)) \quad \text{for any } t \geq 0 \text{ and } \Delta t > 0. \quad (29)$$

We use this last implication to introduce some additional results that will turn useful when dealing with switching topology systems.

Denote by $\mathcal{V}_\infty = \lim_{t \rightarrow \infty} (x_{i_1}(t) - x_{i_n}(t))$ the final value of $\mathcal{V}(x(t))$. Observe that for some network $G(\Gamma, E)$ and initial state $x(0)$, there may exist some disturbance realizations $\{d(t) \in D : t \geq 0\}$ such that even if $\mathcal{V}(x(0)) > \mathcal{V}_\infty$, the value $\mathcal{V}(x(t))$ may be constant over some finite time interval before reaching its final value \mathcal{V}_∞ . More specifically, there may exist t and Δt such that $\mathcal{V}(x(t)) = \mathcal{V}(x(t + \Delta t)) > \mathcal{V}_\infty$.

Example 3 Consider a networks of six agents with chain topology depicted in Fig. 1. The initial state is $x(0) = [100, 100, 100, 0, 0, 0]^T$ and disturbances are $d_{12} = d_{21} = d_{23} = d_{32} = d_{34} = 1$ and $d_{43} = d_{45} = d_{54} = d_{56} = d_{65} = -1$. Figure 2 (a) shows the time plot of the evolution of the state $x(t)$ for $0 \leq t \leq 20$ and as it can be seen, trajectories converge to the equilibrium $x^* = [63, 61, 55, 45, 39, 37]^T$ with $\epsilon = 26$ (note that the initial deviation between maximum and minimum value of the state is 100). Figure 2 (b) displays a zoom of the trajectories for $0 \leq t \leq 3$ pointing out that $\mathcal{V}(x(t)) = \mathcal{V}(x(0))$ for $0 \leq t \leq 0.5$.

In the following we prove that we can always sample the state trajectory in such a way that the sequence of values for $\mathcal{V}(\cdot)$ is strictly decreasing on time until the state is “almost” in $P(D, E)$.

To this end, given the system (1) on $G = (\Gamma, E)$ and disturbance realizations $\{d(t) \in D : t \geq 0\}$, with a little abuse of notation, we denote by $\mathcal{V}(x) = \max_{i \in \Gamma} \{x_i\} - \min_{i \in \Gamma} \{x_i\}$ the maximum difference between two components of vector x , for any $x \in \mathbb{R}^n$. Also, we denote by $P(D, E) + \nu = \{x \in \mathbb{R}^n : \exists y \in P(D, E) \text{ s.t. } \|x - y\|_\infty \leq \nu\}$, with $\nu > 0$, the set of points whose distance from set $P(D, E)$ is not greater than ν , according the \mathcal{L}_∞ norm; and by $V(x) = \frac{1}{2} \sum_{(i,j) \in E} (x_i - x_j)^2$ the Lyapunov function considered in the proof of Theorem 1.

The following lemma holds

Lemma 5 Given the system (1) on $G = (\Gamma, E)$. Let \hat{x} and \bar{x} in \mathbb{R}^n , if $V(\bar{x}) \leq \frac{4\gamma^2 V(\hat{x})}{n^2(n-1)}$ then $\mathcal{V}(\bar{x}) \leq \gamma \mathcal{V}(\hat{x})$, for any $0 < \gamma < 1$.

Proof. First we determine the bounds for the values of $V(x)$ for $x \in \mathbb{R}^n$ such that $\mathcal{V}(x) = \mathcal{V} = \text{const}$. Denote by $\hat{E} = \{(i, j) : i < j, i, j \in \Gamma\}$ the edgeset of the complete network induced by vertices in Γ . Observe that if $x \in \mathbb{R}^n$ and $\mathcal{V}(x) = \mathcal{V}(x^1)$ then

$$V(x) \leq \max_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \sum_{(i,j) \in \hat{E}} (x_i - x_j)^2 : \mathcal{V}(x) = \mathcal{V} \right\} = \frac{n^2 \mathcal{V}^2}{8}.$$

The last equality holds as $V(x)$, for fixed $\mathcal{V}(x)$, is maximum when is maximum the number of couples of elements of x whose difference is equal to $\mathcal{V}(x)$. On the other hand, denote by $\tilde{E} = \{(i, i+1) : i, i+1 \in \Gamma\}$ the edgeset of a chain network induced by vertices in Γ . Then

$$V(x) \geq \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \sum_{(i,j) \in \tilde{E}} (x_i - x_j)^2 : \mathcal{V}(x) = \mathcal{V} \right\} = \frac{\mathcal{V}^2}{2(n-1)}.$$

The latter equality holds as there surely exists a chain network defined by \tilde{E} such that $\sum_{(i,j) \in \tilde{E}} (x_i - x_j) = \mathcal{V}$. The previous inequality holds as $\sum_{(i,j) \in E} (x_i - x_j) \geq \mathcal{V}$ because E defines a connected network on G , hence there exists a path on E from the agent with the maximum value of the state and the agent with the minimum value of the state.

We can now affirm that $\frac{\mathcal{V}^2(\hat{x})}{2(n-1)} \leq V(\hat{x}) \leq \frac{n^2 \mathcal{V}^2(\hat{x})}{8}$ and that we have $\mathcal{V}(\bar{x}) < \gamma \mathcal{V}(\hat{x})$ if $V(\bar{x}) \leq \frac{(\gamma \mathcal{V}(\hat{x}))^2}{2(n-1)}$. The latter situation certainly occurs when $V(\bar{x}) \leq \frac{4\gamma^2 V(\hat{x})^2}{n^2(n-1)}$. □

Theorem 3 *Given the system (1) on $G = (\Gamma, E)$, implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3). For each $\nu > 0$, $0 < \gamma < 1$ there exists a finite $q(\nu, \gamma) > 0$ such that the values assumed by the state trajectory $x(t)$ satisfy the following condition: either $\mathcal{V}(x(t + q(\nu, \gamma))) < \gamma \mathcal{V}(x(t))$ or $x(t + q(\nu, \gamma)) \in P(D, E) + \nu$, for any $t \geq 0$ and for any disturbance realization $\{d(t) \in D : t \geq 0\}$.*

Proof. We know that, when we implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (3), the system trajectory $x(t)$ converges to a point in $P(D, E)$ for any $x(0) \in \mathbb{R}^n$. We also know that $V(x(t))$ is strictly decreasing along $x(t)$. Then, for any $x(t) \in \mathbb{R}^n$, there exists a finite $\hat{q}(x(t), \nu, \gamma) \geq 0$ such that either $\mathcal{V}(x(t + \hat{q}(x(t), \nu, \gamma))) < \gamma \mathcal{V}(x(t))$ or $x(t + \hat{q}(x(t), \nu, \gamma)) \in P(D, E) + \nu$, for any $t \geq 0$ and for any disturbance realization $\{d(t) \in D : t \geq 0\}$. As the system converges (exponentially) for any $x(0) \in \mathbb{R}^n$, the value $\hat{q}(x, \nu, \gamma)$ is finitely bounded for $\|x\| \rightarrow \infty$. Hence, the theorem is proved by defining $q(\nu, \gamma) = \max_{x \in \mathbb{R}^n} \{\hat{q}(x, \nu, \gamma)\}$. □

The above result is strictly related to the ϵ -consensus problem stated at the beginning of the paper (see Problem 1). Actually, the above result means that the state converges in finite time to a tube of radius $\epsilon = \max\{\mathcal{V}(x) : x \in P(D, E) + \nu\}$.

5 Switching Topology.

In the following, we generalize the results obtained in Section 4 to networks with switching topologies. Consider a network $G_{\sigma(t)} = (\Gamma, E_{\sigma(t)})$ that has a time variant edgeset $E_{\sigma(t)} \in \mathcal{E}$. We define an edgeset E_k as recurrent for a given realization of $\sigma(t)$ if for all $t \geq 0$, there exists $t_k \geq t$ such that $\sigma(t_k) = k$. As \mathcal{E} is finite there exists at least a recurrent edgeset for any realization of $\sigma(t)$. Throughout the rest of the paper we assume that all the edgesets in \mathcal{E} are recurrent over time for any realization of $\sigma(t)$, more formally

Assumption 2 *Any realization $\sigma(t)$ is such that, for all $t \geq 0$, there exists $t_k \geq t$ such that $\sigma(t_k) = k$ for all $k \in \mathcal{I}$.*

When we say that the edgeset $E_{\sigma(t)}$ is time variant we understand that Assumption 2 holds and all the edgesets in \mathcal{E} are recurrent. Note that there is no loss of generality in Assumption 2, if it were false, the results of this section would hold for the subset $\hat{\mathcal{E}} \in \mathcal{E}$ of recurrent edgesets.

A basic observations is that even in presence of switches $\mathcal{V}(x(t))$ is not increasing on t as stated in (29). To see this note that the protocol $u_{\sigma(t)}(\cdot)$ induces a continuous and bounded state trajectory even for a network with switching topology (1) on $G_{\sigma(t)} = (\Gamma, E_{\sigma(t)})$. Then, let $t_k < t_{k+1}$ be two generic consecutive switching times. Corollary 4 applied at time $t = t_k$ instead of $t = 0$ guarantees that $x_{i_n}(t_k) \leq x_i(t) \leq x_{i_1}(t_k)$ for all $i \in \Gamma$ and for all $t_k < t < t_{k+1}$. As $x(t)$ is bounded, then $u_{\sigma(t)}(\cdot)$ is also bounded in the same time interval. Hence the first condition in (3) implies that $x(t)$ is continuous for $t_k < t < t_{k+1}$, whereas, the second conditions in (3) imposes the continuity of the state trajectory in t_k as $x(t_k^-) = x(t_k^+)$ and in t_{k+1} as $x(t_{k+1}^-) = x(t_{k+1}^+)$. As a consequence, for all $t \geq 0$, we also have that $x_{i_n}(0) \leq x_i(t) \leq x_{i_1}(0)$ for all $i \in \Gamma$ and $\mathcal{V}(x(t))$ is not increasing.

We also need to redefine the value $\mu(\cdot)$ initially introduced as (16). In particular, for given realizations $d(t)$ and $\sigma(t)$, and a subset Q of D , we define

$$\mu(Q, E_k, t_1, t_2) = \max\{\Delta : t_1 \leq \tilde{t} \leq \tilde{t} + \Delta \leq t_2 \text{ s.t. } d(t) \in Q \text{ and } \sigma(t) = k \text{ for all } \tilde{t} \leq t \leq \tilde{t} + \Delta\}. \quad (30)$$

In other words, given a time interval $[t_1, t_2]$, the value $\mu(Q, E_k, t_1, t_2)$ is the length of the longest subinterval where $d(t)$ remains in Q and $E_{\sigma(t)}$ is equal to E_k .

The following lemma generalizes Lemma 2 and Corollary 1.

Lemma 6 *Given the switched system (1) on $G_{\sigma(t)} = (\Gamma, E_{\sigma(t)})$, implement a distributed and stationary protocol $u_{\sigma(t)}(\cdot)$ whose components have the feedback form (3). Consider a disturbance realization $\{d(t) \in D, t \geq 0\}$ and a finite set $\mathcal{Q} = \{Q_1, Q_2, \dots\}$ of boxes of D . Assume that there exist two nonnegative finite numbers M and δ , such that $\mu(Q_r, E_k, t, t + M) > \delta$, for all $Q_r \in \mathcal{Q}$, for all $E_k \in \mathcal{E}$, and for all $t \geq 0$. Then the equilibrium points x exist and belong to $\bigcap_{E_k \in \mathcal{E}} \bigcap_{Q_r \in \mathcal{Q}} P(Q_r, E_k)$.*

Proof. Equilibrium points exists as $\{\pi \mathbf{1}\} = \bigcap_{E_k \in \mathcal{E}} \bigcap_{d \in D} P(d, E_k)$ holds. Then, we can prove that any equilibrium point must belong to $\bigcap_{E_k \in \mathcal{E}} \bigcap_{Q_r \in \mathcal{Q}} P(Q_r, E_k)$ using the argument in the proof of Lemma 2 for each couple (Q_r, E_k) , for all $E_k \in \mathcal{E}$ and all $Q_r \in \mathcal{Q}$. □

The results of Lemma 3 still hold in each subinterval between two consecutive switches, and then apply even in the switching case.

To generalize the convergence results of Theorem 1, we need to introduce the following notations, for each $Q \subseteq D$: $L(Q, E_k) = \max_{x \in P(Q, E_k)} \mathcal{V}(x)$ the maximum value of $\mathcal{V}(x)$ for points in $P(Q, E_k)$; $S(Q, E_k, \nu) = \{x \in \mathbb{R}^n : \mathcal{V}(x) \leq L(Q, E_k) + 2\nu\}$ the set of points whose maximum difference between two components does not exceed $L(Q, E_k) + 2\nu$. It is worth to be noted that $S(Q, E_k, \nu)$ are tubes of radius less than or equal to $L(Q, E_k) + 2\nu$ and then, $S(Q, E_k, \nu) \subseteq S(Q, E_{\hat{k}}, \nu)$ whenever $L(Q, E_k) \leq L(Q, E_{\hat{k}})$. Also, observe that, by definition, it holds that $P(Q, E_k) \subseteq P(Q, E_k) + \nu \subseteq S(Q, E_k, \nu)$. Finally, we introduce a minimum dwell time $\tau(\nu, \gamma) = \max_{k \in \mathcal{I}} \{q_k(\nu, \gamma)\}$. In other words, the minimum length of the switching intervals is equal to the maximal value over the different $E_k \in \mathcal{E}$ of the times $q(\nu, \gamma)$ introduced by Theorem 3.

Theorem 4 *Given the switched system (1) on $G_{\sigma(t)} = (\Gamma, E_{\sigma(t)})$, with a minimum dwell time $\hat{\tau}(\nu, \gamma)$, implement a distributed and stationary protocol $u_{\sigma(t)}(\cdot)$ whose components have the feedback form (3) with $\tilde{y}^{(i)}$ as in (8). Assume that two values $\nu > 0$ and $0 < \gamma < 1$ are also given. The state is driven in finite time to the tube*

$$T = \bigcap_{E_k \in \mathcal{E}} S(D, E_k, \nu). \quad (31)$$

Proof. We already know that the system trajectory $x(t)$ is continuous and that $\mathcal{V}(x(t))$ is not increasing. Denote by t_s and t_{s+1} with $t_{s+1} \geq t_s + \tau(\nu, \gamma)$ two generic consecutive switching times such that $\sigma(t) = k$ for all $t_s \leq t < t_{s+1}$. From Theorem 3, we deduce that either $\mathcal{V}(x(t_{s+1})) \leq \gamma \mathcal{V}(x(t_s))$ or $x(t_{s+1}) \in P(D, E_k) + \nu$. The fact that $\mathcal{V}(x(t))$ is not increasing and all the edgesets $E_k \in \mathcal{E}$ are recurrent implies that there exists $\bar{t} \geq 0$ such that, if $t_s \geq \bar{t}$, we have $x(t_{s+1}) \in P(D, E_k) + \nu$. Again, a not increasing $\mathcal{V}(x(t))$ implies that $x(t) \in S(D, E_k, \nu)$, for all $t \geq t_{s+1} \geq \bar{t}$. As we can apply the above argument for all the edgesets $E_k \in \mathcal{E}$, the theorem thesis follows. □

This last result gives an answer to the ϵ -consensus problem stated at the beginning of this paper (see Problem 1). Indeed, convergence to T , as in (31), means that the agents have reached ϵ -consensus with $\epsilon = \min\{\epsilon_k\}$ where ϵ_k is the radius of tubes $S(D, E_k, \nu)$.

Also note that the above theorem does not guarantee the convergence to an equilibrium point. Actually, switching systems may oscillate as shown by the following example.

Example 4 Consider a family of chain networks G on the set of agents $\Gamma = \{1, 2, 3\}$ and edgsets $E_1 = \{(1, 2), (2, 3)\}$ and $E_2 = \{(1, 2), (1, 3)\}$ (see Fig. 3). Let $x_1(0) = 2$, $x_2(0) = 0$, and $x_3(0) = 1$, and $\{d(t) = d = \text{const} \in D : t \geq 0\}$, with in particular $d_{12} = d_{13} = d_{31} = 1$ and $d_{21} = d_{23} = d_{32} = -1$. If the switching time intervals are sufficiently long the systems trajectory oscillates in \mathbb{R}^3 along the segment delimited by points $[2, 0, 0]$ and $[2, 0, 2]$. Note that only the state of agent 3 changes over time.

Finally, note that we could generalize Theorem 2 and Corollary 3 only in the assumption that, for all $Q_r \in \mathcal{Q}$, for all $E_k \in \mathcal{E}$, the values $\mu(Q_r, E_k, t, t + M)$ define sufficiently long intervals $[t_s, t_{s+1})$ so that either $\mathcal{V}(x(t_{s+1}))$ is finitely smaller than $\mathcal{V}(x(t_s))$ of a finite value or $x(t_{s+1}) \in P(Q_r, E_k) + \nu$. In this case we have the system trajectory eventually assume values in $\bigcap_{E_k \in \mathcal{E}} \bigcap_{Q_r \in \mathcal{Q}} S(Q_r, E_k, \nu)$.

6 Conclusions.

Despite the literature on consensus is now becoming extensive, only few approaches have considered a disturbance affecting the measurements. In our approach we have assumed an UBB noise in the neighbors' state feedback as it requires the least amount of a-priori knowledge on the disturbance. Only the knowledge of a bound on the realization is assumed, and no statistical properties need to be satisfied. Because of the presence of UBB disturbances convergence to equilibria with all equal components is, in general, not possible. Therefore, the main contribution has been the introduction and solution of the ϵ -consensus problem, where the states converge in a tube of ray ϵ asymptotically or in finite time. In solving the ϵ -consensus problem we have focused on linear protocols and presented a rule for estimating the average from a compact set of candidate points.

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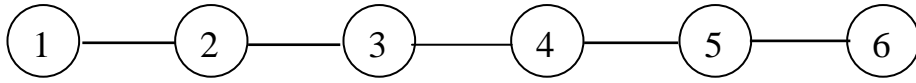


Figure 1: Chain of six agents.

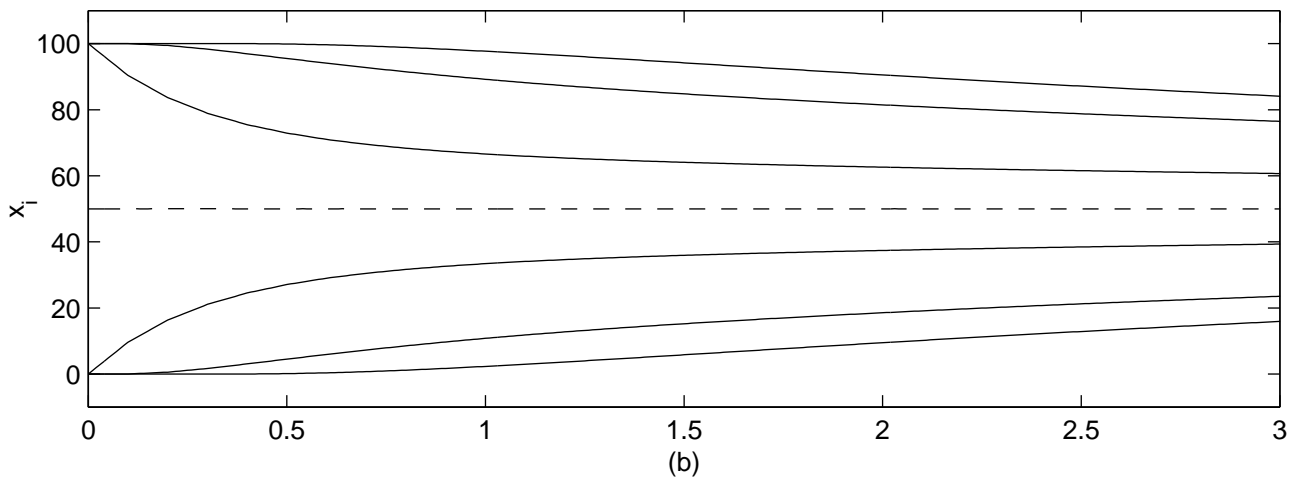
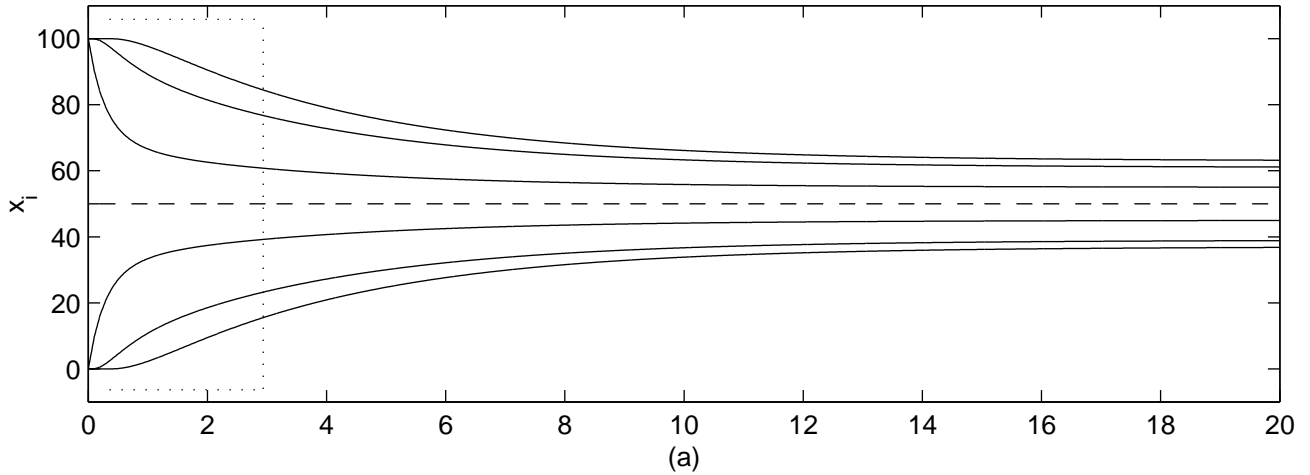


Figure 2: (a) Time plot of the state x for an initial state $x(0) = [100, 100, 100, 0, 0, 0]^T$. Trajectories converge to the equilibrium $x^* = [63, 61, 55, 45, 39, 37]^T$ with $\epsilon = 26$; (b) zoom of the first time instants (dotted rectangle in (a)), which highlights $\mathcal{V}(x(t)) = \mathcal{V}(x(0))$ for $0 \leq t \leq 0.5$.

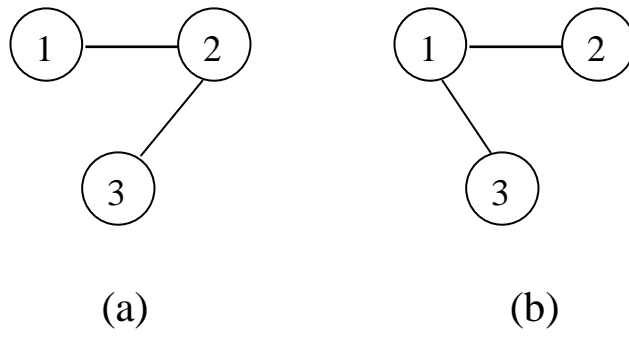


Figure 3: Family of chain networks of three agents.