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Mixed integer predictive control and shortest path reformulation

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March 16, 2010

Abstract

Mixed integer predictive control deals with optimizing integer and real control variables over a receding horizon. The mixed integer nature of controls might be a cause of intractability for instances of larger dimensions. To tackle this little issue, we propose a decomposition method which turns the original n -dimensional problem into n independent scalar problems of lot sizing form. Each scalar problem is then reformulated as a shortest path one and solved through linear programming over a receding horizon. This last reformulation step mirrors a standard procedure in mixed integer programming. The approximation introduced by the decomposition can be lowered if we operate in accordance with the predictive control technique: i) optimize controls over the horizon ii) apply the first control iii) provide measurement updates of other states and repeat the procedure.

1 Introduction

Mixed integer predictive control arises when optimizing integer and real control variables in a receding horizon context [1]. For this reason, many authors see it as a specific field in the broader area of optimal hybrid control [3]. Optimal integer control problems have been receiving a growing attention and are often categorized under different names. See, for instance, the literature on finite alphabet control [5, 9]. Integer control requires a bit more than standard convex optimization techniques. From the literature we know that new properties come into play. As an example, look at *multimodularity* presented as the counterpart of convexity in discrete action spaces [4]. When talking about mixed integer variables, it is, of course, not possible not to mention the more than vast literature on mixed integer programming [7]. It is exactly in this context that we have found inspiration as clarified in more details next.

In this paper, we have moved our steps along the line of [8] which surveys solution methods for mixed integer lot sizing models. Indeed, decomposing an n -dimensional dynamic system into n independent lot sizing systems is almost all about this paper is centered around. The approximation introduced by the decomposition can be reduced if we operate in accordance with the predictive control technique: i) optimize controls for each independent system all over a prediction horizon, ii) apply the first control to each independent system, iii) provide measurement updates of other states and repeat the procedure. The main contribution of this work is to reformulate the mixed integer problem of point i) as a shortest path problem and solve this last through linear programming. This approach mirrors the method surveyed in [8] with the differences that here the shortest path problems run iteratively forward in time over a receding horizon. Reframing the method in a receding horizon context is an element of novelty and presents some additional and new issues which are discussed and overcome throughout the paper.

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22 This paper differs from [1] as we focus on a smaller class of problems that can be solved exactly and do
 23 not require advanced relaxation methods which, in turn, are a main topic in [1]. To bring our discussion
 24 back to hybrid control, the lot sizing like model used here has much to do with the inventory example briefly
 25 mentioned in [3]. There, the authors simply include the example in the large list of hybrid optimal control
 26 problems but do not address the issue of how to fit general methods to this specific problem. On the contrary,
 27 this work cannot emphasize enough the computational benefits deriving from the “nice structure” of the lot
 28 sizing constraints matrix. Binary variables, used to model impulses, match linear programming in a previous
 29 work of the same author [2]. There, the linear reformulation is a straightforward derivation of the (*inverse*)
 30 *dwell time* conditions appeared first in [6]. Analogies with [2] are, for instance, the use of total unimodularity
 31 to prove the exactness of the linear programming reformulation. Differences are in the procedure itself upon
 32 which the linear program is built up. The shortest path model is an additional element which distinguishes
 33 the present approach from [2].

34 This paper is organized as follows. We state the problem in Section 2. We then move to present the
 35 decomposition method in Section 3. In Section 4, we turn to introducing the shortest path reformulation and
 36 the linear program. We dedicate the last Section 5 to support our theoretical analysis with some numerical
 37 results.

38 2 Mixed integer predictive control

In mixed integer control we usually have continuous state $x(k) \in \mathbb{R}^n$, continuous controls $u(k) \in \mathbb{R}^n$ and
 disturbances $w(k) \in \mathbb{R}^n$, discrete controls $y(k) \in \{0, 1\}^n$ (see e.g., [1]). Evolution of the state over a finite
 horizon of length N is described by a linear discrete time dynamics in the general form (1), where A and E
 are matrices of compatible dimensions:

$$x(k+1) = Ax(k) + Ew(k) + u(k) \geq 0, \quad x(0) = x(N) = 0. \quad (1)$$

39 The above dynamics is characterized by one discrete and continuous control variable per each state, and this
 40 reflects the idea that we may wish to control independently each state component. Also, starting from initial
 41 state at zero, we wish to drive the final state to zero which is a typical requirement when controlling a system
 42 over a finite horizon. On this purpose, we have added equality constraints on the final states. Also, we force
 43 the states to remain confined within a desired region, take for it the positive orthant, which may describe a
 44 safety region in engineering applications or the desire of preventing shortcomings in inventory applications.

Continuous and discrete controls are linked together by general *capacity constraints* (2), where the pa-
 rameter C is an upper bound on control:

$$0 \leq u(k) \leq Cy(k), \quad y(k) \in \{0, 1\}^n. \quad (2)$$

45 For clarity reasons, $y(k)$ is the decision of controlling or not the system, and $u(k)$ is the control action. So if
 46 we decide not to control the system then the control action is null, otherwise this last is any value between
 47 zero and its upper bound C .

48 The following assumption helps us to describe the common situation where the disturbance seeks to push
 49 the state out of the desired region.

Assumption 1 (Unstabilizing disturbance effects)

$$Ew(k) < 0. \quad (3)$$

50 At this point, the non negative nature of controls $u(k)$ should become much clearer. Actually, control
 51 actions are used to push the state far from boundaries into the positive orthant thus to counterbalance the

52 unstabilizing effects of disturbances over a certain period to come. However, controlling the system has a
 53 cost and “over acting” on it is punished by introducing a cost/objective function as explained next.

The objective function to minimize with respect to $y(k)$ and $u(k)$ is a linear one including proportional, holding and fixed cost terms expressed by parameters p^k , h^k , and f^k respectively:

$$\sum_{k=0}^{N-1} (p^k u(k) + h^k x(k) + f^k y(k)). \quad (4)$$

54 Conditions (1)-(4) introduced so far describe coincisely the problem of interest. In the next section, we
 55 recall a standard method to convert the problem of interest (1)-(4) into a mixed integer linear program
 56 returning the exact solution in terms of optimal control actions $u(k)$ and $y(k)$.

57 **Remark 1** For sake of simplicity disturbances $w(k)$ are deterministic and apriori known. The approach
 58 presented below is still valid if we drop this assumption and turn to consider unknown disturbances. Only, we
 59 should carefully repropose problem (1)-(4) in a receding horizon form with iterative measuments updates and
 60 control optimization forward in time all over the horizon.

61 2.1 Mixed integer linear program and exact solution.

62 The mixed integer nature of the above program makes it intractable for increasing number of variables and
 63 horizon length. So, the topic presented below is motivated mainly by comparisons reasons and applies only
 64 to problems of relatively small dimensions.

Before introducing the mixed integer linear program we need to define the following notation. Let us start
 by collecting states, continuous and discrete controls, proportional, holding and fixed costs all in oportune
 vectors as shown below:

$$x = [x(0)^T \dots x(N)^T]^T, \quad u = [u(0)^T \dots u(N-1)^T]^T, \quad y = [y(0)^T \dots y(N-1)^T]^T,$$

$$p = [(p^0)^T \dots (p^{N-1})^T]^T, \quad h = [(h^0)^T \dots (h^{N-1})^T]^T, \quad f = [(f^0)^T \dots (f^{N-1})^T]^T.$$

Furthermore, to put dynamics (1) into “constraints” form, let us introduce matrices \mathbf{A} , \mathbf{B} and vector \mathbf{b}
 defined as

$$\mathbf{A} = \begin{bmatrix} -I & 0 & 0 & \dots & 0 & 0 \\ A & -I & 0 & \dots & 0 & 0 \\ 0 & A & -I & \dots & 0 & 0 \\ 0 & 0 & A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A & -I \\ 0 & 0 & 0 & \dots & 0 & -I \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B \\ 0 & 0 & \dots & 0 \end{bmatrix}; \quad \mathbf{b} = [-\xi_0^T \ (Ew(0))^T \ \dots \ (Ew(N))^T \ -\xi_f^T]^T.$$

65 Notice that once we take for ξ_0 and ξ_f the value zero, the first and last rows in the aforementioned matrices
 66 restate the constraints on initial and final state of (1).

Finally, we are in the condition to establish that problem (1)-(4) can be solved exactly through the
 following mixed integer linear program:

$$(MIPC) \quad \min_{u,y} \quad J(u,y) = pu + hx + fy \quad (5)$$

$$\mathbf{A}x + \mathbf{B}u = \mathbf{b} \quad (6)$$

$$0 \leq u \leq Cy, \quad y \in \{0, 1\}^{nN}. \quad (7)$$

67 The mixed integer linear program (5)-(7) is the most natural mathematical programming representation
 68 of the problem of interest (1)-(4). For this reason, throughout this paper we will almost always refer to (5)-(7)
 69 when we wish to bring back the discussion to the source problem (1)-(4) and its exact solution.

70 To overcome the intractability of the mixed integer linear program (5)-(7), we propose a new method
 71 whose underlying idea is to bring back dynamics (1) to the lot sizing model [8]. To do this, we introduce
 72 some additional assumptions on the structure of matrix A which simplify the tractability and affect in no
 73 way the generality of the results. This argument is dealt with in details in the next section.

74 2.2 Introducing some structure on A

75 Our main goal in this section is to rewrite (1) in a “nice” form. With “nice form” we mean a form that
 76 emphasizes the analogies with standard lot sizing models [8]. “Stop beating around the bush”, we will
 77 henceforth refer to the following dynamics in state of (1):

$$x(k+1) = x(k) + \Delta x(k) + Ew(k) + u(k) \geq 0. \quad (8)$$

78 The reasons why expression (8) is a nice one is that it isolates the dependence of one component state on
 79 the other ones. To tell it differently we have separated the influence of all other states on state i . It will be
 80 soon clearer that turning our attention to the new expression (8) is a prelude in view of the decomposition
 81 approach discussed later on.

82 Once clarified the reasons, we need next to clarify how to go from (1) to (8) and what is the underlying
 83 assumption that allows us to do that. Before doing this let us denote with $I \in \mathbb{R}^{n \times n}$ the identity matrix and
 84 a_{ij} the dependence of state i on state j . So, we can make the following assumption.

Assumption 2 *Matrix A can be decomposed as*

$$A = I + \Delta, \quad \Delta = \begin{bmatrix} 0 & a_{12} & \dots & a_{1,n-1} & a_{1n} \\ a_{21} & 0 & \dots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} & 0 \end{bmatrix}.$$

85 The reader may notice that (8) is a straightforward derivation of (1) once we take for good Assumption 2.

86 Our secondary goal in this section is to preserve the nature of the game which has stabilizing control
 87 actions playing against unstabilizing disturbances. To do this, in our next assumption we do consider the
 88 case where the influence of other states on state i is relatively “weak” in comparison to the unstabilizing
 89 effects of disturbances.

Assumption 3 (Weakly coupling)

$$\Delta x(k) + Ew(k) < 0. \quad (9)$$

90 Notice that the above assumption preserves the nature of the game by bounding the effects of mutual
 91 dependence of state components represented by the term $\Delta x(k)$. A closer look at (3) and (9) sounds like the
 92 term $\Delta x(k)$ do not counterbalance the effects of $Ew(k)$. States mutual dependence only emphasize or reduce
 93 “weakly” the unstabilizing effects of disturbances.

94 We end this section by noticing that (8) is not yet in “lot sizing” form [8]. In the next section, we present
 95 a decomposition approach that translate dynamics (8) into n scalar dynamics in “lot sizing” form [8].

3 Robust decomposition

With the term “decomposition” we mean a mathematical manipulation through which the original dynamics (8) is replaced by n independent dynamics of the form:

$$x_i(k+1) = x_i(k) - d_i(k) + u_i(k). \quad (10)$$

The above dynamics is in a typical lot sizing form in the sense that the (inventory) state tomorrow $x_i(k+1)$ is equal to the (inventory) state today $x_i(k)$ plus the discrepancy between today demand $d_i(k)$ and today reordered quantity $u_i(k)$. Changing (8) with (10) is possible once we relate the demand $d_i(k)$ to the current values of all other state components and disturbances as expressed below:

$$\begin{aligned} d_i(k) &= - \left[\sum_{j=1, j \neq i}^n A_{ij} x_j(k) + \sum_{j=1}^n E_{ij} w_j(k) \right] \\ &= - [\Delta_{i\bullet} x(k) + E_{i\bullet} w(k)]. \end{aligned} \quad (11)$$

To tell it differently, we do assume that the influence that all other states have on state i enters into equation (10) through demand $d_i(k)$ defined in (11). Our next step is to make the n dynamics in the form (10) mutually independent. This is possible by replacing the current state values $x_j(k)$, $j \neq i$ with their estimated values on the part of agent i which we denote by $\tilde{x}_j(k)$, $j \neq i$. Still with reference to (10), this implies to replace the current demand $d_i(k)$ by the “estimated” demand $\tilde{d}_i(k)$ defined as in (12) where X^k is the set of admissible state vectors $x(k)$:

$$\tilde{d}_i(k) = \max_{\xi \in X^k} \{-\Delta_{i\bullet} \xi - E_{i\bullet} w(k)\}. \quad (12)$$

The idea behind (12) is to take for estimated value the worst admissible demand, i.e., the demand that would push the state out of the positive orthant in a fewest time and such a demand is of course the maximal one. However, it must be noted that we cannot see any drawbacks in combining other decomposition methods with the approach presented in the rest of the paper. To complete the decomposition, it is left to turn the objective function (4) into n independent components

$$J_i(u_i, y_i) = \sum_{k=0}^{N-1} (p_i^k u_i(k) + h_i^k x_i(k) + f_i^k y_i(k)).$$

Note that because of the linear structure of $J(u, y)$ in (5), it turns $J(u, y) = \sum_{i=1}^n J_i(u_i, y_i)$. So, in the end we have translated our original problem into n independent mixed integer linear minimization problems of the form (13)-(15) as requested at the beginning of this section. In the spirit of predictive control, each minimization problem is then solved forwardly in time all over the horizon. So, for $\tau = 0, \dots, N-1$ we need to solve

$$(MIPC_i) \quad \min_{u_i, y_i} \sum_{k=\tau}^{N-1} (p_i^k u_i(k) + h_i^k x_i(k) + f_i^k y_i(k)) \quad (13)$$

$$x_i(k+1) = x_i(k) - \tilde{d}_i(k) + u_i(k) \geq 0, \quad x_i(\tau) = \xi_i^0, \quad x_i(N) = 0 \quad (14)$$

$$0 \leq u_i(k) \leq C y_i(k), \quad y_i(k) \in \{0, 1\}. \quad (15)$$

It is worth to be noted that non null initial states, which materialize in values of ξ_i^0 strictly greater than zero in constraints (14) might induce infeasibility of $(MIPC_i)$. So, moving from $(MIPC)$ to $(MIPC_i)$ has this little drawback that we will discuss in more details later on in Section 4.3 together with some other issues concerned with the receding implementation of our method.

4 Shortest path and linear programming

So far, we have first formulated the problem of interest and then decomposed it into n independent scalar problems. By the way, decomposition is only the first step of our solution approach. Actually, the mixed

116 integer nature of variables in (13)-(15) is still an issue to be dealt with. This second part of the work focuses on
 117 the relaxation of the integer constraints $y_i(k) \in \{0, 1\}$ which would facilitate the tractability of the problem.
 118 It is well known that relaxation introduces, in general, some approximation in the solution. The main result
 119 of this work establishes that, for the problem at hand, relaxing and massaging the problem in a certain
 120 manner, will lead to a shortest path reformulation of the original problem. This is a great result as, it is
 121 well known that shortest path problem are in turn easily tractable and solvable through linear programming.
 122 Shortest path formulations are based on the notion of regeneration interval discussed in details in the next
 123 section.

124 4.1 Regeneration interval $[\alpha, \beta]$

125 Let us start by introducing a formal definition of *regeneration interval* which represents the central topic in
 126 this section. The definition, available in the literature for scalar lot sizing models, is borrowed from [8] and
 127 adapted to each single (scalar) dynamics i of our decomposed n -dimensional model. So, with reference to
 128 the generic minimization problem i expressed by (13)-(15), let us state what follows.

129 **Definition 1 (Pochet and Wolsey 1993)** *A pair of periods $[\alpha, \beta]$ form a regeneration interval for (x_i, u_i, y_i)*
 130 *if $x_i(\alpha - 1) = x_i(\beta) = 0$ and $x_i(k) > 0$ for $k = \alpha, \alpha + 1, \dots, \beta - 1$.*

131 Given a regeneration interval $[\alpha, \beta]$, we can define the accumulated demand over the interval $d_i^{\alpha\beta}$, and
 132 the residual demand $r_i^{\alpha\beta}$ as

$$d_i^{\alpha\beta} = \sum_{k=\alpha}^{\beta} \tilde{d}_i(k), \quad r_i^{\alpha\beta} = d_i^{\alpha\beta} - \left\lfloor \frac{d_i^{\alpha\beta}}{C} \right\rfloor C. \quad (16)$$

133 Our idea is now to translate problem (13)-(15) into new variables. More formally, let us consider variables
 134 $y_i^{\alpha\beta}(k)$ and $\epsilon_i^{\alpha\beta}(k)$ defined in (17) with the following meaning. Variable $y_i^{\alpha\beta}(k)$ is equal to one in presence of
 135 a saturated control on time k and zero otherwise. Similarly, variable $\epsilon_i^{\alpha\beta}(k)$ is equal to one in presence of a
 136 non saturated control on time k and zero otherwise:

$$y_i^{\alpha\beta}(k) = \begin{cases} 1 & \text{if } u_i(k) = C \\ 0 & \text{otherwise.} \end{cases} \quad \epsilon_i^{\alpha\beta}(k) = \begin{cases} 1 & \text{if } 0 < u_i(k) < C \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

137 To translate the meaning of $y_i^{\alpha\beta}(k)$ and $\epsilon_i^{\alpha\beta}(k)$ in a lot sizing context, such variables tell us on which period
 138 full or partial batches are ordered.

At this point and with in mind the above variable transformation, we can rely on well known results in
 the lot sizing literature which convert the original mixed integer problem (13)-(15) into a number of linear
 programs $(LP_i^{\alpha\beta})$, each one associated to a specific regeneration interval. Regeneration intervals and the
 associated linear programs are mutually related in a way that gives raise to a shortest path problem, which
 will be the central topic in the next section. For now, we simply repropose below the linear programming
 problem associated to a single regeneration interval $[\alpha, \beta]$. Denoting by $e_i^k = p_i^k + \sum_{j=k+1}^{N-1} h_i^j$ and after some

standard manipulation, the linear program for fixed regeneration interval $[\alpha, \beta]$ appears as:

$$(LP_i^{\alpha\beta}) \quad \min_{y_i^{\alpha,\beta}, u_i^{\alpha,\beta}} \sum_{k=\alpha}^{\beta} (Ce_i^k + f_i^k) y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{\beta} (r_i^{\alpha\beta} e_i^k + f_i^k) \epsilon_i^{\alpha\beta}(k) \quad (18)$$

$$\sum_{k=\alpha}^{\beta} y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{\beta} \epsilon_i^{\alpha\beta}(k) = \left\lceil \frac{d_i^{\alpha\beta}}{C} \right\rceil \quad (19)$$

$$\sum_{k=\alpha}^t y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^t \epsilon_i^{\alpha\beta}(k) \geq \left\lceil \frac{d_i^{\alpha t}}{C} \right\rceil, \quad t = \alpha, \dots, \beta - 1 \quad (20)$$

$$\sum_{k=\alpha}^{\beta} y_i^{\alpha\beta}(k) = \left\lceil \frac{d_i^{\alpha\beta} - r_i^{\alpha\beta}}{C} \right\rceil \quad (21)$$

$$\sum_{k=\alpha}^t y_i^{\alpha\beta}(k) \geq \left\lceil \frac{d_i^{\alpha t} - r_i^{\alpha t}}{C} \right\rceil, \quad t = \alpha, \dots, \beta - 1 \quad (22)$$

$$y_i^{\alpha\beta}(k), \epsilon_i^{\alpha\beta}(k) \geq 0, \quad k = \alpha, \dots, \beta. \quad (23)$$

139 The above model is extensively used in the lot sizing context. We can limit ourselves to a pair of comments
 140 on the underlying idea of the constraints. So, let us start by focusing on the equality constraints (19) and
 141 (21). These constraints tell us that the ordered quantity over the interval has to be equal to the accumulated
 142 demand over the same interval. This makes sense as initial and final state of a regeneration interval are null
 143 by definition. Let us turn our attention to the inequality constraints (20) and (22). There, we impose that
 144 the accumulated demand in any subinterval may not exceed the ordered quantity over the same subinterval.
 145 Again, this is due to the condition that states are nonnegative at any period of a regeneration interval.
 146 Finally, the objective function (18) is simply a rearrangement of (13) induced by the variable transformation
 147 seen above and specialized to the regeneration interval $[\alpha, \beta]$ rather than on the entire horizon $[0, N]$.

148 We are ready to recall the following “nice property” of $(LP_i^{\alpha\beta})$ presented first by Pochet and Wolsey in
 149 [8].

150 **Theorem 1 (Total unimodularity)** *The optimal solution of $(LP_i^{\alpha\beta})$ is feasible.*

151 *Proof.* The proof is based on the observation that the constraint matrix of $(LP_i^{\alpha\beta})$ is a 0 – 1 matrix. We
 152 can reorder the constraints in a certain manner, so that matrix has the consecutive 1’s property on each
 153 column and turns to be totally unimodular. It follows that $y_i^{\alpha,\beta}$ and $\epsilon_i^{\alpha,\beta}$ are 0 – 1 in any extreme solution.
 154 □

155 The above theorem represents a first step in the process of converting the mixed integer problem $(MIPC_i)$
 156 into a linear programming one.

157 4.2 Shortest path

158 In the previous section we have introduced a linear programming problem associated to a specific regeneration
 159 interval. In this section, we resort to well known results on lot sizing to come up with a shortest path model
 160 which links together the linear programming problems of all possible regeneration intervals. Actually, it must
 161 be noted that the solution of (13) -(15) can be expressed as a unique regeneration interval $[0, N]$ or as a list
 162 of regeneration intervals.

163 So, let us define variables $z_i^{\alpha\beta} \in \{0, 1\}$ which tell us one or zero whenever a regeneration interval $[\alpha, \beta]$
 164 appears or not in the solution of (13) -(15). The linear programming problem solving (13) -(15) takes on the
 165 form below. For $\tau = 0, \dots, N - 1$, solve

$$(LP_i) \quad \min_{y_i^{\alpha\beta}, u_i^{\alpha\beta}, z_i^{\alpha\beta}} \sum_{\alpha=\tau+1}^{N-1} \sum_{\beta=\alpha}^{N-1} \sum_{k=\alpha}^{\beta} \left[(Ce_i^k + f_i^k) y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{\beta} (r^{\alpha\beta} e_i^k + f_i^k) \epsilon_i^{\alpha\beta}(k) \right] \quad (24)$$

$$\sum_{\beta=\tau+1}^N z_i^{\tau+1\beta} = 1 \quad (25)$$

$$\sum_{\alpha=\tau+1}^{t-1} z_i^{\alpha,t-1} - \sum_{\beta=t}^N z_i^{t\beta} = 0 \quad t = \tau+2, \dots, N, \quad \tau+1 \leq \alpha \leq \beta \leq N \quad (26)$$

$$\sum_{k=\alpha}^{\beta} y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^{\beta} \epsilon_i^{\alpha\beta}(k) = \left\lceil \frac{d_i^{\alpha\beta}}{C} \right\rceil z_i^{\alpha\beta}, \quad \tau+1 \leq \alpha \leq \beta \leq N \quad (27)$$

$$\sum_{k=\alpha}^t y_i^{\alpha\beta}(k) + \sum_{k=\alpha}^t \epsilon_i^{\alpha\beta}(k) \geq \left\lceil \frac{d_i^{\alpha t}}{C} \right\rceil z_i^{\alpha\beta}, \quad t = \alpha, \dots, \beta-1, \quad \tau+1 \leq \alpha \leq \beta \leq N \quad (28)$$

$$\sum_{k=\alpha}^{\beta} y_i^{\alpha\beta}(k) = \left\lceil \frac{d_i^{\alpha\beta} - r_i^{\alpha\beta}}{C} \right\rceil z_i^{\alpha\beta} \quad \tau+1 \leq \alpha \leq \beta \leq N \quad (29)$$

$$\sum_{k=\alpha}^t y_i^{\alpha\beta}(k) \geq \left\lceil \frac{d_i^{\alpha t} - r_i^{\alpha t}}{C} \right\rceil z_i^{\alpha\beta}, \quad t = \alpha, \dots, \beta-1, \quad \tau+1 \leq \alpha \leq \beta \leq N \quad (30)$$

$$y_i^{\alpha\beta}(k), \epsilon_i^{\alpha\beta}(k), z_i^{\alpha\beta} \geq 0, \quad k = \alpha, \dots, \beta. \quad (31)$$

166 Let us spend a couple of words on the meaning of the above linear program. Constraints (27)-(31) should
 167 be familiar to the reader as they already appeared in (19)-(23). The only difference is that, now, because of
 168 the presence of $z_i^{\alpha\beta}$ in the right hand term, the constraints referring to a given regeneration interval come
 169 into play only if that interval is chosen as part of the solution, that is, whenever $z_i^{\alpha\beta}$ is set equal to one.
 170 Furthermore, a new class of constraints appear in (25)-(26). These constraints are typical of shortest path
 171 problems and in this specific case help us to force the variables $z_i^{\alpha\beta}(k)$ to describe a path from 0 to N .
 172 Finally, note that for $\tau = 0$, the linear program (LP_i) coincide with the linear program presented by Pochet
 173 and Wolsey in [8].

174 At this point, we are in a position to recall the crucial result established in [8].

175 **Theorem 2 (Pochet and Wolsey, 1993)** *The linear program (LP_i) solves $(MIPC_i)$.*

176 *Proof.* (Sketch) It turns out that the linear program (LP_i) is a shortest path problem on variables $z_i^{\alpha,\beta}$. Arcs
 177 are all associated to a different regeneration interval $[\alpha, \beta]$ and the respective costs are the optimal values of
 178 the objective functions of the corresponding linear programs $(LP_i^{\alpha,\beta})$. We refer the reader to [8] for further
 179 details. □

180 4.3 Receding horizon implementation of (LP_i)

This section is dedicated to certain issues concerned with the implementation of (LP_i) in a receding horizon
 context as typical of predictive control. As the reader may know, in predictive control we solve (LP_i)
 iteratively and forward in time all over the horizon. In the formulation of (LP_i) , this is stated clearly when

we specify that τ goes from 0 to $N - 1$ and for each value of τ we obtain a new linear program of type (LP_i) . After we solve (LP_i) for $\tau = 0$, we apply the first control to the system, update initial states according to the last available measurements at time $\tau = 1$ and move to solve a new (LP_i) starting at $\tau = 1$. We repeat this procedure until the end of the horizon, $\tau = N - 1$. So, consecutive linear programs are linked together by initial state condition expressed in (14), and which we rewrite below

$$x_i(\tau) = \xi_i^0.$$

At this point, we would restate with emphasis the fact that dealing with non null initial states is a main difference between the linear program (LP_i) and the linear program used in the lot sizing literature [8]. To counter this little issue, we need to elaborate more on how to compute the accumulated demand in (16). Actually, take for $[\tau, t]$ any interval with $x(\tau) = \xi_i^0 > s_0$. Then, condition (16) needs to be revised as

$$d_i^{\tau t} = \max \left\{ \sum_{k=\tau}^t \tilde{d}_i(k) - \xi_i^0, 0 \right\}. \quad (32)$$

The rationale behind the above formula has an immediate interpretation in the lot sizing context. Actually, the effective demand over an interval is the accumulated demand reduced by the inventory stored and initially available at the warehouse. From a computational standpoint, the revised formula (32) has a different effect depending on the cases where the accumulated demand exceeds the initial state or not as discussed next.

1. $\sum_{k=\alpha}^{\beta} \tilde{d}_i(k) \geq \xi_i^0$: the mixed linear program (MPC_i) with initial state $x(\tau) = \xi_i^0 > 0$ and accumulated demand $\sum_{k=\alpha}^{\beta} \tilde{d}_i(k)$ is turned into an (LP_i) characterized by null initial state $x(\alpha - 1) = 0$ and effective demand $d_i^{\alpha\beta} = \sum_{k=\alpha}^{\beta} \tilde{d}_i(k) - \xi_i^0$ as in the example below:

$$(MPC_i) \quad \sum_{k=\alpha}^{\beta} \tilde{d}_i(k) = 12, \quad x(\tau) = \xi_i^0 = 10 \quad \implies \quad (LP_i) \quad x(\alpha - 1) = 0, \quad d_i^{\alpha\beta} = 2;$$

2. $\sum_{k=\alpha}^{\beta} \tilde{d}_i(k) < \xi_i^0$: the mixed linear program (MPC_i) with initial state $x(\tau) = \xi_i^0 > 0$ and accumulated demand $\sum_{k=\alpha}^{\beta} \tilde{d}_i(k)$ is unfeasible. The solution obtained at previous period $\tau - 1$ applies. A second example is shown next:

$$(MPC_i) \quad \sum_{k=\alpha}^{\beta} \tilde{d}_i(k) = 7, \quad x(\tau) = \xi_i^0 = 10 \quad \implies \quad (LP_i) \text{ unfeasible.}$$

In both cases, the revised formula (32) helps us to generalize the linear program (LP_i) to cases where the initial state is non null and this is a crucial point when applying the lot sizing model in a receding horizon form.

5 Numerical example

In this specific example, dynamics (1) takes on the form expressed below. Such a dynamics is particularly significant as it reproduces the typical influence between position and velocity in a sampled second-order system. Initial and final states are null and state values must remain in the positive quadrant all over the horizon. More specifically, denoting by x_1 the position and $x_2(k)$ an opposite in sign velocity, the dynamics appears as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -\kappa \\ \kappa & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} - \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix} + \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \geq 0, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(N) \\ x_2(N) \end{bmatrix} = 0. \quad (33)$$

A closer look at the first equation reveals that a greater velocity $x_2(k)$ reflects into a faster decrease of position $x_1(k+1)$. Similarly, the second equation tells us that a greater position $x_1(k)$ induces a faster increase of

206 velocity $x_2(k+1)$ because of some elastic reaction. In both equations, the non negative disturbances $w_i(k) \leq 0$
 207 seek to push the states $x_i(k)$ out of the positive quadrant in accordance to Assumption 3. Their effect is
 208 counterbalanced by positive control actions u_i . Notice that matrix A can be decomposed as described
 209 in Assumption 2. Also, acting on parameter κ we can easily guarantee the “weakly coupling” condition
 210 expressed in Assumption 3.

Turning to the capacity constraints (2), for this two-dimensional example, these constraints can be rewritten as:

$$0 \leq \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \leq C \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}, \quad \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \in \{0, 1\}^2.$$

It is left to comment on the objective function (4). We consider the case where fixed costs are much more relevant than proportional and holding ones. This materializes in choosing a high value for f^k in comparison to values of parameters p^k, h^k as shown in the next linear objective function:

$$J(u, y) = \sum_{k=0}^{N-1} (\mathbf{1}^n u(k) + \mathbf{1}^n x(k) + \mathbf{100}^n y(k)).$$

211 This choice makes sense for two reasons. First, all the work is centered around issues deriving from the
 212 integer nature of $y(k)$. So, high values of f^k emphasize the role of integer variables in the objective function.
 213 Second, high fixed costs incentivate solutions with the fewest number of control actions and this facilitate
 214 the validation and interpretation of the simulated results.

The next step is to decompose dynamics (33) in scalar lot sizing form (14) which we rewrite below:

$$x_i(k+1) = x_i(k) - \tilde{d}_i(k) + u_i(k).$$

215 When it comes to the discussion on how to compute the estimated demand \tilde{d}_i , a natural choice is to set \tilde{d}_i as
 216 below, where we have denoted by $\tilde{x}_1(k)$ (respectively $\tilde{x}_2(k)$) the estimated value of state $x_1(k)$ (respectively
 217 $x_2(k)$) available to agent 2 (agent 1):

$$\begin{bmatrix} \tilde{d}_1(k) \\ \tilde{d}_2(k) \end{bmatrix} = \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix}. \quad (34)$$

218 Now, the question is: which expression should we use to represent the set of admissible state vectors X^k
 219 appearing in equation (12)? This question has much to do with another one: how does agent 1 predict \tilde{x}_2
 220 and the same for agent 2 with respect to state \tilde{x}_1 ? A possible answer is shown next:

$$\begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \kappa \bar{x}_1 \end{bmatrix} - \begin{bmatrix} 0 \\ w_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ C \end{bmatrix}, \quad \begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ \tilde{x}_2(0) \end{bmatrix}. \quad (35)$$

221 Let us elaborate more on the above equations. Regarding to variable $\tilde{x}_2(k)$, this is used in the evolution of
 222 $\tilde{d}_1(k)$ as in the first equation of (34). Because of the positive contribution of the term $\kappa \tilde{x}_2(k)$ on $\tilde{d}_1(k)$, a
 223 conservative approach would suggest to take for $\tilde{x}_2(k)$ a possible upper bound of $x_2(k)$ and this is exactly
 224 the spirit behind the evolution of $\tilde{x}_2(k)$ as expressed in the second equation of (35). Here, \bar{x}_1 is an average
 225 value for x_1 . A similar reasoning applies to $\tilde{x}_1(k)$, used in the evolution of $\tilde{d}_2(k)$ as in the second equation of
 226 (34). We now observe a negative contribution of the term $-\kappa \tilde{x}_1(k)$ on $\tilde{d}_2(k)$ and therefore take for $\tilde{x}_1(k)$ a
 227 possible lower bound of $x_1(k)$ as shown in the first equation of (35).

228 We can now move to show and comment our simulated results. We have carried out two different set
 229 of experiments whose parameters are displayed in Table 1. In the line of the weakly coupling assumption
 230 (see Assumption 3), we have set κ small enough and in the range equal from 0.01 to 0.225. Such a range
 231 works good as we will see that $|\kappa x_i|$ is always less than w_i , which also means $\Delta x(k) + Ew(k) < 0$. For sake
 232 of simplicity and without loss of generality, capacity C is set to three, disturbances w_i are unitary and \bar{x}_1
 233 is equal to one. Unitary disturbances facilitate the check out and interpretation of the results as when the
 234 accumulated demand over the horizon turns to be very close to the horizon length. The two experiments
 235 differ also in the horizon length N for the reasons clarified next.

	N	κ	C	$w_1(k)$	$w_2(k)$	\bar{x}_1
I	1 ... 10	0.1	3	1	1	1
II	6	{0.01, 0.2, 0.225}	3	1	1	1

Table 1: Simulation parameters chosen for the two experiments.

236 The first set of experiments aims at analysing the computational benefits of decomposition and relaxation
237 upon which our solution method is based. So, we consider horizon lengths N from one to ten. We do not need
238 to consider larger values of N as even in this small range of values, differences in the computational times
239 are already evident enough as clearly illustrated in Fig. 1. Here, we plot the average computational time vs.
240 the horizon length N of the mixed integer predictive control problem (solid diamonds), of the decomposed
241 problem ($MIPC_i$) (dashed squares), and of the linear program (LP_i). Average computational time means
242 the average time for one agent to make a single decision (the total time is about $2N$ times the average one).
243 As the reader may notice, the computational time of the linear program (LP_i) is a fraction either of the one
244 requested by the (MPC) or of the one required by the ($MIPC_i$).

245 (Figure 1 about here)

In a second set of simulations, we have inspected how the percentage error

$$\epsilon\% = \frac{\text{optimal cost of } (MPC_i) - \text{optimal cost of } (MPC)}{\text{optimal cost of } (MPC)}\%$$

246 varies with different values of the elastic coefficient κ . The role of κ is crucial as we recall that κ describes the
247 effective tightness and coupling between different states $x_1(k)$ and $x_2(k)$. We do expect that small values for
248 coefficient κ , which means weak coupling of state components, may lead to small errors $\epsilon\%$. Differently, high
249 values of κ , describing a strong coupling between state components, are supposed to induce higher values of
250 $\epsilon\%$.

251 This is in line with what we can observe in Fig. 3 where we plot the error $\epsilon\%$ as function of coefficient κ .
252 For a relatively small values of κ in the range from 0 to 0.2, we observe a percentage error not exceeding the
253 one percent, $\epsilon\% \leq 1$. A discontinuity at around $\kappa = 0.2$ causes the error $\epsilon\%$ to go from about 1% to 20%.

254 (Figure 2 about here)

255 We might not be surprised as discontinuity of errors is typical in mixed integer programs and we try to
256 clarify this in more details in the plot of Fig. 4. Here, for a horizon length $N = 6$ and for a relatively high
257 value of $\kappa = 0.225$, we display the exact solution (dashed squares) and approximate solution (solid triangles)
258 returned by the mixed integer linear program ($MIPC$) and by the linear program (LP_i) respectively. The
259 solution is in terms of the time plot of states $x_i(k)$, continuous controls $u_i(k)$ and discrete controls $y_i(k)$.
260 Dotted lines represent predicted trajectories in earlier periods of the receding horizon implementation. At a
261 first check, and this is in accordance with what we do expect, we note that controls $u_i(k)$ never exceed the
262 capacity and are always associated to unitary control actions $y_i(k)$. Now, with a look at the behaviour of
263 discrete controls $y_1(k)$, it can be observed that the approximate solution presents four control actions (four
264 peaks at one), whereas the exact solution has control $y_1(k)$ acting on the system only three times (three
265 peaks at one). One peak out of four represents an increase in the use of control actions of about 25 percent
266 which reflects into an approximate increase in the percentage error of 20%. A last observation concerning
267 the exact plot of $y_i(k)$ is that the number of control actions are as minimal as possible, i.e., three for $y_1(k)$
268 and two for $y_2(k)$. This makes sense as the accumulated demand over the horizon approximates by above the
269 horizon length. This implies that the minimum number of control actions can be roughly obtained dividing
270 the accumulated demand (about something above six) by the capacity C (equal to three) and rounding the
271 fractional result up to the next integer.

272

(Figure 3 about here)

273 Let us move to compare exact and approximate solutions for a smaller value of $\kappa = 0.2$. With reference
 274 to Fig. 4, we observe that, differently from above, discrete controls $y_i(k)$ coincide. However, we still have
 275 notable differences in the plot of continuous controls $u_1(k)$ which cause distinct state trajectories for $x_1(k)$.
 276 Small differences can be noted for $u_2(k)$ and $x_2(k)$ as well. The observed differences still cause a reduced
 277 percentage error $\epsilon\% = 1$.

278

(Figure 4 about here)

279 We conclude our simulations by showing that the percentage error $\epsilon\%$ is around zero when we reduce further
 280 the value of κ to 0.01. This is evident if we look at Fig. 5, where plots of different styles overlap which means
 281 that exact and approximate solutions coincide.

282

(Figure 5 about here)

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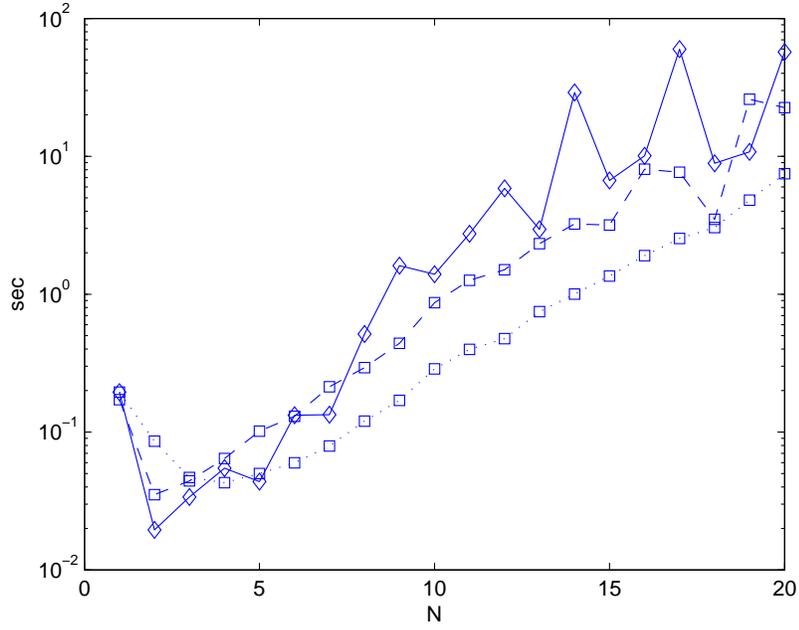


Figure 1: Average computational time vs. horizon length N of the mixed integer predictive control problem (solid diamonds), of the decomposed problem ($MIPC_i$) (dashed squares), and of the linear program (LP_i).

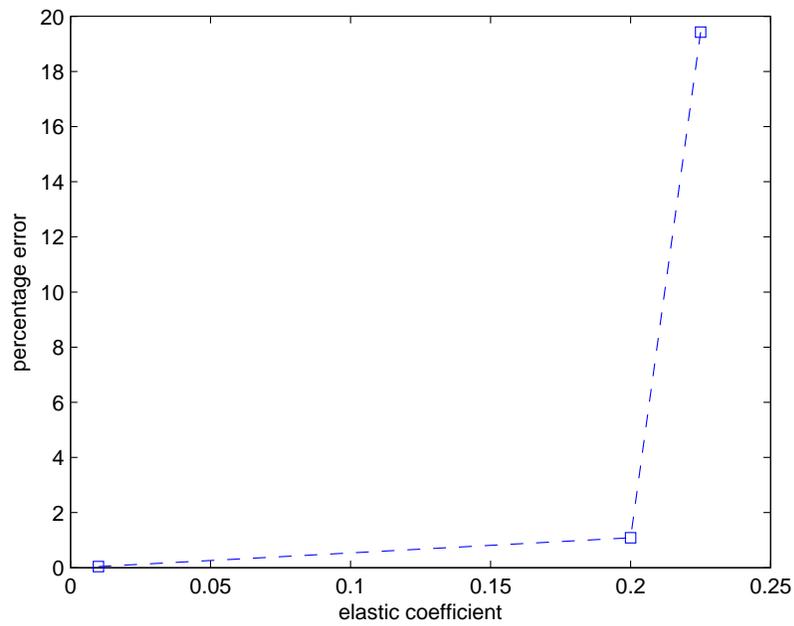


Figure 2: Percentage error $\epsilon\%$ for different values of the elastic coefficient k .

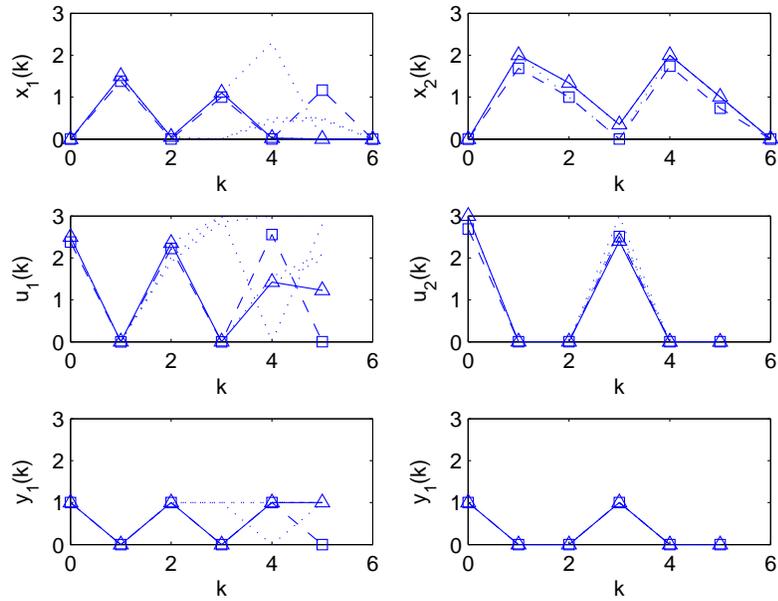


Figure 3: Elastic coefficient $\kappa = 0.225$. Exact solution (dashed squares) and approximate solution (solid triangles) returned by the mixed integer linear program (*MIPC*) and by the linear program (*LP_i*) respectively. Horizon length $N = 6$. Time plot of states $x_i(k)$, continuous controls $u_i(k)$ and discrete controls $y_i(k)$.

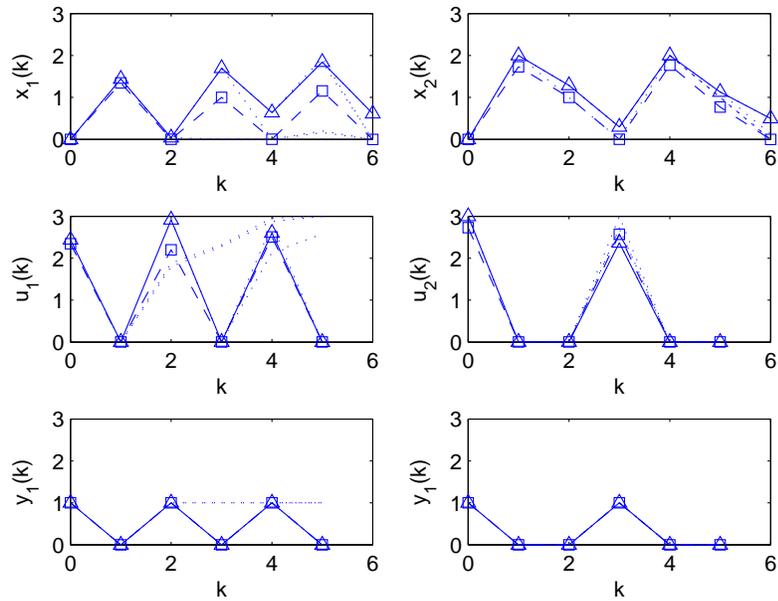


Figure 4: Elastic coefficient $\kappa = 0.20$. Exact solution (dashed squares) and approximate solution (solid triangles) returned by the mixed integer linear program (*MIPC*) and by the linear program (*LP_i*) respectively. Horizon length $N = 6$. Time plot of states $x_i(k)$, continuous controls $u_i(k)$ and discrete controls $y_i(k)$.

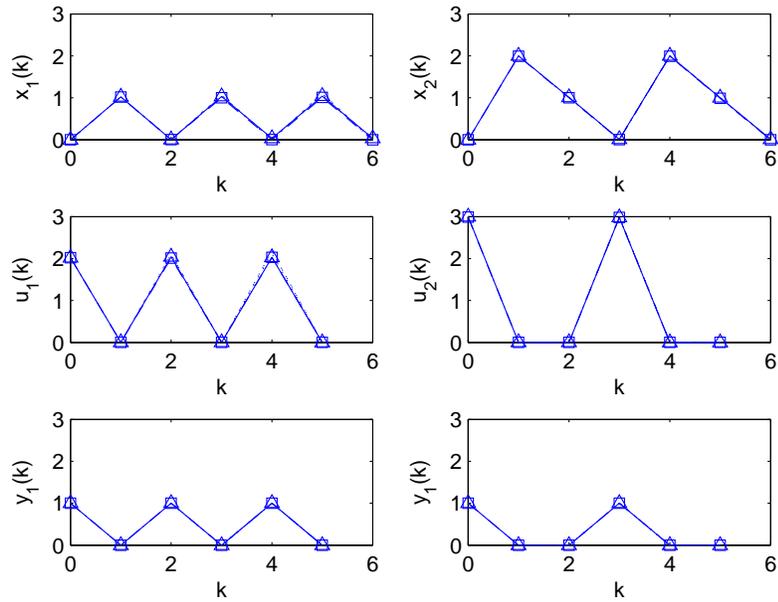


Figure 5: Elastic coefficient $\kappa = 0.001$. Exact solution (dashed squares) and approximate solution (solid triangles) returned by the mixed integer linear program (*MIPC*) and by the linear program (*LP_i*) respectively. Horizon length $N = 6$. Time plot of states $x_i(k)$, continuous controls $u_i(k)$ and discrete controls $y_i(k)$.