Article:

The linear saturated decentralized strategy for constrained flow control is asymptotically optimal

Dario Bauso [b] Franco Blanchini [a] Laura Giarré [c] Raffaele Pesenti [d]

[a]*Department of Mathematics and Computer Science, University of Udine, Via delle Scienze, 208 - 33100 Udine, Italy (e-mail: blanchini@uniud.it)*

[b]*Dip. di Ingegneria Chimica, Gestionale, Informatica e Meccanica, Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy (e-mail:dario.bauso@unipa.it)*

[c]*Dip. di Dipartimento di Ingegneria Elettrica, Elettronica e delle Telecomunicazioni, di tecnologie Chimiche, Automatica e modelli Matematici, Viale delle Scienze, I-90128, Palermo, Italy, (e.mail: laura.giarre@unipa.it)*

[d]*Dip. di Management, Università Ca' Foscari - Cannaregio 873 - 30121 Venezia, Italy (e-mail: pesenti@unive.it)*

Abstract

We present an algorithm for constrained network flow control in the presence of an unknown demand. Our algorithm is decentralized in the sense that it is implemented by a team of agents, each controlling just the flow on a single arc of the network based only on the buffer levels at the nodes at the extremes of the arc, while ignoring the actions of other agents and the network topology. We prove that our algorithm is also stabilizing and steady-state optimal. Specifically, we show that it asymptotically produces the minimum–norm flow. We finally generalize our algorithm to networks with a linear dynamics and we prove that certain least-square optimality properties still hold.

1 Introduction and motivation

Constrained minimum–norm flows play a central role in flow network optimization and find important applications in communication [11,22], water distribution control [19], traffic [12,21], manufacturing [9,27]. In particular, decentralized strategies are appealing in the control of large networks [29,2,12–15] in which the existence of a centralized supervisor implies a cost or it is even unrealistic.

In this paper, we look for a decentralized control capable of robustly stabilizing the network in the presence of an unknown demand. Although the literature on flow networks is quite wide, "robustness" has been brought into the picture only recently [1,4,7,20,23]. A robust decentralized strategy has been proposed in [8], where the authors present a Lyapunov based control that guarantees robustness against uncertain demand in the presence of both buffer and arc flow constraints. However, [8] is only concerned with stability while, here, we are mainly concerned with flow optimality. Long–term optimality, in flow networks with uncertain demand has been considered in [6], but the provided solutions are *centralized*.

A related work is [25] where the authors design the edges of a graph to optimize the spectrum of the corresponding weighted adjacency matrix. This paper is also related to the consensus seeking problem in [24]. There, there the same type of edge interaction is considered but with a different goal.

As main contribution of this work, we show that, under some necessary and sufficient conditions, a simple linear saturated control has the following properties:

- it assures practical stability, namely it drives the inventory levels arbitrarily close to an assigned reference, under flow capacity constraints;
- it is fully decentralized, namely it can be implemented by a set of agents, each of which controls just the flow on a single arc of the network based only on the buffer levels (*states*) at the nodes at the extremes of the arc while ignoring the actions of the other agents and the network topology.
- it is robust with respect to failures;
- it assures minimum–norm steady state flow.

Finally, we extend our results to a class of networks with *natural dynamics*. The results in this paper have been presented in a preliminary form in [3].
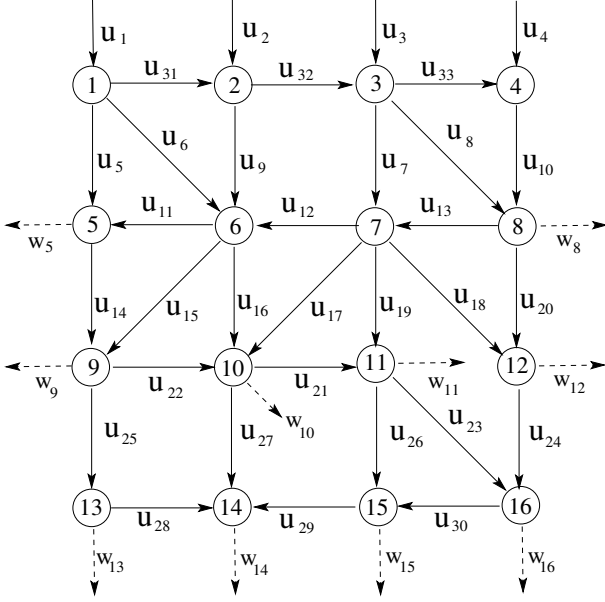
Fig. 1. The flow network with buffers

1.1 Motivating example

Consider the network in Fig. 1 with 16 buffers represented by nodes and 33 controlled flows represented by plain arcs. This system has the typical structure of a water distribution network in which the flow has to compensate the demand w represented by dashed arcs. The problem is how to stabilize this system assuming that each arc is governed by an agent who has only information about the source and destination buffers. For instance "agent 17" that governs arc $(7,10)$ must decide the instantaneous flow $u_{17}(t)$, compatibly with the constraints, based only on the head and tail nodes: $x_7(t)$ and $x_{10}(t)$. The agent ignores the demand, the other agents' controls, the network topology, including possible failures. Therefore, for agent 17, the cause of a possible decreasing/increasing trend of the buffer levels $x_7(t)$ or $x_{10}(t)$, such as a variation of demand w_{12} compensated by u_{18}, is unknown. Similarly, agent 17 cannot know how the consequences of its decisions propagate through the network.

2 Problem statement and preliminary results

Consider the system

$$\dot{x}(t) = Bu(t) - w(t), \quad \forall t \geq 0, \tag{1}$$

where $B \in \mathbb{R}^{n \times m}$, $u(t) \in \mathbb{R}^m$ is the controlled flow, henceforth the *control*, and $w(t) \in \mathbb{R}^n$ an uncontrolled flow, henceforth *demand*. The state variables $x_i(t)$ represent the buffer levels at the nodes, e.g., the amount of inventory or stored product at warehouses.

Models like (1) often arise in network flow [1] (as in Fig. 1), inventory [5–7], and supply-chain applications [26] and

traffic [2,14]. In general, matrix B can be *any real matrix*. Its generic entry B_{ij} indicates that the flow u_j causes the instantaneous variation $B_{ij}u_j$ in the state of node i. As we do not require the flow conservation, the sum of the entries of each column of B may be different from 0. In the example in Subsection 1.1, the first column $B_{\bullet 1} = [1\ 0\ 0 \ldots]^T$ of the B matrix associated to the network models the dependence of the state of node 1 on a flow coming from outside.

The control and demand are assumed subject to capacity constraints

$$u(t) \in \mathcal{U}, \quad w(t) \in \mathcal{W}, \tag{2}$$

where $\mathcal{U} \doteq \{u \in \mathbb{R}^m, u_i^- \leq u_i \leq u_i^+, \forall i\}$ is a box which constraints each component u_i of the control between a lower bound u_i^- and an upper bound u_i^+; differently $\mathcal{W}$ is a general convex and compact set.

The following assumptions are considered.

Assumption 1 *Matrix $B \in \mathbb{R}^{n \times m}$ is full row rank and set $\mathcal{W}$ is in the interior of $B\mathcal{U}$, that is,*

$$\mathcal{W} \subset int\{B\mathcal{U}\}. \tag{3}$$

Assumption 2 *The demand w is constant and not available for the synthesis of control $u(t)$.*

Assumption 1 implies that the network described by B is connected, and may receive and give flow to the external world. The whole assumption is required in view of the following result.

Proposition 1 *[8] Condition (3) is necessary and sufficient for the existence of a stabilizing control under constraints (2).*

Assumption 2 is required to prove flow optimality at steady state, though it is not necessary for practical stability.

As previously explained, any control component u_j affects only the nodes corresponding to the non–zero entries of the jth column $B_{\bullet j}$ (arc) of matrix B. Let us call "extremes" the corresponding nodes. Formally, we say that a control is decentralized if it satisfies the following definition.

Definition 2.1 Decentralized control. *The control strategy $u = \Phi(x)$ is decentralized if the control flow $u_j(t) \in [u_j^-, u_j^+]$ of each arc j depends only on the values of the buffer levels $x_i(t)$ of the nodes i at its extremes (i.e., the nodes such that $B_{ij} \neq 0$).*

For instance if $B_{\bullet 1} = [0\ 1\ 1\ 0\ -1]^T$, then the control u_1 must be function only of the state of nodes 2, 3 and 5.

We are finally in the position of formally stating the problem. Assume that a reference point $\hat{x}$ is assigned and assume, without loss of generality, $\hat{x} = 0$.

Problem 1 *Given a tolerance $\varepsilon > 0$, a demand $w \in \mathcal{W}$ and a reference value $\hat{x}$ for the buffer levels, find a decentralized control law $u(t) = \Phi(x(t))$, not depending on w, ensuring*

- Practical stability*: $x(t)$ is ultimately bounded in the ε– ball centered in 0, namely, for all $x(0)$ there exists T such that $\|x(t)\| \leq \varepsilon$ for $t \geq T$.*
- Optimality at steady–state*: the control at the equilibrium, denoted by $\bar{u} \doteq \lim_{t \to \infty} u(t)$ is of minimum–norm [1], i.e.,*

$$\bar{u} = u^* \doteq \arg\min_{u \in \mathcal{U}} \frac{1}{2} u^T u : \quad Bu = w. \qquad (4)$$

Remark 2.1 *No agent can estimate the demand from its extreme node inventory levels neither in the case of known topology nor in the case of constant demand. For instance agent 17 in the example in Subsection 1.1, controls nodes 7 and 10 and can estimate neither w nor the whole state x from the outputs $y_1 = x_7$ and $y_2 = x_{10}$, because, as it is easy to verify, the observability matrix would not have full rank. Unfortunately the control cannot change this situation, unless we admit, against our assumptions, that an agent can know exactly both the controls applied by other agents and their link bounds.*

2.1 Towards the optimum of Problem 1

As a preliminary result, we observe that problem (4) has a strictly convex objective function $\frac{1}{2} u^T u$ continuous over all the compact convex domain $D = \{u \in \mathcal{U} : Bu = w\}$, hence (4) is a convex optimization problem with a unique optimal solution u^*. Specifically, Problem (4) is a particular quadratic programming problem [10] pp. 302-304).

A saturation function $sat[.] : \mathbb{R}^m \to \mathbb{R}^m$ is defined componentwise as follows (see Fig. 2)

$$u_i = sat[y_i] \doteq \begin{cases} u_i^- & \text{if } y_i < u_i^- \\ y_i & \text{if } u_i^- \leq y_i \leq u_i^+ \\ u_i^+ & \text{if } y_i > u_i^+ \end{cases}.$$

Lemma 2.1 *There exists $\xi^* \in \mathbb{R}^n$ such that $u^* = sat[B^T \xi^*]$.*

Proof. Let ∇ be the gradient operator with respect to u. Recall that in Problem (4): the objective function is a continuously differentiable convex function over the compact convex domain D and the constraints are continuously differentiable affine functions. Then, the following Karush-Khun-Tucker conditions are necessary and sufficient to identify

---

Problem (4) unique optimal solution u^* (see [10] pp. 243-246):

$$\nabla \frac{1}{2} u^T u =$$
$$\nabla(Bu - w)^T \xi - \nabla(u^- - u)^T \lambda - \nabla(u - u^+)^T \nu \qquad (5a)$$
$$Bu = w \quad u^- \leq u \leq u^+ \qquad (5b)$$
$$\lambda^T(u^- - u) = 0, \quad \nu^T(u - u^+) = 0 \qquad (5c)$$
$$\lambda, \nu \geq 0 \qquad (5d)$$
$$u, \lambda, \nu \in \mathbb{R}^m, \quad \xi \in \mathbb{R}^n \qquad (5e)$$

Note that (5a) is equivalent to $u = B^T \xi + I\lambda - I\nu$.

As an optimal solution u^* exists for problem (4), then there also exists a solution $(u^*, \xi^*, \lambda^*, \nu^*)$ satisfying the above conditions. The next step is to note that this solution satisfies

$$u^* = sat[B^T \xi^*] \qquad (6a)$$
$$\lambda^* = \max\{sat[B^T \xi^*] - B^T \xi^*, 0\} \qquad (6b)$$
$$\nu^* = \max\{B^T \xi^* - sat[B^T \xi^*], 0\} \qquad (6c)$$

Indeed, conditions (5) trivially imply conditions (6), for all components u_i^* such that $u_i^- < u_i^* < u_i^+$. Now assume that $u_i^* = u_i^-$ and hence $\lambda_i^* \geq 0$ and $\nu_i^* = 0$. Then, (5a) implies $u_i^- = (B^T \xi^*)_i + \lambda_i^*$. As $\lambda_i^* \geq 0$, we have $(B^T \xi^*)_i \leq u_i^-$, more specifically $(B^T \xi^*)_i \leq u_i^- = sat[(B^T \xi^*)_i] = u_i^*$, that is, condition (6a). Condition (6b) holds as $\lambda_i^* = u_i^- - (B^T \xi^*)_i = sat[(B^T \xi^*)_i] - (B^T \xi^*)_i \geq 0$. Finally, condition (6c) holds as $\nu^* = 0 = \max\{(B^T \xi^*)_i - sat[(B^T \xi^*)_i], 0\}$.

The proof is completed observing that a symmetrical argument holds if $u_i^* = u_i^+$. ∎

In general, we can state the following theorem.

Theorem 2.1 *For all vector $\xi \in \mathbb{R}^n$, such that $Bsat(B^T \xi) = w$, the control $u = sat[B^T \xi]$ is the optimal solution of (4).*

Proof. It follows from Lemma 2.1, because for any vector ξ satisfying the hypotheses of this theorem, we can take $u = sat[B^T \xi]$, $\lambda = \max\{sat[B^T \xi] - B^T \xi, 0\}$, $\nu = \max\{B^T \xi - sat[B^T \xi], 0\}$. It is easily verified that (u, ξ, λ, ν) satisfy the necessary and sufficient Karush-Khun-Tucker conditions (5) introduced in the previous lemma. ∎

3 Decentralized linear saturated control

We now exploit the structure of the optimum of problem (4) described in Theorem 2.1 to prove convergence of system (1) to an equilibrium point $\bar{x}$ under the linear saturated control

$$u(t) = sat\left[-\frac{B^T}{\gamma} x(t)\right], \quad \gamma > 0 \qquad (7)$$

which is *decentralized* as in Definition 2.1. As a prelude we analyze the steady–state condition for the system with control (7). Specifically, we collect all states x satisfying the steady–state conditions in the set $\mathscr{L}_\gamma$:

$$\mathscr{L}_\gamma \doteq \left\{ x : Bsat\left[-\frac{B^T}{\gamma}x\right] = w \right\}. \tag{8}$$

We also notice that, under Assumption 1, there exists $\beta > 0$ such that the following condition holds for all $x \in \mathbb{R}^n$ [8]

$$\min_{u \in \mathscr{U}} x^T Bu - x^T w \leq -\beta \|x\|, \tag{9}$$

The minimizer $u(x) = \Phi_{BB}(x) = \arg\min_{u \in \mathscr{U}} x^T Bu$ has the expression ($B_{\bullet j}$ is the jth column of B)

$$u_j = \Phi_{BBj}(x) = \begin{cases} u_j^+ & \text{if } -B_{j\bullet}^T x > 0 \\ u_j^- & \text{if } -B_{j\bullet}^T x < 0 \ , \\ u_j \in [u^-, u^+] & \text{otherwise} \end{cases} \tag{10}$$

(controllers of this type have been adopted in [2,14,15]). Control $\Phi_{BB}(x)$ can be seen as the limit of (7) for $\gamma \to 0$.

Remark 3.1 *Differently from control (7), control $\Phi_{BB}(x)$ is not continuous, and in general causes chattering, therefore cannot assure $u(t) \to u^*$ as required. The small price we pay is that we have to accept practical stability which seems more than reasonable in the context of networks.*

Lemma 3.1 *Under Assumption 1, $\mathscr{L}_\gamma$ is bounded and convex for any $\gamma > 0$. Moreover, if we denote by $\mathscr{L}_1$ the set of solutions of (8) with $\gamma = 1$, the set $\mathscr{L}_\gamma$ is achieved by scaling $\mathscr{L}_1$ as*

$$\mathscr{L}_\gamma = \gamma \mathscr{L}_1 = \{\gamma x : \ x \in \mathscr{L}_1\}.$$

Proof. We first prove the second part. If $y \in \mathscr{L}_1$ then $x = \gamma y \in \mathscr{L}_\gamma$. Indeed, as we can write $y = \frac{x}{\gamma}$, then $Bsat\left[-B^T y\right] = w$ implies $Bsat\left[-\frac{B^T}{\gamma}x\right] = w$. Symmetrically, if $x \in \mathscr{L}_\gamma$, then $y = \frac{x}{\gamma} \in \mathscr{L}_1$. As we can write $x = \gamma y$, then $Bsat\left[-\frac{B^T}{\gamma}x\right] = w$ implies $Bsat\left[-B^T y\right] = w$.

Now we prove that $\mathscr{L}_1$ is convex (which proves that $\mathscr{L}_\gamma$ is such). To this end, we recall that (4) has a unique optimal solution and that $u = sat\left[-B^T x\right]$ satisfies the hypotheses of Theorem 2.1, for all $x \in \mathscr{L}_1$. Then, for all x^1 and $x^2 \in \mathscr{L}_1$, we have

$$sat\left[-B^T x^1\right] = sat\left[-B^T x^2\right].$$

As a consequence, for all unsaturated components i for which $u_i^- < -B_{i\bullet}^T x^1 = -B_{i\bullet}^T x^2 < u_i^+$ holds, also $u_i^- < -B_{i\bullet}^T(\lambda x^1 + (1-\lambda)x^2) < u_i^+$ holds, for any $0 \leq \lambda \leq 1$. On the other hand, the saturated components i, which are characterized by $-B_{i\bullet}^T x^k \geq u_i^+$, respectively $-B_{i\bullet}^T x^k \leq u_i^-$, for

$k = 1,2$, satisfy the condition $-B_{i\bullet}^T(\lambda x^1 + (1-\lambda)x^2) \geq u_i^+$, respectively $-B_{i\bullet}^T(\lambda x^1 + (1-\lambda)x^2) \leq u_i^-$, for any $0 \leq \lambda \leq 1$. Therefore, $\mathscr{L}_1$ is convex as for all x^1 and $x^2 \in \mathscr{L}_1$, also $x = \lambda x^1 + (1-\lambda)x^2$ is in $\mathscr{L}_1$ for any $0 \leq \lambda \leq 1$.

To prove boundedness of $\mathscr{L}_1$ (and then of $\mathscr{L}_\gamma$) we take an arbitrary $\hat{x}$ and $x \neq 0$ both in $\mathscr{L}_1$, and we define

$$\sigma \doteq \sup \ \{\lambda \geq 0 \ \text{ such that } \ \hat{x} + \lambda x \in \mathscr{L}_\gamma\}.$$

Note that boundedness of $\mathscr{L}_1$ is ensured if σ is finite.

By contradiction assume $\sigma = +\infty$. By the definition of $\mathscr{L}_1$, condition (9) and the properties of $\Phi_{BB}(x)$ to write

$$\begin{aligned} 0 = \ & x^T B \ sat[-B^T(\hat{x}+\lambda x)] - x^T w = \\ & x^T B \ \Phi_{BB}(x) - x^T w + x^T B \ sat[-B^T(\hat{x}+\lambda x)] \\ & -x^T B \ \Phi_{BB}(x) \leq -\beta \|x\| + \phi(\hat{x}, x, \lambda) \end{aligned} \tag{11}$$

where

$$\phi(\hat{x}, x, \lambda) = x^T B \ sat[-B^T(\hat{x}+\lambda x)] - x^T B \ \Phi_{BB}(x).$$

The first equality in (11) must hold for all $\lambda \geq 0$. In addition, we get

$$\phi(\hat{x}, x, \lambda) = \sum_{j=1}^m \ x^T B_{\bullet j}(sat[-B_{j\bullet}^T(\hat{x}+\lambda x)] - \Phi_{BBj}(x)),$$

where each of the terms in the sum is null for λ large. Therefore, the sequence of (in)equalities (11) returns the contradiction $0 \leq -\beta \|x\|$ for $x \neq 0$ for λ large enough. $\blacksquare$

The previous lemma leads to the following convergence result.

Theorem 3.1 *Under Assumption 1, the control (7), with arbitrary $\gamma > 0$, is such that:*

- *$x(t)$ is bounded,*
- *$x(t) \to \mathscr{L}_\gamma$,*
- *$u(t) \to \bar{u} = sat\left[-\frac{B^T}{\gamma}\bar{x}\right] = u^*$ the optimal solution of (4).*

Proof. Denote by $\bar{x} \in \mathscr{L}_\gamma$ any solution of (8). Consider the Lyapunov function

$$\Psi(x - \bar{x}) = (x - \bar{x})^T(x - \bar{x})/2 = \|x - \bar{x}\|^2/2.$$

If we take for $\bar{x}$ the barycenter of $\mathscr{L}_\gamma$, then $\Psi(\cdot)$ can be seen as a distance of x from $\mathscr{L}_\gamma$. Denoting by $v \doteq -B^T x/\gamma$ and

$\bar{v} = -B^T\bar{x}/\gamma$, the derivative of $\Psi(\cdot)$ takes the form

$$\dot{\Psi}(x-\bar{x}) = (x-\bar{x})^T \left(Bsat\left[-\frac{B^T}{\gamma}x\right] - w \right) =$$

$$-\gamma \left((x-\bar{x})^T \left(-\frac{B}{\gamma}\right) \right) \left(sat\left[-\frac{B^T}{\gamma}x\right] - sat\left[-\frac{B^T}{\gamma}\bar{x}\right] \right)$$

$$= -\gamma(v-\bar{v})^T \left[sat[v] - sat[\bar{v}] \right]$$

$$= -\gamma \sum_{i=1}^{m} (v_i - \bar{v}_i) \left[sat[v_i] - sat[\bar{v}_i] \right].$$

It is easy to see that each function $(v_i - \bar{v}_i)\left[sat[v_i] - sat[\bar{v}_i]\right]$ is non negative, and it is positive for $v_i \neq \bar{v}_i$ if $\bar{v}_i$ is an interior point of the interval $[u_i^-, u_i^+]$ (see Fig. 2). Then $\|x(t) - \bar{x}\| \leq \|x(0) - \bar{x}\|$ so that $x(t)$ is bounded.
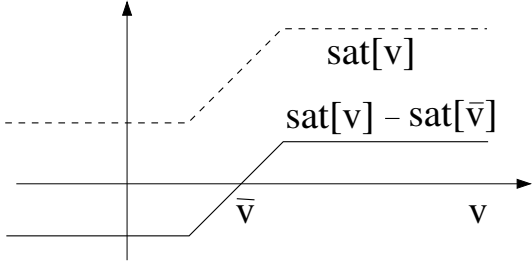

Fig. 2. The sat–function and its translation

Now note that the condition $\dot{\Psi}(x-\bar{x}) = 0$ holds either when $\left(Bsat\left[-\frac{B^T}{\gamma}x\right] - w \right) = 0$ or $x = \bar{x}$. Therefore, since $\bar{x} \in \mathscr{L}_\gamma$, $\mathscr{L}_\gamma$ is the set of all x for which $\dot{\Psi}(x-\bar{x}) = 0$. By the LaSalle Theorem (see [16], Th. 3.4) $x(t)$ converges to the largest invariant subset of $\mathscr{L}_\gamma$ [2], and then $x(t) \to \mathscr{L}_\gamma$.

From Theorem 2.1, and the continuity of the adopted control, we can immediately conclude that $u(t) \to u^*$. ∎

The following corollary proves that the linear saturated control (7) guarantees practical stability of system (1) even under time–varying $w(t) \in \mathscr{W}$.

Corollary 3.1 *Given any arbitrary small neighborhood of the origin, the linear saturated control (7), makes the state x of system (1) converge to such neighborhood provided that $\gamma > 0$ is small enough. Moreover, convergence is assured even for time–varying $w(t) \in \mathscr{W}$.*

Proof. If w is constant, the proof immediately follows from the fact that $\mathscr{L}_\gamma$ is arbitrarily small and $x(t) \to \mathscr{L}_\gamma$. In the case of time–varying $w(t)$, control (10) assures convergence to the origin with Lyapunov function $V(x) = x^T x$ [8]. On the other hand, control (7) converges to control (10) and therefore, by

_____
[2] in our case we have a stronger condition since $\dot{x} = 0$ for $x \in \mathscr{L}_\gamma$, so $\mathscr{L}_\gamma$ is invariant

invoking [17] Section 11.2.4, the former assures convergence to an arbitrarily small neighborhood of the origin for $\gamma > 0$ small. ∎

Under the uniqueness of $\bar{x}$ we have the following.

Corollary 3.2 *If the equilibrium state $\bar{x}$ is the unique solution of (8) it is globally uniformly asymptotically stable.*

If $\mathscr{L}_\gamma$ is not as singleton, then the system converge to an equilibrium state, because any point of $\mathscr{L}_\gamma$ is a possible steady state. Consider for instance $\dot{x} = u_1 + u_2 - w$, and $0 \leq u_1 \leq 1$, $2 \leq u_1 \leq 3$, $1 \leq w \leq 4$. It is easy to see that for $\gamma = 1$ and $w = 3$ we have the interval $\mathscr{L}_\gamma = [1,2]$. Then the state $x(t)$ will converge to the upper bound $x = 2$ for $x(0) > 2$ and to the lower bound $x = 1$ for $x(0) < 1$. It will be constant for $1 < x(0) < 2$.

Remark 3.2 (Unbounded control) *In the case $\mathscr{U} = \mathbb{R}^m$, the control (7) collapses to the linear feedback control $u(t) = -\frac{B^T}{\gamma}x(t)$. At the equilibrium the control is $\bar{u} = B^T(BB^T)^{-1}w$, which corresponds to the minimum 2-norm solution of $Bu = w$, and does not depend on γ. As a possible physical interpretation, we can see the buffers as capacitors and the links as identical resistors. In this framework, the equilibrium control correspond to the minimal–dissipation solution. We skip the details for brevity.*

Remark 3.3 (Failures) *The control is robust against failures as long as the necessary and sufficient condition (3) remains satisfied. Indeed, assume that the link must work at a reduced capacity. This corresponds to changing arc j bounds from u_j^-, u_j^+ to $\tilde{u}_j^- \geq u_j^-$ or $\tilde{u}_j^+ \leq u_j^+$. Complete flow interruptions on the arc are captured by setting $\tilde{u}_j^- = \tilde{u}_j^+ = 0$. Even if the occurrence of a failure on arc j is sensed only by the agent acting on the arc (information on failures is local), the decentralized strategy assures steady-state optimality (though the minimum-norm flow may have changed in consequence of the failures).*

Finally we notice that dealing with bounds of the form

$$x^- \leq x \leq x^+$$

(where x^- is a negative vector since 0 is the reference) is quite easy in this context. First of all the considered Lyapunov function $V(x) = (x - \bar{x})^T(x - \bar{x})$ assures a domain of attraction (a sphere) internal to such bounds $\mathscr{S}_\kappa = \{x : V(x) \leq \kappa\}$ for a proper κ. For any initial condition inside this set, the bounds are not violated. It is also possible to "enlarge" such a domain of attraction by adopting the Lyapunov function proposed in [8].

3.1 Ultimate bounds for the solution

In the view of Theorem 3.1 and of Corollary 3.1, and given a tolerance $\varepsilon > 0$, we here face the problem of selecting a

value γ ensuring $||\mathscr{L}_\gamma|| < \varepsilon$ without an a-priori knowledge of w, that is knowing only $\mathscr{W}$ (or, even, only knowing $\mathscr{U}$). To this end, we initially study the set $\mathscr{L}_1(w) = \{x \in \mathbb{R}^n : Bsat[-B^T x] = w\}$ for a generic $w \in \mathscr{W}$. In view of Theorem 2.1 we have that

$$\mathscr{L}_1(w) = \{x : sat[-B^T x] = u^*\}$$

where u^* is the optimal solution. Then $\mathscr{L}_1(w)$ is a polyhedron defined by the following equalities and inequalities

$$(-B^T x)_j = u_j^* \quad \forall j \in I^U, \tag{12}$$
$$(-B^T x)_j \geq u_j^+ \ \forall j \in I^+, \ (-B^T x)_j \leq u_j^- \quad \forall j \in I^- \tag{13}$$

where I^U, I^+ I^- denote the subset of indexes where the components u^* are, respectively unconstrained, are at the upper bound $u_j = u_j^+$ or are at the lower bound $u_j = u_j^-$. The above argument and Theorem 3.1 imply that $\mathscr{L}_1(w)$ is a bounded polyhedron and that we can determine a bound for such a set in each principal direction x_i by solving an linear programming problem

$$\mu_i^+ (\text{resp. } \mu_i^-) \doteq \max \ (\text{resp. min})x_i : \ x \in \mathscr{L}_1(w) \tag{14}$$

Then we have $||x||_\infty \leq \max_i \{\max\{|\mu_i^+|, |\mu_i^-|\}\}$ for all $x \in \mathscr{L}_1(w)$. In view of Lemma 3.1, the condition $||\mathscr{L}_\gamma(w)|| < \varepsilon$ is assured if

$$\gamma \leq \varepsilon \sqrt{n} \max_i \{\max\{|\mu_i^+|, |\mu_i^-|\}\} \tag{15}$$

To get some bounds $\hat{\mu}_i^+$, $\hat{\mu}_i^-$ which are valid for all w we relax the equality (12) to achieve a superset $\hat{\mathscr{L}}_1(w) \supseteq \mathscr{L}_1(w)$

$$\hat{\mathscr{L}}_1(w) = \left\{x \in \mathbb{R}^n : u_j^- \leq (-B^T x)_j \leq u_j^+, \forall j \in I^U, \text{ and } (13)\right\}$$

Note that the boundedness of $\mathscr{L}_1(w)$ implies the boundedness of $\hat{\mathscr{L}}_1(w)$ [3]. Note also that the superset $\hat{\mathscr{L}}_1(w)$ depends *only* on the partition $\{I^U, I^+, I^-\}$ of the index set associated with $u^*(w)$, so there is a finite number of possible supersets $\hat{\mathscr{L}}_1(w)$. This immediately implies that we can evaluate the bounds $\hat{\mu}_i^+$, $\hat{\mu}_i^-$ and hence achieving an expression of the form (15) by enumerating all the possible partitions $\{I^U, I^+, I^-\}$ of the index set and then solving linear programming problems similar to (14) for the associated sets $\hat{\mathscr{L}}_1$, paying attention to disregard the problems with unbounded solutions.

4 Networks with natural dynamics

The results of the previous sections can be extended to more general systems. In particular, consider linear systems evolv-

---

[3] since a linear equality $px = q$ has been replaced by two inequalities $q^- \leq px \leq q^-$, with $q^- \leq q \leq q^+$

ing according to the equation

$$\dot{x} = -Lx + Bu - w \tag{16}$$

where L is a $n \times n$ symmetric positive semidefinite matrix and B is, again, the $n \times m$ network matrix. The term $-Lx$ describes "the natural system dynamics", whereas Bu is the "forced flow". For example, in water distribution networks, the natural system flows are induced by the potentials due to the differences among buffer levels. In this cases, L is the graph Laplacian matrix. Due to the presence of L, we can relax Assumption 1 as follows.

Assumption 3 $\mathscr{W} \subset int\{Ra[L] + B\mathscr{U}\}$ where $Ra[L]$ is the range of matrix L.

The structural interpretation of the assumption is basically that if we decompose $L = \sum_1^M l_j l_j^T$, with l_i non–zero vectors, we may consider these vectors as inducing corresponding columns of B: $[B \ l_1 \ldots l_M]$ with infinite capacity in which "nature" has already placed the controls $u_j = -l_i^T x$, $j = m+1, \ldots, m+M$.

Theorem 4.1 *Assumption 3 is necessary and sufficient for the existence of a strategy which keeps the state bounded.*

Proof. We prove necessity, since sufficiency will be proved constructively later on. If the assumption fails there exists a vector w not included in $Ra[L] + B\mathscr{U}$. Consider the following square matrix, whose columns form a basis of the vector space, $[T^L \ T^\perp]$ in which T^L represents the components along $Ra[L]$ and $T^\perp$ the orthogonal to T^L. Let $[R \ S]^T$ be its inverse and write

$$\begin{bmatrix} x_L \\ x_\perp \end{bmatrix} = \begin{bmatrix} R^T \\ S^T \end{bmatrix} x$$

so that we get

$$\dot{x}_L = -R^T Lx + R^T Bu - R^T w, \qquad \dot{x}_\perp = S^T Bu - S^T w.$$

Now, note that Assumption 3 is equivalent to Assumption 1 applied to the second subsystem, namely $S^T \mathscr{W} \subset S^T B\mathscr{U}$ which is necessary to keep $x_\perp$ bounded. $\blacksquare$

We seek a stabilizing control law $u \in \mathscr{U}$ which is optimal at steady–state, precisely which minimizes

$$J_L = \frac{1}{2} x^T Lx + \frac{1}{2} u^T u, \quad s.t. \quad -Lx + Bu = w, \tag{17}$$

where the term $x^T Lx$ represents the energy of the natural flow and the term $u^T u$ is the energy of the forced flow. For instance, in the case of water distribution, $x^T Lx$ represents the sum of the squares of the difference of potentials at the nodes.

Theorem 4.2 *The decentralized control $u = sat[-B^T x]$ asymptotically minimizes (17).*

Proof. Given any factorization of L, $L = CC^T$, consider the auxiliary system

$$\dot{x} = Cv + Bu - w \qquad (18)$$

whose control components are u and v. Despite the usual assumption on u being bounded in $\mathscr{U}$, we let v be unbounded. Now, we reformulate Problem 1 in terms of the controls u and v

$$\tilde{J}_L = \frac{1}{2} v^T v + \frac{1}{2} u^T u, \qquad s.t. \quad Cv + Bu = w. \qquad (19)$$

According to the derivations obtained in Section 3, and specifically reported in Theorem 3.1, the optimum for problem (19) takes on the form

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} sat[-C^T x] \\ sat[-B^T x] \end{bmatrix} = \begin{bmatrix} -C^T x \\ sat[-B^T x] \end{bmatrix}, \qquad (20)$$

where the first equality holds because v is unbounded. The minimum $v = -C^T x$ is a feasible solution, and hence an optimal one, for the original problem (17). Indeed, if we replace control (20) in (18), we get the stable closed–loop system achieved by (16) when $u = sat[-B^T x]$. This last system asymptotically reaches the equilibrium condition $-Lx + Bu - w = 0$ with x and u optimal in the sense of (17). ∎

5 Example

Let us reconsider the network proposed in Subsection 1.1, with the following data:

$$u^+ = [\,4\ 4\ 4\ 2\ 2\ 2\ 1\ 2\ 3\ 2\ 1\ 1\ 1\ 2\ 2\ 2\ 1$$
$$2\ 2\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 3\ 1\ 1\ 1\,]^T$$

$$u^- = [\,1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0$$
$$0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]^T.$$

We assumed for simulation purposes the following demand:

$$w = [\,0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\,]^T$$

In Fig. 3 we plot the buffer evolution from the initial condition

$$x(0) = [\ 10\ -10\ 10\ 20\ -10\ 10\ -10\ 20\ 0$$
$$10\ 10\ -10\ -20\ 10\ -20\ -10]^T$$

converging to $\bar{x}$ whose norm is $\|\bar{x}\| = 0.2210$.

The minimal norm solution computed by a standard convex
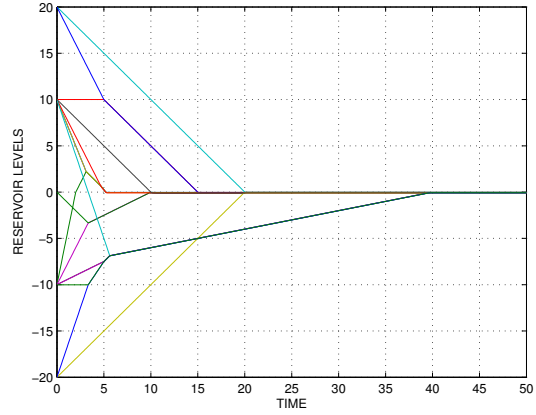

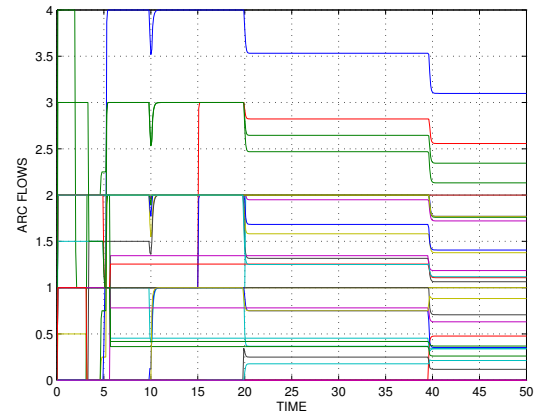Fig. 3. The buffer transient which converges to u_{opt}

Fig. 4. The flow transient

optimization routine is

$$u_{min} = [\ 3.098\ 2.344\ 2.557\ 2.000\ 1.721\ 1.378\ 1.000$$
$$1.770\ 2.132\ 2.000\ 0.343\ 0.000\ 1.000\ 1.064$$
$$1.407\ 1.760\ 0.477\ 0.338\ 1.185\ 1.7700.707$$
$$0.353\ 0.261\ 1.108\ 1.118\ 0.631\ 0.882\ 0.118$$
$$0.000\ 0.369\ 0.000\ 0.213\ 0.000\ ]$$

In Fig. 4 it is shown the flow transient evolving to such value numerically checked with a tolerance of 10^{-14}.

6 Conclusions

We have considered a decentralized linear–saturated control for constrained network flow. The main result of the paper shows that such a control is optimal at steady state, in the sense that it returns the minimum–norm flow that satisfies the demand. Finally we have proved optimality at steady state even in those cases in which the network has a natural dynamics induced from node potentials.

References

[1] A. Atamturk and M. Zhang, "Two–stage robust network flow and design under demand uncertainty," *Operations Research*, vol. 55, no. 4, pp. 662–673, 2007.

[2] B. Ataslar and A. Iftar, "A decentralized control approach for transportation networks," *Proc. of the 8th IFAC Symposium. on Large Scale Systems*, pp. 348-353, 1998.

[3] D. Bauso, F. Blanchini, L. Giarre, L. Pesenti, "A decentralized solution for the constrained minimum cost flow", Proceedings of the 49th IEEE Conference on Decision and Control (CDC), pp. 661 - 666, Atlanta, Dec. 2010.

[4] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski, "Foreword: special issue on robust optimization," *Mathematical Programming*, Ser. B, vol. 107, pp. 1–3, 2006.

[5] D. Bertsimas and A. Thiele, "A Robust Optimization Approach to Inventory Theory," *Operations Research*, vol. 54, no. 1, pp. 150–168, 2006.

[6] D. Bauso, F. Blanchini, and R. Pesenti, "Optimization of Long-run Average-Flow Cost in Networks with Time-Varying Unknown Demand," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 20–31, 2010.

[7] F. Blanchini, F. Rinaldi, and W. Ukovich, "Least Inventory Control of Multi-Storage Systems with Non-Stochastic Unknown Input," *IEEE Transactions on Robotics and Automation*, vol. 13, pp. 633–645, 1997.

[8] F. Blanchini, S. Miani, and W. Ukovich, "Control of production-distribution systems with unknown inputs and system failures," *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1072–1081, 2000.

[9] E. K. Boukas, H. Yang, and Q. Zhang, "Minimax production planning in failure–prone manufacturing systems," *Journal of Optimization Theory and Applications*, vol. 82, no. 2, pp. 269–286, 1995.

[10] S. Boyd, L. Vandenberghe, *Convex optimization*, Cambridge University Press, Cambridge, UK 2004

[11] A. Ephremides and S. Verdú, "Control and Optimisation Methods in Communication Networks," *IEEE Transactions on Automatic Control*, vol. 34, pp. 930–942, 1989.

[12] A. Iftar, "A linear programming based decentralized routing controller for congested highways," *Automatica*, vol. 35, no. 2, pp. 279–292, 1999.

[13] A. Iftar and E. J. Davison, "Decentralized Robust Control for Dynamic Routing of Large Scale Networks," *Proc. of the American Control Conference*, San Diego, CA, pp. 441–446, 1990.

[14] A. Iftar, "A decentralized routing control strategy for semi-congested highways," *Proc. of the 13th IFAC World Congress*, Vol. P, pp. 319-324, 1996.

[15] A. Iftar and E.J. Davison, "Decentralized control strategies for dynamic routing," *Optimal Control Applications and Methods*, vol. 23, pp. 329-355, 2002.

[16] H. Khalil, *Nonlinear systems*, Prentice Hall, Upper Saddle River, NJ, 2004.

[17] T. Hu and Z. Lin, *Control of systems with actuator saturation*, Birkhauser, Boston, MA, 2001.

[18] M. K. Kozlov, S. P. Tarasov and L. G. Khachiyan,"Polynomial solvability of convex quadratic programming," Soviet Mathematics - Doklady , vol. 20, pp. 1108-1111, 1979.

[19] R. E. Larson and W. G. Keckler, "Applications of Dynamic Programming to the Control of Water Resource Systems," *Automatica*, vol. 5, pp. 15–26, 1969.

[20] S. Mudchanatongsuk, F. Ordóñez, and J. Liu, "Robust Solutions for Network Design under Transportation Cost and Demand Uncertainty," *Journal of the Operational Research Society*, vol. 59, 2008, pp. 652–662.

[21] J. C. Moreno and M. Papageorgiou, "A Linear Programming Approach to Large-scale Linear Optimal Control Problems," *IEEE Transactions on Automatic Control*, vol. 40, pp. 971–977, 1995.

[22] F. H. Moss and A. Segall, "An Optimal Control Approach to Dynamic Routing in Data Communication Networks," *IEEE Transactions on Automatic Control*, pp. 1–7, 1978.

[23] F. Ordóñez and J. Zhao, "Robust capacity expansion of network flows," *Network*, vol. 38, pp. 136–145, 2007.

[24] W. Ren, R. W. Beard, and E. M. Atkins, Information consensus in multivehicle cooperative control: Collective group behavior through local interaction, IEEE Control Syst. Mag., vol. 27, no. 2, pp. 7182, Apr. 2007.

[25] S. Roy, Y, Wan and A. Saberi, "'On time-scale designs for networks", International Journal of Control, Vol. 82, no. 7, pp. 1313 1325, 2009.

[26] H. Sarimveisa, P. Patrinos, C. D. Tarantilis, and C. T. Kiranoudis, "Dynamic modeling and control of supply chain systems: A review," *Computers & Operations Research* vol. 35, pp. 3530–3561, 2008.

[27] E.A. Silver and R. Peterson, *Decision System for Inventory Management and Production Planning*, Wiley, New York, NY, 1985.

[28] V. I. Vorotnikov, *Partial stability and control*, Birkhauser, Boston, MA, 1998.

[29] S.H Wang and E. Davison, "On the stabilization of decentralized control systems", IEEE Transactions on Automatic Control, vol. 18, no. 5, pp. 473–478, 1973.