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Bandwagon effect in mean-field games

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Abstract— This paper provides a mean-field game theoretic model of the bandwagon effect in social networks. The latter phenomenon can be observed whenever individuals tend to align their own opinions to a mainstream opinion. The contribution is three-fold. First, we provide a mean-field games framework that describes the opinion propagation under local interaction. Second, we establish *mean-field equilibrium* strategies in the case where the mainstream opinion is stationary. Such strategies are shown to have a threshold structure. Third, we study conditions under which a given opinion distribution is stationary if agents implement optimal non-idle and threshold strategies.

I. INTRODUCTION

Nowadays social networks have been shown to have an effect also on political and socio-economic events. Thus a rigorous study of the mutual influence between individuals' opinions and population's mainstream opinion has involved scientists in different disciplines such as engineering, economics, finance and game theory, just to name a few.

A common observation is that in most cases opinions evolve following so-called averaging processes [11]. It has been shown that when the interaction is global, every agents interact with all other agents, opinions may converge to a unique consensus-value. On the other hand, if agents have local interactions, that is, agents talk only with those "who think similarly", the macroscopic behavior yields clusters of opinions, representing separate groups, parties, or communities [8]. The literature offers a variety of Lagrangian and Eulerian models to model opinion dynamics [1]. This paper provides a mean-field game theoretic perspective on the problem.

Main results. First, we provide a mean-field games framework that describes the opinion propagation under local interaction. The model assumes that the agents adjust their opinions based on a local measure of the mainstream opinion. Changes in the opinion involve fixed and quadratic costs. The model borrows concepts from mean-field games, statistical physics, and optimal control.

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Second, we establish *mean-field equilibrium* strategies in the case where the mainstream opinion is stationary. Such strategies are characterized by the property that no player can benefit from a unilateral deviation similarly to the definition of Nash equilibrium strategies in noncooperative n-player games [5]. In particular, we show that mean-field equilibrium strategies are non-idle, that is, the controls of a given agent can switch from non null to null only but not viceversa. It is shown that non-idleness is justified by the fixed costs. Such strategies are also shown to be thresholds strategies, that is, the control is non null only if the target point (the mainstream opinion) is sufficiently far from the current opinion. This corresponds to saying that for a control to be non null the distance between mainstream and agent's opinion must be higher than a given time-varying threshold.

Third, for the general case of time-varying mainstream opinion, and forcing the control strategies to be non-idle, we provide a detailed macroscopic analysis of the time evolution of the opinions' distribution. More specifically, we address the question on what initial distributions are stationary under optimal non-idle and threshold strategies.

Related literature on mean-field games. The theory of mean-field games was first presented by Lasry and Lions in [14], [4]. This theory studies interactions between indistinguishable individuals and the population around them. Several application domains accommodates mean-field game theoretic models such as economics, physics, biology, and network engineering (see [3], [10], [12], [13], [16], [18], [19]). Decision problems with mean-field coupling terms have also been formalized and studied in [7]. The classical structure of mean-field games consists of two partial differential equations (PDEs). The first PDE is the Hamilton-Jacobi-Bellman equation, which returns the optimal individual response to the population mean-field behavior. The second PDE is the Fokker-Planck equation which describes the density evolution of the players as a results of the implementation of the individual optimal strategies [14], [17]. Robustness and risk-sensitivity are also open lines of research as evidenced recently in [6], [17].

The paper is organized as follows. In Section II, we discuss the problem. In Section III, we prove some properties of the optimal control strategy in the case of a stationary mainstream opinion. In Section IV, we study the system response in consequence to the application of a threshold policy. In Section V, we provide numerical illustrations and draw conclusions in Section VI.

Notation We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. We let $\mathcal{B}$ be a finite-dimensional standard Brownian motion process defined on this probability space. We define

$\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, its natural filtration augmented by all the $\mathbb{P}$−null sets (sets of measure-zero with the respect $\mathbb{P}$). We write ∂_x and ∂_{xx}^2 to stand respectively for the first and second derivatives with respect to x.

II. THE GAME

Consider a population of homogeneous agents (players), each one characterized by an initial opinion $X(0) \in \mathbb{R}$ and an opinion $X(t) \in \mathbb{R}$ at time $t \in [0, T]$, where $[0, T]$ is the time horizon window.

The control variable is a measurable function of time $u(\cdot)$, $t \mapsto u(t) \in \mathbb{R}$ and establishes the rate of variation of an agent's opinion. It turns out that the opinion dynamics can be written in the form

$$\begin{cases} dX(t) = udt + \sigma dB(t), \ t \in [0, T], \\ X(0) = x. \end{cases} \tag{1}$$

Consider a probability density function $m : \mathbb{R} \times [0, T] \to \mathbb{R}$, $(x, t) \mapsto m(x, t)$, representing the percentage of agents in state x at time t, which satisfies $\int_{\mathbb{R}} m(x, t) dx = 1$ for every t. Let us also denote the *mainstream opinion* at time t as $\overline{m}(t)$, and the *mainstream opinion as perceived by* a player in state $X(t)$, as $\overline{m}[X(t)](t)$. Generally speaking, both $\overline{m}(t)$ and $\overline{m}[X(t)](t)$ may be any statistics of $m(\cdot)$.

The objective of an agent with opinion x is to adjust its opinion based on the perceived mainstream opinion $\overline{m}[x](t)$. This describes typical emulating effects where "to think similarly" is rewarding.

Then, for the agents, consider a running cost $g : \mathbb{R}^3 \to [0, +\infty[$, $(x, \overline{m}[x], u) \mapsto g(x, \overline{m}[x], u)$ of the form:

$$g(x, \overline{m}[x], u) = \frac{1}{2} \left[q \left(\overline{m}[x] - x \right)^2 + ru^2 \right] + K\delta(u), \tag{2}$$

where q, r and K are constant positive values; and $\delta : \mathbb{R} \to \{0, 1\}$ is defined as

$$\delta(u) = \begin{cases} 0 & \text{if } u = 0 \\ 1 & \text{otherwise} \end{cases}. \tag{3}$$

Also consider a final cost $\Psi : \mathbb{R}^2 \to [0, +\infty[$, $(\overline{m}[x], x) \mapsto \Psi(\overline{m}[x], x)$ of the form

$$\Psi(\overline{m}[x], x) = \frac{1}{2} S \left(\overline{m}[x] - x \right)^2, \tag{4}$$

where S is scalar and positive.

Problem statement. The problem in its generic form is the following: given a finite horizon $T > 0$, an initial distribution of opinions m_0, a suitable running cost: g, as in (2); a final cost Ψ, as in (4), and given a suitable dynamics for X as in (1), minimize over $\mathcal{U}$ the following cost functional,

$$\begin{aligned} J(x, t, u(\cdot)) &= \mathbb{E}\{ \int_t^T g(X(s), \overline{m}[X(s)](s), u(s)) ds + \\ &\quad + \Psi(\overline{m}[X(T)](T), X(T)) \} \end{aligned} \tag{5}$$

where $\overline{m}(\cdot)$ as time-dependent function is the evolution of the mainstream opinion when every agent behaves optimally and $\mathcal{U}$ is the set of all non-idle measurable functions from $[0, T]$ to $\mathbb{R}$. We say that a control function is non-idle if

no switching time exists, $\hat{s} > 0$, such that $u(\hat{s}) = 0$ and $u(s) \neq 0$ for $s > \hat{s}$. If we call a player *active* (*inactive*) if its control is nonnull (null), then a non-idle strategy forces a player in one of the three following situations i) the player is always active, ii) the player is always inactive, iii) the player first is active and then inactive. In no case a player can switch from being inactive to active. We call *switching time instant* the time, if it exists, in which an optimal control turns null.

III. STATIONARY AND GLOBAL INTERACTION

In this section we assume for the mainstream opinion $\overline{m}(\cdot) = \overline{m}$, where $\overline{m}$ is constant. In other words, the mainstream opinion is constant all over the horizon window. In addition to this, we denote the value function $\min_{u \in \mathcal{U}} J(x, t, u(\cdot))$ as $v(x, t)$ and denote $v_0(x, t) := J(x, t, 0)$, the latter being the cost corresponding to null controls over the horizon $[t, T]$.

Given the above notation, the Hamilton-Jacobi-Bellmann conditions that characterize the optimal controls are

$$\partial_t v(x, t) + \frac{\sigma^2}{2} \partial_{xx}^2 v(x, t) = \tag{6a}$$

$$= - \inf_{u \in \mathbb{R}} \left\{ u \partial_x v(x, t) + \frac{1}{2} \left[q \left(\overline{m} - x \right)^2 + ru^2 \right] + K\delta(u) \right\}$$

$$v(x, T) = \frac{1}{2} S \left(\overline{m} - x \right)^2. \tag{6b}$$

Rearranging and isolating the cases where the minimizer is $u^* = 0$ or $u^* \neq 0$, we obtain

$$\partial_t v(x, t) + \frac{\sigma^2}{2} \partial_{xx}^2 v(x, t) = \tag{7}$$

$$= \begin{cases} -\frac{1}{2} q \left(\overline{m} - x \right)^2 & \text{for } u^* = 0 \\ -u^* \partial_x v(x, t) - \frac{q}{2} \left(\overline{m} - x \right)^2 - \frac{r}{2} u^{*2} - K, \ u^* \neq 0 \end{cases}.$$

Hence, if the optimal control remains null from t to the end of the horizon, i.e., $u^*(x, s) = 0$ for $t \leq s \leq T$, we obtain

$$v_0(x, s) = \frac{\sigma^2 S}{2}(T-s) + \frac{\sigma^2 q}{4}(T-s)^2 + \frac{q(T-s)+S}{2}(\overline{m}-x)^2.$$

The above equality is useful as it provides an explicit computation of the cost which we can plug into the Hamilton-Jacobi-Bellmann equation.

Theorem 3.1 (Threshold optimal policy): There exists a time-varying threshold function $\lambda : [0, T) \to \mathbb{R}$ such that an agent optimal policy has structure:

$$u^*(X(t)) = \begin{cases} 0 & \text{if } |\overline{m}[X(t)](t) - X(t)| \leq \lambda(t), \\ \neq 0 & \text{otherwise.} \end{cases} \tag{8}$$

Function $\lambda(\cdot)$ is increasing over time and is equal to

$$\lambda(t) = \frac{\sqrt{2Kr}}{q(T-t)+S}, \quad 0 \leq t \leq T. \tag{9}$$

Proof: Let s be the switching time instant, that is, $u(t) \neq 0$ for $t < s$ and $u(t) = 0$ for $t > s$. Then, in s the following condition must hold

$$\inf_u \{ g(x(s), \overline{m}, u) + v_0(x(s) + dx, s + dt) \} \geq v_0(x(s), s)$$

that is, for all $u \in \mathbb{R}$,

$$\partial_s v_0(x,s) + \frac{\sigma^2}{2}\partial_{xx}^2 v_0(x,s) + \tag{10}$$
$$+ u\partial_x v_0(x,s) + \frac{q}{2}\left(\overline{m} - x\right)^2 + \frac{r}{2}u^2 + K \geq 0.$$

To see why (10) holds true, note that by dynamic programming we have the following: the null control $u^* \equiv 0$ is optimal for (x,t) if and only if, for every $\tau > 0$,

$$v_0(x,t) = \inf_{u(\cdot)} \mathbb{E}\left(\int_t^{t+\tau} g(x(s),\overline{m},u(s))ds \right.$$
$$\left. + v(x(t+\tau),t+\tau)\right),$$

where v is the "true" value function. This can be written as

$$\inf_{u(\cdot)}\left(\mathbb{E}\left(\int_t^{t+\tau} g(x(s),\overline{m},u(s))ds\right)\right.$$
$$+\mathbb{E}\left((v(x(t+\tau),t+\tau) - v_0(x,t))\right) = 0,$$

and, being $v \leq v_0$, this implies

$$\inf_{u(\cdot)}\left(\mathbb{E}\left(\int_t^{t+\tau} g(x(s),\overline{m},u(s))ds\right)\right.$$
$$+\mathbb{E}\left((v_0(x(t+\tau),t+\tau) - v_0(x,t))\right)$$
$$\geq \inf_{u(\cdot)}\left(\mathbb{E}\left(\int_t^{t+\tau} g(x(s),\overline{m},u(s))ds\right)\right.$$
$$+\mathbb{E}\left((v(x(t+\tau),t+\tau) - v_0(x,t))\right) = 0$$

Since v_0 is the cost associated to the null control $u^* \equiv 0$, we have

$$\mathbb{E}\left(\int_t^{t+\tau} g(x(s),\overline{m},0)ds\right)$$
$$+\mathbb{E}\left((v_0(x(t+\tau),t+\tau) - v_0(x,t)\right) = 0,$$

which implies the following necessary and sufficient condition for the optimality of u^*:

$$\inf_{u(\cdot)}\left(\mathbb{E}\left(\int_t^{t+\tau} g(x(s),\overline{m},u(s))ds\right)\right.$$
$$+\mathbb{E}\left((v_0(x(t+\tau),t+\tau) - v_0(x,t))\right) \geq 0.$$

Dividing by $\tau > 0$, passing to the limit as $\tau \to 0^+$, and using the expression (and the regularity) of v_0, we then get

$$\frac{r}{2}u^2 - (q(T-s)+S)(\overline{m}-x)u + K \geq 0.$$

This last inequality holds for all u if $|\overline{m}-x| \leq \frac{\sqrt{2Kr}}{q(T-s)+S}$.

The structure of $\lambda(\cdot)$ as defined in (9) can be immediately derived by determining the values of x for which condition the above condition holds as an equality.

Notice that if $|\overline{m} - X(0)| \leq \frac{\sqrt{2Kr}}{q(T)+S}$ then the corresponding control policy yields controls constantly null over the horizon. On the other hand, if $|\overline{m} - X(T)| > \frac{\sqrt{2Kr}}{S}$ then the corresponding control policy yields controls constantly nonnull over the horizon. ■

IV. THRESHOLD POLICY

Here, we study the consequence of the application of threshold strategies in the case where $\overline{m}(\cdot)$ is time-varying. This corresponds to saying that players are myopic in that they make their decisions as if the mainstream opinion would remain constant. Specifically, we model the *band-wagon effect*, that is the situation in which "the probability of any individual adopting [an opinion] increases with the proportion who have already done so" [2].

More formally, in the following we assume that

1) the players implement a non-idle threshold policy as in (8) in spite of the possible time-varying nature of $\overline{m}(\cdot)$;
2) the mainstream opinion as perceived by a player in state $X(t)$ is a (distorted) mode defined as $\overline{m}[X(t)] = \arg\max\{h(|X(t) - y|)m(y)\}$, where $m(y)$ is the density of the players' states in y and $h : [0,+\infty[\to [0,+\infty[$ is a continuous fading function. Specifically, $h(\cdot)$ is a non-increasing such that $h(0) = 1$. In presence of multiple distorted modes then function $\arg(\cdot)$ returns the minimum among the distorted modes closest to $X(t)$.

We are interested in determining the initial distributions m_0 such that the initial controls are null for all the players, i.e., $u(x;0) = 0$ for all x in the support of m_0. This condition in turn implies that the controls, given the non-idleness of the strategies considered, are then null all over the horizon, provided that we assume by definition $u(x;0) = 0$ also for all x not in the support of m_0. Hereafter, we refer to such distributions as Null Control Inducing (NCI) distributions.

Lemma 4.1: Let a threshold policy (8) be implemented and Assumptions 1 and 2 hold. A distribution m_0 is NCI iff

$$\forall x \, \exists y : \tag{11}$$
$$|y - x| \leq \lambda(t), \; m(y)h(|y-x|) \geq m(z)h(|z-x|), \forall z.$$

Proof: By definition of threshold policy (8), $u^*(x,t) = 0$ iff $|\overline{m}[x] - x| \leq \lambda(t)$. Since $m(y)h(|y-x|) \geq m(z)h(|z-x|)$ for all z then we have $y = \overline{m}[x]$ and $|y - x| \leq \lambda(t)$ and the lemma is proved. ■

We can restate the above lemma saying that m_0 is NCI, if each player with opinion x thinks that the mainstream opinion is within $I(x)$, where $I(x)$ is the neighborhood of x with radius $\lambda(t)$. Then, it is immediate to verify the uniform distribution is NCI as well as any distribution with a single point mass (a Dirac impulsion a single point). In the former case each player considers its opinion the mainstream one. In the latter case all the players share the same opinion.

In the following, we introduce some particular non trivial NCI distributions, in presence of fading functions of the following types:

- linear: $h(q) = \max\{1 - \alpha q, 0\}$, with $\alpha > 0$;
- exponential: $h(q) = e^{-\alpha q}$, with $\alpha > 0$;
- Gaussian: $h(q) = e^{-\alpha q^2/2}$, with $\alpha > 0$.

We will show that m_0 is NCI if it is either sufficiently smooth or sufficiently "peaky".

A. $m(x)$ Lipschitz function

Let $m(x)$ be such that $m(x) = 0$ outside a compact set $\mathcal{C}$, that $\inf_{\mathcal{C}} m(x) > \varepsilon > 0$ for a suitable ε, and that it is a Lipschitz function on $\mathcal{C}$, that is there exists L such that $|m(x) - m(y)| \le L|x - y|$ for all $x, y \in \mathcal{C}$.

If $h(\cdot)$ is linear, $m(x)$ is NCI if $\alpha \ge \frac{L}{\inf\{m(x)\}}$, where $\inf(\cdot)$ is taken over the support of $m(x)$.

Under the above hypotheses, condition (11) holds for $y = x$, i.e., each player considers its opinion the mainstream one. Indeed, for $y = x$, $|y - x| \le \lambda(t)$ trivially holds, and the second condition can be rewritten as

$$m(x) \ge m(z)h(|z - x|), \quad \forall z. \tag{12}$$

As $m(x)$ is a Lipschitz function, we have $m(x) \ge m(z) - L|x - z|$. Hence, (12) certainly holds as, for all z,

$$m(z) - L|x - z| \ge m(z)h(|z - x|) \quad \Leftrightarrow$$
$$L|x - z| \le m(z)\alpha|z - x| \quad \Leftrightarrow$$
$$L \le m(z)\alpha,$$

and the last condition is certainly true as $\alpha \ge \frac{L}{\inf\{m(x)\}}$.

B. $\log(m(x))$ Lipschitz function

Let $\log(m(x))$ be a Lipschitz function, i.e., $|\log(m(y)) - \log(m(x))| \le L|x - y|$. If $h(\cdot)$ is linear or exponential $m(x)$ is NCI if $\alpha \ge L$.

Even under these hypotheses, condition (11) holds for $y = x$. In fact, (12) certainly holds as, for all z,

$$|\log(m(x)) - \log(m(z))| \le L|x - z| \quad \Leftrightarrow$$
$$m(x) \ge m(z)e^{-L|z-x|}. \tag{13}$$

Hence, $m(x) \ge m(z)h(|z - x|)$ holds if

$$m(z)e^{-L|z-x|} \ge m(z)h(|z - x|) \quad \Leftrightarrow$$
$$e^{-L|z-x|} \ge h(|z - x|) \quad \Leftrightarrow$$
$$e^{-L|z-x|} \ge 1 - \alpha|z - x| \text{ or } e^{-L|z-x|} \ge e^{-\alpha|z-x|} \quad \Leftrightarrow$$
$$L \le \alpha. \tag{14}$$

Now, if $\log(m(x))$ is a Lipschitz function and differentiable, then distribution $m(x)$ is NCI also if $h(q)$ is Gaussian kernel and $L \le \alpha\lambda(t)$ as, for all x, we have $\overline{m}[x] = \arg\max\{m(y)e^{-\alpha(x-y)^2/2}\} \in I(x)$. Indeed, we have

$$\frac{\partial m(y)e^{-\alpha(x-y)^2/2}}{\partial y} =$$
$$e^{-\alpha(x-y)^2/2}(m'(y) + \alpha(x - y)m(y)) = 0 \Leftrightarrow$$
$$m'(y, t) + \alpha(x - y)m(y) = 0.$$

Hence, $\hat{y} = \overline{m}[x]$ is such that $\frac{m'(\hat{y})}{m(\hat{y})} = \alpha(\hat{y} - x)$

As $\log(m(x))$ is a Lipschitz function, we have $\left|\frac{m'(y)}{m(y)}\right| \le L$ for all y. Hence, $y \in I(x)$ if $|\hat{y} - x| \le \frac{L}{\alpha} \le \lambda(t)$.

C. $m(x)$ "peaky" function

Let $m(x)$ be characterized by a set $\Gamma = \{x^1, x^2, \ldots, x^n : x^1 < x^2 < \ldots < x^n\}$ of local maxima such that:
i) each $x^k \in \Gamma$ is an absolute maximum of function $m(x)h(|x - x^k|)$, that is, each player in x^k feels itself a leader and considers its opinion the mainstream one;
ii) for all $x \in \partial I(x^k)$, $\overline{m}[x] \ge sup_{y \in I(x^k)} m(y, t)h(|y - x|)$, for all $x^k \in \Gamma$, that is, each player on the frontier of $I(x^k)$ thinks that the mainstream opinion is not in $I(x^k)$;
iii) $\bigcup_{x^k \in \Gamma} I(x^k) \supseteq [x^0, x^{n+1}]$, where $[x^0 \ge 0, x^{n+1} \le 1]$ is the minimum interval including the support set of $m_0 = m(x)$, that is, the neighborhoods of the leaders cover all the the possible opinions.
We have that $m(x)$ is NCI if $\log h(\cdot)$ is sublinear, that is, for any $p, q \ge 0$ $h(p + q) \le h(p)h(q)$.

To prove such a result we need to show that the following critical properties hold true:

- *first critical property*: if $x \in [x^k, x^{k+1}]$ then $\overline{m}[x] \in [x^k, x^{k+1}]$.
- *second critical property*: if $x \in [x^k, x^{k+1}]$, then either both x and $\overline{m}[x]$ are in $I(x^k)$ or both of them are in $I(x^{k+1})$.

If both properties hold we have that if $x \in [x^k, x^{k+1}]$ then $|\overline{m}[x] - x| \le \lambda(t)$, as assumption iii) implies that $x^{k+1} - x^k \le 2\lambda(t)$ and $I(x^k)$ (respectively $I(x^{k+1})$) has radius $\lambda(t)$ and is centered in x^k (respectively is x^{k+1}). Consequently, if both such properties hold, condition (11) holds true as well since, from assumption iii), all x in the support of $m(x)$ belong to some interval $[x^k, x^{k+1}]$. Hence $m(x)$ is NCI.

To prove the first critical property, we observe that, given $x \in [x^k, x^{k+1}]$, if $y = \overline{m}[x] < x^k$ were true, then it would hold $m(y)h(|y - x|) > m(x^k)h(|x^k - x|)$ in contradiction with $m(y)h(|y - x|) \le m(y)h(|y - x^k|)h(|x^k - x|) \le m(x^k)h(|x^k - x|)$, where the first inequality holds for the logarithmic sublinearity of $h(\cdot)$ and the second inequality for assumption i). Symmetric argument applies to $y > x^{k+1}$.

To prove the second critical property, we observe that $x \in [x^k, x^{k+1}]$ implies that either

1) $x \in I(x^k) \setminus I(x^{k+1})$ or
2) $x \in I(x^k) \cap I(x^{k+1})$ or
3) $x \in I(x^{k+1}) \setminus I(x^k)$.

If condition 2) holds the property is proved since the first critical property implies $\overline{m}[x] \in I(x^k) \cup I(x^{k+1})$. Condition 3) is symmetrical to condition 1), so we need to verify the second critical property only when the latter condition holds.

Let z be the only element on the frontier of $I(x^{k+1})$ in $[x^k, x^{k+1}]$, $\hat{z} = \overline{m}[z]$, $\hat{x} = \overline{m}[x]$. By assumption ii) we have $x^k \le \hat{z} \le z$. If condition 1) holds then $x^k \le x < z$ and we have to prove that also $x^k \le \hat{x} \le z$ is true. We demonstrate the latter inequalities by contradiction, i.e., assuming that $z \le \hat{x} \le x^{k+1}$.

By definition of $\hat{z}$ we have

$$m(\hat{z})h(z - \hat{z}) \ge m(\hat{x})h(\hat{x} - z).$$

n	x_{min}	x_{max}	r	q	T	$\overline{m}_{01}$	$\overline{m}_{02}$	σ
10^3	0	1	1	2	10	0.25	0.75	0.001

TABLE I

CONSTANT SIMULATION PARAMETERS.

	α	λ	$std(m_{01}) = std(m_{02})$
I	2	$\{0.01, 0.06, 0.1\}$	0.09
II	0	$\{0.01, 0.06, 0.1\}$	0.05

TABLE II

VARYING SIMULATION PARAMETERS WITH DIFFERENT REGIMES.

If $x^k \leq \hat{z} \leq x$ and $z \leq \hat{x} \leq x^{k+1}$, we also have

$$m(\hat{z})h(x - \hat{z}) \geq m(\hat{z})h(z - \hat{z}) \geq$$
$$\geq m(\hat{x})h(\hat{x} - z) \geq m(\hat{x})h(\hat{x} - x)$$

in contradiction with the definition of $\hat{x}$ that implies $m(\hat{z})h(x - \hat{z}) < m(\hat{x})h(\hat{x} - x)$.

The proof concludes if also $x < \hat{z} \leq x$ and $z \leq \hat{x} \leq x^{k+1}$ lead to a contradiction. Specifically, under these conditions, we can write

$$m(\hat{z})h(\hat{z} - x) \geq m(\hat{z})h(\hat{z} - x)h(z - \hat{z}) \geq$$
$$\geq m(\hat{x})h(\hat{x} - z)h(\hat{z} - x) \geq m(\hat{x})h(\hat{x} - x),$$

where the first inequality holds as $h(z - \hat{z}) \leq 1$, the second inequality holds by definition of $\hat{z}$, the third inequality holds by logarithmic sublinearity of the fading function. Again, the above chain of inequalities is in contradiction with the definition of $\hat{x}$ that implies $m(\hat{z})h(\hat{z} - x) < m(\hat{x})h(\hat{x} - x)$.

V. NUMERICAL EXAMPLES

Examples show two main evolution regimes (see Fig. 1-4). The first regime presents how the system evolves in the case of a bimodal distribution with linear fading function. From an initial almost uniform distribution, because of the influence of α, all agents do not converge to a single limit, instead two limits are formed. The second regime simulates two separate Gaussians that converge to a single point when $\alpha = 0$.

Simulations have been performed using the algorithm below and the following parameters, also shown in Tables I-II. The number of agents is set to $n = 10^3$. The set of states is a discretization of the interval $[0, 1[$ with step size $dx = 10^{-4}$, i.e. $\mathcal{X} = \{x_{min}, x_{min} + 0.001, \ldots, x_{max}\}$. The horizon length is $T = 10$, large enough to show convergence of the population regimes. As regards the initial distribution, Parameter σ is set to 0.001.

Regime I. The first set of simulations highlights the convergence to two separate clusters in case of α different from zero, although agents' opinions start from an almost uniform distribution. In Fig. 1, the graphics show the distribution evolution from a macro perspective of all the opinions vs. time. Figure 2 displays the distribution evolution $m(t)$ at

Algorithm

Input: Set of parameters as in Tables I-II.
Output: Distribution function $m(t)$
  1 : **Initialize.** Generate x_0 from two Gaussian distributions with $\overline{m}_{01}$, $\overline{m}_{02}$ and $std(m_0)$
  2 : generate Brownian Motion $d\mathcal{B}(t)$,
  3 : **for** time $t = 0, 1, \ldots, T - 1$ **do**
  4 :   **if** $t > 0$, **then** compute distribution $m(t)$,
  5 :   **end if**
  6 :   **for** agent $i = 1, 2, \ldots, n$ **do**
  7 :   compute mode $\overline{m}[X(t)](t)$, control $u^*(X(t))$ from (8), and $X(t+1)$ from (1)
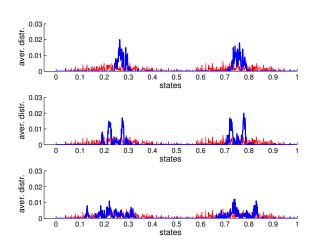 8 :   **end for**
  9 : **end for**
  10 : **STOP**

Fig. 1. Regime I, macro perspective: starting from an almost uniform distribution, the contribute of α is to separate the opinions into two clusters.

the beginning of the horizon, $t = 0$, and at the end of the horizon, $t = 10$.

The initial distribution is a bimodal which is obtained as the sum of two Gaussian distributions with mean $\overline{m}_{01} = 0.25$ and $\overline{m}_{02} = 0.75$, respectively, and standard deviation $std(m_{01}) = std(m_{02}) = 0.09$. This value of standard deviation is chosen to let the two Gaussians to almost collide at $t = 0$, and such that the effect of α is visible at different times, especially in T. From top to bottom, the threshold is set to $\lambda = 0.01$ (top), $\lambda = 0.06$ (middle) and $\lambda = 0.1$ (bottom).

Regime II. The second set of simulations shows that, when α is set to zero all agents come to an agreement, i.e., all opinions paths converge to a single point.

The graphics show the distribution evolution from a macro perspective of all the opinions vs. time in Fig. 1 and the distribution evolution m_t at time $t = 0$ and $t = T$ in Fig. 2. Likewise in Regime I, the initial distribution is a bimodal which is the sum of two Gaussians $\bar{m}_{01} = 0.25$ and
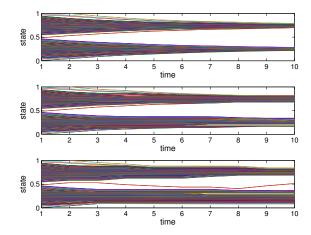
Fig. 2. Regime I, micro perspective: starting from an almost uniform distribution, the contribute of α is to separate the opinions into two clusters.

Fig. 4. Regime II, micro perspective: the distribution evolution converges to a limit, starting from a two separate clusters, from a micro perspective.

$\bar{m}_{02} = 0.75$ respectively and standard deviation $std(m_{01}) = std(m_{02}) = 0.05$. From top to bottom, the threshold takes on the values $\lambda = 0.01$ (top), $\lambda = 0.06$ (middle) and $\lambda = 0.1$ (bottom).
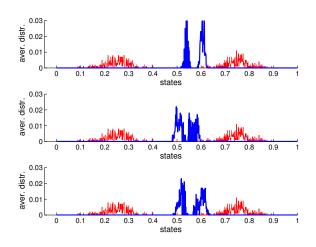

Fig. 3. Regime II, macro perspective: the distribution evolution converges to a limit, starting from a two separate clusters, from a macro perspective.

VI. CONCLUSIONS

This paper aims at an understanding of the bandwagon effect using a mean-field game theoretic perspective. The contribution includes the investigation of *mean-field equilibrium* strategies under local interactions. Future research will address cases where agents' opinions set the "anti-mode" as target point.

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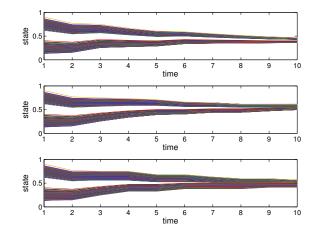[2] A. Colman, Oxford Dictionary of Psychology, New York, Oxford University Press, 2003.

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