Macroscopic Thermal Entanglement Due to Radiation Pressure

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Can entanglement and the quantum behavior in physical systems survive at arbitrary high temperatures? In this Letter we show that this is the case for a electromagnetic field mode in an optical cavity with a movable mirror in a thermal state. We also identify two different dynamical regimes of generation of entanglement separated by a critical coupling strength.

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Recently there has been a lot of interest in macroscopic systems which can support nonzero temperature entanglement [1–3]. These works suggest that entanglement not only persists in these conditions but that it is also crucial in understanding the macroscopic proprieties of the physical systems. On the other hand, there is also a possibility that, under some circumstances (very large number of atoms, very large temperature, etc.), the physical systems become effectively classical and we no longer require a quantum description.

It is usually believed that decoherence explains the transition from the quantum world to the classical world. Several other alternatives, such as spontaneous wave-function collapse models [4] and gravitationally induced decoherence [5] are also found in the literature. Different schemes to create and probe macroscopical superpositions, which can give us important clues about the quantum to classical transition, have already been proposed for different physical systems [6–9]. For instance, in Ref. [9], like in the original *gedanken* experiment proposed by Schrödinger, a single photon state induces quantum superpositions of a mirror and in Ref. [7] multicomponent cats of a cavity are created in the interaction of a cavity field with a movable mirror.

Here we proceed in a different direction, shedding light into the problem of how quantum features may survive in the macroscopic world. Recent research on this problem has shown that in systems with finite Hilbert space dimension in equilibrium with a thermal bath, bipartite entanglement vanishes above a critical temperature [10]. The following question arises, is this behavior general, or are there some systems where entanglement is robust against temperature?

In this Letter we will show that the entanglement between macroscopic mirror and a cavity mode field can arise due to radiation pressure at arbitrarily high temperatures as the system evolves in time. This is very surprising because it is commonly believed that high temperature completely destroys entanglement. We will study entanglement in the time domain using a discrete variable method and identify its dependence on the relevant physical parameters, such as the strength of the radiation pressure coupling and temperature.

In particular, we consider a perfect optical cavity with a movable mirror with mass m in one end, modeled as a mechanical harmonic oscillator whose quivering motion is quantized. Identical systems have been considered in the studying of decoherence [5] and nonclassical states of the cavity field [7]. The general Hamiltonian of this system has been extensively studied by Law [11]. Following the same approach as Law, we consider the adiabatic limit where the resonant frequency of the mirror is much slower than the frequency of the cavity mode $w_m \ll 2\pi nc/L$, where L is the length of the cavity when the mirror is in equilibrium, n is the order of the longitudinal cavity mode, and c is the speed of light. Henceforth, the coupling between different cavity field modes (leading to the Casimir effect, etc.) can be neglected. For cavities with very high quality factor Q the damping is negligible, as it occurs on a time scale much longer than it takes for the photons to perform several round trips. Under these conditions the Hamiltonian includes only the free terms of both the field and the mirror plus the interaction term due to the radiation pressure (which causes the displacement of the mirror),

$$H = \hbar w_0 a^\dagger a + \hbar w_m b^\dagger b - \hbar g a^\dagger a(b + b^\dagger), \qquad (1)$$

where a is the annihilation operator of the cavity mode, b is the annihilation operator of the mirror, w_0 is the frequency of the cavity mode, and $g = w_0\sqrt{\hbar}/(L\sqrt{m w_m})$ is the coupling constant.

The electromagnetic field is prepared in a coherent state, $|\alpha\rangle$, using a driving laser tuned to resonance with the cavity mode, whereas the mirror is initially in a Gibbs state with temperature T. Then, the composite state of the system is

$$\rho(t_0) = \frac{1}{Z}\int \frac{d^2 z}{\pi} e^{-|z|^2/\bar{n}} |\alpha\rangle\langle\alpha| \otimes |z\rangle\langle z|,$$

where $\bar{n} = 1/(e^{\hbar w_m/K_B T} - 1)$ is the mean number of excitations, Z is the mirror partition function, and z represents all the possible coherent states of the mirror.

The evolution operator associated with the Hamiltonian (1) has a closed formula and it was derived in Ref. [6], using the Campbell-Baker-Hausdorff formula for the Lie algebra, and in Ref. [7], using operator algebra methods:

$$U(t) = e^{-iw_0 a^\dagger a t} e^{ik^2 (a^\dagger a)^2 \Lambda(t)} D_m[\eta(t) k a^\dagger a] e^{-iw_m b^\dagger b t},$$

where $\Lambda(t) = w_m t - \sin(w_m t)$, $\eta(t) = 1 - e^{-iw_m t}$, $k = g/w_m$, and $D_m[\eta(t) k a^\dagger a] = e^{k a^\dagger a[\eta(t) b^\dagger - \eta(t)^* b]}$ is the displacement operator of the mirror, $D_m(\gamma)|0\rangle = |\gamma\rangle$. Since the system is periodic we only need to investigate entanglement in the time interval $[0, 2\pi/w_m]$.

The interaction term of the Hamiltonian has the potential to entangle the cavity field modes with the vibrational modes of the mirror. Heuristically, the generation of entanglement in this physical system is better understood by considering the cavity in the initial state $|0\rangle + |1\rangle$ and the mirror in the vacuum state $|0\rangle_m$ (which is a good assumption for $T \approx 0$). The state of the composite system evolves according to (up to a normalization factor)

$$(|0\rangle + |1\rangle) \otimes |0\rangle_m \to |0\rangle \otimes |0\rangle_m + e^{if(t)} |1\rangle \otimes |k\eta(t)\rangle_m,$$

where $f(t)$ is a phase, resulting in an entangled state for $0 < t < 2\pi/w_m$. At $t = 0$ and $= 2\pi/w_m$, $\eta(t)$ becomes null and the system returns to the initial separable state. The entanglement results from the evolution of the term $|1\rangle \otimes |0\rangle_m$, which can be interpreted as the transference of momentum from the photon $|1\rangle$ to the mirror $|0\rangle_m$, as the photon kicks the mirror. Though it is easy to produce the state of light $|0\rangle + |1\rangle$ experimentally, the radiation pressure in this case is so small that it is virtually impossible to detect any entanglement using present day technology. However, as we will show below, a detectable amount of entanglement is expected when the cavity is initially in a coherent state with sufficiently high amplitude.

Proceeding to the general case, we introduce the discrete variable method which will allow the study of the entanglement in the system. The density matrix $\rho(t)$ expressed in the Fock basis of both cavity field and mirror reads

$$\rho = \sum_{\mu,\nu,n,m} \rho_{\mu\nu nm} |n\rangle\langle m| \otimes |\mu\rangle\langle\nu|, \tag{2}$$

where the Latin indexes refer to the radiation and the Greek indexes refer to the mirror. The elements of the density matrix are calculated to be

$$\rho_{\mu\nu nm} = \begin{cases} \mu!\delta_{\mu\nu}\Phi_{nm\mu\nu}(t)e^{-\beta\hbar w_m(\mu+1)}, & \text{for } t = 2\pi k/w_m \quad \text{or } n = m = 0 \\ (-1)^\mu \Phi_{nm\mu\nu}(t) e^{-\beta\hbar w_m(\mu+1)+\Omega_{nm}(t)} W^*_{mn}(t)^{\nu-\mu} H_U[-\mu, 1+\nu-\mu, z_{nm}(t)], & \text{elsewhere.} \end{cases} \tag{3}$$

$H_U[a, b, c]$ is the hypergeometric confluent function and

$$\Phi_{nm\mu\nu}(t) = \frac{\alpha^n \alpha^{*m} e^{i\Lambda(t)(n^2-m^2)-|\alpha|^2}}{Z(n!m!\mu!\nu!)^{1/2}},$$
$$W_{nm}(t) = \Delta_n(t) - e^{-\beta\hbar w_m}[\Delta_n(t) + \Delta_m(t)]/2,$$
$$z_{nm}(t) = -e^{\beta\hbar w_m} W_{nm}(t) W^*_{mn}(t),$$
$$\Omega_{nm}(t) = [|\Delta_n(t)|^2 + |\Delta_m(t)|^2]/2 - e^{-\beta\hbar w_m}|\Delta_n(t) + \Delta_m(t)|^2/4,$$

where $\Delta_n(t) = kn\eta(t)$.

Quantifying entanglement in mixed states is a nontrivial problem, except for bipartite two level systems, where the Peres criterium is sufficient [12] and necessary [13] to guarantee entanglement. In this Letter a new approach inspired by Ref. [14] is followed. First, we project the original density matrix (2) into a 2×2 subspace, which corresponds to a local action ($P_{2\times2}\rho P_{2\times2}$) thus not increasing the amount of entanglement $E(\rho)$ in the overall system [15], i.e., $E(\rho) \geq E(P_{2\times2}\rho P_{2\times2})$. Afterwards different measures and markers of entanglement for 2×2 systems can be applied. In particular, the *tangle* is an entanglement monotone valid only for bipartite pure states [16]. The *negativity* is also an entanglement monotone but valid for both pure and mixed bipartite states [17] $\mathcal{N}(\rho) = (\sum_i |\lambda_i| - 1)/2$, where λ_i are the normalized eigenvalues of the partial transposed matrix ρ^{T_P}. The relation between *negativity* and the Peres criterium is clear: it measures by how much the partial transposed density matrix fails to be positive, i.e., separable. Therefore, to show the existence of entanglement in the overall system it is sufficient to verify that $E(P_{2\times2}\rho P_{2\times2}) > 0$. From now on we will use the notation where each subspace of the density matrix is represented by $[\mu, \nu; n, m]$ where μ, ν refer to the number of excitations of the mirror and n, m refer to the number of excitations of the cavity field.

For $T = 0$ the system is in a pure state and we can investigate entanglement using the *tangle* [16], $\tau(t)$, which is much simpler to compute than the *negativity*. Figure 1

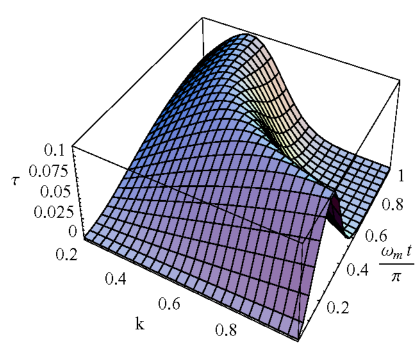

FIG. 1 (color online). *Tangle* τ as a function of k and the scaled time for $T = 0$ K, $\alpha = 1$, and subspace $[1, 2; 1, 2]$.

shows that the system is always entangled except for $t = 0$ and $t = 2\pi/w_m$ (only the range $t \in [0, \pi/w_m]$ has been plotted since the *tangle* has reflection symmetry around $t = \pi/w_m$), when the mirror returns to its initial state. For small k, the system reaches the maximum of entanglement at $t = \pi/w_m$, simultaneously with the maximum displacement of the mirror. For k above a critical value, k_c, the maximum of entanglement is achieved before $t = \pi/w_m$. The time of maximum entanglement depends on the balance between the interaction time t_{int} ($t_{\text{int}} \sim 1/g$), i.e., the time scale of the interaction term in the Hamiltonian, and the time of oscillation of the mirror, t_m ($t_m \sim 1/w_m$).

It is interesting to try to understand the importance of the amplitude of the coherent state, α, in establishing the value of k_c. We expect that when increasing α the value of k_c should decrease because there are more photons interacting with the mirror resulting in a larger effective coupling $(g\langle a^\dagger a\rangle)$. Surprisingly, this is not the case: the value of k_c increases with α. This can be understood as follows. The ratio between the weight of the $|n+1\rangle$ number state and the weight of $|n\rangle$ number state in the expansion of the coherent state, being given by $\alpha/(\sqrt{n+1})$, increases with α, weakening the entanglement generated after interaction with the mirror. The best situation occurs when the weights of states are the most equally distributed. Hence a higher coupling helps the entanglement generation to have the same efficiency when α is increased.

Considering the subspace $[1, 2; 1, 2]$, k_c can be calculated as solution of the transcendental equation

$$-1 + 14k_c^2 + 24k_c^4 = 2\alpha^2(1 + 4k_c^2 + 96k_c^4)e^{-12k_c^2} \geq 0. \tag{4}$$

The right-hand side of Eq. (4) is non-negative resulting in a restriction for k_c, i.e., k_c is lower bounded. Also we can see from Eq. (4) that k_c increases with α. This confirms, at least for this subspace, that a higher coupling is necessary for reaching the maximum of entanglement before $t = \pi/w_m$ if the amplitude of the cavity field is increased.

Since $\tau(\pi)$ is the maximum of entanglement achieved during the evolution of the system for $k \leq k_c$, and k_c depends on α, the value of $\tau(\pi)$ is maximized for

$$\alpha_{\max} = \frac{e^{6k^2}\sqrt{1 + 2k^2}}{\sqrt{2}\sqrt{1 + 8k^2}}, \tag{5}$$

which can only be defined for $k_c(\alpha_{\max}) \geq k$. Equation (4) indicates that the ideal value of α increases with the coupling. It is useful to define α_c as the value of α, for a given k, such that $k_c = k$ and to notice that $k_c(\alpha_{\max}) \geq k$ is equivalent to the condition $\alpha_c \geq \alpha_{\max}$. By squaring (4) and dividing it by (5) (with $k_c \to k$ and $\alpha \to \alpha_c$) the condition of validity of (5) yields $k \geq 1/2$. This behavior is independent of the particular subspace considered. For example, the asymptotic behavior of the *tangle* at $t = \pi$ as a function of α is always $\tau(\pi) \sim |\alpha|^2$, for $\alpha \ll \alpha_{\max}$, and $\tau(\pi) \sim |\alpha|^{-2}$, for $\alpha \gg \alpha_{\max}$.

Figure 1 shows how the maximum of entanglement in the considered subspace is shifted to earlier times when k increases.

For temperatures slightly above $T = 0$ this behavior is not significantly altered. At temperatures $T > 0$ the system is in a mixed state and the entanglement must be investigated using the *negativity*. It can be inferred from the plots of the *negativity* obtained from the simulations that, for the subspace $[1, 2; 1, 2]$, the value of k_c increases slowly with the temperature. The investigation of the *negativity* also shows, as expected, that the entanglement decreases with temperature in a given subspace.

For higher temperatures, i.e., higher number of excitations of the mirror, the *negativity* cannot be computed because the numerical calculation of the hypergeometric confluent function is ill conditioned. However, it is still possible to verify the existence of entanglement for different values of α, k, and temperature. This can be accomplished by introducing a marker of entanglement based on the Peres criterium, which consists simply in verifying whether the determinant of the partial transposed density matrix ρ^{T_P} is negative, thus indicating the existence of a negative eigenvalue. This is, of course, a very weak marker of entanglement, which can only detect entanglement if ρ^{T_P} has a odd number of negative eigenvalues, but has the useful advantage of being easy to calculate.

The determinant of ρ^{T_P} for the subspace $[0, 1; 0, 1]$ is $\Upsilon_{[0,1;0,1]}(\alpha, b, x) = -G(\alpha, b, x)H(b, x)$, where $H(b,x) = 16x^2e^{b^2x} + x^2[-4 + b^2(-2 + x)(-1 + e^{b^2/2})]^2 - e^{b^2x/2}\{b^4(-2 + x)^4 + 32x^2 + 4b^2x^2(-1 + e^{b^2/2})[4 + (-3 + e^{b^2/2})x]\}$, $G(\alpha, b, x) = x^4|\alpha|^4e^{b^2(x-2)-4|\alpha|^2}/(16\bar{n}^4)$ is always positive, $b \equiv b(t) = \sqrt{2}k\sqrt{1 - \cos(w_m t)}$, and $x = \bar{n}/(1 + \bar{n})$. The sign of Υ is determined by $H(b, x)$ and is plotted as a function of $(2/\pi)\arctan[b(t)]$ and x in Fig. 2, where it is clear that the entanglement occurs for all parameters except for high values of $b(t)$ and low temperature (region I) and for low values of $b(t)$ and high temperatures (region II).

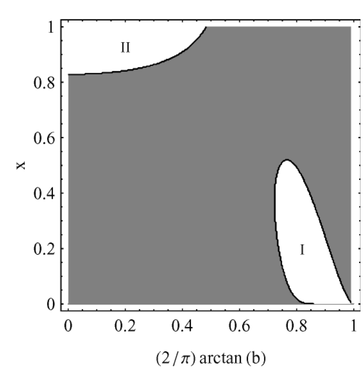

FIG. 2. Marker of entanglement, Υ, as a function of $(2/\pi) \times \arctan[b(t)]$ and x for the subspace $[0, 1; 0, 1]$. Entanglement (gray) exists for all parameter space except in regions I and II (white).

In the parameter region I, the lack of entanglement is not important since, during the evolution of the system, the value of $b(t)$ ranges between 0 and $b_{\max} = 2k$ and, for sufficiently short times, the system is entangled independently of how large k is. In the parameter region II the proof of existence of entanglement in the system is not as easy because, in principle, the system could always be separable for small values of k. However, by rewriting Eq. (3) as follows

$$\rho_{\mu\nu nm} = \Phi_{nm\mu\nu}(t) \int \frac{d^2 z}{\pi} F_n(z)^\mu F_m^*(z)^\nu e^{K_{nm}(z)},$$

where $K_{nm}(z) = -[|F_n(z)|^2 + |F_m(z)|^2]/2 - |z|^2/\bar{n}$ and $F_n(z) = z + kn\eta(t)$, it is clear that k is always multiplied either by n or by m. Then it is straightforward to verify that $\Upsilon_{[0,1;0,s]}$ it is proportional to $\Upsilon_{[0,1;0,1]}$ if in the former the coupling $k_s = k/s$ is chosen; i.e.,

$$|\alpha|^{-2s} s! \Upsilon_{[0,1;0,s]}(k_s) = |\alpha|^{-2} \Upsilon_{[0,1;0,1]}(k).$$

For high temperatures, if we choose a large k capable of producing entanglement in the subspace $[0, 1; 0, 1]$, then there must be entanglement in the subspace $[0, 1; 0, s]$ for the coupling constant $k_s = k/s$, even though k_s might not lead to entanglement in the subspace $[0, 1; 0, 1]$. Therefore, for systems where $b(t)$ varies in region II there is always a s such that entanglement occurs in the subspace $[0, 1; 0, s]$.

Although entanglement occurs at any finite temperature this does not imply the existence of entanglement in the limit of infinite temperature. In fact, making use of the general inequality $E(\sum_i p_i \rho^i_{AB}) \leq \sum_i p_i E(\rho^i_{AB})$ to write $E[P\rho(t)P] \leq \int dz e^{-|z|^2/\bar{n}} E(PU|\alpha\rangle\langle\alpha| \otimes |z\rangle\langle z|U^\dagger P)/\bar{n}\pi$ and taking into account that (i) the limit of infinite temperature is equivalent to the limit $\bar{n} \to \infty$ and (ii) entanglement is bounded in any subspace, yields $\lim_{\bar{n}\to\infty} E[P\rho(t)P] = 0$.

The entanglement appears naturally in the dynamics of the system for any value of coupling between the mirror and the cavity field at any finite temperature, independently of the value of α (notice that H does not depend on α, at least for the class of subspaces considered here), in spite of α playing an important role in determining the amount of entanglement in the system. An argument will be given to infer this role. We define the normalized mutual information as $I \equiv I_{12}/(S_1 + S_2)$, where $I_{12} \equiv S_1 + S_2 - S$, S_i is the linear entropy of the mirror ($i = 1$) and the cavity field ($i = 2$). $S = 1 - 1/(2\bar{n} + 1)$ is the linear entropy of the composite system and

$$S_i = 1 - e^{-2|\alpha|^2} f_i(\bar{n}) \sum_{p,q=0}^{\infty} \frac{|\alpha|^{2(p+q)}}{p!q!} e^{-g_i(\alpha,k,\bar{n},t)(p-q)^2},$$

for $t \neq 0, 2\pi/w_m$, where $f_1(\bar{n}) = 1/[\bar{n}^2(a^2 - 1)]$, $f_2(\bar{n}) = 1$, $g_1(\alpha, k, \bar{n}, t) = 2k^2\xi(t)(1 - b)$, $b = (1 - 1/a)^2 + 1/a$, $a = 1 + 1/\bar{n}$, $\xi(t) = 1 - \cos(w_m t)$, and $g_2(\alpha, k, \bar{n}, t) = k^2\xi(t)(\bar{n} + 2)$. Since $g_i(\alpha, k, \bar{n}, t) > 0$ (for $k \neq 0$) we have $\partial I/\partial|\alpha| > 0$, which shows that quantum correlations increase monotonously with α as mentioned before. Hence, a detectable amount of entanglement is expected for sufficiently high α. Finally, we give a numerical value for I for realistic physical parameters $T = 1$ K, $w_m = 10$ MHz, $k = 1$, and $\alpha = 10^6$, $I - 1/2 \gtrsim 0.19 \times 10^{-4} > 0$. The Araki-Lieb inequality [18] guarantees that quantum correlations are present whenever $I > 1/2$. This is consistent with our previous conclusions that entanglement persists even in the high temperature limit.

In this Letter the generation of entanglement between a cavity mode and a movable mirror via radiation pressure was analyzed by a projection method. Since in the process of projection of the density matrix much of the entanglement is lost, this method does not allow us to assess the entanglement of the overall system but it is still robust enough to identify two dynamical regimes depending on the coupling between the mirror and the cavity field. For $k \leq k_c$ the maximum of entanglement is achieved at $t = \pi$ and for $k > k_c$ is achieved before and, furthermore, demonstrate the remarkable result that entanglement does occur for any finite temperature.

We hope that our results stimulate further investigations on entanglement in the macroscopic domain. We gratefully acknowledge M. Aspelmeyer, Č. Brukner, S. Gigan, J. Kofler, E. Lage, P. Vieira, and M. Wieśniak. A. F. is supported by FCT (Portugal) through Grant PRAXIS No. SFRH/BD/18292/04. V. V. acknowledges funding from EPSRC and the European Union.

[1] V. Vedral, New J. Phys. **6**, 102 (2004).
[2] G. C. Ghirardi *et al.*, Phys. Rev. D **34**, 470 (1986).
[3] M. Pinard *et al.*, Europhys. Lett. **72**, 747 (2005).
[4] S. Ghosh *et al.*, Nature (London) **425**, 48 (2003).
[5] R. Penrose, in *Mathematical Physics 2000*, edited by A. Fokas *et al.* (Imperial College, London, 2000).
[6] S. Mancini, V. I. Man'ko, and P. Tombesi, Phys. Rev. A **55**, 3042 (1997).
[7] S. Bose, K. Jacobs, and P. L. Knight, Phys. Rev. A **56**, 4175 (1997).
[8] J. Ruostekoski *et al.*, Phys. Rev. A **57**, 511 (1998); J. I. Cirac *et al.*, Phys. Rev. A **57**, 1208 (1998).
[9] W. Marshall *et al.*, Phys. Rev. Lett. **91**, 130401 (2003).
[10] B. V. Fine, F. Mintert, and A. Buchleitner, Phys. Rev. B **71**, 153105 (2005).
[11] C. K. Law, Phys. Rev. A **51**, 2537 (1995).
[12] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[13] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[14] S. Bose *et al.*, Phys. Rev. Lett. **87**, 050401 (2001).
[15] V. Vedral *et al.*, Phys. Rev. Lett. **78**, 2275 (1997); V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[16] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[17] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[18] A. Wherl, Rev. Mod. Phys. **50**, 221 (1978).