Proceedings Paper:

Al-Jazzar, SO, Strangeways, HJ and McLernon, DC (2014) 2-D angle of arrival estimation using a one-dimensional antenna array. In: European Signal Processing Conference (EUSIPCO). European Signal Processing Conference, 01-05 Sep 2014, Lisbon. IEEE , 1905 - 1909. ISBN 9780992862619

2-D Angle of Arrival Estimation Using a One-Dimensional Antenna Array

Saleh O. Al-Jazzar* **Hal J. Strangeways†** **Desmond C. McLernon‡**
*Al-Zaytoonah University of Jordan, Amman, Jordan.
†Newcastle University, Newcastle, UK.
‡University of Leeds, Leeds, UK.

***Abstract*—In this paper, a two-dimensional (2-D) angle of arrival (AOA) estimator is presented for vertically polarised waves in which a one-dimensional (1-D) antenna array is used. Many 2-D AOA estimators were previously developed to estimate elevation and azimuth angles. These estimators require a 2-D antenna array setup such as the L-shaped or parallel antenna 1-D arrays. In this paper a 2-D AOA estimator is presented which requires only a 1-D antenna array. This presented method is named Estimation of 2-D Angle of arrival using Reduced antenna array dimension (EAR). The EAR estimator utilises the antenna radiation pattern factor to reduce the required antenna array dimensionality. Thus, 2-D AOA estimation is possible using antenna arrays of reduced size and with a minimum of two elements only, which is very beneficial in applications with size and space limitations. Simulation results are presented to show the performance of the presented method.**

keywords: 2-D AOA Estimation, Statistical Signal Processing, Subspace Methods

I. Introduction

Angle of arrival (AOA) estimation is useful in many applications such as positioning and signal enhancement. Many estimators were developed in the literature to estimate the AOAs. These methods were initially developed to estimate the AOAs in one plane. These methods utilised different antenna array structures. The most popular of these antenna array structures is the uniform linear antenna array (ULA).

In many practical applications it is required to estimate both the elevation and azimuth AOAs. So, many methods were developed initially to estimate the elevation and azimuth angles separately using two ULAs (such as L-shaped or parallel ULAs). Then, another method was applied to pair up these estimated elevation and azimuth angles [1]–[7]. Other methods were also developed in which the elevation and azimuth angles were estimated jointly without the need for any pairing technique. Still, these methods require the use of two ULAs [1], [8]. For example, in the L-shaped ULAs, one ULA will be placed on one axis (say the y-axis) and the other will be placed on a perpendicular axis (say the z-axis). This means that to estimate the elevation and azimuth angles we were required to have a two-dimensional (2-D) antenna array structure and a minimum of three elements (with one reference element at the origin point, one on the y-axis and one on the z-axis). But in many applications, there are size and space limitations on antenna array structure. One example for such an application is the downlink between a base station (BS) and a mobile station (MS). In such a scenario, estimating the 2-D AOAs at the MS will help in locating it. But due to MS size limitations it is desirable to estimate the 2-D AOAs using a small size antenna array structure.

In this paper, a 2-D AOA estimator is developed which requires the antenna elements to be placed along one axis only to perform the estimation. This will reduce the size and space of the required antenna arrays. The method utilises the radiation pattern factor of different antennas to help reduce the required antenna array dimensionality. Thus, a minimum of two antennas can be used to perform the estimation. The method is named Estimation of 2-D Angle of arrival using Reduced antenna array dimension (EAR). The reason for naming it thus is that human ears can distinguish the elevation and azimuth angles of arriving sound waves using only two ears. This is true for two reasons. The first one is that the distance between the two ears is similar in concept to the distance between the antenna elements. The second reason is the different directionality of each ear which is the basic idea of the EAR method presented in this paper. To achieve the different directionality it is not required (although possible) to have different antenna elements types. This different directionality can be achieved by rotating the direction of the axis of some of the antenna elements in a ULA away from the direction of the axis of the other element(s). So for instance in a two element ULA of short dipoles, by placing the first one with its axis aligned vertically (z-direction) and the other aligned at an angle γ from the z-direction, then both elements will have different radiation pattern factor equations with respect to the angles upon which they depend, i.e., different directionality (although both elements are similar). This will be explained latter in Section III. Since the basic array for the EAR method requires two elements, this setup of two elements is named the EAR-block in reference to its function as a basic building block for constructing a new antenna array for the EAR method. The dipoles considered in this paper are assumed to be short dipoles. Also, the incident signal(s) considered in this paper are assumed to be vertically polarised.

The paper is organised as follows: Section II introduces the system model that forms the foundation for the proposed estimator. The proposed EAR estimator is explained in Section III. The Cramer-Rao bound (CRB) for the elevation and azimuth AOA estimation is presented in Section IV. The simulated performance of the EAR estimator is presented in Section V. Finally, conclusions are given in Section VI.

II. System Model

The antenna array is formed from M uniform antenna EAR-blocks (i.e., $2M$ total antenna elements) as shown in Fig 1. Each antenna EAR-block is formed from two antenna elements. One element has its axis aligned vertically (z-direction) and the other is aligned with a rotated axis ($\tilde{y}$) which has an angle γ with the z-axis rotated towards the y-axis in the y-z plane, with a spacing d between their centres. The antenna array, formed by a set of M EAR-blocks that are arranged along one axis, is called the EAR-array (as shown in Fig. 1).

Assume K sources signals, $\{\check{s}_k(t)\}_{k=1}^{K}$, impinging upon the EAR-array. The signal $\check{s}_k(t)$ is represented as $\check{s}_k(t) = s_k(t)\cos(2\pi f_c t) = \Re\{s_k(t)\exp(j2\pi f_c t)\}$ where f_c is the carrier frequency and $s_k(t) = \sum_i \alpha_k(i) g(t - iT)$, where i is an integer which represents the time index and $s_k(t)$ is the low-pass equivalent of $\check{s}_k(t)$, T is the symbol period, $\alpha_k(i) = A_k \beta_k(i)$ where $\beta_k(i) \in (-1, +1)$ which represents the symbol parity and A_k is a positive constant which represents the amplitude of $s_k(t)$ and $g(t)$ is the (raised cosine) pulse shaping function with

$$g(nT) = \left\{ \begin{array}{ll} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{array} \right\} \quad (1)$$

where n is an integer so that $s_k(nT) = \alpha_k(n)$.

The sampled received signal at the antenna elements (after the matched filter stage) for the antenna structure shown in Fig. 1 is given by:

$$\mathbf{r}(i) = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\phi})\boldsymbol{\alpha}(i) + \mathbf{n}(i) \quad (2)$$

where

$$\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\phi}) = [\, \mathbf{a}_1(\boldsymbol{\theta}, \boldsymbol{\phi}) \quad \cdots \quad \mathbf{a}_K(\boldsymbol{\theta}, \boldsymbol{\phi}) \,] \quad (3)$$

with

$$\mathbf{a}_k(\boldsymbol{\theta}, \boldsymbol{\phi}) = \begin{bmatrix} F_1(\theta_k, \phi_k) \\ \eta(\gamma) F_2(\theta_k, \phi_k) e^{-j\frac{2\pi d}{\lambda}\sin\theta_k\cos\phi_k} \\ F_1(\theta_k, \phi_k) e^{-j2\frac{2\pi d}{\lambda}\sin\theta_k\cos\phi_k} \\ \eta(\gamma) F_2(\theta_k, \phi_k) e^{-j3\frac{2\pi d}{\lambda}\sin\theta_k\cos\phi_k} \\ \vdots \\ F_1(\theta_k, \phi_k) e^{-j(2M-2)\frac{2\pi d}{\lambda}\sin\theta_k\cos\phi_k} \\ \eta(\gamma) F_2(\theta_k, \phi_k) e^{-j(2M-1)\frac{2\pi d}{\lambda}\sin\theta_k\cos\phi_k} \end{bmatrix} \quad (4)$$

where $F_1(\theta_k, \phi_k)$ and $F_2(\theta_k, \phi_k)$ are the radiation pattern factors for the first and second antenna elements of the EAR-block. Also,

$$\boldsymbol{\alpha}(i) = [\, \alpha_1(i) \quad \cdots \quad \alpha_K(i) \,]^T$$

with

$$\boldsymbol{\theta} = [\, \theta_1 \quad \cdots \quad \theta_K \,]^T$$

and

$$\boldsymbol{\phi} = [\, \phi_1 \quad \cdots \quad \phi_K \,]^T \quad .$$

Also, λ is the signal wavelength, d is the distance between the consecutive antenna elements centres and (θ_k, ϕ_k) are elevation and azimuth angles, respectively, for the k^{th} signal. In addition, $\mathbf{n}(i)$ is the noise vector added to the received signal at the antenna elements, which is additive white Gaussian noise (AWGN) of zero mean and has a covariance matrix of $\sigma^2 \mathbf{I}_{2M\times 2M}$, where $\mathbf{I}_{2M\times 2M}$ is the $2M \times 2M$ identity matrix and $(.)^T$ represents the transpose operation. Now, $\eta(\gamma)$ is the polarisation mismatch loss factor for the second antenna elements of the EAR-block. The reason for considering the polarisation mismatch is because the received signal is assumed to be vertically polarised whereas the second antenna element at the EAR-block is rotated from the vertical (z-axis) by an angle γ. Thus, a polarisation mismatch loss will occur [9].

III. The EAR method

The EAR method utilises the different antenna radiation pattern factors for the different antenna elements in the array. As an illustration, consider the EAR-block in Fig. 1. This EAR-block can have a different setup by rotating the second element by an angle γ towards the x-axis instead of the y-axis. But in this paper the case of Fig. 1 is considered. This single EAR-block is useful for a single source 2-D AOA estimation such as in the downlink between the BS and the MS. The EAR-array can be used as well in the single source case, but for sake of illustration of the ability of 2-D AOA estimation using two elements only (i.e., using the EAR-block) the EAR-block will be used. Another possibility considered is the case when it is required to estimate the 2-D AOAs for multiple sources. In this case, an array of the EAR-block is used for 2-D AOA estimation for those multiple sources.

Since the EAR-block case shown here is formed of two short dipoles, then for far field case, the radiation pattern factors are given as follows [9]:

$$F_1(\theta_1, \phi_1) = F_1(\theta_1) = \sin\theta_1 \quad (5)$$

and

$$F_2(\theta_1, \phi_1) = F_2(\tilde{\phi}_1) = \sin\tilde{\phi}_1 \quad (6)$$

where $\tilde{\phi}_1$ is the angle between the direction of the received signal and the second antenna element axis ($\tilde{y}$). Also, it can be shown that the relation between $\tilde{\phi}_1$ with θ_1 and ϕ_1 is

$$\tilde{\phi}_1 = \cos^{-1}(\sin\gamma\sin\theta_1\cos\phi_1 + \cos\gamma\cos\theta_1) \quad . \quad (7)$$

The proof of (7) is shown in the Appendix section.

It is clear form (5) and (6) that $F_1(\theta_1, \phi_1)$ and $F_2(\theta_1, \phi_1)$ depend on θ_1 and $\tilde{\phi}_1$, respectively. So, we will use the notations $F_1(\theta_1)$ and $F_2(\tilde{\phi}_1)$ instead of $F_1(\theta_1, \phi_1)$ and $F_2(\theta_1, \phi_1)$, respectively. Similarly, for the general k^{th} source, we will use the terms $F_1(\theta_k)$ and $F_2(\tilde{\phi}_k)$ instead of $F_1(\theta_k, \phi_k)$ and $F_2(\theta_k, \phi_k)$, respectively.

For a more general case other than short dipoles, the radiation pattern factors would be different in (5) and (6) depending upon the type of antenna elements used. But, in this paper the antenna elements used are short dipoles.

The polarisation loss factor $(\eta(\gamma))$ is given by [9]

$$\eta(\gamma) = \cos\gamma \quad . \tag{8}$$

For both single source and multiple source 2-D AOA estimation, the MUSIC algorithm is proposed in this paper. Since the EAR-block is a special case of the EAR-array (this is true because the EAR-array will reduce to a EAR-block by setting $M = K = 1$) then we will show how to apply the MUSIC method to the general case (EAR-array multiple sources) and it will be straight forward to apply it for the special case (EAR-block single source). So, consider applying the MUSIC method to the received signal $(\mathbf{r}(i))$ at the EAR-array given in (2). The first step in applying the MUSIC method is to find the autocorrelation matrix $(\mathbf{R})$ for the received signal $(\mathbf{r}(i))$, i.e., find $\mathbf{R} = E\left[\mathbf{r}(i)\mathbf{r}^H(i)\right]$ where $(.)^H$ represents the Hermetian transpose operation. By employing eigenvalue decomposition, then the eigenvalues which correspond to the noise signal are the smallest $2M - K$ eigenvalues and their corresponding eigenvector matrix (which is given the notation $\mathbf{E}_n$) includes the eigenvectors of the noise space in its columns. So, applying the MUSIC method the following vector is formed

$$\mathbf{u}(\theta,\phi) = \begin{bmatrix} F_1(\theta) \\ \eta(\gamma)F_2(\tilde{\phi})e^{-j\frac{2\pi d}{\lambda}\sin\theta\cos\phi} \\ F_1(\theta)e^{-j2\frac{2\pi d}{\lambda}\sin\theta\cos\phi} \\ \eta(\gamma)F_2(\tilde{\phi})e^{-j3\frac{2\pi d}{\lambda}\sin\theta\cos\phi} \\ \vdots \\ F_1(\theta)e^{-j(2M-2)\frac{2\pi d}{\lambda}\sin\theta\cos\phi} \\ \eta(\gamma)F_2(\tilde{\phi})e^{-j(2M-1)\frac{2\pi d}{\lambda}\sin\theta\cos\phi} \end{bmatrix} . \tag{9}$$

The vector $(\mathbf{u}(\theta,\phi))$ varies with θ and ϕ and is orthogonal to the noise subspace $\mathbf{E}_n$ at θ_k and ϕ_k for $\forall k$. Thus, to estimate θ_k and ϕ_k, the MUSIC algorithm searches for the corresponding peaks of the following

$$\frac{\mathbf{u}(\theta,\phi)^H\mathbf{u}(\theta,\phi)}{\mathbf{u}(\theta,\phi)^H\mathbf{E}_n\mathbf{E}_n^H\mathbf{u}(\theta,\phi)} \quad . \tag{10}$$

Thus, the elevation (θ_k) and azimuth (ϕ_k) angles are estimated jointly using antenna elements placed on one axis (the EAR-array) and without the need for any pairing or matching technique. This method is named the EAR-based MUSIC method.

IV. Cramer-Rao Bound

One important step in defining the CRB is to identify the parameters of interest. Since we are concerned with estimating the elevation and azimuth AOAs, then the parameters of interest in this paper are defined in the following vector form

$$\mathbf{\Psi} = [\,\boldsymbol{\theta}^T \quad \boldsymbol{\phi}^T\,]^T \quad . \tag{11}$$

After defining the parameters of interest, we can present the CRB for both the elevation and azimuth AOAs. The CRB for the elevation and azimuth AOAs is derived in [10] as:

$$\mathbf{CRB} = \frac{\sigma^2}{2W} \left(\Re\{\left(\mathbf{D}^H\left(\mathbf{I}_{2M\times 2M} - \mathbf{\Sigma}\right)\mathbf{D}\right) \odot \left(\mathbf{1}_2\mathbf{1}_2^T \otimes \mathbf{\Omega}\right)\}\right)^{-1} \tag{12}$$

where $\mathbf{\Sigma} = \mathbf{A}(\boldsymbol{\theta},\boldsymbol{\phi})\left(\mathbf{A}^H(\boldsymbol{\theta},\boldsymbol{\phi})\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\phi})\right)^{-1}\mathbf{A}^H(\boldsymbol{\theta},\boldsymbol{\phi})$, W is the number of snap shots over which the correlation matrices are estimated, $\odot$ is the Schur-Hadamard element-by-element multiplication, $\otimes$ is the Kronecker product, and $\mathbf{1}_2 = [\,1 \quad 1\,]^T$, with

$$\mathbf{\Omega} = \mathbf{P}\left(\mathbf{A}^H(\boldsymbol{\theta},\boldsymbol{\phi})\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\phi})\mathbf{P} + \sigma^2\mathbf{I}_{K\times K}\right)^{-1} \mathbf{A}^H(\boldsymbol{\theta},\boldsymbol{\phi})\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\phi})\mathbf{P} \tag{13}$$

and

$$\mathbf{P} = E\left[\boldsymbol{\alpha}(i)^H\boldsymbol{\alpha}(i)\right] = \begin{bmatrix} A_1^2 & & 0 \\ & \ddots & \\ 0 & & A_K^2 \end{bmatrix} . \tag{14}$$

Also,

$$\mathbf{D} = [\,\mathbf{A}_{\boldsymbol{\theta}}(\boldsymbol{\theta},\boldsymbol{\phi}) \quad \mathbf{A}_{\boldsymbol{\phi}}(\boldsymbol{\theta},\boldsymbol{\phi})\,] \tag{15}$$

where

$$\mathbf{A}_{\boldsymbol{\theta}}(\boldsymbol{\theta},\boldsymbol{\phi}) = \frac{\partial\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\phi})}{\partial\boldsymbol{\theta}}$$
$$\mathbf{A}_{\boldsymbol{\phi}}(\boldsymbol{\theta},\boldsymbol{\phi}) = \frac{\partial\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\phi})}{\partial\boldsymbol{\phi}} \quad . \tag{16}$$

Thus, the CRB for the elevation and azimuth AOAs of this technique has been derived. Simulation results for the proposed methods are shown in the next section.

V. Simulation Results

Simulations of the proposed EAR estimator for single and multiple sources were completed to assess its performance. The elements of each EAR-block elements were separated by a half-wavelength (i.e., $d = \frac{\lambda}{2}$). The number of snap shots was given the notation W. The results are averaged over 1000 independent ensembles. The step size used for the EAR-based MUSIC was 0.1^o.

Figs. 2 and 3 show the root mean square error (RMSE) estimation in degrees of θ_1 and ϕ_1, respectively, versus SNR in dB for a single source using the EAR-based MUSIC method with $\theta_1 = 60^o$ and $\phi_1 = 40^o$ compared with their corresponding square root CRB. The reason for comparing with the square root CRB rather than the CRB is because we are considering the RMSE and not the mean squared error (MSE) in these figures. Two values of W were considered; $W = 256$ and $W = 1000$. The figures indicate that the performance is enhanced as W is increased. Clearly, the advantage of the proposed method is illustrated by its ability to estimate 2-D AOAs using only two antenna elements which is not feasible using other existing 2-D AOA estimators.

Fig. 4 shows the RMSE of angle estimation in degrees in a two sources case using the EAR-based MUSIC algorithm with $\theta_{1,2} = 60^o, 65^o$ and $\phi_{1,2} = 40^o, 55^o$. This RMSE is evaluated by $\sqrt{E\left[\frac{(\theta_1-\hat{\theta}_1)^2}{2}+\frac{(\phi_1-\hat{\phi}_1)^2}{2}\right]}$ where $\hat{\theta_1}$ and $\hat{\phi_1}$ are the estimated values of θ_1 and ϕ_1, respectively.

Tables I and II show the pairing success rate for the proposed EAR-based MUSIC algorithm compared to the algorithm in [6] and the SVD-based algorithm in [8] for different SNRs in dB with $W = 256$ and $W = 1000$, respectively. In this case, M was set to 10. Both tables indicate that the proposed EAR-array-based MUSIC algorithm managed to pair the elevation and azimuth angles successfully.

VI. Conclusion

In this paper we proposed a 2-D AOA estimator for vertically polarised signals which uses antenna elements placed along a 1-D line. The basic antenna structure, which is named the EAR-block, is composed of two elements. One of the elements of the EAR-block has its axis aligned vertically (along the *z*-direction) and the other element is aligned to an axis $(\tilde{y})$ which has been rotated from the *z*-axis by an angle γ towards the *y*-axis in the *y*-*z* plane. The ability to estimate the 2-D AOAs using only two elements is made possible by utilising the directionality of the different antenna elements in the EAR-block. The fact that only two elements are needed to make the 2-D AOA estimation reduces the space and size of the antenna structure required. The estimation of the elevation and azimuth angles using the EAR method is performed jointly. The proposed method in this paper depends on the MUSIC algorithm to perform the 2-D AOA estimation and is therefore named the EAR-based MUSIC algorithm. The proposed method can also estimate 2-D AOAs for multiple sources using the EAR-array. [1-2] [3-10]

SNR (dB)	[6]	[8]	Proposed method
15	65.7 %	100 %	100 %
20	94.4 %	100 %	100 %
25	97.8 %	100 %	100 %

Table. I. Table of successful pairing rate using the proposed EAR-array-based MUSIC compared to the algorithms in [6] and [8] for different SNRs and $W = 256$.

SNR (dB)	[6]	[8]	Proposed method
15	99.7 %	100 %	100 %
20	100 %	100 %	100 %
25	100 %	100 %	100 %

Table. II. Table of successful pairing rate using the proposed EAR-array-based MUSIC compared to the algorithms in [6] and [8] for different SNRs and $W = 1000$.

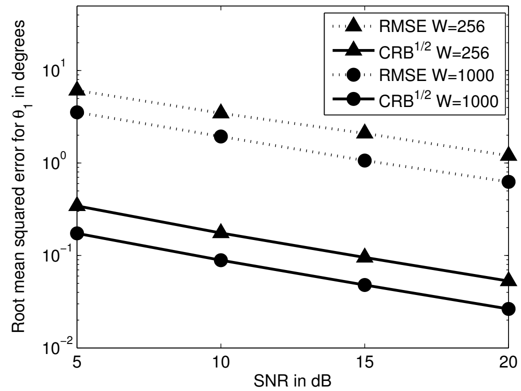

Fig. 2. Root mean square error (RMSE) and square root of the CRB for angle estimation in degrees for θ_1 in degrees versus signal to noise ratio (SNR) in dB for a single source using the EAR-based MUSIC method with $\theta_1 = 60^o$ and $\phi_1 = 40^o$.

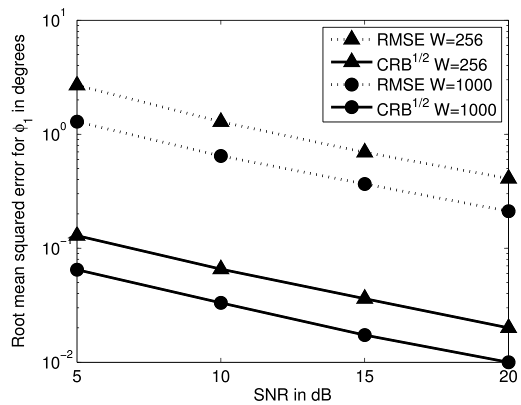

Fig. 3. Root mean square error (RMSE) and square root of the CRB for angle estimation in degrees for ϕ_1 in degrees versus signal to noise ratio (SNR) in dB for a single source using the EAR-based MUSIC method with $\theta_1 = 60^o$ and $\phi_1 = 40^o$.

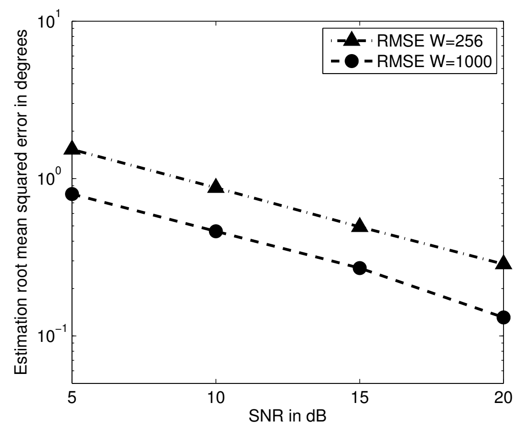

Fig. 4. Root mean square error (RMSE) of angle estimation in degrees versus signal to noise ratio (SNR) in dB for two sources using the EAR-based MUSIC algorithm with $\theta_{1,2} = 60^o, 65^o$ and $\phi_{1,2} = 40^o, 55^o$.

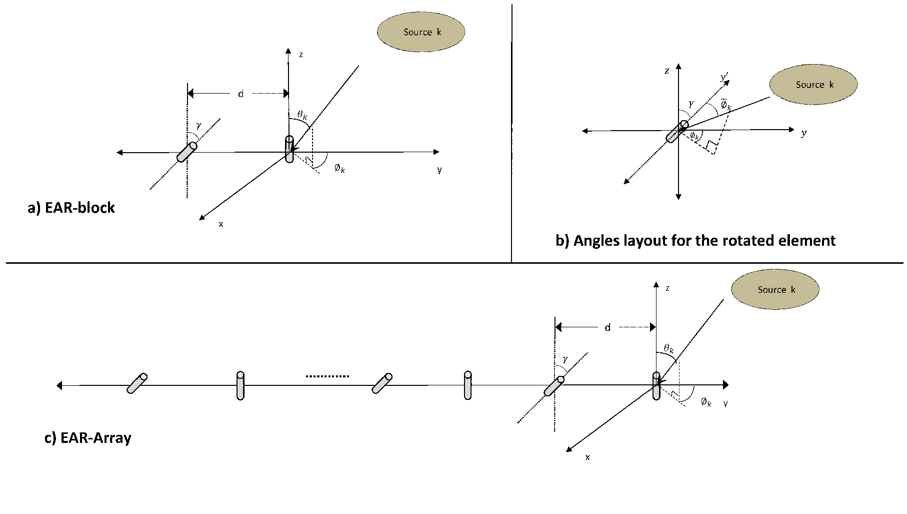

Fig. 1. Antenna elements geometry for a) EAR-block, b) Angles layout for the rotated element, and c) EAR-array.

APPENDIX

To prove (7), consider the vector representing the incoming wave as shown in Fig. 1. This vector is denoted by $\vec{v}$ and can be represented by

$$\vec{v} = v_x \vec{x}^{-} + v_y \vec{y}^{-} + v_z \vec{z}^{-} \tag{17}$$

where $\vec{x}^{-}$, $\vec{y}^{-}$ and $\vec{z}^{-}$ are unit vectors in the negative direction of the x, y, and z axes. Also, v_x, v_y and v_z are the dot product results between $\vec{v}$ and the vectors $\vec{x}^{-}$, $\vec{y}^{-}$ and $\vec{z}^{-}$, respectively. So,

$$v_x = \vec{v} \cdot \vec{x}^{-} = \sin\theta_1 \sin\phi_1 \quad , \tag{18}$$

$$v_y = \vec{v} \cdot \vec{y}^{-} = \sin\theta_1 \cos\phi_1 \tag{19}$$

and

$$v_z = \vec{v} \cdot \vec{z}^{-} = \cos\theta_1 \quad . \tag{20}$$

Now our objective is to find $\tilde{\phi}_1$, which is the angle between the incoming signal vector and the rotated antenna element aligned with the rotated axis $\tilde{y}$. Since $\tilde{\phi}_1$ is an angle between two vectors, then this angle is defined as the cosine inverse of the dot product between these two vectors. These two vectors in this case are $\vec{\tilde{y}}^{-}$ (which is the unit vector in the negative direction of the $\tilde{y}$-axis) and $\vec{v}$. Thus, we can take the dot product between $\vec{v}$ in (17) with $\vec{\tilde{y}}^{-}$ as follows

$$\vec{v} \cdot \vec{\tilde{y}}^{-} = v_x \vec{x}^{-} \cdot \vec{\tilde{y}}^{-} + v_y \vec{y}^{-} \cdot \vec{\tilde{y}}^{-} + v_z \vec{z}^{-} \cdot \vec{\tilde{y}}^{-} \quad . \tag{21}$$

But,

$$\vec{x}^{-} \cdot \vec{\tilde{y}}^{-} = 0 \quad , \tag{22}$$

$$\vec{y}^{-} \cdot \vec{\tilde{y}}^{-} = \sin\gamma \tag{23}$$

and

$$\vec{z}^{-} \cdot \vec{\tilde{y}}^{-} = \cos\gamma \quad . \tag{24}$$

Substituting (18), (19), (20), (22), (23) and (24) in (21), and from the fact that the dot product between $\vec{v}$ and $\vec{\tilde{y}}^{-}$ is actually the cosine of the angle between these two vectors (which is defined as $\tilde{\phi}_1$ as shown in Fig. 1), then one can deduce that

$$\cos\tilde{\phi}_1 = \sin\gamma \sin\theta_1 \cos\phi_1 + \cos\gamma \cos\theta_1 \quad . \tag{25}$$

Then, we can generalise the result in (25) for the general k^{th} signal as follows

$$\cos\tilde{\phi}_k = \sin\gamma \sin\theta_k \cos\phi_k + \cos\gamma \cos\theta_k \quad . \tag{26}$$

Taking the cosine inverse of both sides of (26) we get the same result in (7). Thus, (7) is proved.

REFERENCES

[1] N. Tayem, H. M. Kwon, L-shape 2-dimensional arrival angle estimation with propagator method, IEEE Trans. Antennas Propag. 53 (2005) 1622 1630.

[2] Y.Wu, G. Liao, H.C.So, A fast algorithm for 2-D direction-of-arrival estimation, Signal Process. 83 (2003) 1827 1831.

[3] S. Kikuchi, H. Tsuji, A. Sano, Pair-matching method for estimating 2-D angle of arrival with a cross-correlation matrix, IEEE Antennas and Wireless Propagation Letters, vol. 5, 2006. 5 (2006) 35 – 40.

[4] R. Roy, T. Kailath, ESPRIT estimation of signal parameters via rotational invariance techniques, Opt. Eng. 29 (1990) 296 313.

[5] M. Haardt, J. A. Nossek, Unitary ESPRIT: How to obtain increased estimation accuracy with a reduced computational burden, IEEE Trans. Signal Process. 43 (1995) 1232 1242.

[6] T. H. Liu, J. M. Mendel, Azimuth and elevation direction finding using arbitrary array geometries, IEEE Trans Signal Process. 46 (1998) 2061 – 2065.

[7] T. Xia, Y. Zheng, Q. Wan, X. Wang, Decoupled estimation of 2-D angles of arrival using two parallel uniform linear arrays, IEEE Transactions on Antennas and Propagation 55 (2007) 2627 2632.

[8] S. O. Al-Jazzar, D. Mclernon, M. A. Smadi, SVD-based joint azimuth/elevation estimation with automatic pairing, Signal Processing. 90 (2010) 1669–1675.

[9] C. A. Balanis, Modern Antenna Handbook, JohnWiley & Sons, Inc., 2008.

[10] L. Mailaender, Bounds for 2-D angle-of-arrival estimation with separate and joint processing, EURASIP Journal on Advances in Signal Processing 2011 2011:5.