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Hierarchy of modes in an interacting one-dimensional system—Supplemental material

O. Tsyplyatyev and A. J. Schofield
School of Physics and Astronomy, University of Birmingham, Birmingham, B15 2TT, UK

Y. Jin, M. Moreno, W. K. Tan, C. J. B. Ford, J. P. Griffiths, I. Farrer, G. A. C. Jones, and D. A. Ritchie
Cavendish Laboratory, University of Cambridge, J J Thomson Avenue, Cambridge, CB3 0HE, UK

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Here we provide more details for the theoretical and the experimental parts of the main text. Fig. S1 uses different color scheme for the data in Fig. 3 of the main text giving a better fit of the spin and the charge branches of the low energy part of the spectrum. Section A contains derivations of Eqs. (2,3) of the main text.

Additional data related to this publication are available at the University of Cambridge data repository at http://www.repository.cam.ac.uk/handle/1810/247433 and at the University of Birmingham data repository at http://epapers.bham.ac.uk/1956/.

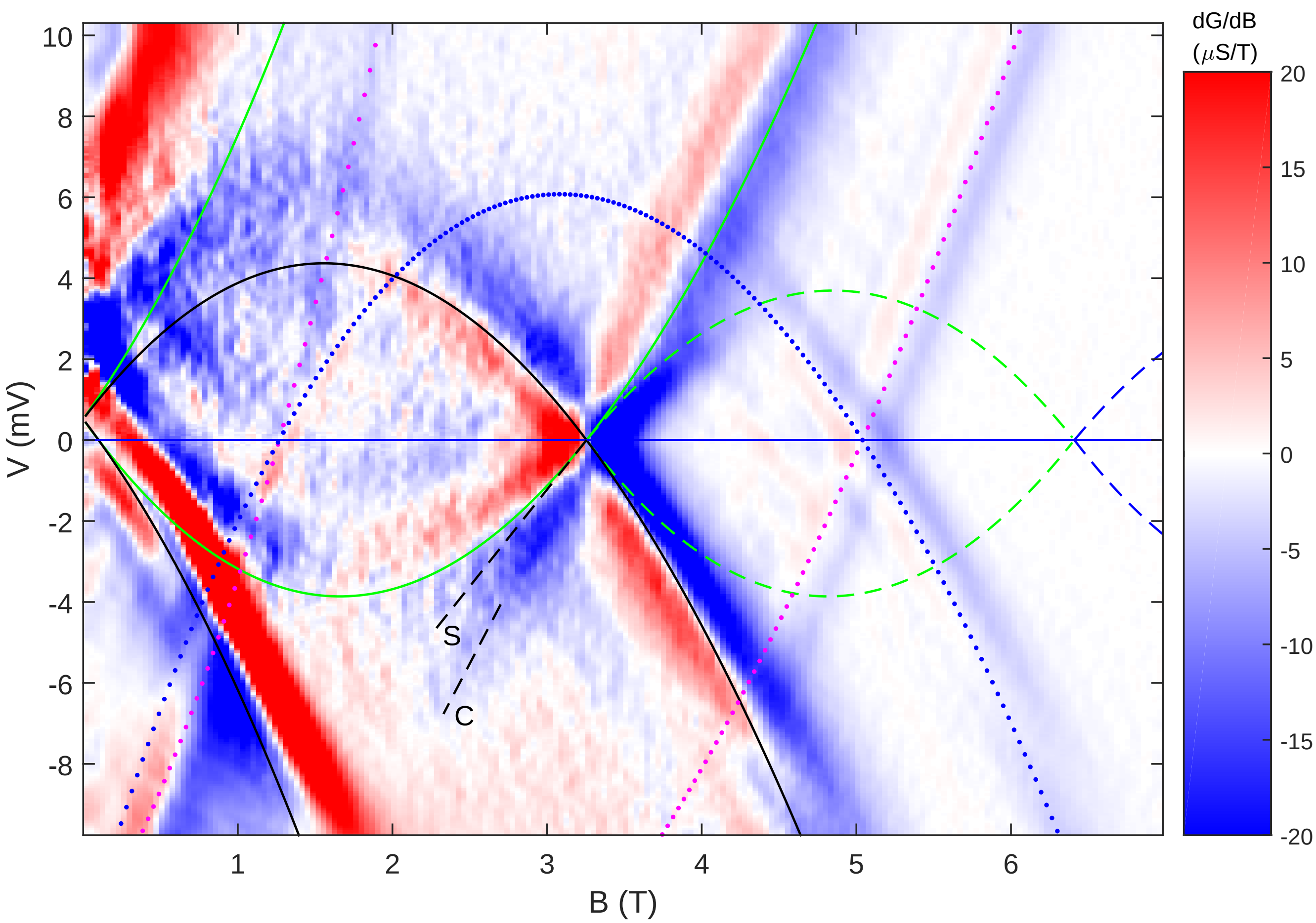

Figure S1. dG/dB for the data in Fig. 3a and b, to show the charge and the spin lines in more detail. Spin (S) and charge (C) modes are indicated with black dashed lines. The solid black line is the 2D dispersion. The green solid line marks a-modes, dashed green lines, b-modes and dashed blue, c-modes; dotted magenta and blue lines are parasitic 2D dispersions. As in Fig. 3, in drawing the parabolae the tunnelling distance is taken to be 34 nm (close to the nominal 32 nm centre-centre distance between the quantum wells) and a very small capacitive correction is applied to account for the variation of the density of each layer with interlayer bias.

A. DERIVATIONS OF EQS. (2,3) OF THE MAIN TEXT

In this section we derive the main technical result of the theoretical part of the paper, Eqs. (2,3) of the main text, using the tools of Bethe ansatz. The derivation of Eq. (2) of the main text is based on our previous work [1] which contains extensive details on the framework that we will use below. The many-body matrix element in Eq. (3) of the main text is based on the results borrowed from spin chains [2–5] in which we evaluate the final determinant expressions explicitly in the regime of the fermionic model in Eq. (1) of the main text. The latter allows analytical analysis of physical observables, which includes establishing of the hierarchy of modes, given in the main text after Eq. (3).

Lattice version of the model in Eq.(1) of the main text reads

$$H=\sum_{j=-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}}\left[\frac{-1}{2m}\left(\psi_j^\dagger\psi_{j+1}+\psi_j^\dagger\psi_{j-1}\right)-U\rho_j\rho_{j+1}\right], \tag{1}$$

where j is the site index on the lattice, the operators obey $\left\{\psi_j,\psi_j^\dagger\right\}=\delta_{ij}$, and $\rho_j=\psi_j^\dagger\psi_j$. In the so-called coordinate representation of Bethe ansatz approach it is diagonalised by N-particle states

$$|\mathbf{k}\rangle=\sum_{\mathcal{P},j_1<\dots<j_N}e^{i\sum_l k_{P_l}j_l+i\sum_{l<l'}\varphi_{P_l,P_{l'}}}\psi_{j_1}^\dagger\dots\psi_{j_N}^\dagger\left|\mathrm{vac}\right\rangle, \tag{2}$$

where $|\mathrm{vac}\rangle$ is the vacuum state, parameterised with sets of N quasimomenta k_j that satisfy the non-linear equations [6]

$$\mathcal{L}k_j-2\sum_{l\neq j}\varphi_{jl}=2\pi I_j, \tag{3}$$

where

$$e^{i2\varphi_{ll'}}=-\frac{e^{i(k_l+k_{l'})}+1-2mUe^{ik_l}}{e^{i(k_l+k_{l'})}+1-2mUe^{ik_{l'}}} \tag{4}$$

are the scattering phases and I_j are sets of non-equal integer numbers.

The dimensionless length of the system $\mathcal{L}=L/\mathcal{R}$ is normalised by the short length-scale $\mathcal{R}$ which is introduced using a lattice (with next-neighbor interaction) as the lattice parameter (and interaction radius) $\mathcal{R}$ that provides microscopically an ultraviolet cutoff for the theory. The latter procedure at high energy is analogous to the point-splitting technique [7] at low energy. For a small $\mathcal{R}$ the scattering phases in Eq. (4) become linear functions of quasimomenta making the non-linear Bethe-ansatz equations a linear system . Solving them for $\mathcal{L}\gg1$ via perturbation theory up to the first subleading order in $1/\mathcal{L}$ we obtain Eq. (2) of the main text,

$$k_j=\frac{2\pi I_j}{\mathcal{L}-\frac{mUN}{mU+1}}-\frac{mU}{mU+1}\sum_{l\neq j}\frac{2\pi I_l}{\left(\mathcal{L}-\frac{mUN}{mU+1}\right)^2}. \tag{5}$$

Note that this is also equivalent to the continuum regime, which corresponds to the thermodynamic ($N,\mathcal{L}\to\infty$, but $N/\mathcal{L}$ is finite) and the long wavelength ($N/\mathcal{L}\ll1$) limits. The corresponding eigenenergy and total momentum (protected by the translational invariance of the system) are $E=\sum_j k_j^2/(2m)$ and $P=\sum_j k_j$.

We now turn to calculation of the form factors using the algebraic form of the Bethe ansatz [6]. Following Ref. 6 we write down the many-body wave functions using operators of a Yang-Baxter algebra as

$$|\mathbf{u}\rangle=\prod_{j=1}^{N}C(u_j)\left|\mathrm{vac}\right\rangle, \tag{6}$$

where u_j are N axillary parameters and the operators $C(u)$ are one of the four operators of the transition matrix

$$T(u)=\begin{pmatrix}A(u)&B(u)\\C(u)&D(u)\end{pmatrix}. \tag{7}$$

which is defined in auxiliary two-element space. This T-matrix satisfies the Yang-Baxter equation

$$R(u-v)\left(T(u)\otimes T(v)\right)=\left(T(v)\otimes T(u)\right)R(u-v), \tag{8}$$

in which the R-matrix that corresponds to the model in Eq. (1) is given as follows

$$R(u)=\begin{pmatrix}1&&&\\&b(u)&c(u)&\\&c(u)&b(u)&\\&&&-1\end{pmatrix}. \tag{9}$$

Here the matrix elements are

$$b\left(u\right)=\frac{\sinh\left(u\right)}{\sinh\left(u+2\eta\right)},\; c\left(u\right)=\frac{\sinh\left(2\eta\right)}{\sinh\left(u+2\eta\right)}, \tag{10}$$

and η is the interaction parameter.

The T-matrix for the model in Eq. (1) can be constructed starting from the following L-operators

$$L_j=\begin{pmatrix}\frac{\cosh\left(u-\eta\left(2c_j^\dagger c_j-1\right)\right)}{\cosh(u-\eta)} & -i\frac{\sinh 2\eta c_j^-}{\cosh(u-\eta)}\\ -i\frac{\sinh 2\eta c_j^\dagger}{\cosh(u-\eta)} & -\frac{\cosh\left(u+\eta\left(2c_j^\dagger c_j-1\right)\right)}{\cosh(u-\eta)}\end{pmatrix} \tag{11}$$

which are defined in the same auxiliary two-element space as the T-matrix but which matrix elements are operators only in the single site subspace of the physical model in Eq. (1). Then, the transition matrix for the whole chain is a product of these operators,

$$T\left(u\right)=\sum_{j=1}^{\mathcal{L}}L_j\left(u\right). \tag{12}$$

The transfer matrix $\tau\left(u\right)$ of the model in Eq. (1) is a super trace of the T-matrix, $\tau\left(u\right)=A\left(u\right)-D\left(u\right)$, which contains all conserved quantities including the Hamiltonian in Eq. (1). Many-body states in Eq. (6) become the eigenstates of the transfer matrix when auxiliary parameters u_j satisfy the following set of non-linear equations

$$\frac{\cosh\left(u_j-\eta\right)^{\mathcal{L}}}{\cosh\left(u_j+\eta\right)^{\mathcal{L}}}=\left(-1\right)^{N-1}\prod_{l=1\neq j}^{N}\frac{\sinh\left(u_j-u_l-2\eta\right)}{\sinh\left(u_j-u_l+2\eta\right)}. \tag{13}$$

This equation follows directly from the condition that a state in Eq. (6) is an eigenstate of the transfer matrix, $\tau\left(u\right)\left|\mathbf{u}\right\rangle=\text{scalar const}\left|\mathbf{u}\right\rangle$.

The equation above is the direct analog of the Bethe ansatz equation (3) in the coordinate representation. Direct mapping between the two is done by the substitution of

$$e^{ik_j}=\frac{\cosh\left(u_j-\eta\right)}{\cosh\left(u_j+\eta\right)},\quad\text{and}\quad mU=\cosh 2\eta. \tag{14}$$

in Eq. (3) and by taking exponential of it. Note that the definitions of the eigenstates in coordinate and the algebraic representations have different normalisation factors. Below we will use the latter definition in Eq. (6).

To proceed we borrow results from Heisenberg chains. The normalisation of Bethe states was obtained in Ref. 2 in the form of a determinant of an $N\times N$ matrix

$$\left\langle\mathbf{u}|\mathbf{u}\right\rangle=\sinh^N\left(2\eta\right)\prod_{i\neq j=1}^{N}\frac{\sinh\left(u_j-u_i+2\eta\right)}{\sinh\left(u_j-u_i\right)}\det\hat{Q} \tag{15}$$

where the matrix elements are

$$Q_{ab}=\begin{cases}-\mathcal{L}\frac{\sinh 2\eta}{\cosh(u_a+\eta)\cosh(u_a-\eta)}-\sum_{j\neq a}\frac{\sinh 4\eta}{\sinh(u_a-u_j-2\eta)\sinh(u_a-u_j+2\eta)} & ,a=b,\\ \frac{\sinh 4\eta}{\sinh(u_b-u_a+2\eta)\sinh(u_b-u_a-2\eta)} & ,a\neq b.\end{cases} \tag{16}$$

The many-body matrix element of the local operator $\psi_0^\dagger$ was obtained in the determinant form in Ref. 3 using Slavnov's formula for scalar products [4] and Drinfield twists [5] to represent the field operators in the basis of Bethe states

$$\left\langle\mathbf{v}|\psi_0^\dagger|\mathbf{u}\right\rangle=\left(-1\right)^{N+1}i\frac{\prod_{j=1}^{N+1}\cosh\left(v_j+\eta\right)}{\prod_{j=1}^{N}\cosh\left(u_j-\eta\right)}\frac{\sinh^{N+1}\left(2\eta\right)\det\hat{M}}{\prod_{j<i=2}^{N}\sinh\left(u_j-u_i\right)\prod_{j<i=2}^{N+1}\sinh\left(v_j-v_i\right)} \tag{17}$$

where the matrix elements are

$$M_{ab}=\frac{\left(-1\right)^{N-1}}{\sinh\left(u_b-v_a\right)}\left(\prod_{j=1\neq b}^{N}\frac{\sinh\left(u_b-u_j-2\eta\right)}{\sinh\left(u_b-u_j+2\eta\right)}\prod_{j=1\neq a}^{N+1}\sinh\left(u_b-v_j+2\eta\right)+\prod_{j=1\neq a}^{N+1}\sinh\left(u_b-v_j-2\eta\right)\right) \tag{18}$$

for $b<N+1$,

$$M_{ab}=\frac{1}{\cosh\left(v_a-\eta\right)\cosh\left(v_a+\eta\right)} \tag{19}$$

for $b=N+1$, and $\mathbf{u}$ and $\mathbf{v}$ are the quasimomenta of two

Bethe states with N and $N+1$ particles respectively. Here anti-commutativity of Fermi particles at different positions is handled added by introducing a fermionic basis for auxiliary space in Eq. (11) in the construction of the algebraic Bethe ansatz [8] that does not alter any major part of the calculations for spin systems but leads only to flipping a sign from minus to plus in front of the second term in brackets in Eq. (18).

In the continuum regime, which corresponds to low particle density, we invert the mapping in Eq. (14), expand the matrix elements in Eq. (16) in $k_j^u \ll 1$, calculate the determinant in Eq. (15) explicitly, and obtain the normalisation factor as

$$\langle \mathbf{u}|\mathbf{u}\rangle = \frac{2^{N^2}\left(-1\right)^N\left(1+mU\right)^{N^2}\left(\mathcal{L}-\frac{mUN}{mU+1}\right)^N}{i^{N(N-1)}\prod_{i\neq j}\left(k_j^u-k_i^u\right)}, \quad (20)$$

where k_j^u are quasimomenta in the coordinate representation of Bethe ansatz that correspond to the parameters u_j. Repeating the same procedure for Eqs. (18, 19) we calculate the determinant in Eq. (17) as well and obtain the following result for the many-body matrix element

$$\left\langle \mathbf{v}|\psi_0^\dagger|\mathbf{u}\right\rangle = \left(-1\right)^N i^{N^2} 2^{N^2+N+\frac{1}{2}} \times \frac{\left(mU+1\right)^{N^2+\frac{1}{2}} m^N U^N \prod_j\left(\Delta P + k_j^u\right)}{\prod_{i,j}\left(k_j^v-k_i^u\right)} \quad (21)$$

where k_j^u and k_j^v are coordinate representations of quasimomenta of the states $\mathbf{u}$ and $\mathbf{v}$ and $\Delta P = \sum_j k_j^u - \sum_j k_j^v$ is difference of two conserved quantities, the total momenta of these two states.

We assume zero temperature so the initial state is the ground state with N particles, $k_j^u = k_j^0$, and the final state is an arbitrary state f with $N+1$ particles, $k_j^v = k_j^f$. Normalising the initial and the final state wave functions using Eq. (20) we evaluate the form factor in Eq. (3) of the main text as $\left|\left\langle f|\psi^\dagger\left(0\right)|0\right\rangle\right|^2 = \left|\left\langle \mathbf{k}^f|\psi_0^\dagger|\mathbf{k}^0\right\rangle\right|^2 \left\langle \mathbf{k}^f|\mathbf{k}^f\right\rangle^{-1}\left\langle \mathbf{k}^0|\mathbf{k}^0\right\rangle^{-1}$ and obtain using Eq. (21)

$$\left|\left\langle f|\psi^\dagger\left(0\right)|0\right\rangle\right|^2 = \frac{Z^{2N}}{\mathcal{L}}\frac{\prod_j\left(k_j^0-P_f\right)^2}{\prod_{i,j}\left(k_j^f-k_i^0\right)^2} \prod_{i<j}\left(k_j^0-k_i^0\right)^2\prod_{i<j}\left(k_j^f-k_i^f\right)^2, \quad (22)$$

where $Z = mU/\left(mU+1\right)/\left(\mathcal{L}-NmU/\left(1+mU\right)\right)$.

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