Modified Flux-Vector-Based Green Element Method for Problems in Steady-State Anisotropic Media - Generalisation to Triangular Elements

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Abstract

This paper is concerned with the generalisation of a numerical technique for solving problems in steady-state anisotropic media, namely the 'flux-vector-based' Green element method ('$\boldsymbol{q}$-based' GEM) for anisotropic media, to triangular elements. The generalisation of the method to triangular elements is based on the same concepts as for a rectangular grid, namely satisfying a nodal flux condition at each node of the mesh and the continuity of the tangential pressure gradient across the elements sharing a node.

Key words: Flux-vector-based Green element method, anisotropy, permeability

1 Introduction

Anisotropic media are widely encountered in nature, for example in oil and gas reservoirs. In many reservoirs, the production of gas and oil is seriously affected by the highly anisotropic and/or heterogeneous structure of the media.

Steady-state problems in anisotropic media can be solved using the modified '$\boldsymbol{q}$-based' GEM for anisotropic media, introduced by Lorinczi et al. [1]. The approach introduced by Lorinczi et al. [1] has been implemented for non-uniform rectangular grids. Lorinczi et al. [2] used this in geological problems in faulted/fractured anisotropic media.

A similar approach has been previously introduced by Lorinczi et al. [3] for isotropic media, and it was applied to highly-heterogeneous isotropic media. The two approaches are using the concept of the

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'$\boldsymbol{q}$-based' GEM (Pecher et al. [4]), which maintains the high-order accuracy of the GEM, diminished by some approximations used in the GEM, see Taigbenu [5].

This paper introduces the '$\boldsymbol{q}$-based' GEM for anisotropic media for triangular elements. Lorinczi et al. [3] showed that the $\boldsymbol{q}$-based' GEM for isotropic media can be naturally extended to triangular finite element grids. The work presented in here is based on satisfying similar conditions at nodes from a triangular mesh as in Lorinczi et al. [3], but considering the medium anisotropy.

2 Mathematical formulation

In this section we present the extension of the modified 'flux-vector-based' GEM for anisotropic media to triangular finite element grids. This is a technique suitable to solve problems in an anisotropic porous medium in which the equation governing the flow in a bounded domain $\Lambda \subset \mathbb{R}^2$ is given by

$$\nabla \cdot (\boldsymbol{K} \nabla p) = -F(\boldsymbol{x}) \quad \text{in} \quad \Lambda \tag{1}$$

where $\boldsymbol{K}$ is the permeability tensor, p is the fluid pressure and $F(\boldsymbol{x})$ is an internal/external source forcing function (term) which incorporates the fluid viscosity. For simplicity, we consider no internal/external source forcing function ($F \equiv 0$).

In the GEM the computational domain is discretised by suitable polygonal elements which collectively represent the shape of the domain. By applying the GEM theory, a more compact system of the following form is obtained:

$$\sum_{e=1}^{N_e} \left(R_{ij}^{(e)} p_j - L_{ij}^{(e)} q_j \right) = 0 \tag{2}$$

where the element matrices R_{ij} and L_{ij} can be found in Lorinczi et al. [1], and p_j and q_j are the pressure and the flux at the node j.

2.1 Internal source node

We denote by N the number of elements sharing an internal source node P in a triangular grid. There are therefore N interfaces intersecting at the node P. Figure 1(a) illustrates an internal node in a triangular grid shared by $N = 5$ elements. The permeability tensors corresponding to each of the N elements are denoted by $K_{i'j,i}$, $i', j = \overline{1, d}$, where $i = \overline{1, N}$ labels the element number. Let $(i, i+1)$ denote the interface between two neighbouring elements (i) and $(i+1)$, and $\boldsymbol{t}^{(i,i+1)}$ and $\boldsymbol{n}^{(i,i+1)}$ the tangent and the normal unit vectors to the interface $(i, i+1)$, for $i = 1, \ldots, N$ (the elements are always counted in a clockwise direction around the node). Using this notation, elements $(N+1)$ and (1) will be the same, and so will be elements (0) and (N). The tangent unit vector $\boldsymbol{t}^{(i,i+1)}$ is oriented toward the source node P, and the normal unit vector $\boldsymbol{n}^{(i,i+1)}$ is oriented outward from the element (i) over which the integration is performed, toward the element $(i+1)$, as represented in Figure 1(b), so that we have $\boldsymbol{t}^{(i,i+1)} = \boldsymbol{t}^{(i+1,i)}$ and $\boldsymbol{n}^{(i,i+1)} = -\boldsymbol{n}^{(i+1,i)}$ (we denote by $\boldsymbol{n}^{(i+1,i)}$ the normal unit vector to the $(i, i+1)$ interface for element $(i+1)$). The flux for an element (i) at node P can be uniquely written in terms of these unit vectors as follows:

$$\boldsymbol{q}^i = q_t^{(i,i+1),i}\boldsymbol{t}^{(i,i+1)} + q_n^{(i,i+1),i}\boldsymbol{n}^{(i,i+1)} \tag{3}$$

or

$$\boldsymbol{q}^i = q_t^{(i-1,i),i}\boldsymbol{t}^{(i,i-1)} + q_n^{(i-1,i),i}\boldsymbol{n}^{(i,i-1)} \tag{4}$$

(for the $(i-1, i)$ interface), where

$$q_n^{(i,i+1),i} = -\left(\frac{\partial p}{\partial n^+}\right)^{(i,i+1),i} = -\sum_{i',j=1}^{d} K_{i'j,i}\cos(n, x_{i'})\left(\frac{\partial p}{\partial x_j}\right)^{(i,i+1),i} \tag{5}$$

and

$$q_t^{(i,i+1),i} = -\left(\frac{\partial p}{\partial t^+}\right)^{(i,i+1),i} = -\sum_{i',j=1}^{d} K_{i'j,i}\cos(t, x_{i'})\left(\frac{\partial p}{\partial x_j}\right)^{(i,i+1),i} \tag{6}$$

denote the tangential and the normal component of $\boldsymbol{q}^i$, respectively, with respect to the interface $(i, i+1)$ on element (i) at P, as indicated in Figure 1(b), with expression (4) following similarly for the interface $(i-1, i)$, and $\cos(n, x_{i'})$ and $\cos(t, x_{i'})$ are the direction cosines of the normal $\boldsymbol{n}$ and the tangent $\boldsymbol{t}$ to the surface Γ, respectively.

Thus each flux component $\boldsymbol{q}^i$ is a linear combination of a tangential and a normal pressure gradient component. Consequently, along each interface there are two tangential and two normal pressure gradient components, arising from the two neighbouring elements.

However, there is an alternative representation of $\boldsymbol{q}^i$ in terms of only the tangential fluxes $q_t^{(i,i+1),i}$ and $q_t^{(i-1,i),i}$ which correspond to the elements $(i-1)$ and $(i+1)$ that neighbour the element (i), namely:

$$\boldsymbol{q}^i = \gamma_t\boldsymbol{t}^{(i,i+1)} + \varrho_t\boldsymbol{t}^{(i-1,i)} \tag{7}$$

where the expression for γ_t and ϱ_t can be found from the system of equations obtained by multiplying equation (7) by $\boldsymbol{t}^{(i,i+1)}$ and $\boldsymbol{t}^{(i-1,i)}$ respectively and taking into account expressions (3) and (4):

$$\begin{cases} \gamma_t + \varrho_t\boldsymbol{t}^{(i-1,i)} \cdot \boldsymbol{t}^{(i,i+1)} = q_t^{(i,i+1),i} \\ \gamma_t\boldsymbol{t}^{(i,i+1)} \cdot \boldsymbol{t}^{(i-1,i)} + \varrho_t = q_t^{(i-1,i),i} \end{cases} \tag{8}$$

The expressions for γ_t and ϱ_t are found to be

$$\gamma_t = \frac{1}{1-(\boldsymbol{t}^{(i-1,i)}\cdot\boldsymbol{t}^{(i,i+1)})^2} q_t^{(i,i+1),i} - \frac{\boldsymbol{t}^{(i-1,i)}\cdot\boldsymbol{t}^{(i,i+1)}}{1-(\boldsymbol{t}^{(i-1,i)}\cdot\boldsymbol{t}^{(i,i+1)})^2} q_t^{(i-1,i),i} \tag{9a}$$

$$\varrho_t = \frac{1}{1-(\boldsymbol{t}^{(i-1,i)}\cdot\boldsymbol{t}^{(i,i+1)})^2} q_t^{(i-1,i),i} - \frac{\boldsymbol{t}^{(i-1,i)}\cdot\boldsymbol{t}^{(i,i+1)}}{1-(\boldsymbol{t}^{(i-1,i)}\cdot\boldsymbol{t}^{(i,i+1)})^2} q_t^{(i,i+1),i} \tag{9b}$$

Similarly, using the alternative representation of $\boldsymbol{q}^i$ in terms of only the normal fluxes $q_n^{(i,i+1),i}$ and $q_n^{(i-1,i),i}$ corresponding to the elements $(i-1)$ and $(i+1)$ that neighbour the element (i), we can write

$$\boldsymbol{q}^i = \gamma_n \boldsymbol{n}^{(i,i+1)} + \varrho_n \boldsymbol{n}^{(i-1,i)} \tag{10}$$

where γ_n and ϱ_n are found in a similar way to γ_t and ϱ_t, and their expression is given by

$$\gamma_n = \frac{1}{1-(\boldsymbol{n}^{(i-1,i)}\cdot\boldsymbol{n}^{(i,i+1)})^2} q_n^{(i,i+1),i} - \frac{\boldsymbol{n}^{(i-1,i)}\cdot\boldsymbol{n}^{(i,i+1)}}{1-(\boldsymbol{n}^{(i-1,i)}\cdot\boldsymbol{n}^{(i,i+1)})^2} q_n^{(i-1,i),i} \tag{11a}$$

$$\varrho_n = \frac{1}{1-(\boldsymbol{n}^{(i-1,i)}\cdot\boldsymbol{n}^{(i,i+1)})^2} q_n^{(i-1,i),i} - \frac{\boldsymbol{n}^{(i-1,i)}\cdot\boldsymbol{n}^{(i,i+1)}}{1-(\boldsymbol{n}^{(i-1,i)}\cdot\boldsymbol{n}^{(i,i+1)})^2} q_n^{(i,i+1),i} \tag{11b}$$

The denominators in equations (9a), (9b), (11a) and (11b) are always non-zero, as the neighbouring tangential or normal unit vectors cannot be parallel.

By using the different representations of $\boldsymbol{q}^i$ given by equations (3), (4), (7) and (10), the following equations can be obtained:

$$q_t^{(i,i+1),i}\boldsymbol{t}^{(i,i+1)} + q_n^{(i,i+1),i}\boldsymbol{n}^{(i,i+1)} = \gamma_t \boldsymbol{t}^{(i,i+1)} + \varrho_t \boldsymbol{t}^{(i-1,i)} \tag{12a}$$

$$q_t^{(i-1,i),i}\boldsymbol{t}^{(i,i-1)} + q_n^{(i-1,i),i}\boldsymbol{n}^{(i-1,i)} = \gamma_n \boldsymbol{n}^{(i,i+1)} + \varrho_n \boldsymbol{n}^{(i-1,i)} \tag{12b}$$

and after all the flux components in these equations are replaced using equations (5) and (6), the two normal pressure gradient components can be determined in terms of the tangential pressure gradient components from these two conditions, in a similar way as they are determined in the case of a rectangular internal source node (see Lorinczi et al. [2]). Thus at each node the number of unknowns at each internal source node reduces to $2N+1$, namely $2N$ tangential pressure gradient components and the pressure.

In a triangular grid, as in the case of a rectangular one, at each internal source node and for every interface occurring at that node there is a condition representing the continuity of the tangential pressure gradient at that interface. Together, these represent N different conditions, which reduce the number of unknowns at each internal source node to $N+1$.

In order to illustrate the conditions at each node, necessary to solve the system of unknowns, we consider a small circle of radius ϵ, centred on the node, and the convex polygon which is formed by the tangents to the circle at the points where the circle intersects the element interfaces, as shown in Figure 1(a). Therefore each side of the polygon is perpendicular to the interface at that point. A generalised nodal flux condition for each node in this case can be expressed by the relation:

$$\sum_{i=1}^{N}(l^{(i,i+1),i}q_t^{(i,i+1),i} + l^{(i,i+1),i+1}q_t^{(i,i+1),i+1})\boldsymbol{t}^{(i,i+1)} = 0 \tag{13}$$

where $l^{(i,i+1),i}$ represents the length of the polygon side intersecting the interface $(i, i+1)$ and situated in element (i).

The nodal flux condition that we use is more restrictive than the mass conservation around a small square enclosing a node. However, it has been shown in a range of problems in Lorinczi et al. [1] and Lorinczi et al. [2] that it does not condition the type of solutions.

Equation (13) can be rewritten using the expressions of the tangential components of the flux and the tangential pressure gradient continuity conditions as follows:

$$\sum_{i=1}^{N}\left[-l^{(i,i+1),i}\sum_{i',j=1}^{d}K_{i'j,i}\cos(t,x_{i'})\left(\frac{\partial p}{\partial x_j}\right)^{(i,i+1),i} - l^{(i,i+1),i+1}\sum_{i',j=1}^{d}K_{i'j,i+1}\cos(t,x_{i'})\left(\frac{\partial p}{\partial x_j}\right)^{(i,i+1),i}\right]\boldsymbol{t}^{(i,i+1)} = 0 \tag{14}$$

Because the polygon edges are tangent to the circle, every two edge segments situated in the same element have the same length, namely

$$l^{(i,i+1),i} = l^{(i-1,i),i}, \quad i = 1, \ldots, N. \tag{15}$$

We can determined these lengths as functions of the circle radius ϵ and the angle between the neigbouring interfaces. If θ_i denotes the angle between the interfaces $(i-1, i)$ and $(i, i+1)$, then we have

$$l^{(i,i+1),i} = \epsilon\tan\left(\frac{\theta_i}{2}\right), \quad i = 1, \ldots, N \tag{16}$$

Using equation (16) in equation (14) gives (as we take the limit as $\epsilon \to 0$)

$$\sum_{i=1}^{N}\left[-\tan\left(\frac{\theta_i}{2}\right)\sum_{i',j=1}^{d}K_{i'j,i}\cos(t,x_{i'})\left(\frac{\partial p}{\partial x_j}\right)^{(i,i+1),i} - \tan\left(\frac{\theta_{i+1}}{2}\right)\sum_{i',j=1}^{d}K_{i'j,i+1}\cos(t,x_{i'})\left(\frac{\partial p}{\partial x_j}\right)^{(i,i+1),i}\right]\boldsymbol{t}^{(i,i+1)} = 0 \tag{17}$$

Two conditions are included in relation (17), and they are obtained by resolving in the x and y directions.

We can further generate $N-1$ equations at each internal source node, by integrating in turn over $N-1$ of the elements from the set of N elements sharing the node. Consequently, at each internal source node, a total of $N+1$ conditions can be generated, sufficient to determine the unknowns at that node. After determining all of these unknowns, the set of tangential and normal components of flux at each interface can be fully determined.

2.2 Side source node

We consider now boundary nodes and we refer to the side source node P in Figure 2(a). If N denotes the number of elements sharing the source node P, then there are $N-1$ internal interfaces intersecting at point P. Elements (1) and (N) are assumed to be the two elements on the boundary of the domain for the node P. For each of these interfaces there will be one independent tangential component of the pressure gradient, since a condition representing the continuity of the tangential pressure gradient at that interface can be used. Consequently, there are $N-1$ unknown tangential components of the flux at the internal boundaries. We can express the normal components of the pressure gradient for all the internal interfaces in terms of the tangential components in the same manner as for the internal source nodes in Section 4.1. For the two boundary interfaces, there is one tangential component of the flux for each, corresponding to the single element to which each boundary interface belongs, and we denote by q_t^1 and q_t^N the fluxes for elements (1) and (N), respectively. The corresponding unit tangent vectors are denoted by $\boldsymbol{t}^{(1)}$ and $\boldsymbol{t}^{(N)} = -\boldsymbol{t}^{(1)}$. The outward normal flux to the boundary, denoted by q_n, is assumed to apply at P for both elements (1) and (N) and is either specified (when the pressure is unknown at P) or is an unknown (when the pressure is known at P). Thus, there are $N+3$ unknowns at the side source node P. The nodal flux condition (similar to condition (14)) incorporates the normal flux q_n and it is expressed in this case as follows:

$$l^{(1,2),1} q_t^1 \boldsymbol{t}^{(1)} + \sum_{i=1}^{N-1} \left(l^{(i,i+1),i} q_t^{(i,i+1),i} + l^{(i,i+1),i+1} q_t^{(i,i+1),i+1} \right) \boldsymbol{t}^{(i,i+1)} + l^{(N-1,N),N} q_t^N \boldsymbol{t}^{(N)} = 2\epsilon q_n \boldsymbol{n} \tag{18}$$

which can be reformulated as

$$-\tan\left(\frac{\theta_1}{2}\right) \sum_{i',j=1}^{d} K_{i'j,1} \cos(t, x_{i'}) \left(\frac{\partial p}{\partial x_j}\right)^{(1)} \boldsymbol{t}^{(1)} + \sum_{i=1}^{N-1} \left[-\tan\left(\frac{\theta_i}{2}\right) \sum_{i',j=1}^{d} K_{i'j,i} \cos(t, x_{i'}) \left(\frac{\partial p}{\partial x_j}\right)^{(i,i+1),i} - \tan\left(\frac{\theta_{i+1}}{2}\right) \sum_{i',j=1}^{d} K_{i'j,i+1} \cos(t, x_{i'}) \left(\frac{\partial p}{\partial x_j}\right)^{(i,i+1),i} \right] \boldsymbol{t}^{(i,i+1)} - \tan\left(\frac{\theta_N}{2}\right) \sum_{i',j=1}^{d} K_{i'j,N} \cos(t, x_{i'}) \left(\frac{\partial p}{\partial x_j}\right)^{(N)} \boldsymbol{t}^{(N)} = 2q_n \boldsymbol{n} \tag{19}$$

From equation (19) we can generate two conditions (in the x and y directions). Together with the conditions obtained by integrating in turn over all of the elements sharing the side source node P and the boundary condition at P, these represent the necessary $N+3$ conditions for the unknowns at P to be determined. The approach can be directly applied to corner points also.

3 Conclusions

In this article we have presented the generalisation of a numerical technique for solving steady-state flow problems, namely the modified '$\boldsymbol{q}$-based' GEM for anisotropic media, to triangular elements. This is an extension of the modified '$\boldsymbol{q}$-based' GEM has been developed for rectangular grids (Lorinczi et al. [1]), the two techniques being based on the same concepts- satisfying a nodal flux condition at each node and on the continuity of the tangential pressure gradient across the element boundaries.

The extension of the modified '$\boldsymbol{q}$-based' GEM for anisotropic media to triangular grids is still to be tested in practice. However, this represents a useful theoretical advancement among the numerical solution techniques for reservoir simulation.

Acknowledgement

Piroska Lorinczi would like to acknowledge the financial support received from the ORS and the Rock Deformation Research Group, University of Leeds, UK.

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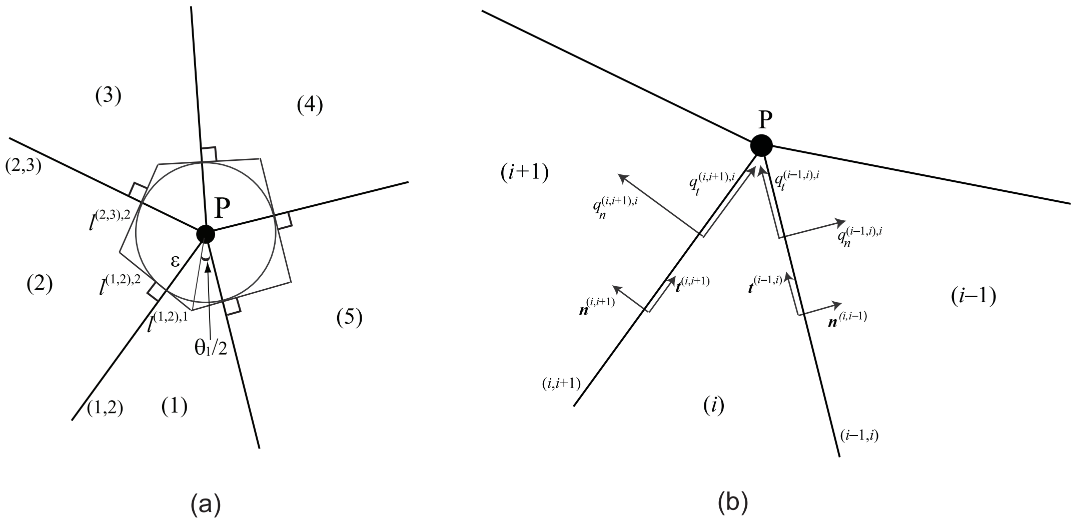

Fig. 1. (a) An internal node in a triangular grid and (b) representation of the flux components and of unit vectors for an element (i) in a triangular grid.

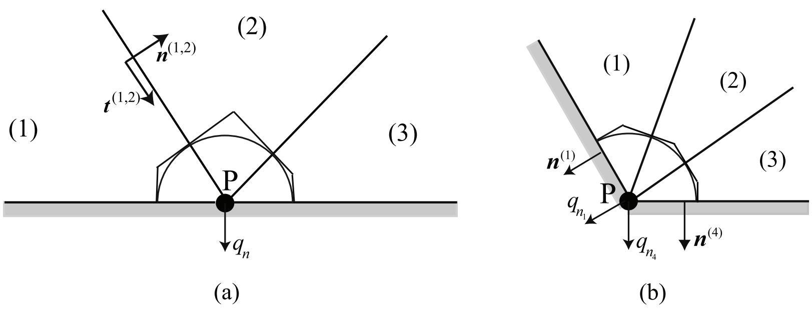

Fig. 2. (a) A side source node and (b) a corner source node in a triangular grid.