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ON THE PATH INDEPENDENCE CONDITIONS FOR DISCRETE-CONTINUOUS DEMAND

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ABSTRACT

We consider the manner in which the well-established path independence conditions apply to Small & Rosen's (1981) problem of discrete-continuous demand, focussing especially upon the restricted case of discrete choice (probabilistic) demand. We note that the consumer surplus measure promoted by Small & Rosen, which is specific to the probabilistic demand, imposes path independence to price changes *a priori*. We find that path independence to income changes can further be imposed provided a numeraire good is available in the consumption set. We show that, for practical purposes, McFadden's (1981) 'residual income' specification of the conditional indirect utility function offers an appropriate means of representing path independence to price and income changes.

JEL CLASSIFICATION: B41 Economic Methodology; D01 Microeconomic Behaviour, Underlying Principles

KEYWORDS: Path independence, discrete-continuous demand, discrete choice, consumer surplus, residual income

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1. INTRODUCTION

In common with colleagues applying continuous demand models, economists practised in discrete choice modelling have an interest in the impacts of price and income changes on demand and welfare. The paper by Small & Rosen (1981) (referred to henceforth as 'S&R') has been particularly influential in exploring the interface between continuous demand models - which might be regarded as the convention - and discrete choice models. S&R outline a model of discrete-continuous demand, whereby an individual selects from a set of mutually exclusive alternatives and, conditioned by that choice, consumes a positive quantity of the selected good. Within the context of this model, S&R isolate the consumer surplus change specific to the discrete choice (probabilistic) demand, associated with a change in price, income or some other qualitative attribute of the good in question.

When measuring consumer surplus in any demand context - discrete choice or otherwise - an issue of particular relevance is the welfare impact of income changes, following from a lump sum income supplement/reduction and/or an increase/decrease in real income associated with a price change. As is well established in the literature, the change in Marshallian consumer surplus, which derives from the integration of the Marshallian demand function with respect to the relevant price and income changes, is sensitive to the path of integration (i.e. the sequence of price and income changes). By contrast, the integral of the Hicksian demand function is independent of the path of integration.

S&R's consumer surplus measure is defined in terms of a representative consumer (Gorman, 1953), and conveniently allows the aggregation of discrete choices across repetitions and/or individuals. However, as is widely acknowledged, a limiting property of S&R's measure is that non-linear income effects[1] of price and lump sum income changes are excluded. This property straightforwardly ensures path independence (see Morey (1984) for a discussion of path independence more generally), but is somewhat crude, and potentially introduces bias into the resulting measure of surplus. Recognising this limitation, a number of contributors (e.g. Dagsvik & Karlström, 2005; Hau, 1985; Herriges & Kling, 1999; Jara-Díaz & Videla,

[1] That is to say, income effects which entail a non-linear income expansion path.

1989, 1990; Karlström, 1999; Karlström & Morey, 2001; McFadden, 1995) have explored methods for estimating the Hicksian compensating variation. The attraction of the compensating variation - relative to S&R's measure - is that it elicits a path independent measure of consumer surplus, even when non-linear income effects are present.

Despite this interest in Hicksian surplus measures, the extant literature offers no authoritative commentary on the path independence conditions for discrete choice. The present paper endeavours to fill this gap in the literature. The specific objectives of the paper are:

- To outline the path independence conditions applicable to the discrete-continuous demand in general, and the probabilistic demand (associated with discrete choice) in particular.
- To relate these conditions to the assumptions underpinning the derivation of S&R's consumer surplus measure.
- To draw implications for the practical specification of discrete choice models.

2. DERIVING CONSUMER SURPLUS FROM A MODEL OF DISCRETE-CONTINUOUS DEMAND

This section will introduce notation and, for the benefit of readers unfamiliar with the subject area, briefly summarise the salient features of S&R's model of discrete-continuous demand. Readers already initiated in S&R may wish to proceed directly to section 3.

2.1 S&R's model of discrete-continuous demand

Following S&R, consider a maximisation problem wherein the individual consumes non-negative quantities of three goods. Let us assume that goods 1 and 2 are mutually exclusive, whilst the third good - which we refer to as good *n* - acts as a numeraire. We might think of the latter, more intuitively, as 'all other goods'.

Defining notation: u is direct utility; $\mathbf{x}=(x_1,x_2,x_n)$ is a bundle comprising the quantities of goods 1, 2 and the numeraire good; $\mathbf{p}=(p_1,p_2,1)$ is the associated vector of prices of goods 1, 2 and n (noting that the price of the numeraire good is normalised to one); y is total income; and y_{1+2} is the income share available to goods 1 and 2 once the numeraire good has been accounted for (i.e. $y_{1+2}=y-x_n$, alluding to the potential for combining good 1 or 2 with good n to form composite goods). We are now equipped to formalise S&R's maximisation problem, as follows:

$$\begin{aligned} \text{Max} \quad & u=u(\mathbf{x}) \\ \text{s.t.} \quad & p_1x_1+p_2x_2=y_{1+2} \\ & x_1x_2=0 \\ & \mathbf{x}\geq 0 \end{aligned} \tag{1}$$

where $y_{1+2}=y-x_n$

An important feature of (1) is the constraint $x_1x_2=0$, which precludes joint consumption of goods 1 and 2. Indeed, S&R conceptualise (1) as a problem of discrete-continuous demand, whereby the individual first chooses between goods 1 and 2 according to which yields the greater utility:

$$u^*(\mathbf{x})=v^*(\mathbf{p},y)=\tilde{v}_k(p_k,y)=\text{Max}\{\tilde{v}_1(p_1,y),\tilde{v}_2(p_2,y)\} \tag{2}$$

where u^* is the maximum direct utility both unconditionally and conditionally given income y, v^* is the maximum indirect utility, $\tilde{v}_k$ is the conditional indirect utility, and k indexes the chosen (i.e. utility maximising) good, i.e. $k=1$ if $\tilde{v}_1\geq\tilde{v}_2$, or $k=2$ otherwise. Having chosen between goods 1 and 2, the individual selects a positive quantity of the chosen good, as well as a non-negative quantity of the numeraire good. If income is devoted entirely to goods 1 and 2, then $y=y_{1+2}$ and consumption of the numeraire good will be zero.

As the annex to the present paper shows, if we solve (1) for the uncompensated demands for goods 1 and 2 then, unlike more conventional continuous demand models, Roy's identity derives the demands for goods 1 and 2 *conditional* upon the discrete choice between goods 1 and 2:

$$-\frac{\partial v^{*}(\mathbf{p},y)/\partial p_1}{\partial v^{*}(\mathbf{p},y)/\partial y}=\begin{cases}-\dfrac{\partial \tilde{v}_1(p_1,y)/\partial p_1}{\partial \tilde{v}_1(p_1,y)/\partial y}=\tilde{x}_1 & \text{if } k=1\\ -\dfrac{\partial \tilde{v}_2(p_2,y)/\partial p_1}{\partial \tilde{v}_2(p_2,y)/\partial y}=0 & \text{if } k=2\end{cases}$$

(3)

$$-\frac{\partial v^{*}(\mathbf{p},y)/\partial p_2}{\partial v^{*}(\mathbf{p},y)/\partial y}=\begin{cases}-\dfrac{\partial \tilde{v}_1(p_1,y)/\partial p_2}{\partial \tilde{v}_1(p_1,y)/\partial y}=0 & \text{if } k=1\\ -\dfrac{\partial \tilde{v}_2(p_2,y)/\partial p_2}{\partial \tilde{v}_2(p_2,y)/\partial y}=\tilde{x}_2 & \text{if } k=2\end{cases}$$

where $\tilde{x}_j$ is uncompensated demand conditional upon the choice of good $j=1,2$.

Whilst this notion of conditional demand is central to S&R's analysis, we should acknowledge that the demand revealed empirically is not the conditional demand, but rather the unconditional demand. S&R reconcile the unconditional and conditional demands via the following construct, which represents unconditional demand as the expected demand:

$$x_j(\mathbf{p},y)=\pi_j(\mathbf{p},y)\cdot\tilde{x}_j(p_j,y) \quad \text{for } j=1,2 \tag{4}$$

where π_j denotes the uncompensated probabilistic demand for good j, and $\tilde{x}_j$ denotes the uncompensated conditional demand for good j. In this context, the probability of choosing good *j* is represented as a demand function in its own right[2]. The probabilistic demand can be estimated empirically at either the individual level (if we have data on multiple occurrences of the same consumption decision by a given individual) or the aggregate level (if we have data on a given consumption decision by multiple individuals).

2.2 Econometric specification of the probabilistic demand

For purposes of econometric implementation, convention is to specify the probabilistic demand in the form of the Random Utility Model (RUM). With reference

[2] See Hau (1985, 1987) for a derivation of the probabilistic demand from first principles.

to Marschak (1960) and Block & Marschak (1960), who conceived RUM, and Daly & Zachary (1978), who formalised RUM in mathematical terms, define:

$$\left.\begin{aligned}\pi_j &= \Pr\left\{\tilde{v}_j\left(p_j,y\right) \geq \tilde{v}_m\left(p_m,y\right)\right\} = \Pr\left\{W_j\left(p_j,y\right)+\varepsilon_j \geq W_m\left(p_m,y\right)+\varepsilon_m\right\} \\ &= \varphi\left(W_j\left(p_j,y\right)-W_m\left(p_m,y\right)\right)\end{aligned}\right\} \quad \text{for } j,m=1,2 \quad m \neq j \tag{5}$$

where the conditional indirect utility $\tilde{v}_j$ arises from the sum of deterministic utility W_j and the random term ε_j, and φ is the distribution function of $\varepsilon_m - \varepsilon_j$. In the context of consumer surplus measurement, an important feature of (5) is that W_j is dependent on prices and income, whilst ε_j is independent of prices and income.

Informed by previous contributions (including S&R, as well as Hanemann (1982, 1999) and Hau (1985, 1987) among others), Batley & Ibáñez (2013) showed that, in order to comply with the fundamental properties of demand functions (i.e. adding-up, negativity, homogeneity and symmetry), (5) must observe five assumptions, namely:

- **Assumption I:** unit conditional[3] demand for goods 1 and 2, i.e. $\tilde{x}_j = 1 \quad \text{for } j=1,2$
- **Assumption II:** for each alternative, equivalence (in absolute value) between the conditional marginal utilities of income and price, i.e. $-\partial W_j/\partial p_j = \partial W_j/\partial y \quad \text{for } j=1,2$
- **Assumption III:** common conditional marginal utility of income across alternatives, i.e. $\partial W_j/\partial y = \lambda_j = \lambda \quad \text{for } j=1,2$
- **Assumption IV:** common conditional marginal utility of price across alternatives, i.e. $\partial W_j/\partial p_j = -\lambda_j = -\lambda \quad \text{for } j=1,2$
- **Assumption V:** the conditional marginal utility of income is independent of the prices of goods 1 and 2, i.e. $\partial \lambda_j/\partial p_j = 0 \quad \text{for } j=1,2$

Given Roy's identity, the assumptions are inter-related in the following manner:

[3] That is to say, *conditional* upon discrete choice.

- **Assumption I $\leftrightarrow$ Assumption II**
- **(Assumption II + Assumption III) $\mapsto$ Assumption IV**
- **(Assumption II + Assumption IV) $\mapsto$ Assumption III**

Thus, in effect, Assumptions I to V reduce to three independent assumptions.

2.3 Deriving consumer surplus from the discrete-continuous demand

Completing our discussion of preliminary concepts, let us consider the derivation of consumer surplus measures from (1). In particular, consider the change in consumer surplus arising from changes in both prices and income between two states (denoted by the superscripts 0 and 1, respectively), which in Marshallian terms can be written[4]:

$$\Delta E(v) = \int_{(\mathbf{p}^0, y^0)}^{(\mathbf{p}^1, y^1)} \left(\sum_{j=1,2} \left(\lambda(\mathbf{p}, y) \cdot \pi_j(\mathbf{p}, y) \cdot \tilde{x}_j(p_j, y) \right) \right) d\mathbf{p} dy \tag{6}$$

where $\lambda(\mathbf{p}, y)$ is the marginal utility of income, functional upon prices and income.

S&R effectively impose Assumptions I to V on (6), and thereby isolate the consumer surplus change associated with the probabilistic demand, thus:

$$\begin{aligned} &\Delta E(v)\big|_{\text{AI-AV}} = \int_{W_j^0}^{W_j^1} \pi_j(\mathbf{p}, y)\, dW_j \\ &\text{where} \quad \pi_j(\mathbf{p}, y) = \pi_j\left(W_j(p_j, y) - W_m(p_m, y)\right) \quad \text{for } j, m = 1,2 \quad m \neq j \end{aligned} \tag{7}$$

Three features of (7) might be noted in relation to aggregation. First, (7) adopts the perspective of the 'representative consumer' (Gorman, 1953), and yields measures of Marshallian and Hicksian consumer surplus that are exactly equivalent. This ensures path independence, thereby avoiding any complications of aggregation across individuals. Second, RUM (5) embodies the 'translational invariance' property, meaning that only the difference between deterministic utilities $(W_j - W_m)$

[4] See for example Johansson (1987) for a formal statement of the change in Marshallian consumer surplus, which in this case we adapt to S&R's expected demand function. Although (6) measures consumer surplus change in utils, this can be straightforwardly converted into a money measure by dividing through by the marginal utility of income, i.e. $\Delta E(e) = \Delta E(v)/\lambda$, where e is the 'dual' expenditure function to the 'primal' direct utility function.

affects probability, and not the absolute utilities. On this basis, and because we are operating within the binary case, (7) can be measured from the perspective of either good 1 or good 2, thereby avoiding any complications of aggregation across goods 1 and 2. Third, drawing upon the comments at the end of section 2.1, relaxation of Assumption I would admit aggregation across multiple occurrences of a given choice between goods 1 and 2 (i.e. $\tilde{x}_j \neq 1$ for $j = 1,2$ in (6)).

Applying (7) in practice, S&R further assume *'...on the basis of purely empirical considerations...which are likely to be valid in many applications...that the discrete goods are sufficiently unimportant to the consumer so that income effects…are negligible, i.e. that the compensated demand…is adequately approximated by the Marshallian demand function'* (p124). In what follows, we will relate Assumptions I to V to the path independence conditions for discrete-continuous demand, and in so doing clarify the role of S&R's additional assumption concerning income effects.

3. THE PATH INDEPENDENCE CONDITIONS

As is widely understood and accepted, the integral of the Marshallian demand with respect to changes in prices and income - representing the change in consumer surplus associated with the price/income changes - is in general sensitive to the *sequence* of price/income changes. The subsequent discussion will consider the so-called 'path independence' conditions; the conditions under which the integral of the Marshallian demand is unaffected by the path of integration. The derivation of the path independence conditions is comprehensively documented in the literature (e.g. Johansson, 1987), and we will not therefore devote attention to such matters ourselves. Rather we will proceed by simply stating the conditions, as they apply to goods 1 and 2, thus[5]:

$$\frac{\partial\left(\lambda(\mathbf{p},y)x_1\right)}{\partial p_2} = \frac{\partial\left(\lambda(\mathbf{p},y)x_2\right)}{\partial p_1} \qquad (8)$$

[5] For notational brevity, the remainder of the paper will suppress functional dependencies where this is opportune and does not impinge upon the clarity of the analysis.

$$\frac{\partial\left(\lambda(\mathbf{p},y)x_j\right)}{\partial y}=\frac{\partial\lambda(\mathbf{p},y)}{\partial p_j}\quad \text{for } j=1,2 \tag{9}$$

3.1 Path independence conditions for the expected demand

Substituting for S&R's expected demand (4) in (8) and (9), and applying the product rule of differentiation, the path independence conditions become:

$$\begin{aligned}&\pi_1\tilde{x}_1\frac{\partial\lambda(\mathbf{p},y)}{\partial p_2}+\lambda(\mathbf{p},y)\frac{\partial\left(\pi_1\tilde{x}_1\right)}{\partial p_2}=\\&\pi_2\tilde{x}_2\frac{\partial\lambda(\mathbf{p},y)}{\partial p_1}+\lambda(\mathbf{p},y)\frac{\partial\left(\pi_2\tilde{x}_2\right)}{\partial p_1}\end{aligned} \tag{10}$$

$$\pi_j\tilde{x}_j\frac{\partial\lambda(\mathbf{p},y)}{\partial y}+\lambda(\mathbf{p},y)\frac{\partial\left(\pi_j\tilde{x}_j\right)}{\partial y}=\frac{\partial\lambda(\mathbf{p},y)}{\partial p_j}\quad \text{for } j=1,2 \tag{11}$$

Expanding the expected demands, again using the product rule, (10) and (11) can be re-stated, respectively:

$$\begin{aligned}&\pi_1\tilde{x}_1\frac{\partial\lambda(\mathbf{p},y)}{\partial p_2}+\lambda(\mathbf{p},y)\left[\pi_1\frac{\partial\tilde{x}_1}{\partial p_2}+\tilde{x}_1\frac{\partial\pi_1}{\partial p_2}\right]=\\&\pi_2\tilde{x}_2\frac{\partial\lambda(\mathbf{p},y)}{\partial p_1}+\lambda(\mathbf{p},y)\left[\pi_2\frac{\partial\tilde{x}_2}{\partial p_1}+\tilde{x}_2\frac{\partial\pi_2}{\partial p_1}\right]\end{aligned} \tag{12}$$

$$\pi_j\tilde{x}_j\frac{\partial\lambda(\mathbf{p},y)}{\partial y}+\lambda(\mathbf{p},y)\left[\pi_j\frac{\partial\tilde{x}_j}{\partial y}+\tilde{x}_j\frac{\partial\pi_j}{\partial y}\right]=\frac{\partial\lambda(\mathbf{p},y)}{\partial p_j}\quad \text{for } j=1,2 \tag{13}$$

A priori, compliance with (12) and (13) is not guaranteed. The literature has therefore considered restrictions on the analysis that will ensure path independence (see Deaton & Muellbauer (1980) or Johansson (1987) for a useful summary). In this regard, two points might be noted.

- First, we can assume that the marginal utility of income is independent of the prices of all goods but not income, or independent of income and the price of all but one good, but we cannot (in general) assume that the marginal utility of income is independent of the prices of all goods and income; this would violate homogeneity. Another way of rationalising this would be to observe that, in the first order conditions for solving (1), independence of the marginal utility of income from the prices of all goods and income would imply that consumption is unconstrained by budget.

- Second, we can assume zero income effects in relation to all but one good, but we cannot (in general) assume zero income effects in relation to all goods; this would violate adding-up.

Let us illustrate these two points, by considering some possible restrictions on (12) and (13).

Case 1: Let the marginal utility of income be independent of the price of good 1 and income, but dependent on the price of good 2, i.e. $\partial\lambda(\mathbf{p},y)/\partial p_1 = \partial\lambda(\mathbf{p},y)/\partial y = 0$ and $\partial\lambda(\mathbf{p},y)/\partial p_2 \neq 0$. In this case, the path independence conditions (12) and (13) simplify to the following:

$$\left[\pi_1 \frac{\partial \tilde{x}_1}{\partial p_2} + \tilde{x}_1 \frac{\partial \pi_1}{\partial p_2}\right] - \left[\pi_2 \frac{\partial \tilde{x}_2}{\partial p_1} + \tilde{x}_2 \frac{\partial \pi_2}{\partial p_1}\right] = -\frac{\pi_1 \tilde{x}_1}{\lambda(p_2)} \frac{\partial \lambda(p_2)}{\partial p_2}$$

$$\lambda(p_2)\left[\pi_1 \frac{\partial \tilde{x}_1}{\partial y} + \tilde{x}_1 \frac{\partial \pi_1}{\partial y}\right] = \frac{\partial \lambda(p_2)}{\partial p_1} = 0$$

$$\lambda(p_2)\left[\pi_2 \frac{\partial \tilde{x}_2}{\partial y} + \tilde{x}_2 \frac{\partial \pi_2}{\partial y}\right] = \frac{\partial \lambda(p_2)}{\partial p_2}$$

Under this restriction, the income effects of price and/or income changes manifest entirely in terms of the demand for good 2. Therefore, if we were to draw an indifference map in terms of goods 1 and 2, and specify good 2 on the horizontal (vertical) dimension, the income expansion path would be a horizontal (vertical) line. That is to say, Case 1 would give rise to an indifference map embodying quasi-linear preferences; see Batley (2013) for a diagrammatic exposition of this case.

Case 2: Let the marginal utility of income be independent of the prices of goods 1 and 2, but dependent on income, i.e. $\partial\lambda(\mathbf{p},y)/\partial p_1 = \partial\lambda(\mathbf{p},y)/\partial p_2 = 0$ and $\partial\lambda(\mathbf{p},y)/\partial y \neq 0$. In this case, (12) and (13) simplify as follows:

$$\pi_1 \frac{\partial \tilde{x}_1}{\partial p_2} + \tilde{x}_1 \frac{\partial \pi_1}{\partial p_2} = \pi_2 \frac{\partial \tilde{x}_2}{\partial p_1} + \tilde{x}_2 \frac{\partial \pi_2}{\partial p_1}$$

$$\pi_j \tilde{x}_j \frac{\partial \lambda(y)}{\partial y} + \lambda(y)\left[\pi_j \frac{\partial \tilde{x}_j}{\partial y} + \tilde{x}_j \frac{\partial \pi_j}{\partial y}\right] = 0 \quad \text{for } j = 1,2$$

Under this restriction, the income effects of a price change are zero for both goods (meaning that a price change leads to a pure substitution effect), but the income effects of a lump sum income change are non-zero. Case 2 would give rise to an indifference map exhibiting homothetic preferences, i.e. the income expansion path would be a straight line from the origin, such that goods 1 and 2 are consumed in fixed proportion.

Whereas Cases 1 and 2 are reasonably well documented in the literature - indeed these cases were identified in the seminal work of Samuelson (1942) on the path independence conditions - Case 3 which follows is especially pertinent to the demand problem considered by S&R.

Case 3: With reference to the specification of the budget constraint in (1), we will now return to our earlier remark concerning composite goods, by distinguishing between the sub-budgets allocated to goods 1 and 2 (i.e. y_{1+2}) and to the numeraire good (i.e. x_n) respectively[6]. On this basis, we can re-state the path independence conditions (12) and (13) as follows:

$$\begin{aligned} &\pi_1 \tilde{x}_1 \frac{\partial \lambda(\mathbf{p},y)}{\partial p_2} + \lambda(\mathbf{p},y)\left[\pi_1 \frac{\partial \tilde{x}_1}{\partial p_2} + \tilde{x}_1 \frac{\partial \pi_1}{\partial p_2}\right] = \\ &\pi_2 \tilde{x}_2 \frac{\partial \lambda(\mathbf{p},y)}{\partial p_1} + \lambda(\mathbf{p},y)\left[\pi_2 \frac{\partial \tilde{x}_2}{\partial p_1} + \tilde{x}_2 \frac{\partial \pi_2}{\partial p_1}\right] \end{aligned} \tag{14}$$

$$\pi_j \tilde{x}_j \frac{\partial \lambda(\mathbf{p},y)}{\partial y_{1+2}} + \lambda(\mathbf{p},y)\left[\pi_j \frac{\partial \tilde{x}_j}{\partial y_{1+2}} + \tilde{x}_j \frac{\partial \pi_j}{\partial y_{1+2}}\right] = \frac{\partial \lambda(\mathbf{p},y)}{\partial p_j} \quad \text{for } j = 1,2 \tag{15}$$

Progressing Case 3, let the marginal utility of income be independent of the prices of goods 1 and 2 and the budget allocation to goods 1 and 2, but dependent on the

[6] In effect, Cases 1 and 2 have implicitly assumed that $x_n = 0$ and $y = y_{1+2}$, such that budget is devoted entirely to goods 1 and 2, whereas Case 3 assumes $x_n \neq 0$ and $y = y_{1+2} + x_n$.

budget allocation to the numeraire good, i.e. $\partial\lambda(\mathbf{p},y)/\partial p_1 = \partial\lambda(\mathbf{p},y)/\partial p_2 = \partial\lambda(\mathbf{p},y)/\partial y_{1+2} = 0$ and $\partial\lambda(\mathbf{p},y)/\partial x_n \neq 0$. This case permits simplification of (14) and (15), thus:

$$\pi_1 \frac{\partial \tilde{x}_1}{\partial p_2} + \tilde{x}_1 \frac{\partial \pi_1}{\partial p_2} = \pi_2 \frac{\partial \tilde{x}_2}{\partial p_1} + \tilde{x}_2 \frac{\partial \pi_2}{\partial p_1}$$

$$\lambda(\mathbf{p},y)\left[\pi_j \frac{\partial \tilde{x}_j}{\partial y_{1+2}} + \tilde{x}_j \frac{\partial \pi_j}{\partial y_{1+2}}\right] = 0 \quad \text{for } j = 1,2$$

On this basis, changes in prices and income will have no income effects on goods 1 and 2, but may have an income effect on the numeraire good. Provided the change in consumer surplus is path independent with respect to the numeraire (as applies to Marshall's (1920) definition of the numeraire in the context of partial equilibrium analysis[7]), then homogeneity and adding-up will be observed.

Otherwise, where the expected demands for goods 1 and 2 are subject to income effects of price and/or income changes, (12) and (13) may not hold. If (12) and (13) do not hold then the consumer surplus measure emanating from the expected demands will exhibit path dependence. In such cases, it is instructive to consider the attribution of these income effects to the component parts of the expected demand, namely the conditional and probabilistic demands for goods 1 and 2 (as well as any attribution to the numeraire good). With this interest in mind, let us now consider the path independence conditions for the probabilistic demand in particular, since these conditions are pertinent to S&R's consumer surplus measure (7).

3.2 Path independence conditions for the probabilistic demand

If we impose Assumption I from section 2, acknowledge that this further implies that the conditional demand is fixed at one and therefore independent of price and income (i.e. $\partial\tilde{x}_j/\partial p_j, \partial\tilde{x}_j/\partial y = 0$ for $j = 1,2$), and condition the marginal utility of

[7] This was pointed out to the authors by Robert Cochrane in the course of private communication.

income by choice[8] (i.e. $\lambda_j\left(p_j,y\right)$ for $j=1,2$), then we can restrict the path independence conditions (12) and (13) to the case of a single discrete choice (i.e. probabilistic demand), thus:

$$\pi_1\frac{\partial\lambda_1\left(p_1,y\right)}{\partial p_2}+\lambda_1\left(p_1,y\right)\frac{\partial\pi_1}{\partial p_2}=\pi_2\frac{\partial\lambda_2\left(p_2,y\right)}{\partial p_1}+\lambda_2\left(p_2,y\right)\frac{\partial\pi_2}{\partial p_1} \tag{16}$$

$$\pi_j\frac{\partial\lambda_j\left(p_j,y\right)}{\partial y}+\lambda_j\left(p_j,y\right)\frac{\partial\pi_j}{\partial y}=\frac{\partial\lambda_j\left(p_j,y\right)}{\partial p_j}\quad\text{for } j=1,2 \tag{17}$$

If we further impose Assumptions III and V, then (16) and (17) simplify, respectively:

$$\frac{\partial\pi_1}{\partial p_2}=\frac{\partial\pi_2}{\partial p_1} \tag{18}$$

$$\pi_j\frac{\partial\lambda_j\left(p_j,y\right)}{\partial y}+\lambda_j\left(p_j,y\right)\frac{\partial\pi_j}{\partial y}=0\quad\text{for } j=1,2 \tag{19}$$

For purposes of econometric implementation, the probabilistic demand π_j is typically specified as RUM (i.e. as (5)). Since RUM embodies the Jacobian condition $\partial\pi_1/\partial W_1=\partial\pi_2/\partial W_2$ (as discussed in footnote 27 of S&R, but see Daly & Zachary (1978) for a proof), (18) simplifies to:

$$\frac{\partial W_1}{\partial p_1}=\frac{\partial W_2}{\partial p_2} \tag{20}$$

We arrive thus at a rationale for Assumption IV. Moreover, we can conclude that, given Assumptions I to V[9], path independence in relation to price changes will be imposed *a priori*.

In a similar vein, if we draw upon the econometric specification (5), then we can re-write (19):

[8] Indeed, if the marginal utility were not conditioned by choice, then the probabilistic demand would potentially give rise to the 'mother logit' (McFadden *et al.*, 1978) class of models. This is where the conditional indirect utility of good 1 is a function of the conditional indirect utility of good 2, and *vice versa*. It can be shown that 'mother logit' is inconsistent with RUM (Ibáñez & Batley, 2011).

[9] This is consistent with our assertion in section 2 that Assumption I implies II, and that II and III together imply IV.

$$\frac{\partial W_j}{\partial y} = -\pi_j \frac{\partial \lambda_j\left(p_j, y\right)}{\partial y}\left[\frac{\partial \pi_j}{\partial y}\right]^{-1} \quad \text{for } j = 1,2 \tag{21}$$

If we further apply Assumption III, then the following equality arises from (21):

$$\pi_1 \frac{\partial \lambda_1\left(p_1, y\right)}{\partial y}\left[\frac{\partial \pi_1}{\partial y}\right]^{-1} = \pi_2 \frac{\partial \lambda_2\left(p_2, y\right)}{\partial y}\left[\frac{\partial \pi_2}{\partial y}\right]^{-1} \tag{22}$$

Given translational invariance, we can simplify (22) to:

$$\pi_1 \frac{\partial \lambda_1\left(p_1, y\right)}{\partial y} = \pi_2 \frac{\partial \lambda_2\left(p_2, y\right)}{\partial y} \tag{23}$$

Equipped with (20) and (23), let us now apply the path independence conditions for the probabilistic demand to Cases 1, 2 and 3 introduced in section 3.1. Note that, since Assumption V imposes path independence to price changes *a priori*, we need only consider cases where consumer surplus will (additionally) be path independent to lump sum income changes.

Case 1: Unlike the discrete-continuous demand, Case 1 cannot apply to the probabilistic demand, since it is effectively excluded by Assumption V.

Case 2: Noting Assumption III, and further assuming that the rate of change of the conditional marginal utility of income with respect to income is common across goods[10], we can write: $\partial \lambda_j\left(p_j, y\right)/\partial y = \partial \lambda/\partial y \neq 0$ for $j = 1,2$. On this basis, (23) will in principle hold provided $\pi_1 = \pi_2 = 0.5$, implying a particular form of homothetic preferences where goods 1 and 2 are perfect substitutes and have an equal chance of being chosen. If budget is devoted entirely to goods 1 and 2, and demand for the numeraire good is zero, then equi-probability further implies that $y = p_1 = p_2$; see Batley & Ibáñez (2013) for discussion of this point in the context of homogeneity and symmetry[11]. In practice, however, Assumption I eliminates the possibility that

[10] If we were to admit different rates of change by good, then this would seem to call for a more fundamental re-statement of the path independence conditions for the probabilistic demand.

[11] The property of common prices is not as restrictive as it might seem, since it follows from the manner in which the 'goods' and the 'budget' are defined. Consider for example a choice between a vacation, with an actual price of £2000, and a 'staycation' with an actual price of £0; let us assume that consumption of the vacation exhausts the available budget. Since the foregoing of the vacation will - in effect - release £2000 for consumption of the numeraire good, we can conceptualise good 1 as the vacation (at a unit price of £2000) and good 2 as the numeraire consumption associated with

increased income could be devoted to increased consumption (even along the income expansion path for $\pi_1 = \pi_2 = 0.5$). Thus, like Case 1, Case 2 is effectively excluded by assumption.

Case 3: In an analogous fashion to Case 3 of the discrete-continuous demand, let us now distinguish between the sub-budgets allocated to goods 1 and 2 and to the numeraire good respectively, and admit income effects only through the latter. On this basis, let us consider the separate impacts of income and price changes.

If income changes but Assumptions I to V hold, we would expect goods 1 and 2 to have common choice shares (i.e. $\pi_1 = \pi_2 = 0.5$), in the same manner as Case 2. Unlike Case 2, however, the prices of goods 1 and 2 will now be equal to the budget allocation to goods 1 and 2 (rather than equal to total budget, i.e. $y_{1+2} = p_1 = p_2$), and any lump sum income change will be directed entirely to the numeraire good. In terms of S&R's measure of consumer surplus change (7), which is defined on goods 1 and 2, the income change will manifest as a constant of integration.

Digressing slightly, we can also reason that if $\partial\pi_j / \partial y_{1+2} = 0$ (as would apply if we admitted S&R's 'empirical' assumption *'...that the discrete goods are sufficiently unimportant to the consumer so that income effects…are negligible'* (p124) to Case 3), then (19) simplifies to:

$$\pi_j \frac{\partial\lambda_j\left(p_j, y\right)}{\partial y_{1+2}} = 0 \quad \text{for } j = 1,2 \qquad (24)$$

Since probability must be non-zero for at least one of the goods (and non-negative for both goods), (24) will hold only if $\partial\lambda_j\left(p_j, y\right) / \partial y_{1+2} = 0$ for $j = 1,2$, i.e. only if the probabilistic demand embodies path independence to lump sum income changes (as well as path independence to price changes). On this basis, S&R's 'empirical' assumption is consistent with Case 3.

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the staycation (also at a *notional* unit price of £2000). On this basis, one unit of either good will exhaust the budget.

If relative prices change but Assumptions I to V hold, any income effect will again be directed entirely to the numeraire good[12]. Unlike a lump sum income change, however, this entails the reconstitution of good 1 or 2 as a composite good (as mentioned at the beginning of section 2). For example, if the price of good 2 falls relative to the price of good 1, then Assumption I precludes additional consumption of good 2, but does not preclude the establishment of the composite good $\left(\tilde{x}_2 + x_n/p_2\right)$, which can proxy for the demand response to the price reduction. On this basis, the probabilistic demand must now be defined in terms of good 2 and the composite good, and would be expected to give rise to different choice shares (i.e. $\pi_1 \neq \pi_2$). In terms of S&R's measure of consumer surplus change (7), the price change will, unlike the income change, manifest within the integration.

3.3 Practical implications for model specification

Drawing together sections 3.1 and 3.2, let us now consider the implications of the path independence conditions for the practical specification of RUM models. Table 1 introduces six specifications of the conditional indirect utility function; all have seen practical usage in the literature, although some more commonly than others. The first column of the table labels the models A to F, the second column gives the precise specification of conditional indirect utility that each model entails, and the third column notes whether the model relates to the discrete-continuous demand or the probabilistic demand. The fourth column derives the conditional marginal utility of income, whilst the fifth and sixth columns differentiate the conditional marginal utility of income with respect to price and income respectively. The final column summarises the properties of the model in terms of path independence (with particular reference to Cases 1, 2 and 3 discussed above).

Model A is McFadden's (1981) 'residual income' form, which complies with Assumptions I to V detailed in section 2. Note furthermore that, since $-\partial\tilde{v}_j/\partial p_j = \partial\tilde{v}_j/\partial y$ for $j = 1,2$, Roy's identity (3) will yield conditional demands of

[12] Since this demand response embodies quasi-linear preferences, it could instead be interpreted as a pure substitution effect. See Case C of Batley (2013) for a diagrammatic exposition.

one unit for both goods 1 and 2, meaning that Model A is applicable to the context of probabilistic demand. With regards to the path independence conditions, note that the conditional marginal utility of income is independent of prices and income. On this basis, we can reason that Model A is consistent with Case 3 of the path independence conditions for the probabilistic demand; this is to say, if $y > p_j$ then any income effects (associated with homothetic or quasi-linear preferences) can be capitalised in terms of the demand for the numeraire good.

Model B is a slight adjustment to Model A that introduces a power term on income, common to goods 1 and 2. If this power term is significantly different from one then income will have a non-linear effect on conditional indirect utility. Furthermore, since $-\partial \tilde{v}_j / \partial p_j \neq \partial \tilde{v}_j / \partial y$ for $j = 1,2$, Roy's identity will yield conditional demands that differ from one, meaning that Model B is applicable to the context of discrete-continuous demand. Since Model B embodies path independence with respect to prices but path dependence with respect to income, we can reason that it is consistent with Case 2 of the path independence conditions for the discrete-continuous demand.

Model C also introduces a power term to Model A, but this time on price rather than income; again we assume that the power term is common to goods 1 and 2. If this power term is significant, then Model C will yield discrete-continuous demands for goods 1 and 2, whilst preserving the path independence properties of Model A. In other words, the conditional marginal utility of income is independent of prices and income, consistent with Case 3.

Model D specifies price as a ratio of income (see for example Hau (1985)). If $y > p_j$ for $j = 1,2$, this model will again yield discrete-continuous demands. In the case of Model D, however, the conditional marginal utility of income is dependent upon the prices of both goods 1 and 2, and dependent upon income. The path independence conditions for the discrete-continuous demand are therefore violated.

Model E applies a power term to residual income as a whole rather than to income or price individually. Although Karlström & Morey (2001) have claimed that this specification admits non-linear income effects of price and income changes, Model E embodies two features which are mutually inconsistent; it admits path dependence but restricts conditional demand to a single discrete choice. In the case of a single discrete choice, theory dictates that any income effects should be linear and admitted through the numeraire good. We conclude therefore that Model E is not theoretically valid.

Finally, Model F employs a Cobb-Douglas-type specification. If $y > p_j$ then this will give rise to discrete-continuous demands. Furthermore, if the power terms are significant, then Model F will embody path dependence with respect to prices and income.

4. SUMMARY AND CONCLUSIONS

With reference to the three objectives introduced in section 1, we will finish by summarising the principal outcomes, and drawing conclusions.

In response to the first objective, we introduced the problem of discrete-continuous demand proposed by S&R, involving two mutually exclusive goods (goods 1 and 2) and a 'numeraire' good (good *n*). We discussed the derivation of consumer surplus from the discrete-continuous demand in general, and the discrete choice (or probabilistic) demand in particular. We then considered the conditions under which a change in consumer surplus will be independent of the sequence of price and/or income changes. With regards to the discrete-continuous demand, we identified three such cases:

Case 1: Let the marginal utility of income be independent of the price of good 1 and income, i.e. $\partial\lambda(\mathbf{p},y)/\partial p_1 = 0$ and $\partial\lambda(\mathbf{p},y)/\partial y = 0$. This case will give rise to an indifference map embodying quasi-linear preferences.

Case 2: Let the marginal utility of income be independent of the prices of goods 1 and 2, i.e. $\partial\lambda(\mathbf{p},y)/\partial p_j = 0$ for $j = 1,2$. This case will give rise to an indifference map that exhibits homothetic preferences.

Case 3: Let the marginal utility of income be independent of the prices of goods 1 and 2 as well as the income share available to goods 1 and 2 (once the numeraire good has been accounted for), i.e. $\partial\lambda(\mathbf{p},y)/\partial p_j = 0$ for $j = 1,2$ and

$\partial \lambda(\mathbf{p}, y) / \partial y_{1+2} = 0$. In this case, any income change will be entirely devoted to/abstracted from the numeraire good.

In response to the second objective of the paper, we noted that the probabilistic demand, which might be seen as a restriction upon the discrete-continuous demand, gives rise to S&R's consumer surplus measure (7). This measure is underpinned by Assumptions I to V (section 2 above), which impose path independence to price changes *a priori*. On this basis, the paper devoted particular attention to Case 3, since this is the only case where changes in consumer surplus will (additionally) be path independent to changes in income. More specifically, if $\partial \lambda_j(p_j, y) / \partial y_{1+2} = 0$ for $j = 1,2$ and a numeraire good is present in the consumption bundle then, with certain qualifications (Marshall, 1920), S&R's consumer surplus measure will be path independent to income changes as well as to price changes, whilst observing homogeneity and adding-up.

In response to the third objective, we illustrated several specifications of the conditional indirect utility function commonly applied in practice, and reconciled them with Cases 1, 2 and 3 above. With particular reference to the probabilistic demand, we showed that McFadden's (1981) 'residual income' specification is an appropriate means of implementing Case 3.

ANNEX: DERIVATION OF THE DISCRETE-CONTINUOUS DEMAND FUNCTION

In this annex, we will derive[13] the Marshallian discrete-continuous demands for goods 1 and 2 from S&R's (1981) consumption problem. Slightly adjusting (1), so as to explicate the role of the numeraire good, S&R's problem is given by:

$$\begin{aligned} \text{Max} \quad & u = u(x_1, x_2, x_n) \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 + x_n = y \quad (\lambda) \\ & x_1 x_2 = 0 \quad (\mu) \\ & x_1, x_2, x_n \geq 0 \end{aligned} \tag{A1}$$

[13] This is adapted from a derivation in Johansson (1987).

Using the Lagrangean method, (A1) translates to the following maximization problem:

$$\text{Max} \quad L = u(x_1, x_2, x_n) + \lambda(y - p_1 x_1 - p_2 x_2 - x_n) + \mu(x_1 x_2) \tag{A2}$$

Differentiating (A2) for the first order conditions:

$$\frac{\partial L}{\partial x_1} = \frac{\partial u}{\partial x_1} - \lambda p_1 + \mu x_2 = 0 \tag{A3}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial u}{\partial x_2} - \lambda p_2 + \mu x_1 = 0 \tag{A4}$$

$$\frac{\partial L}{\partial x_n} = \frac{\partial u}{\partial x_n} - \lambda = 0 \tag{A5}$$

$$\frac{\partial L}{\partial y} = \lambda \tag{A6}$$

$$y - p_1 x_1 - p_2 x_2 - x_n = 0 \tag{A7}$$

$$x_1 x_2 = 0 \tag{A8}$$

As discussed in section 2, S&R conceptualise (A1) as a problem of discrete-continuous demand. The individual first chooses between goods 1 and 2 according to which yields the greater utility. Having made that choice, the individual then consumes a positive continuous quantity of only the chosen good. On this basis, note that if $x_1 > 0$, then $x_2 = 0$, and (A4) will become redundant. If instead $x_2 > 0$, then $x_1 = 0$, and (A3) will become redundant. Motivated by this rationale, let us define the conditional indirect utility functions, which might be seen as an extension of (2):

$$\tilde{v}_1(p_1, y) = \tilde{u}_1\left(\tilde{x}_1(p_1, y), x_n(p_1, y)\Big|_{\tilde{x}_1 > 0}\right) \tag{A9}$$

$$\tilde{v}_2(p_2, y) = \tilde{u}_2\left(\tilde{x}_2(p_2, y), x_n(p_2, y)\Big|_{\tilde{x}_2 > 0}\right) \tag{A10}$$

where the tilda notation distinguishes the conditional direct utility, indirect utility and demand functions for goods 1 and 2 from their unconditional counterparts. It is important to recognise that any residual income not devoted to the chosen good will be devoted to the numeraire good, and that the extent of this residual may be different depending on which good is chosen.

On the basis of (A9) and (A10), and making use of (A3) to (A6), we can restate the first order conditions in terms of the conditional indirect utility functions:

$$\frac{\partial \tilde{v}_1}{\partial p_1} = \frac{\partial \tilde{u}_1}{\partial \tilde{x}_1}\frac{\partial \tilde{x}_1}{\partial p_1} + \frac{\partial \tilde{u}_1}{\partial x_n|_{\tilde{x}_1>0}}\frac{\partial x_n|_{\tilde{x}_1>0}}{\partial p_1} = \lambda_1\left(p_1\frac{\partial \tilde{x}_1}{\partial p_1} + \frac{\partial x_n|_{\tilde{x}_1>0}}{\partial p_1}\right) \quad \text{(A11)}$$

$$\frac{\partial \tilde{v}_2}{\partial p_2} = \frac{\partial \tilde{u}_2}{\partial \tilde{x}_2}\frac{\partial \tilde{x}_2}{\partial p_2} + \frac{\partial \tilde{u}_2}{\partial x_n|_{\tilde{x}_2>0}}\frac{\partial x_n|_{\tilde{x}_2>0}}{\partial p_2} = \lambda_2\left(p_2\frac{\partial \tilde{x}_2}{\partial p_2} + \frac{\partial x_n|_{\tilde{x}_2>0}}{\partial p_2}\right) \quad \text{(A12)}$$

$$\frac{\partial \tilde{v}_1}{\partial y} = \frac{\partial \tilde{u}_1}{\partial \tilde{x}_1}\frac{\partial \tilde{x}_1}{\partial y} + \frac{\partial \tilde{u}_1}{\partial x_n|_{\tilde{x}_1>0}}\frac{\partial x_n|_{\tilde{x}_1>0}}{\partial y} = \lambda_1\left(p_1\frac{\partial \tilde{x}_1}{\partial y} + \frac{\partial x_n|_{\tilde{x}_1>0}}{\partial y}\right) \quad \text{(A13)}$$

$$\frac{\partial \tilde{v}_2}{\partial y} = \frac{\partial \tilde{u}_2}{\partial \tilde{x}_2}\frac{\partial \tilde{x}_2}{\partial y} + \frac{\partial \tilde{u}_2}{\partial x_n|_{\tilde{x}_2>0}}\frac{\partial x_n|_{\tilde{x}_2>0}}{\partial y} = \lambda_2\left(p_2\frac{\partial \tilde{x}_2}{\partial y} + \frac{\partial x_n|_{\tilde{x}_2>0}}{\partial y}\right) \quad \text{(A14)}$$

where the marginal utility of income is now conditioned by choice (see footnote 8), hence the notation λ_j for $j = 1,2$.

Given the budget constraint (A7), and again accounting for conditioning by choice, it must hold that:

$$\left.\frac{\partial y}{\partial p_1}\right|_{\tilde{x}_1>0} = \tilde{x}_1 + p_1\frac{\partial \tilde{x}_1}{\partial p_1} + \frac{\partial x_n|_{\tilde{x}_1>0}}{\partial p_1} = 0 \quad \text{(A15)}$$

$$\left.\frac{\partial y}{\partial p_2}\right|_{\tilde{x}_2>0} = \tilde{x}_2 + p_2\frac{\partial \tilde{x}_2}{\partial p_2} + \frac{\partial x_n|_{\tilde{x}_2>0}}{\partial p_2} = 0 \quad \text{(A16)}$$

$$\left.\frac{\partial y}{\partial y}\right|_{\tilde{x}_1>0} = p_1\frac{\partial \tilde{x}_1}{\partial y} + \frac{\partial x_n|_{\tilde{x}_1>0}}{\partial y} = 1 \quad \text{(A17)}$$

$$\left.\frac{\partial y}{\partial y}\right|_{\tilde{x}_2>0} = p_2\frac{\partial \tilde{x}_2}{\partial y} + \frac{\partial x_n|_{\tilde{x}_2>0}}{\partial y} = 1 \quad \text{(A18)}$$

Substituting for the bracketed terms in (A11), (A12), (A13) and (A14), using (A15), (A16), (A17) and (A18) respectively, we have that:

$$\frac{\partial \tilde{v}_1}{\partial p_1} = -\lambda_1 \tilde{x}_1 \tag{A19}$$

$$\frac{\partial \tilde{v}_2}{\partial p_2} = -\lambda_2 \tilde{x}_2 \tag{A20}$$

$$\frac{\partial \tilde{v}_1}{\partial y} = \lambda_1 \tag{A21}$$

$$\frac{\partial \tilde{v}_2}{\partial y} = \lambda_2 \tag{A22}$$

Equipped with (A19) to (A22), we can write Roy's identity for the case of discrete-continuous consumption, and thereby derive the conditional demand for goods 1 and 2:

$$-\frac{\partial u^* / \partial p_1}{\partial u^* / \partial y} = \begin{cases} -\dfrac{\partial \tilde{v}_1 / \partial p_1}{\partial \tilde{v}_1 / \partial y} = \tilde{x}_1 & \text{if } k = 1 \\ -\dfrac{\partial \tilde{v}_2 / \partial p_1}{\partial \tilde{v}_2 / \partial y} = 0 & \text{if } k = 2 \end{cases} \tag{A23}$$

$$-\frac{\partial u^* / \partial p_2}{\partial u^* / \partial y} = \begin{cases} -\dfrac{\partial \tilde{v}_1 / \partial p_2}{\partial \tilde{v}_1 / \partial y} = 0 & \text{if } k = 1 \\ -\dfrac{\partial \tilde{v}_2 / \partial p_2}{\partial \tilde{v}_2 / \partial y} = \tilde{x}_2 & \text{if } k = 2 \end{cases} \tag{A24}$$

Note that if we restrict (A1) to the context of a single discrete choice, then (A23) and (A24) will elicit conditional demands $\tilde{x}_1 = \tilde{x}_2 = 1$.

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Table 1: Some common practical model specifications, and their properties in terms of path independence

Model	Conditional indirect utility	Implication of Roy's identity	$\partial\tilde{v}_j/\partial y =$	$\partial(\partial\tilde{v}_j/\partial y)/\partial p_j =$	$\partial(\partial\tilde{v}_j/\partial y)/\partial y =$	Properties
A	$\tilde{v}_j = \lambda(y - p_j) + \varepsilon_j$	Probabilistic	λ	0	0	Path independent (Case 3)
B	$\tilde{v}_j = \beta_j(y^\alpha - p_j) + \varepsilon_j$	Discrete-continuous	$\alpha \cdot \beta_j \cdot y^{\alpha-1}$	0	$(\alpha-1)\cdot\alpha\cdot\beta_j\cdot y^{\alpha-2}$	Path independent (Case 2)
C	$\tilde{v}_j = \lambda_j(y - p_j^\alpha) + \varepsilon_j$	Discrete-continuous	λ_j	0	0	Path independent (Case 3)
D	$\tilde{v}_j = \beta_j(p_j/y) + \varepsilon_j$	Discrete-continuous	$-\beta_j \cdot p_j \cdot y^{-2}$	$-\beta_j \cdot y^{-2}$	$2\beta_j \cdot p_j \cdot y^{-3}$	Path dependent on prices and income
E	$\tilde{v}_j = \beta_j(y - p_j)^\alpha + \varepsilon_j$	Probabilistic	$\alpha\cdot\beta_j(y-p_j)^{\alpha-1}$	$-(\alpha-1)\cdot\alpha\cdot\beta_j(y-p_j)^{\alpha-2}$	$(\alpha-1)\cdot\alpha\cdot\beta_j(y-p_j)^{\alpha-2}$	Path dependent on prices, income and numeraire
F	$\tilde{v}_j = \beta_j \cdot y^\alpha \cdot p_j^\gamma + \varepsilon_j$	Discrete-continuous	$\alpha\cdot\beta_j\cdot y^{\alpha-1}\cdot p_j^\gamma$	$\gamma\cdot\alpha\cdot\beta_j\cdot y^{\alpha-1}\cdot p_j^{\gamma-1}$	$(\alpha-1)\cdot\alpha\cdot\beta_j\cdot y^{\alpha-2}\cdot p_j^\gamma$	Path dependent on prices and income