Monograph:
Owens, D.H. (1976) A Note on Diagonal Dominance and the Stability of Multi-variable Feedback Systems. Research Report. ACSE Research Report 50 . Department of Automatic Control and Systems Engineering

A NOTE ON
DIAGONAL DOMINANCE AND THE STABILITY OF
MULTIVARIABLE FEEDBACK SYSTEMS

by

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Research Report No. 50

November 1976.

Abstract

Diagonal dominance plays a fundamental role in the design of multivariable feedback control systems by the method of dyadic expansion and the Inverse Nyquist array by providing a systematic procedure for the structural simplification of the return-difference determinent. It is shown that, by the use of equivalent transformations and origin shift methods, sufficient conditions for closed-loop stability using diagonal dominance methods can be obtained which remove many of the difficulties arising in previous formulations.

1. Introduction

The relevance of the concept of diagonal dominance as a theoretical tool for the design of unity negative feedback systems for the control of a linear time-invariant system described by an mxm transfer function matrix G(s) was first established by Rosenbrock (1969) in the form of the inverse Nyquist array stability criterion which has since been developed (Rosenbrock 1970, 1974) into a systematic method for computer aided control systems design. In essence (MacFarlane, 1970), diagonal dominance makes possible the simple evaluation and modification of the number of encirclements of the return-difference determinant evaluated on the standard contour in the complex plane. A more recent development (Owens 1975a) indicates how a combination of the techniques of equivalence transformation and the ideas of diagonal dominance can provide an approach to feedback design by systematic manipulation of the system characteristic loci (MacFarlane and Bellettrutti, 1973) over a selected set of frequency intervals. Major practical problems in the application of these techniques are, in the case of the inverse Nyquist array, the choice of a simple precompensator structure to achieve diagonal dominance at all frequencies and, in the case of the method of dyadic expansion (Owens, 1975a), the maximization of the frequency interval over which the transformed system is diagonally dominant. This paper illustrates how equivalence transformations and origin shift methods can be used to derive sufficient conditions for closed-loop stability which contains previous results as special cases and which can remove many of the difficulties arising in their application.

2. Feedback Stability

Consider a unity negative feedback configuration for the control of an invertible system described by the mxm transfer function matrix G(s) and let K(s) be the mxm forward path controller. Write K(s) in the form similar to that used by Owens (1976) in the analysis of multivariable root-loci,

$$K(s) = K_p(s)\ \{K_1(s) + K_2(s)\} \qquad \ldots (1)$$

It is well known that the stability of the closed-loop system is described by the zeros of the return difference determinant $|I_m + G(s)\ K(s)|$. More precisely, if $\rho_o(s)$ is the open-loop characteristic polynomial and $\rho_c(s)$ is the closed-loop characteristic polynomial,

$$\frac{\rho_c(s)}{\rho_o(s)} = |I_m + G(s)\ K(s)| \qquad \ldots (2)$$

If $|G(s)| \not\equiv 0$, $|K_p(s)| \not\equiv 0$ and $|K_1(s)| \not\equiv 0$ and, following Rosenbrock (1969, 1970, 1974), denoting the inverse of a transfer function matrix L(s) (wherever it exists) by $\hat{L}(s)$, then, from (2),

$$\begin{aligned} \rho_c(s) &= \rho_o(s)\ |I_m + G(s)\ K_p(s)\ (K_1(s) + K_2(s))| \\ &= \rho_o(s)\ |G(s)\ K_p(s)| \cdot |\hat{K}_p(s)\ \hat{G}(s) + K_1(s) + K_2(s)| \\ &= \rho_o(s)\ |G(s)\ K_p(s)\ K_1(s)| \cdot |I_m + \hat{K}_1(s)\ \{\hat{K}_p(s)\hat{G}(s) + K_2(s)\}| \end{aligned} \qquad \ldots (3)$$

It is assumed that $K_p(s)K_1(s)$ and $K_2(s)$ are minimum phase and stable. Let D be the usual Nyquist contour in the complex plane consisting of the imaginary axis (with suitable indentations to exclude zeros and poles of various terms in (3)) and a large semi-circle in the right-half complex plane. The following equality now follows directly by application of standard encirclement theorems (Rosenbrock 1974)

$$n_c = n_z + n_r \qquad \ldots (4)$$

where n_c is equal to the number of poles of $\rho_c(s)$ within D, n_z is the number of system zeros of G(s) within D and n_r is the number of clockwise encirclements of $|I + \hat{K}_1(\hat{K}_p\hat{G} + K_2)|$ about the origin of the complex plane as s varies over D in a clockwise manner.

Suppose now that $I + \hat{K}_1(\hat{K}_p\hat{G} + K_2)$ is diagonally dominant on D then (Rosenbrock 1969, 1970, 1974)

$$n_r = \sum_{i=1}^{m} n_i \qquad \ldots (5)$$

where n_i is the number of clockwise encirclements of the i^{th} diagonal term of $\hat{K}_1(\hat{K}_p\hat{G} + K_2)$ about the (-1,0) point of the complex plane i.e.

$$n_c = n_z + \sum_{i=1}^{m} n_i \qquad \ldots (6)$$

from which the closed-loop system is asymptotically stable if, and only if, $n_c = o$ or, equivalently,

$$n_z + \sum_{i=1}^{m} n_i = 0 \qquad \ldots (7)$$

If $K_2(s) \equiv 0$ then this stability criterion is identical to that used in the inverse Nyquist array technique. If $K_2(s) \not\equiv 0$, then the criterion is identical to the inverse Nyquist array criterion applied to a unity negative feedback system with forward path inverse transfer function matrix $\hat{Q} = \hat{K}_1(\hat{K}_p\hat{G} + K_2)$. It is in this context that the results play a useful role in attaining stability of the closed-loop system, in the sense that $K_2(s)$ provides additional degrees of freedom to achieve diagonal dominance. The following procedure is a practical approach to regulator design.

STEP ONE: Compute $\hat{G}(s)$ and choose the precompensator $K_p(s)$ such that the inverse Nyquist array of $\hat{K}_p\hat{G}$ is diagonally dominant over some frequency interval .

STEP TWO: Choose $K_2(s)$, if necessary, to improve the degree of diagonal dominance of $K_pG + K_2$. For example, in many practical applications it is difficult to choose a simple K_p to ensure that $\hat{K}_p\hat{G}$ is diagonally dominant on the whole of D. In many cases it is only possible to attain dominance at high frequencies (say). The choice of a suitable proper K_2 can improve the degree of diagonal dominance at low frequencies without spoiling the diagonal dominance achieved at high frequencies. In effect, $K_2(s)$ 'shifts the origin' of each element of $\hat{K}_p\hat{G}$ to improve the degree of diagonal dominance.

STEP THREE: Apply the inverse Nyquist array technique in the form of equation (6) - (7) to the composite system $\hat{K}_p\hat{G} + K_2$ to choose a diagonal controller $K_1(s)$ satisfies the stability criterion (7).

It is noted that the technique is similar in structure to the cancellation of off-diagonal terms method suggested by Rosenbrock (1974) using a non-unity feedback matrix F(s). The above technique achieves a similar objective but includes the 'cancellation component' $K_2(s)$ in the forward path controller. This approach allows a greater deal of freedom in the choice of $K_2(s)$ without the posśible transient performance difficulties induced by the inclusion of a non-unity F(s). To illustrate this point, note that the closed-loop transfer function matrix H_c is given by the relation

$$\hat{H}_c = \widehat{(K_1 + K_2)}\; K_1 \; (I + \hat{K}_1(\hat{K}_p \; \hat{G} + K_2))$$

$$\simeq (I + \hat{K}_1(\hat{K}_p \; \hat{G} + K_2)) \qquad \text{... (8)}$$

if the gains in K_1 are much larger than those in K_2. Equivalently, if the system $(I + \hat{K}_1(\hat{K}_p\hat{G} + K_2))$ is highly non-interacting and the gains in K_1 are much larger than those in K_2, then the resulting closed-loop system represented by H_c is highly non-interacting.

To illustrate the simplicity of the method, consider the problem of the regulation of a plant described by

$$\hat{G}(s) = \begin{bmatrix} s^3 + 2s^2 + 2s + 1 & s^2 + 3s + 2 \\ & \\ s^2 + 3s + 2 & s^3 + 2s^2 + 2s + 1 \end{bmatrix} \qquad \text{... (9)}$$

Due to the symmetry of the system, the inverse Nyquist array can be represented by the (1,1) element together with its Gershgorin circles (Fig. 1) It is noted that the system is diagonally dominant at high frequencies but highly nondominant at low frequencies. Choosing $K_p(s) = I_2$ to retain the dominance of the system at high frequencies and choosing $K_2(s)$ to improve the dominance at low frequencies,

$$K_2(s) = \frac{(2 + 3s)}{(1+0.15s)} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \text{... (10)}$$

the inverse nyquist array $\hat{K}_p \; \hat{G} + K_2$ takes the form shown in Fig. 2. Note that the degree of dominance has been significantly improved and the design can continue to improve the stability characteristics by suitable choice of a diagonal controller $K_1(s) = \mathrm{diag}\ \{k_1(s), k_2(s)\}$.

3. The Use of Equivalence Transformations

The range of applicability of the methods of diagonal dominance for the regulation of multivariable systems can be greatly increased by the use of transformation techniques. Suppose that $\hat{G}$ can be written in the form

$$\hat{G}(s) = P_1 \hat{H}(s) P_2 \qquad \dots (11)$$

where P_1, P_2 are constant nonsingular matrices. It is supposed that P_1, P_2 are such that the structural properties of $\hat{H}$ are much simpler than those of $\hat{G}$. For example, if G(s) is a dyadic transfer function matrix (Owens 1975a) it is possible to choose $\hat{H}$ to be diagonal. Alternatively, it is possible to choose P_1, P_2 such that $\hat{H}$ is diagonal at a specified frequency of interest and diagonally dominant in the vicinity of that frequency by application of the method of dyadic expansion (Owens 1975). Applying the techniques of section 2 to the transformed system $\hat{H}(s)$ will produce a controller for H(s) of the form $\tilde{K}(s) = K_p(s) \{K_1(s) + K_2(s)\}$ satisfies the stability criterion. It follows directly from the identity

$$|I + H\tilde{K}| = |I + P_2GP_1\tilde{K}| = |I + GP_1\tilde{K}P_2| \qquad \dots (12)$$

that a suitable controller for G(s) is

$$K(s) = P_1K_p(s) \{K_1(s) + K_2(s)\} P_2 \qquad \dots (13)$$

in the sense that the resulting closed-loop system $(I + GK)^{-1}GK$ is asymptotically stable.

To illustrate the application of the technique consider the system

$$\hat{G}(s) = \begin{bmatrix} s^2 + 3s + 1 & s + 2 \\ s + 3 & 2s + 1 \end{bmatrix} \qquad \dots (14)$$

and choose P_1, P_2 by inspection from the identity

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and choose P_1, P_2 by inspection from the identity

$$\hat{G}(s) = P_1 \hat{H}(s) P_2$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.4s^2 + s - 0.5 & 1.5 \\ 0.5 & s + 0.5 \end{bmatrix} \begin{bmatrix} 2.5 & 0 \\ 0.5 & 1.0 \end{bmatrix} \qquad \ldots (15)$$

Note that $\hat{H}(s)$ is diagonally dominant at high frequencies but not at low frequencies. Choosing $K_p(s) = I_2$ to retain the high frequency dominance and (say)

$$K_2(s) = \begin{bmatrix} 1.0 & -1.5 \\ -0.5 & 0 \end{bmatrix} \qquad \ldots (16)$$

then

$$\hat{K}_p \hat{H} + K_2 = \begin{bmatrix} 0.4s^2 + s + 0.5 & 0 \\ 0 & s + 0.5 \end{bmatrix} \qquad \ldots (17)$$

which is diagonal. The choice of a suitable diagonal controller $K_1(s)$ to to ensure that the system is stable is now a trivial matter by application of single-loop methods to the individual diagonal terms of (17).

6. Conclusions

It has been demonstrated that the concept of diagonal dominance can play a fundamental role in the regulation and design of multivariable feedback systems. Noting that a common practical problem in the application of the inverse Nyquist array technique is the choice of a suitable precompensator to achieve diagonal dominance over the frequency range of interest, two techniques have been demonstrated to be a valuable aid in the stability analysis. The first technique makes use of a more general controller structure (equation (1)) first used by Owens (1976) in the analysis of multivariable root-loci. The term $K_2(s)$ is new and is used to structure the inverse Nyquist array by linear operations to improve the degree of diagonal dominance. The second technique makes use of the invariance of stability relations under similarity transformation and the use of equivalence transformations (similar to those used in the method of dyadic expansion) to improve the degree of dominance in the system and hence ensure closed-loop stability.

An intersting point arising from the analysis is the observation that the inverse Nyquist array technique and the method of dyadic expansion are closely related. Bearing in mind that the method of dyadic expansion has close links with the Characteristic Locus design method, it is seen that all three methods can be partially unified in the framework of diagonal dominance and transformation techniques.

7. References

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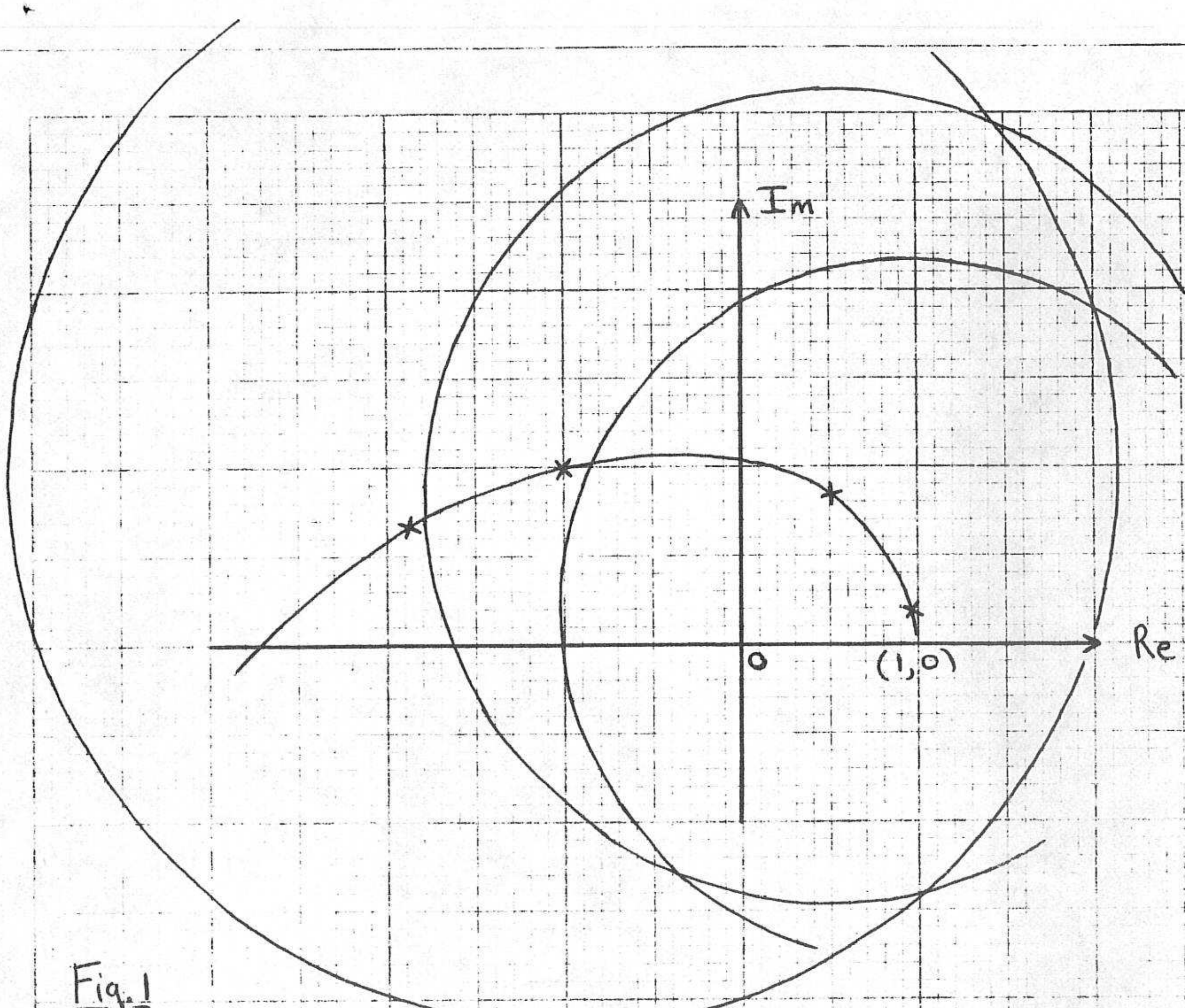

Fig. 1

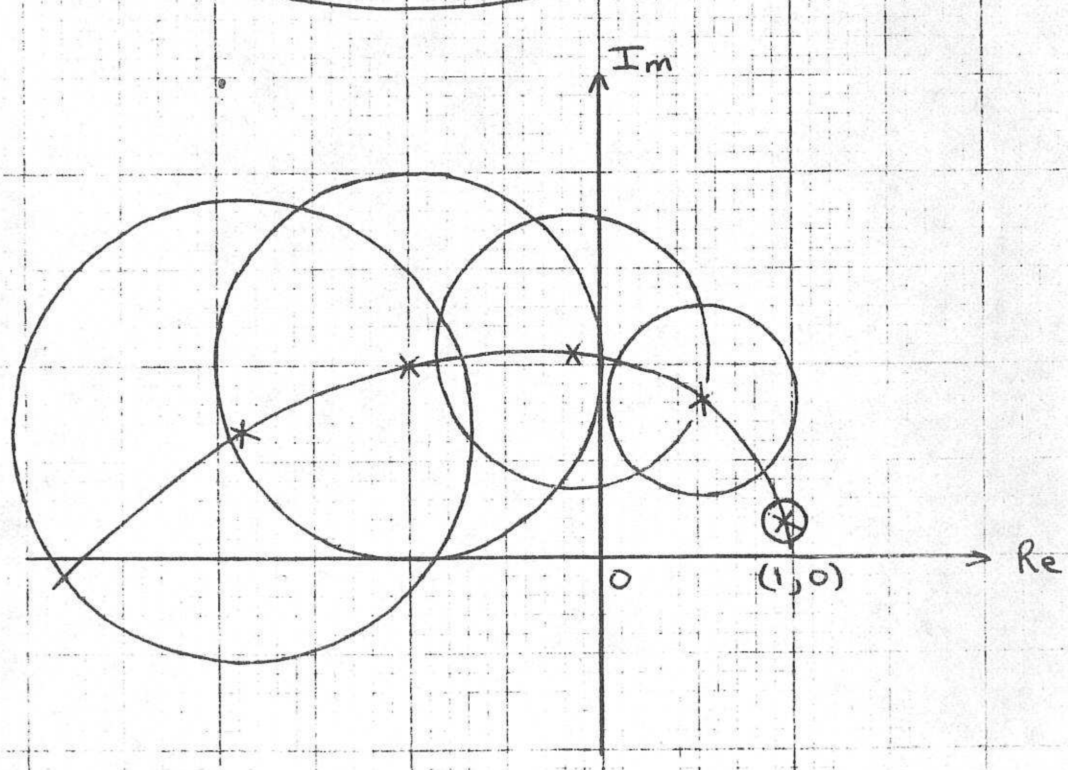

Fig. 2.