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FIRST ORDER TYPE MODELS FOR MULTIVARIABLE PROCESS CONTROL

by

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Abstract

It is well known in classical feedback control that many high order linear time-invariant systems can be approximated, for the purpose of feedback design, by a low order state-space model due to the presence of approximately cancelling poles and zeros in the system transfer function. The paper presents an equivalent technique in the case of a multivariable system described by a strictly proper mxm, minimum-phase and invertible transfer function matrix G(s) by the application of the contraction mapping theorem. It is shown that, in many cases of practical interest, a multivariable first-order type model is adequate for the purposes of control system design and that such a model can be determined directly from transient response data or, equivalently, by the analysis of the high and low frequency characteristics of the system. The application of the technique is illustrated by the analysis of a high order binary distillation column model and the dynamics of a counter-flow heat exchanger.

1. Introduction

In recent years, several techniques[1-4] have been suggested for the design of unity negative feedback systems for a plant described by an mxm strictly proper and invertible transfer function matrix (TFM) G(s) using the concept of frequency response analysis. A powerful feature of each technique is its generality and the ability to systematically design forward path compensation networks to improve the overall feedback system response. However, by analogy with classical feedback theory, it is anticipated that many high-order linear time-invariant systems can be approximated, for the purposes of controller design, by a low order state space model due to the presence of approximately cancelling pole-zero pairs. Furthermore, the validity of the approximation will improve in the closed-loop situation due to the attraction of poles to the system zeros[7]. This paper uses the contraction mapping theorem[8] in terms of the inverse transfer function matrix[9] to present and interpret the use and validity of low order dynamic models for feedback controller design. The results are presented in a form easily adapted for computer-aided graphical design.

The authors can see little justification, other than computational ease, in the use of reduced order models in frequency response analysis unless the reduced model deepens insight into system structure or enables the use of direct analytic techniques in the choice of controller structure. In this context, it is demonstrated that the scope of applications of multivariable first-order type systems[5,6] can be enlarged to provide a technique for the systematic design of feedback controller for a large class of multivariable process plant. The first order model can be deduced directly from transient response data or by the analysis of high and low frequency characteristics of the system, and a suitable feedback controller can be obtained by direct inspection of the reduced system parameters, and hence removes the need for trial and error design using frequency response methods. The techniques are illustrated by application to a high order binary distillation column model and a liquid-liquid counter-flow heat exchanger.

2. Feedback Stability and Reduced Order Models

Consider a unity negative feedback system for the control of a strictly proper, invertible system described by the mxm TFM $G(s)$ and let $K(s)$ be the proper, invertible mxm forward path controller. The closed-loop system is described by the relation

$$y(s) = \{I_m + G(s)K(s)\}^{-1} G(s)K(s)r(s) \qquad (1)$$

where $y(s)$ is the vector of output transforms and $r(s)$ is the vector of demand signal transforms. The stability of the closed-loop system can be assessed[1-4] by the analysis of the return-difference determinant $|I_m + G(s)K(s)|$. However, in the analysis of reduced order models, it is more convenient to use the techniques suggested by Freeman[8] in the form used by Owens[9]. Defining $Q(s) = G(s)K(s)$, equation (1) takes the form

$$y(s) = -Q^{-1}(s)\ y(s) + r(s) \qquad (2)$$

If $G_A(s)$ is a strictly proper, invertible reduced model of $G(s)$ and $Q_A(s) = G_A(s)K(s)$ the equation (2) can be rewritten in the form,

$$y(s) = \{I + Q_A{}^{-1}(s)\}^{-1}\{\{Q_A{}^{-1}(s) - Q^{-1}(s)\}y(s) + r(s)\} \qquad (3)$$

Let D be the usual Nyquist contour in the complex plane consisting of the imaginary axis and a large semi-circle in the right-half plane. Assuming that $Q^{-1}(s)$ and Q_A^{-1} are bounded on D and analytic in its interior and that $\{I_m + Q_A^{-1}(s)\}^{-1} = \{I + Q_A(s)\}^{-1}Q_A(s)$ is stable, then a sufficient condition for the closed-loop system defined by equation (1) to be stable can be obtained by application of the contraction mapping theorem[9] i.e.

$$\max_{1 \leq i \leq m} \ \sup_{s \in D} \sum_{j=1}^{m} \left| (\{I + Q_A^{-1}(s)\}^{-1}\{Q_A^{-1}(s) - Q^{-1}(s)\})_{ij} \right| < 1 \qquad (4)$$

In terms of controller design the result outlined about states that if a minimum phase multivariable system $G(s)$ is approximated by a minimum phase reduced model $G_A(s)$ and a forward path controller K(s) is designed to ensure that the reduced closed-loop system $\{I_m + Q_A(s)\}^{-1}\{Q_A(s)\} = \{I_m + Q_A^{-1}(s)\}^{-1}$ is stable, then application of K(s) to G(s) yields a stable closed-loop system (equation (1)), provided condition (4) is satisfied. The result is of great generality and has potentially wide application, but it does leave open questions of the best technique for approximating G(s) and the possibility of satisfying equation (4). In practice however, given a choice of $G_A(s)$ the stability, condition (4) can easily be checked by graphical analysis of the frequency responses $\{I_m + Q_A^{-1}(s)\}^{-1}\{Q_A^{-1}(s) - Q^{-1}(s)\}$, or, more probably, by simulation of the closed-loop system and its reduced form.

It can be argued that there is little justification for the use of reduced order models in frequency response analysis unless the reduced model provides insight into system structure or enables direct evaluation of a suitable controller structure. In this context it is natural to consider the use of reduced models for which there exists a known analytic design method. Examples of such models are the first and second order type multivariable systems[5,6]. Consider, for example, the use of a first order type reduced model. A multivariable first order lag can be defined[5,6] to be a mxm invertible system with inverse TFM

$$G_A^{-1}(s) = A_o s + A_1 \quad , \ |A_o| \neq 0 \tag{5}$$

It has been shown[5,6] that the use of a proportional controller of the form

$$K(s) = k\,A_o - A_1 \quad , \ (k \text{ scalar}) \tag{6}$$

yields a closed-loop system with TFM

$$\{I_m + G_A(s)K(s)\}^{-1}G_A(s)K(s) = \frac{k}{s+k}\{I_m - k^{-1}A_o^{-1}A_1\} \tag{7}$$

so that, by suitable choice of gain parameter k, the closed-loop system can be designed to exhibit arbitrarily small steady state errors and transient interaction effects and to possess an arbitrarily fast response speed.

Consider the use of a first order model for the representation of a plant G(s) of the form

$$G^{-1}(s) = s A_o + A_1 + A_o H(s)$$
$$|A_o| \neq o \quad , H(o) = o \quad , H(s) \text{ proper and stable} \tag{8}$$

and define $G_A(s)$ by equation (5). The matrices A_o, A_1 can be deduced directly from G(s) using the formulae,

$$A_o^{-1} = \lim_{s \to \infty} s G(s)$$
$$A_1 = \lim_{s \to o} G^{-1}(s) \tag{9}$$

or, if G(o) is finite and non-singular ,

$$A_1^{-1} = \lim_{s \to o} G(s) = G(o) \tag{10}$$

Equivalently A_1^{-1}, A_o^{-1} represent the initial rate and steady state values respectively of the system in response to unit step inputs. They can hence be evaluated analytically, by simulation of a realization of G(s) or estimated from experimental data.

Choosing, for example, the controller of equation (6) to ensure that $G_A(s)$ is adequately controlled, the stability of the closed-loop system of equation (1) can be assessed by examination of condition (4). Noting that by using equations (5)-(8),

$$\{I_m + Q_A^{-1}(s)\}^{-1} \{Q_A^{-1}(s) - Q^{-1}(s)\}$$
$$= \{K(s) + G_A^{-1}(s)\}^{-1} K(s) K^{-1}(s) \{G_A^{-1}(s) - G^{-1}(s)\}$$
$$= \frac{(-1)}{s+k} H(s) \tag{11}$$

condition (4) becomes,

$$\max_{1 \leq i \leq m} \sup_{s \in D} \sum_{j=1}^{m} \frac{|H_{ij}(s)|}{|s+k|} < 1 \tag{12}$$

It is easily verified that it is always possible to choose k > o to satisfy equation (12) and hence guarantee the stability of the closed-loop system by choosing

$$k > \max_{1 \leq i \leq m} \sup_{s \in D} \sum_{j=1}^{m} |H_{ij}(s)| \tag{13}$$

In particular, if H(s) is small in the sense that the right-hand-side of equation (13) is small, then $G_A(s)$ will be a good approximation to G(s) in both the closed and open-loop and by analogy with classical terminology, the system can be regarded as possessing a high degree of pole-zero cancellation. In more general situations, H(s) may be significant and (equation (13)) higher gains are required to ensure stability. The matching of open-loop responses in this case will be poor, but it is anticipated that these errors will be reduced in the closed-loop as is the case in classical feedback systems.

In summary, given an mxm invertible proper, minimum-phase process plant G(s) of the form of equation (8), the following simple procedure can enable the design of a high performance closed-loop control system,

STEP ONE: Compute $G^{-1}(s)$ and hence (equation (8)) A_o, A_1, H(s). Alternatively compute A_o, A_1 from frequency or transient response data.

STEP TWO: Construct the reduced model $G_A^{-1}(s) = A_o s + A_1$ and compare the open-loop response of G(s) and $G_A(s)$ to obtain an estimate of the effect of H(s).

STEP THREE: Choose a controller of the form defined by equation (6) (or the equivalent proportional plus integral controller[5-6]) and, by examination of equation (7) estimate a suitable gain to ensure the required response speed, steady state errors and interaction effects.

STEP FOUR: Check the stability of the closed-loop system $\{I_m+G(s)K(s)\}^{-1}G(s)K(s)$. either by a direct check of relation (12) or by direct simulation.

Note that the technique is one of control synthesis on the basis of a reduced order model and hence removes the need for detailed frequency response analysis. In this sense the method is easy to apply and, for the cases defined, it is always possible to ensure closed-loop stability. The technique will not cope with any system however, as $G^{-1}(s)$ must take the form defined by equation (8) and it is possible to envisage situations where the control gains required to satisfy equation (12) may be too high for practical application. In such a situation it is necessary to use a more general design technique[1-4]. It is felt however, that the technique will be valuable in the analysis of many industrial processes (for example thermal and chemical processes) possessing only a small number of dominant modes. Two such examples are illustrated in the following sections.

3. Application to a Long Binary Distillation Column

Such processes usually comprise a vertical sequence of spatially discrete stages (trays), for separating a binary mixture into its two components. (We here assume N+1 trays above and M+1 trays below the feed entry point). Complete separation is usually uneconomic and the object of plant and control-system design is to regulate the fractional compositions $X(N)$ and $X^1(M)$ at or near pre-specified reference values close to 1.0 and 0.0 respectively. If n and m are general tray numbers above and below the feed-point respectively and if N and M >> 1.0, then the N + M - 2 material balance equation - one for each non-terminal tray, may be reduced[10,11] to a pair of partial differential equation, (p.d.e's), in n and m, now regarded as continuous variables. These are

$$\left.\begin{aligned} \partial x/\partial t &= (L/H)\partial^2 x/\partial n^2 + (G_r/H)(\ell - v\alpha_r) \\ \partial x^1/\partial t &= (L^*/H)\partial^2 x^1/\partial m^2 + (G_r^1/H)(v\alpha_s - \ell) \end{aligned}\right\} \qquad (14)$$

where $x(n,t)$ and $x^1(m,t)$ are small perturbations in $X(n,t)$ and $X^1(m,t)$. $\ell(t)$ and $v(t)$ are perturbations in the liquid and vapour flows $L(t)$ and $V(t)$ circulating within the column and are the manipulable process forcing-functions. H denotes the capacity of each tray, G_r and G_r^1 the steady-state quiescent distribution $\partial X/\partial n$ and $\partial X^1/\partial m$ (constants for well-designed plant) and $L^* = L + F$ where F is the feed flow into the plant ($F \ll L,V$). α_r and α_s are mixture parameters marginally <1.0 and >1.0 respectively.

Material balances at the column boundaries (i.e. at n=N, m=M, n=o and m=o) yield the following boundary conditions(11,12).

$$\left.\begin{aligned}
&H\partial x(N)/\partial t + L\{\partial x(N)/\partial n + x(N) - x_t\} = G_r(\ell - v\alpha_r) \\
\text{where}\quad &(H_a/V)\partial x_t/\partial t + x_t = \alpha_r\, x(N) \\
&H_b\partial x^1(M)/\partial t + L^*\{\partial x^1(M)/\partial m + (\alpha_s-1)x^1(M)/\alpha_s\} = G_r^1(v\alpha_s-\ell) \\
&H\partial x(o)/\partial t + L^*\{x(o) - x^1(o) - \partial x(o)/\partial n\} = G_r(\ell-v\alpha_r) \\
&H\partial x^1(o)/\partial t + L^*\{x^1(o) - x(o) - \partial x^1(o)/\partial m\} + Fx(o) = G_r\ell - G_r^1 v\alpha_s
\end{aligned}\right\} \quad (15)$$

x_t being a small perturbation in top product x_t and H_a and H_b, the capacities of the terminating vessels (accumulator and reboiler) at the top and bottom of the column respectively.

Equations 14 and 15 provide a complete description permitting the computation of the process inverse T.F.M. $G^{-1}(s)$ if necessary (step 1). Although for $s = j\omega$ such a procedure would be tedious for a wide range of ω, the determination of A_1 and A_o is quite simple. Ignoring the time derivatives, equation 14 yields two ordinary differential equations, (d.e's), in n and m which may be easily solved numerically or even analytically[11,12] subject to the now algebraic boundary conditions (15) so generating the matrix A_1. High frequency analysis is even easier since, on taking Laplace transforms of equation 14 (in s w.r.t. t), we obtain:

$$\lim_{s\to\infty} \left\{ s \begin{bmatrix} \tilde{x}(s) \\ \tilde{x}^1(s) \end{bmatrix} \right\} = \frac{1}{} \begin{bmatrix} -\alpha_r G_r , & G_r \\ \alpha_s G_r^1 , & -G_r^1 \end{bmatrix} \begin{bmatrix} \tilde{v}(s) \\ \tilde{\ell}(s) \end{bmatrix}$$

so that the inverse T.F.M , $A_o s$, is clearly spatially independent and the boundary conditions can be disregarded in its determination.

Figures 1 and 2 show respectively the open-loop responses at n=m=4 to steps in v and d{perturbation in top-product flow rate, D, (=V-L)} of a column having N=8, M=10, V=2.5, L=2.0, F=1.0, α_s=1.2, α_r=0.8 (making G_r = 0.0382 and G_r^1 = -0.0355), $H=H_a=H_b$ = 1.0. These are computed from the 22 original d.e's of the process and are presented alongside those of the multivariable first-order lag process $G_A(s)$ obtained from equation 5 (step 2) from A_1 and A_o matrices determined in the manner indicated immediately above. Note the identical steady-states and initial rates of rise : as expected. Nearly identical responses are obtained for $H_a, H_b < H$ but when $H << H_a, H_b$, the responses exhibit a much longer settling time (due to the mismatched boundary capacitances) after the predicted initial rate of rise. Responses for $H_a = H_b = 10H = 10$ are also presented in figures 1 and 2 showing the larger deviations from those of $G_A(s)$. Of more importance however are the discrepancies in the closed-loop environment.

The A_1 and A_o matrices for both columns (inputs v and d) are

$$A_1 = \begin{bmatrix} 0.0670 , & -\ 0.630 \\ -0.0592 , & -\ 0.550 \end{bmatrix} , \quad A_o = \begin{bmatrix} 0.00770 , & -\ 0.0382 \\ -0.00731 , & -\ 0.0355 \end{bmatrix} \tag{17}$$

and from equation 6,(step 3), the controller matrix K is found to be, for instance:

$$K = \begin{bmatrix} 8.645 , & -\ 8.788 \\ -2.509 , & -\ 2.581 \end{bmatrix} ,k=0.25 \text{ or,} K= \begin{bmatrix} 56.835 , & -\ 60.643 \\ -12.433 , & -\ 13.033 \end{bmatrix} , \ k=1.0 \tag{18}$$

The closed-loop transfer-function from equation 7 is therefore

$$\{k/(s+k)\} \left[\begin{bmatrix} 1 , & 0 \\ 0 , & 1 \end{bmatrix} - k^{-1} \begin{bmatrix} 0.088, & -0.031 \\ -0.026, & 0.094 \end{bmatrix} \right]$$

predicting the behaviour specified in table I

TABLE I

Predicted Closed-Loop Column Behaviour

k	Time Constant	Steady-state error	Interaction
0.25	4.0	32 to 36%	8 to 12%
1.00	1.0	8 to 9%	2 to 3%

The closed-loop responses obtained (step 4) for the 22nd-order process are stable as shown in Figure 3 for the cases $H = H_a = H_b = 1.0$ and $H_a = H_b = 10H = 10$. These, in the first case, are almost indistinguishable from those obtained by use of $G_A(s)$, (similarly controlled), and the effect of boundary capacitance mismatch, so evident on open-loop, is clearly unimportant when control (of the type proposed) is applied. In fact the multivariable first order approximation has resulted, in both cases, in a highly satisfactory closed-loop design.

4. Application to a Counterflow Liquid/liquid Heat-Exchanger

Although transient responses are informative, for a comprehensive insight into the approximation involved, we now consider a distributed process governed by much simpler boundary conditions which permit the analytical determination of $G(j\omega)$ for $-\infty \lessgtr \omega \lessgtr \infty$ and therefore a direct comparison with $G_A(j\omega)$, in the frequency domain, under open-and closed-loop conditions.

Heat balance consideration applied to the heat exchanger concerned produce the following p.d.e. description[10,12] in terms of distance x and time t:

$$\left.\begin{aligned} T\partial\phi_1/\partial t &= -X\partial\phi_1/\partial x + \phi_2 - \phi_1 + f_1 \\ T\partial\phi_2/\partial t &= X\partial\phi_2/\partial x + \phi_1 - \phi_2 + f_2 \end{aligned}\right\} \qquad (19)$$

where

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = (\Delta\theta/W) \begin{bmatrix} 0.6\,, & -0.4 \\ 0.4\,, & -0.6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad (20)$$

$\phi_1(x,t)$ and $\phi_2(x,t)$, the dependent variables, being perturbations in the temperatures θ_1 and θ_2 of the two liquid streams and $w_1(t)$, $w_2(t)$ being manipulable perturbations in the two mass flow rates W_1 and W_2. X and T are parameters dependent on the quiescent state of the process and may be assumed constant for small variations. $\Delta\theta = \theta_1 - \theta_2$ = a constant for a well-designed plant in which, inter alia, $W_1 = W_2 = W$. Inlet temperatures being constant, the boundary conditions are simply

$$\phi_1(o,t) = \phi_2(L,t) = 0 \qquad (21)$$

where L is the effective length of the heat-exchanger. Taking Laplace transforms, (in s w.r.t. t), of equation 19 and solving the resulting d.e.'s subject to equation 21 yields†:

$$\begin{bmatrix}\phi_1(L,s)\\ \phi_2(0,s)\end{bmatrix} = H(s)\begin{bmatrix}\tilde{f}_1(s)\\ \tilde{f}_2(s)\end{bmatrix} = \begin{bmatrix}h_1+h_2\ , & h_1\\ h_1\ , & h_1+h_2\end{bmatrix}\begin{bmatrix}\tilde{f}_1(s)\\ \tilde{f}_2(s)\end{bmatrix} \quad (22)$$

where $h_1(s) = (\cosh\alpha L - 1) / \{X^2\alpha(\alpha\cosh\alpha L + \beta_1\sinh\alpha L)\}$

and $h_2(s) = \{-2\beta_2 + (\alpha+\beta_2)\exp(\alpha L)-(\alpha-\beta_2)\exp(-\alpha L)\}/\{2X\alpha(\alpha\cosh\alpha L + \beta_1\sinh\alpha L)\}$ (23)

in which $\alpha = \{(sTL/X)^2 + 2sTL^2/X^2\}^{0.5}/L$, $\beta_1=(1+Ts)/X$ and $\beta_2 = Ts/X$ (24)

Alternatively, in terms of inputs w_1 and w_2,

$$\begin{bmatrix}\tilde{\phi}_1(L,s)\\ \tilde{\phi}_2(0,s)\end{bmatrix} = G(s)\begin{bmatrix}\tilde{w}_1(s)\\ \tilde{w}_2(s)\end{bmatrix} = \begin{bmatrix}g_1, g_2\\ -g_2, -g_1\end{bmatrix}\begin{bmatrix}\tilde{w}_1(s)\\ \tilde{w}_2(s)\end{bmatrix} \quad (25)$$

where, from (20) and (22)

$g_1(s) = (\Delta\theta/w)\{h_1(s) + 0.6h_2(s)\}$ and $g_2(s) = -(\Delta\theta/w)\{h_1(s) + 0.4h_2(s)\}$ (26)

For the approximate first-order lag model we deduce, (step 1), from the Laplace transform of equation 19 that

$$\lim_{s\to\infty} \{s[\tilde{\phi}_1(x,s),\tilde{\phi}_2(x,s)]^T = T^{-1}[\tilde{f}_1(s),\ \tilde{f}_2(s)]^T \quad (27)$$

and, neglecting the time derivatives in (19) and solving subject to (21) yeilds:

$$\lim_{s\to o}\begin{bmatrix}\tilde{\phi}_1(L,s)\\ \tilde{\phi}_2(o,s)\end{bmatrix} = 2\begin{bmatrix}1+2X/L\ , & -1\\ -1, & 1+2X/L\end{bmatrix}^{-1}\begin{bmatrix}\tilde{f}_1(s)\\ \tilde{f}_2(s)\end{bmatrix} \quad (28)$$

Synthesising the reduced-order model, (step 2), from functions 5,20,27 and 28 with w_1 and w_2 as inputs therefore gives

$$G_A(s) = \begin{bmatrix}g_{A1}, g_{A2}\\ -g_{A2}, -g_{A1}\end{bmatrix} = \frac{\Delta\theta}{W}\begin{bmatrix}0.5+X/L+Ts\ , & -0.5\\ -0.5\ , & 0.5+X/L+Ts\end{bmatrix}^{-1}\begin{bmatrix}0.6, -0.4\\ 0.4, -0.6\end{bmatrix} \quad (29)$$

For an open-loop comparison of the systems the analysis of section 2 can be simplified by expressing G(s) and $G_A(s)$ in the equivalent forms $U_2G_dU_1$ and $U_2G_{Ad}U_1$ respectively, where

† We here confine attention to outlet temperatures only.

$$G_d = \begin{bmatrix} g_1 + g_2 , & 0 \\ 0, & g_1 - g_2 \end{bmatrix} , \quad G_{Ad} = \begin{bmatrix} g_{A1} + g_{A2} , & 0 \\ 0 , & g_{A1} - g_{A2} \end{bmatrix} \tag{30}$$

$$U_1 = \begin{bmatrix} 1 , & 1 \\ -1, & 1 \end{bmatrix} , \quad U_2 = \begin{bmatrix} 1 , & 0 \\ 0 , & -1 \end{bmatrix} U_1^{-1} \tag{31}$$

and the loci of the elements of $G_d^{-1}(j\omega)$ and $G_{Ad}^{-1}(j\omega)$ compared as in figure 4 which pertain to the case L = 2X. Now if $G_d^{-1}(j\omega)$ is expressed thus:

$$G_d^{-1}(j\omega) = G_{Ad}^{-1}(j\omega)\left[I_2 + \begin{bmatrix} \varepsilon_1(j\omega) , & 0 \\ 0 , & \varepsilon_2(j\omega) \end{bmatrix} \right] \tag{32}$$

then it is obvious from figure 4 that $|\varepsilon_1|$, $|\varepsilon_2|$ <<1.0 for all ω so that, like $G_A(s)$, $G(s)$ is open-loop stable and, furthermore, the transient responses inferred from the loci for the true and approximate systems will be similar. Of more importance however is the comparison of the closed-loop behaviour of the two systems.

Now from equation 29, for L = 2X

$$G_A^{-1}(s), \ (= A_o s + A_1) = \frac{W}{\Delta\theta} \begin{bmatrix} 4 + 3Ts , & -(3.5 + 2Ts) \\ 3.5+2Ts , & -(4 + 3Ts) \end{bmatrix} \tag{33}$$

and, (step 3), choosing a precompensator parameter k = 5/T, (in order to improve the response rate by between 2.5 and 4.0 times), produces through equations 6 and 33 a precompensator:

$$K = \frac{W}{\Delta\theta} \begin{bmatrix} 11, & -6.5 \\ 6.5, & -11 \end{bmatrix} \tag{34}$$

which, from equation 7, 33 and 34 , yields the closed-loop T.F.M.

$$C_A = \begin{bmatrix} c_{A1} , & c_{A2} \\ c_{A2} , & c_{A1} \end{bmatrix} = \{1/(1+0.2s)\} \begin{bmatrix} 0.8 , & 0.06 \\ 0.06 , & 0.8 \end{bmatrix} \tag{35}$$

The locus c_{A1}^{-1} is compared in, (step 4) in figure 5 with its true-system counterpart c_1^{-1}. Interaction is very small in both cases. Furthermore, if $c_{A1}^{-1}(j\omega)$ is expressed as $c_1^{-1}(j\omega)\ \{1 + \varepsilon(j\omega)\}$ then, since figure 5 reveals that $|\varepsilon(j\omega)| << 1.0$, it follows that the transient behaviour of the two systems inferred from the loci is very similar. By way of confirmation, the computed transient responses for the true and approximate closed-loop system are compared in figure 7. Again agreement is better than in the open-loop situation as a comparison of figures 6 and 7 shows. This is as would be expected since a comparison of figures 4 and 5 reveals that $|\varepsilon(j\omega)| < |\varepsilon_1(j\omega)|$ and $|\varepsilon_2(j\omega)|$.

5. Conclusions

By application of the contraction mapping theorem, a design technique has been presented for the feedback control of a multivariable system having a strictly proper mxm minimum-phase, invertible T.F.M., G(s). A multivariable first-order type model $G_A(s)$ has been found to satisfactorily represent G(s), for the purpose of controller design, in the case of two spatially-distributed, (i.e. very high order), processes widely encountered in the process engineering field: (a), a binary distillation column and, (b) a counterflow heat exchanger. The approach has been validated in cases (a) and (b) by computation of the transient responses of G(s) and $G_A(s)$ under open-and closed-loop conditions and also, in case (b), by frequency response analysis.

A feature of the technique is the ease with which the $2m^2$ parameters of $G_A(s)$ can be identified, either from measurement of the initial rate and settling values of the step responses of G(s) or, as in this paper, by the direct analytical solution of the process equation for low and high-frequency inputs. These parameters are sufficient to determine the controller structure, the design exercise being completed using simulation methods.

Finally, it has been shown that good agreement between the closed-loop responses of G(s) and $G_A(s)$ can be obtained despite significant mismatch of the open-loop responses, indicating that the first order type models can be used for closed-loop control despite apparent inadequacies in their description of open-loop behaviour.

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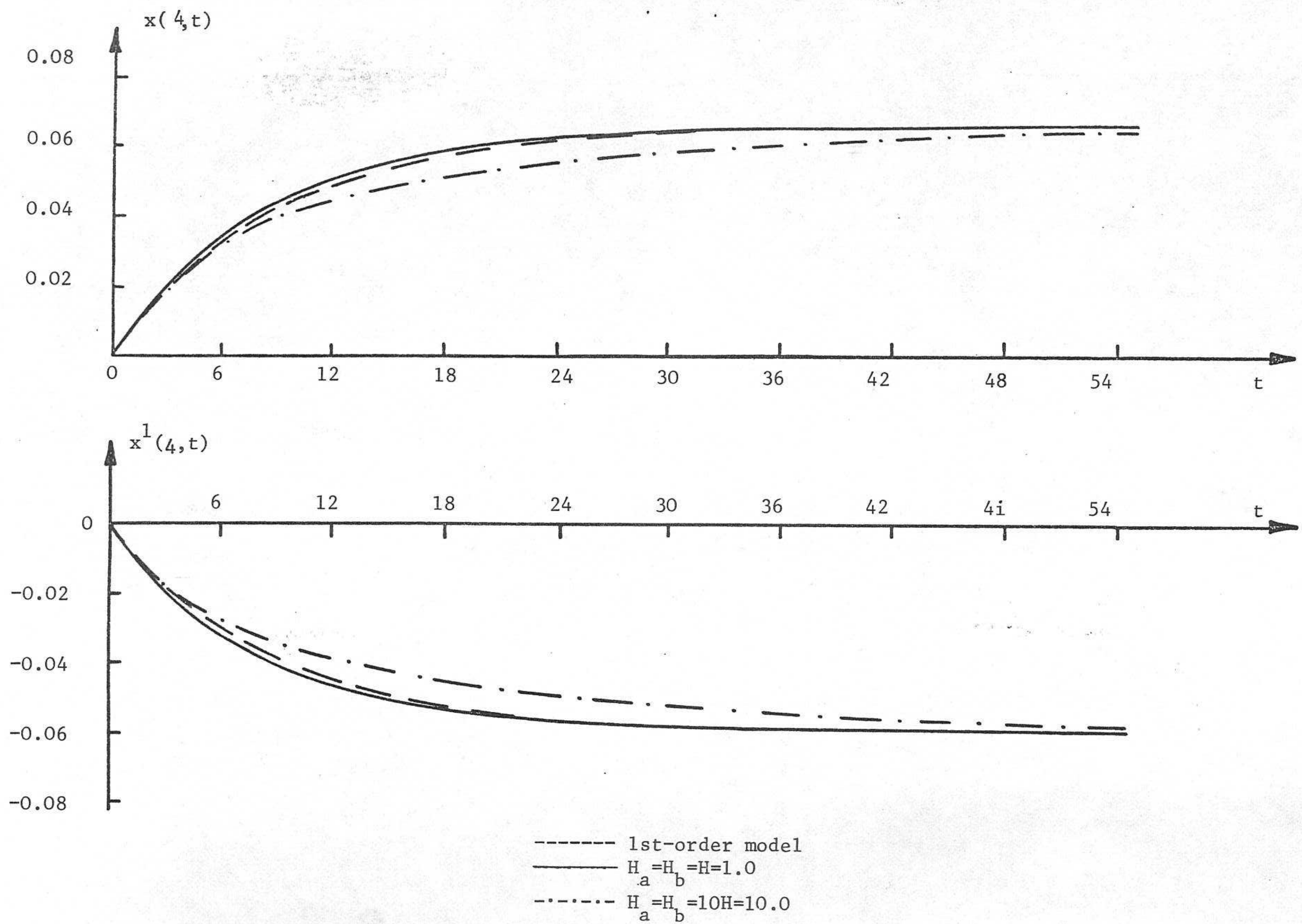


Fig. 1: Open-loop response of column to unit step in v

Fig. 2: Open-loop response of column to unit step in d

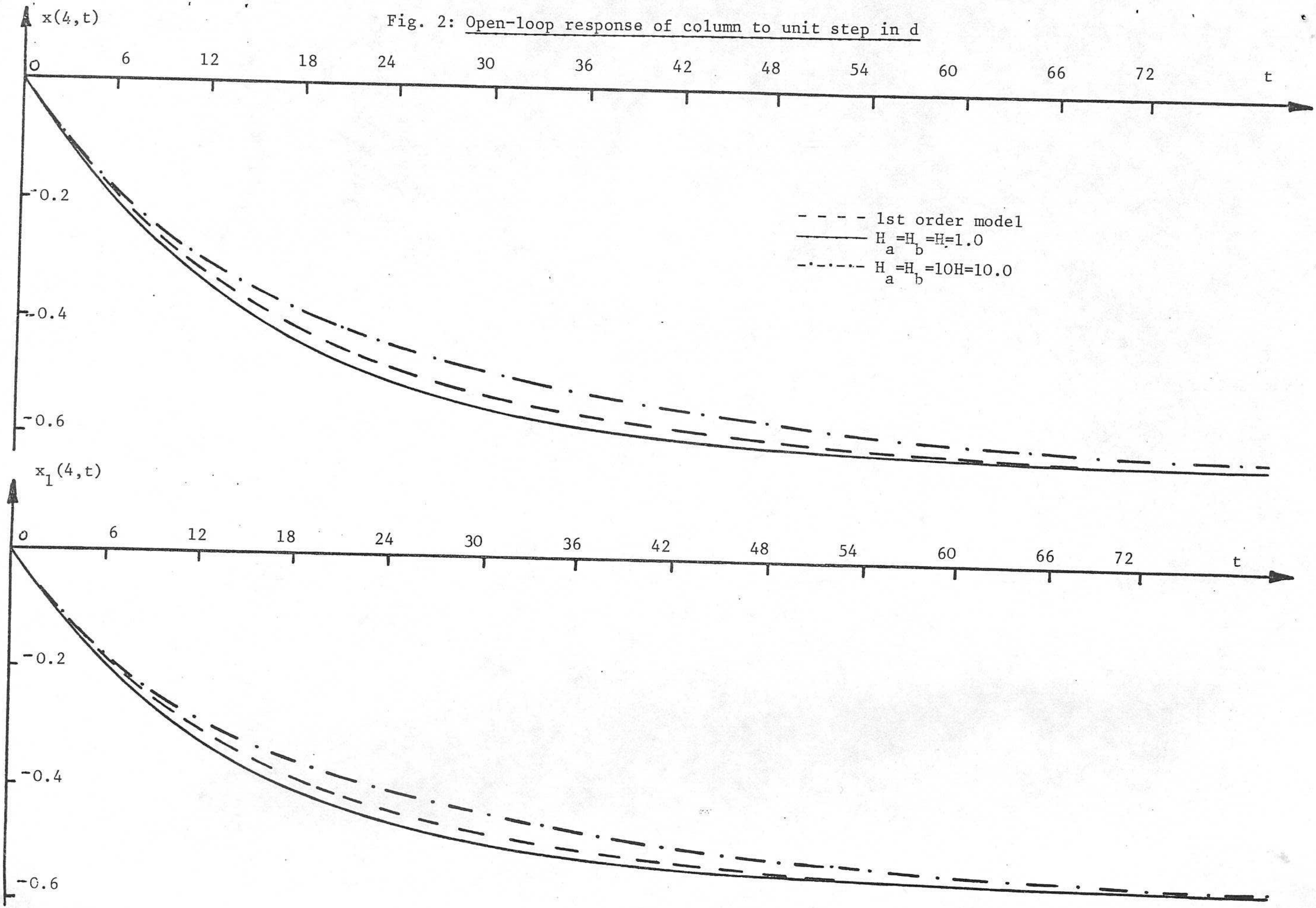


Fig. 3: Closed-loop responses to step inputs

(a) step in $x(4)$ reference } $k = 0.25$
(b) step in $x^1(4)$ reference }

(c) step in $x(4)$ reference } $k = 1.0$
(d) step in $x^1(4)$ reference }

--- reference
—— 1st order model and $H_a = H_b = H = 1.0$
-·-·-· $H_a = H_b = 10H = 10.0$

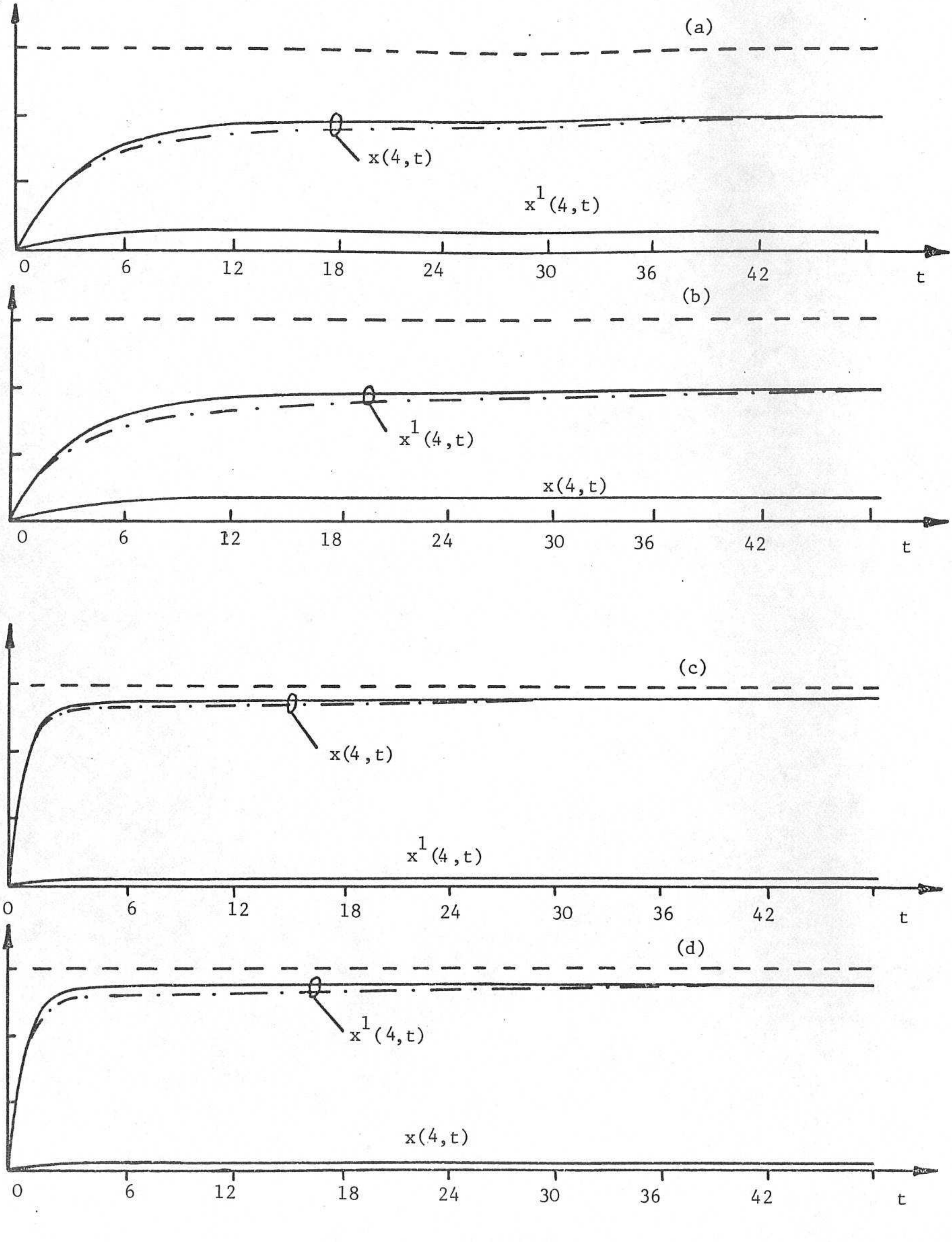
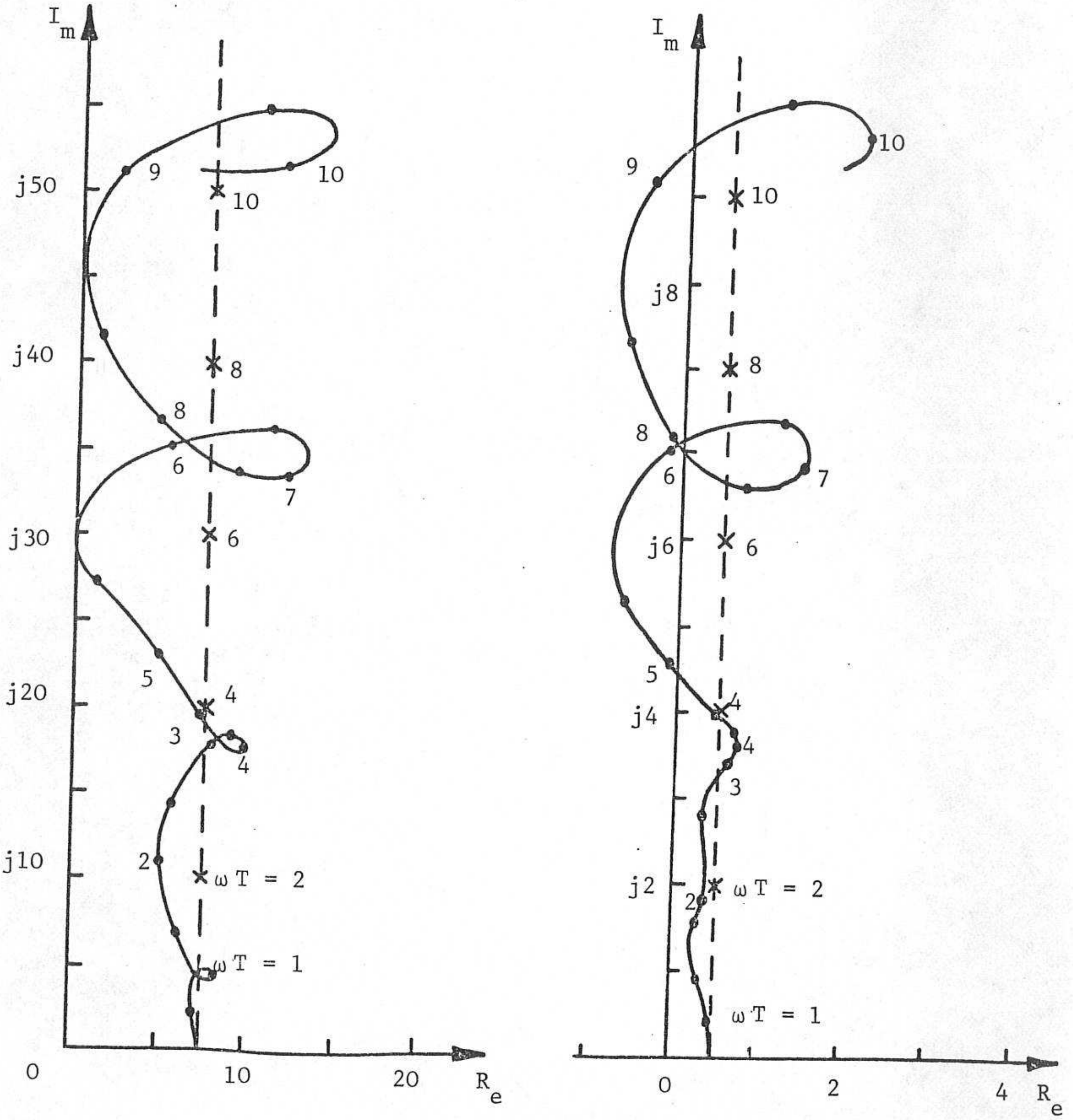


Fig. 4: Inverse Nyquist loci of $G_d(j\omega)$ and $G_{Ad}(j\omega)$



——— $(g_1+g_2)^{-1}\ \Delta\theta/W$

- - - - $(g_{A1}+g_{A2})^{-1}\ \Delta\theta/W$

——— $(g_1-g_2)^{-1}\ \Delta\theta/W$

- - - - - - $(g_{A1}-g_{A2})^{-1}\ \Delta\theta/W$

(5.)

Fig. 5: Inverse Nyquist loci $c_1^{-1}(j\omega)$ and $c_{A1}^{-1}(j\omega)$

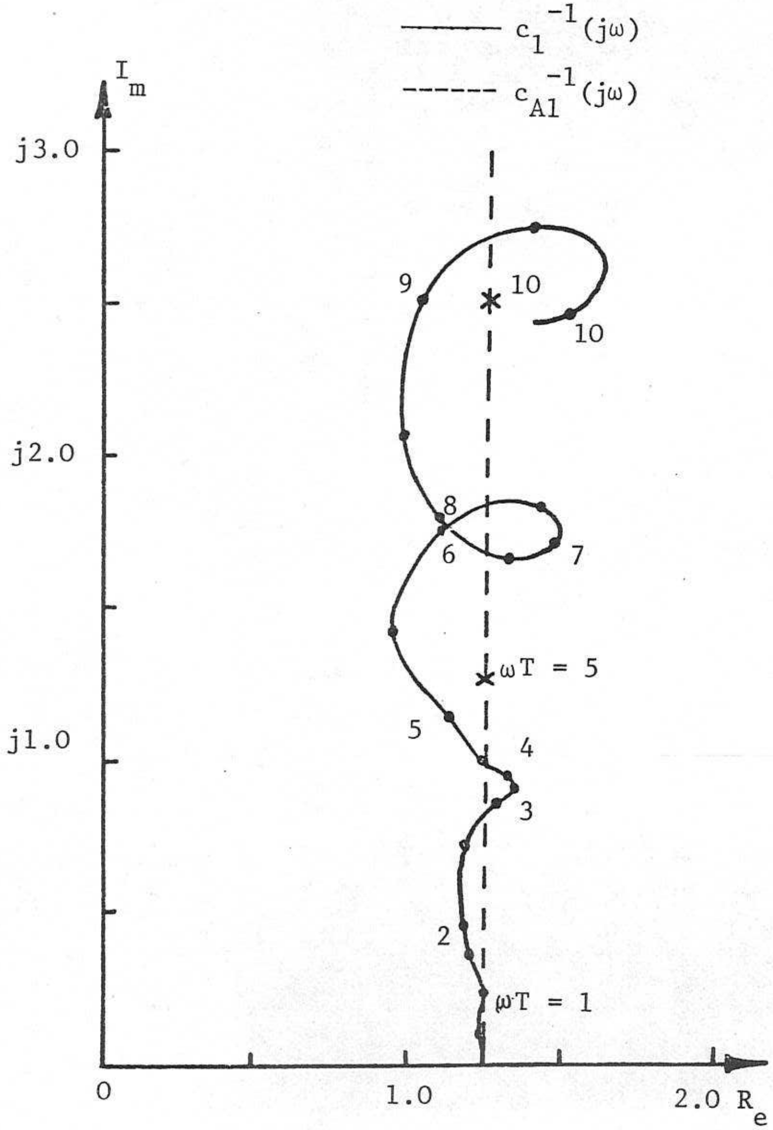
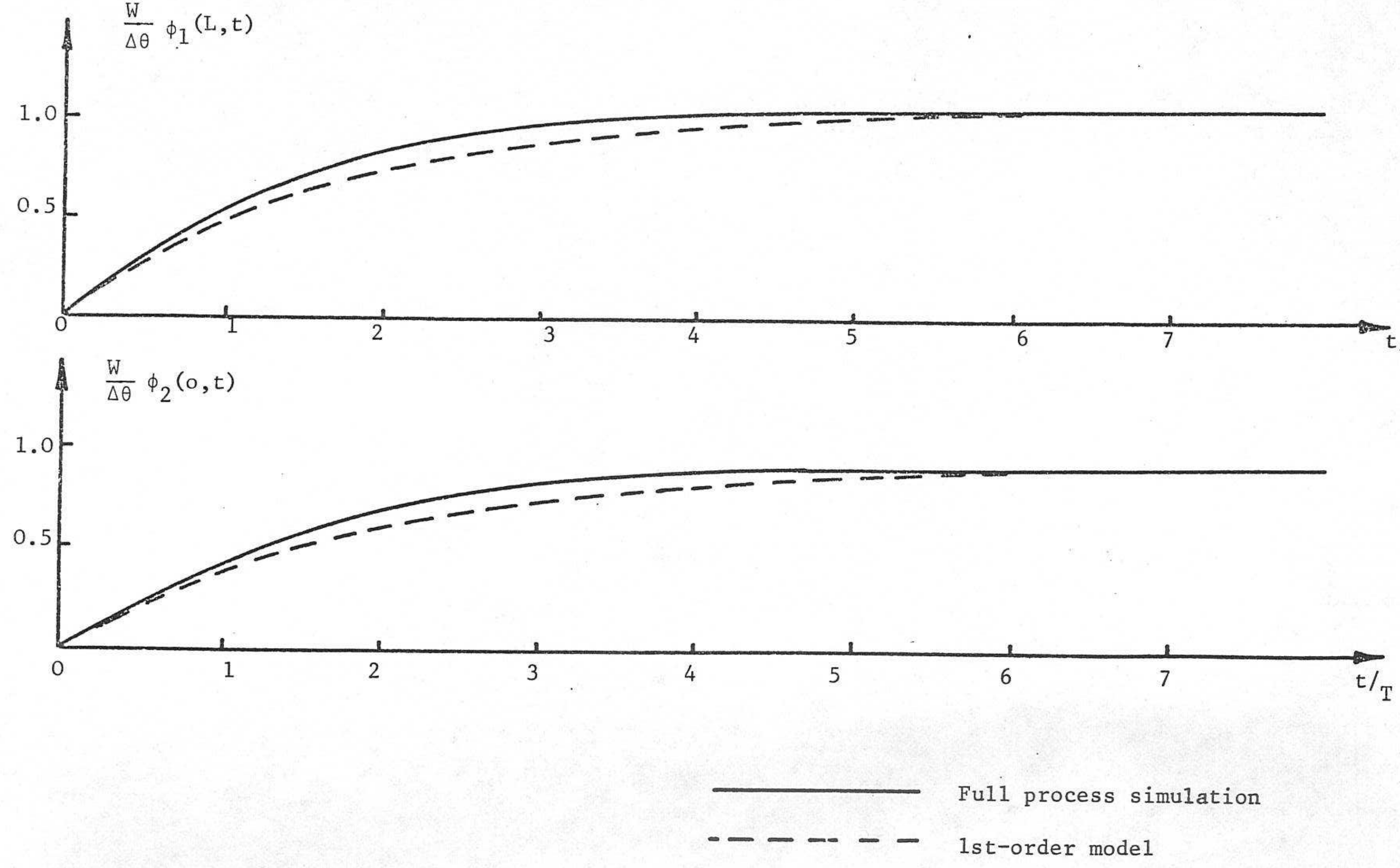
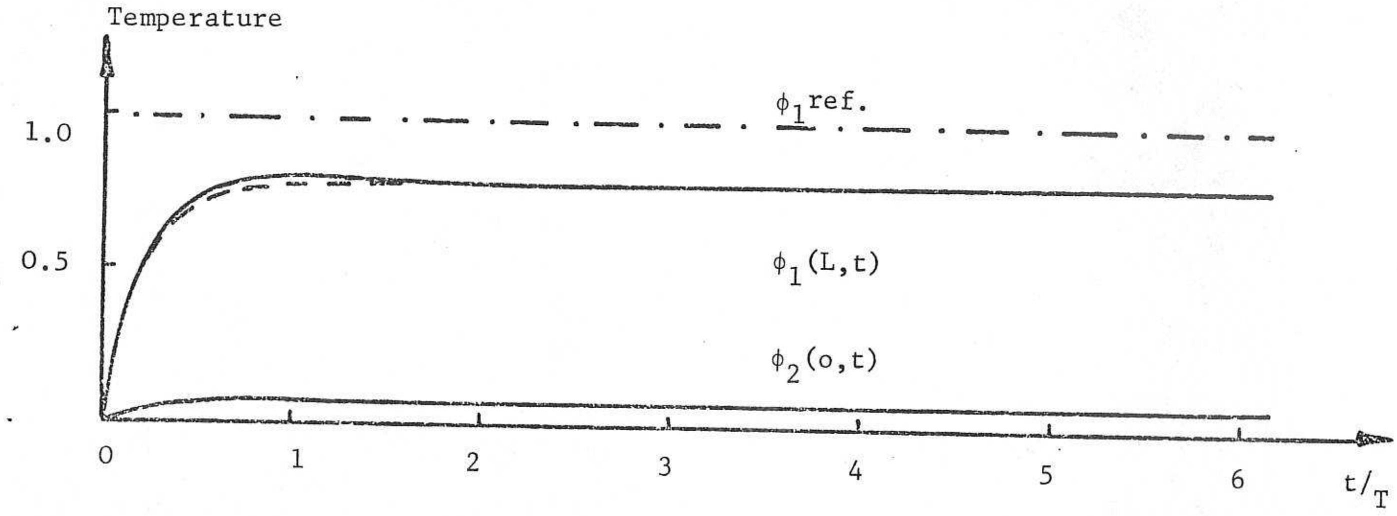


Fig. 6: Open-loop responses of heat exchanger to unit step in w_1



(7)

Fig. 7: Closed-loop responses of heat exchanger to a unit step reference change



Full process simulation

1st order model (where distinguishable)

reference input