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ON STRUCTURAL INVARIANTS AND THE ROOT-LOCI OF LINEAR MULTIVARIABLE SYSTEMS

by

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Abstract

The geometric nature of the infinite zeros of the root-loci of linear multivariable systems is investigated using the canonical form derived by Morse (1973). It is shown that an invertible system S(A,B,C) has integer order infinite zeros in the generic case equal to the controllability indices of a pair (A+KC, B), that suitable choice of proportional output feedback guarantees the absence of other than integer order zeros and that the orders and asymptotic directions of the infinite zeros are independent of constant state feedback and output injection.

1. Introduction

Recent papers (Shaked and Kouvaritakis 1976, Owens 1976, 1977a), have laid the foundation for the theoretical analysis and computation of the asymptotic behaviour of the eigenvalues of the linear time-invariant system $S(A,B,C)$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad u(t) \in R^m \ , \ x(t) \in R^n$$
$$y(t) = Cx(t) \quad , \quad y(t) \in R^m \qquad \ldots(1)$$

when subject to unity negative feedback with scalar gain $p \geq 0$. The closed-loop system takes the form $S(A-pBC, \rho B, C)$

$$\dot{x}(t) = \{A - pBC\}x(t) + \rho Br(t)$$
$$y(t) = Cx(t) \qquad \ldots(2)$$

so that the closed-loop poles are the zeros of the closed-loop characteristic polynomial

$$\rho_c(s) \triangleq |sI_n - A + pBC| \qquad \ldots(3)$$

or, equivalently, the zeros of the return-difference determinant,

$$|T(s)| = |I_m + pG(s)| \qquad \ldots(4)$$

where $G(s) \triangleq C(sI_n-A)^{-1}B$ is the mxm system transfer function matrix. It has been shown (Shaked and Kouvaritakis 1976, Owens 1976) that, under certain well-defined conditions, the unbounded closed-loop poles (ie infinite zeros) of $S(A-pBC,B,C)$ as $p \to +\infty$ have integer order in the sense that $s^{-k}p$ has a finite non-zero limit for some integer $k \geq 1$ on each branch of the root-locus.

This paper uses the canonical structure for $S(A,B,C)$ derived by Morse (1973) to identify the orders of the system infinite zeros in terms of well-defined feedback invariants of $S(A,B,C)$ under the transformation group defined by

$$(C,A,B) \longmapsto (NCT^{-1}, T(A+BF+KC)T^{-1}, TBM) \quad ...(5)$$

where N,T,M are constant nonsingular transformations and F,C are state feedback and output injection maps respectively (Morse, 1973).

2. Mathematical Preliminaries

The basic canonical decomposition of S(A,B,C) used in this paper can be inferred from the paper by Morse (1973). For completeness, the relevant results and concepts are summarized below. It is assumed that

$$\text{rank } B = \text{rank } C = m \quad ...(6)$$

and that the system is invertible in the sense that

$$\nu^* \cap R(B) = \{0\} \quad ...(7)$$

where ν^* is the maximal (A,B)-invariant subspace in N(C). Here R(Q), N(Q) denotes the range and null-space of a transformation Q respectively. It follows directly from previous results (Morse, 1973) that

$$R^n = \nu^* \oplus \mathcal{T} \quad , \quad R(B) \subset \mathcal{T} \quad ...(8)$$

where the subspace $\mathcal{T}$ is obtained via the algorithm,

$$\mathcal{T}_o = \{0\} \quad , \quad \mathcal{T}_i \triangleq A(N(C) \cap \mathcal{T}_{i-1}) + R(B) \quad , \quad i \geq 1$$

$$\mathcal{T} \triangleq \mathcal{T}_n \quad ...(9)$$

Moreover, there exists a feedback map F and an output injection map K such that

$$\{A + BF + KC\} \nu^* \subset \nu^*$$

$$\{A + BF + KC\} \mathcal{T} \subset \mathcal{T} \quad ...(10)$$

and hence a transformation T such that

$$T \{A + BF + KC\} T^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} \quad ...(11)$$

where A_1, A_4 are the matrix representations of the restriction of $A+BF+KC$ to the subspaces ν^*, $\mathcal{T}$ respectively. An important aspect of the structure of $S(A,B,C)$ is that (Morse, 1973) it is always possible to choose T,F,K such that

$$A_4 = \text{block diag }\{J_1, J_2, \ldots, J_m\} \qquad \ldots(12)$$

where

$$J_i = \begin{pmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & & \vdots \\ . & & . & & . \\ . & & & & 0 \\ . & & & & 1 \\ 0 & . & . & . & 0 \end{pmatrix}_{n_i x n_i} , \quad 1 \leq i \leq m \qquad \ldots(13)$$

and such that there exists nonsingular matrices N,M generating the canonical representation

$$NCT^{-1} = \text{block diag }\{C_1, C_2, \ldots, C_m\}$$

$$TBM = \begin{pmatrix} 0_{\dim\nu^* x m} \\ \text{block diag}\{B_1, B_2, \ldots, B_m\} \end{pmatrix} \qquad \ldots(14)$$

where

$$C_i = (1,0,\ldots,0,0)_{1 x n_i} , \qquad 1 \leq i \leq m$$

$$B_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{n_i x 1} , \qquad 1 \leq i \leq m \qquad \ldots(15)$$

The ordered integers $n_1 \leq n_2 \leq \ldots \leq n_m$ are the controllability indices of the pair (A+KC,B), and

$$\sum_{i=1}^{m} n_i = n - \dim \nu^* \qquad \ldots(16)$$

A simple computation yields the canonical form for G(s),

$$N\, C(sI_n - A - BF - KC)^{-1} BM = \text{diag}\{ \frac{1}{s^{n_1}}, \frac{1}{s^{n_2}}, \ldots, \frac{1}{s^{n_m}} \} \qquad \ldots(17)$$

The equivalence class generated by S(A,B,C) under the transformation group defined by (5) is completely defined by structural invariants, namely, the invariant polynomials of A_1 and the ordered list of integers $n_1, n_2, \ldots n_m$. It is known (Kouvaritakis and Shaked, 1976) that the invariant polynomials of A_1, or, more precisely, the zero polynomial,

$$Z(s) \triangleq |sI_{\dim\nu^*} - A_1| \qquad \ldots(18)$$

describes the finite cluster points of the root-locus of S(A,B,C) as $p \to +\infty$. It is shown in the following section that the integers $n_1, n_2, \ldots n_m$ are directly related to the orders of the infinite zeros of S(A,B,C). In general terms, the structural invariants of S(A,B,C) under the transformation group defined by (5) represent a complete description of the asymptotic behaviour of the system root locus.

3. Structural Invariants and the Infinite Zeros of S(A,B,C)

The simplicity of relation (17) suggests that the use of state feedback and output injections maps could be used to simplify the form of equation (4). Taking Laplace transforms of equation (1) with zero initial conditions yields the identities

$$\begin{aligned} x(s) &= (sI_n - A)^{-1} B\, u(s) \\ y(s) &= C\, x(s) \end{aligned} \qquad \ldots(19)$$

It is easily verified that

$$x(s) = (sI_n-A-BF)^{-1}\{-BFx(s) + Bu(s)\} \qquad \text{...(20)}$$

and hence that

$$y(s) = C(sI_n-A-BF)^{-1}B\{I_m - F(sI_n-A)^{-1}B\}u(s) \qquad \text{...(21)}$$

In transfer function matrix terms,

$$G(s) = C(sI_n-A-BF)^{-1}B\ \{I_m + H_1(s)\} \qquad \text{...(22)}$$

where $H_1(s)$ is strictly proper. Similar reasoning applied to the system $S((A+BF)^T, C^T, B^T)$ can be used to prove that

$$C(sI_n-A-BF)^{-1}B = \{I_m+H_2(s)\}C(sI_n-A-BF-KC)^{-1}B \qquad \text{...(23)}$$

where $H_2(s)$ is strictly proper. It follows directly that

$$G(s) = \{I_m+H_2(s)\}C(sI_n-A-BF-KC)^{-1}B\{I_m+H_1(s)\} \qquad \text{...(24)}$$

which is the decomposition used in the following analysis.

Substituting (24) into (4), after some manipulation,

$$|T(s)| = f(s)\ \left|\Gamma + H(s) + \text{pdiag}\left\{\frac{1}{s^{n_1}}, \frac{1}{s^{n_2}}, \ldots, \frac{1}{s^{n_m}}\right\}\right| \qquad \text{...(25)}$$

where

$$f(s) = |I_m+H_1(s)|\cdot|I_m+H_2(s)|\cdot|N^{-1}M^{-1}|$$

$$\Gamma = NM$$

$$H(s) = N\{(I_m+H_1(s))^{-1}(I_m+H_2(s))^{-1}-I_m\}M \qquad \text{...(26)}$$

It is easily verified that Γ is nonsingular and that $H(s)$ is strictly proper.

Suppose that the ordered list $n_1 \leq n_2 \leq .. \leq n_m$ has q distinct entries $m_1 < m_2 < .. < m_q$ each of multiplicity d_i, $1 \leq i \leq q$,

$$\sum_{i=1}^{q} m_i d_i = \sum_{i=1}^{m} n_i \qquad \ldots(27)$$

and write
$$\Gamma = \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{pmatrix}$$

$$H(s) = \begin{pmatrix} H_{11}(s) & \cdots & H_{1q}(s) \\ \vdots & & \vdots \\ H_{q1}(s) & \cdots & H_{qq}(s) \end{pmatrix} \qquad \ldots(28)$$

where Γ_{ij}, $H_{ij}(s)$ are of dimension $d_i x d_j$.

Theorem 1

If the submatrices

$$P_i \triangleq \begin{pmatrix} \Gamma_{ii} & \cdots & \Gamma_{iq} \\ \vdots & & \vdots \\ \Gamma_{qi} & & \Gamma_{qq} \end{pmatrix} \quad , \quad 1 \leq i \leq q \qquad \ldots(29)$$

are nonsingular, then the invertible system S(A,B,C) has $m_i d_i$ infinite zeros of order m_i, $1 \leq i \leq q$.

The theorem is proved later in the section and provides a direct computational technique for the infinite zeros. In essence the result states that, under mild conditions on $\Gamma = NM$, the structural invariants $n_1, \ldots, n_m$ are the orders of the infinite zeros of S(A,B,C). Noting that arbitrarily small perturbations to N,M guarantees the nonsingularity of P_i, $1 \leq i \leq q$, then the result is seen to be generically valid. Moreover, the nonsingularity of P_i, $1 \leq i \leq q$, can be achieved by a suitable choice of output feedback matrix K_o e.g. the system $S(A,BK_o,C)$ with $K_o = M\ N$ generates the matrix $\Gamma = I_m$ and hence $|P_i| = 1$, $1 \leq i \leq q$.

Finally, it follows directly from the construction of the canonical form and the invariance of $n_1, n_2, \ldots, n_m$ under the transformation group defined by (5) that

Corollary

With the assumptions of theorem 1, the asymptotic behaviour of the infinite zeros of $S(A,B,C)$ is invariant under state feedback and output injection transformations of the form

$$A \longmapsto A + BF_1 + K_1 C \qquad \ldots(30)$$

In general terms, the asymptotic behaviour of the infinite zeros depends only upon the structural invariants n_i, $1 \leq i \leq m$, and the matrix $\Gamma = NM$. The corollary may be used to simplify the system for theoretical or computational purposes. For example, choosing F_1 such that $(A+BF_1)\nu^* \subset \nu^*$, the analysis of the infinite zeros of $S(A,B,C)$ can be reduced to the analysis of the infinite zeros of a system $S(\bar{A},\bar{B},\bar{C})$ of state dimension $n - \dim\nu^*$ in the quotient space R^n/ν^*.

Proof of theorem 1:

The proof follows similar lines to the analysis of Shaked and Kouvaritakis (1976), and provides a direct computational solution to the problem in terms of the system canonical representation. As our attention is restricted to the infinite zeros, equation (25) indicates that the return difference determinant can be replaced by the rational polynomial

$$|T_1(s)| \triangleq |P_1 + \Omega_1(s,p) + p \text{ block diag}\{s^{-m_1} I_{d_1}, \ldots, s^{-m_q} I_{d_q}\}| \qquad \ldots(31)$$

and $\Omega_1(s,p) \equiv H(s) \to 0$ as $p \to +\infty$. The nonsingularity of P_1 ensures that ps^{-m_1}

cannot have a cluster point at the origin of the complex plane as $p\to+\infty$. Consider the case when $ps^{-m_i}\to 0$, $i>1$, when $(p\to+\infty)$

$$|T_1(s)| \to |P_1 + p \text{ block diag}\{s^{-m_1}I_{d_1}, 0, .., 0\}| \qquad ...(32)$$

If $q = 1$ then the theorem is proved as $n_1 = n_2 = .. = n_m = m_1$ and $ps^{-m_1}\to\lambda$ where $|P_1+\lambda I_m| = 0$ ie the system has mn_1 n_1^{th} order infinite zeros of the form

$$\lim_{p\to\infty} s^{-n_1}p = -\lambda_i \qquad ...(33)$$

where λ_i is an eigenvalue of $P_1 = \Gamma$. If $q>1$, then P_2 is nonsingular by assumption, and application of Schurs Formula to (32) yields the relation $(p\to+\infty)$

$$|T_1(s)| \to c_1 \left|ps^{-m_1}I_{d_1} + \Gamma_{11} - \{\Gamma_{12}...\Gamma_{1q}\}P_2^{-1}\begin{pmatrix}\Gamma_{21}\\ \vdots \\ \Gamma_{q1}\end{pmatrix}\right| \qquad ...(34)$$

where c_1 is a non-zero constant. Hence, the nonsingularity of P_2 ensures that ps^{-m_1} can only have finite, non-zero cluster points λ generated by solutions of the eigenvalue equation

$$c_1 \left|\lambda I_{d_1} + \Gamma_{11} - \{\Gamma_{12} \cdots \Gamma_{1q}\} P_2^{-1} \begin{pmatrix}\Gamma_{21}\\ \vdots \\ \Gamma_{q1}\end{pmatrix}\right|$$

$$= |P_1 + \lambda\text{diag}\{I_{d_1}, 0, .., 0\}| = 0 \qquad ...(35)$$

Suppose now that $ps^{-m_i}\to 0$, $i>2$, and ps^{-m_1} is unbounded as $p\to+\infty$. Applicationsof Schurs formula to (31) yield the identity,

$$|T_1(s)| = |ps^{-m_1}I_{d_1} + H_{11}(s) + \Gamma_{11}| \times$$

$$|P_2+\Omega_2(s,p) + p \text{ block diag}\{s^{-m_2}I_{d_2},\ldots,s^{-m_q}I_{d_q}\}| \qquad \ldots(36)$$

where $\Omega_2(s,p)\to 0$ is $p\to+\infty$. The first factor is nonsingular at high gain and hence we can replace $|T_1(s)|$ by

$$|T_2(s)| \triangleq |P_2+\Omega_2(s,p)+p \text{ block diag}\{s^{-m_2}I_{d_2},\ldots,s^{-m_q}I_{d_q}\}|$$

$$\to |P_2+p \text{ block diag}\{s^{-m_2}I_{d_2},0,\ldots,0\}| \qquad \ldots(37)$$

If $q = 2$ then the theorem is proved as $ps^{-m_2}\to\lambda$ where $|P_2+\lambda I_{d_2}| = 0$ ie the system has d_2m_2 m_2^{th} order infinite zeros of the form

$$\lim_{p\to\infty} s^{-m_2}p = -\lambda_i \qquad \ldots(38)$$

where λ_i is an eigenvalue of $-P_2$. If $q>2$ then P_3 is nonsingular by assumption and hence, using Schurs formula,

$$|T_2(s)| \to c_2|ps^{-m_2}I_{d_2}+\Gamma_{22}-\{\Gamma_{23}\ldots\Gamma_{2q}\}P_3^{-1}\begin{pmatrix}\Gamma_{32}\\ \vdots\\ \Gamma_{q2}\end{pmatrix}| \qquad \ldots(39)$$

where c_2 is a non-zero constant ie nonsingularity of P_3 ensures that ps^{-m_2} can only have finite, non-zero cluster points λ generated by solutions of the eigenvalue equation,

$$c_2|\lambda I_{d_2} + \Gamma_{22} -\{\Gamma_{23} \ldots \Gamma_{2q}\}P_3^{-1}\begin{pmatrix}\Gamma_{32}\\ \vdots\\ \Gamma_{q2}\end{pmatrix}\}|$$

$$= |P_2 + \lambda \text{ block diag } \{I_{d_2},0,\ldots,0\}| = 0 \qquad \ldots(40)$$

The theorem is now easily proved by induction by noting that (37) takes a similar form to (31). More exactly, if $ps^{-m_i}\to 0$, $i>k$ and ps^{-m_i} are unbounded, $1\leq i<k$, as $p\to+\infty$, and

$$|T_k(s)| \triangleq |P_k + \Omega_k(s,p) + p \text{ block diag } \{s^{-m_k} I_{d_k}, \ldots, s^{-m_q} I_{d_q}\}| \qquad \ldots(41)$$

where $\Omega_k(s,p) \to 0$ as $p \to +\infty$, then the system has $d_k m_k$ m_k^{th} order infinite zeros of the form $ps^{-m_k} \to \lambda$ where

$$|P_k + \text{block diag}\{\lambda I_{d_k}, 0, \ldots, 0\}| = 0 \qquad \ldots(42)$$

It is of particular interest to note that the strictly proper transfer function matrix H(s) can be neglected in the analysis and hence that the asymptotic behaviour of the root-locus can be examined using the approximation

$$|T_1(s)| \simeq |\Gamma + p \text{ block diag}\{s^{-m_1} I_{d_1}, \ldots, s^{-m_q} I_{d_q}\}| \qquad \ldots(43)$$

Equivalently, the asymptotic behaviour of the system root-locus is dictated by the structural invariants $n_1, n_2, \ldots, n_m$ and the matrix $\Gamma = NM$ representing (equation (17)) the input-output couplings in the open-loop system. This result is obvious in the classical single-input-single-output case where $m = 1$ and n_1 is equal to the rank of the system transfer function.

4. Illustrative Example

Consider the invertible system,

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \qquad \ldots(44)$$

and note that $\nu^* = \{0\}$. Choosing

$$F = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K^T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad , \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \ldots(45)$$

from which, choosing $T = I_4$,

$$T(A+BF+KC)T^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$NCT^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad TBM = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \ldots(46)$$

which is in the required canonical form with $n_1 = 1$, $n_2 = 3$. Also, by direct computation,

$$NC(sI_4 - A - BF - KC)^{-1}BM = \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s^3} \end{pmatrix} \qquad \ldots(47)$$

and

$$\Gamma = P_1 = NM = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$P_2 = 1 \qquad \ldots(48)$$

so that the conditions of theorem 1 are satisfied. It follows directly that the system has one first order infinite zero of the form $s^{-1}p \to \lambda_1$ $(p \to \infty)$ where (equation (42), with $k = 1$)

$$\left| P_1 + \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \right| = 1 + \lambda_1 = 0 \qquad \ldots(49)$$

ie $\lambda_1 = -1$. In a similar manner the system has one third order infinite zero of the form $s^{-3}p \to \lambda_2$ $(p\to\infty)$ where (equation (42) with $k = 2$)

$$|P_2 + \lambda_2| = 1 + \lambda_2 = 0 \qquad \ldots(50)$$

ie $\lambda_2 = -1$. These results are easily verified by direct analysis of the closed-loop characteristic polynomial

$$|sI_4 - A + pBC| = s^4 + ps^3 - s^2p + 2ps + p^2 \qquad \ldots(51)$$

and it is noted that, for reasons previously discussed (Owens, 1977b), the algorithm of Shaked and Kouvaritakis (1976) is not strictly applicable to this system.

5. The Case of S(A,B,C) not invertible

If S(A,B,C) is not invertible in the sense that (7) does not hold, it can be deduced from the canonical form due to Morse (1973) that there exists N, M, T, F, K (c.f. equation (17)) such that

$$NC(sI_n - A - BF - KC)^{-1}BM = \mathrm{diag}\left\{ \frac{1}{s^{n_1}}, \frac{1}{s^{n_2}}, \ldots, \frac{1}{s^{n_\ell}}, 0, \ldots, 0 \right\} \qquad \ldots(52)$$

where $\ell = m - \dim \nu^* \cap R(B)$ and the structural invariants $n_1, n_2, \ldots, n_\ell$ are the controllability indices of the pair $(A+KC, B)$. It is easily verified that the results of section 3 remain valid if theorem 1 is reworded,

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Theorem 2

If the submatrices

$$P_i \triangleq \begin{pmatrix} \Gamma_{ii} & \cdots & \Gamma_{i,q+1} \\ \vdots & & \\ \Gamma_{q+1,i} & & \Gamma_{q+1,q+1} \end{pmatrix}, \quad 1 \leq i \leq q+1 \qquad \ldots(53)$$

are nonsingular, then the noninvertible system S(A,B,C) has $m_i d_i$ infinite zeros of order m_i, $1 \leq i \leq q$.

Here, for simplicity of notation, the submatrix $\Gamma_{q+1,q+1}$ is taken to correspond to the zeros in equation (52).

6. Conclusions

It has been shown that the canonical form derived by Morse (1973) can play an important role in the analysis of the properties and structure of multivariable root-loci. More precisely, using the result that a square, invertible system S(A,B,C) has structural invariants (under the transformation group defined by (5)) defined by the invariant polynomials of the restriction of A+BF to ν^* and the controllability indices $n_1, n_2, .., n_m$ of A+KC (for suitable choice of F,K), then, in the generic case, the indices $n_1, n_2, .., n_m$ are simply the orders of the infinite zeros of the root-locus of S(A,B,C). Noting that the zeros of S(A,B,C) (ie the invariant polynomials of $A+BF|\nu^*$) define the finite cluster points of the system root-locus, it is seen that the structural invariants provide a complete description of the asymptotic behaviour of the closed-loop poles.

Of particular interest are the observations:

(i) It is always possible to choose a constant output feedback controller such that the infinite zeros have orders $n_1, n_2, .., n_m$.

(ii) The orders and magnitudes of the infinite zeros are independent of constant state feedback and output injection maps.

(iii) The orders and magnitudes of the infinite zeros depends only upon the invariants $n_1, .., n_m$ and the properties of $\Gamma = NM$ which describes the basic input-output couplings in the system.

The author feels that the canonical form (Morse, 1973) could be a useful tool for setting the theory of root-loci on a firm geometric foundation.

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