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On operads, bimodules and analytic functors

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Abstract

We develop further the theory of operads and analytic functors. In particular, we introduce the bicategory $\mathrm{OpdBim}_{\mathcal{V}}$ of operad bimodules, that has operads as 0-cells, operad bimodules as 1-cells and operad bimodule maps as 2-cells, and prove that it is cartesian closed. In order to obtain this result, we extend the theory of distributors and the formal theory of monads.

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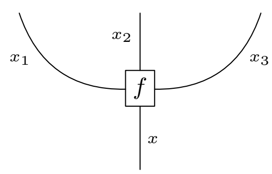
Introduction

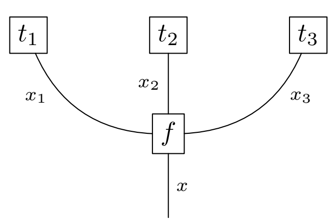
Operads. Operads originated in algebraic topology [**16, 63**] and, while they continue to be very important in that area (see, e.g., [**11, 28, 29, 65**]), they have found applications in several other branches of mathematics, including geometry [**35, 43**], algebra [**57, 64**], combinatorics [**3, 56**] and category theory [**4, 7, 54**]. Recently, operads have started to be considered also in theoretical computer science [**23**]. See [**32, 34, 60, 67**] for recent accounts of the theory of operads.

There are several variants of operads in the literature (symmetric or non-symmetric, many-sorted or single-sorted, enriched or non-enriched). Here, by an operad we mean a many-sorted (sometimes called coloured) symmetric operad, enriched over a fixed symmetric monoidal closed presentable category $\mathcal{V}$. Thus, we call operad what other authors call a $\mathcal{V}$-enriched symmetric multicategory. An operad A has a set of objects $|A|$ and, for every tuple $(x_1, \ldots, x_n, x) \in |A|^{n+1}$, an object $A[x_1, \ldots, x_n; x] \in \mathcal{V}$ of operations with inputs of sort $x_1, \ldots, x_n$ and output of sort x. An operation $f \in A[x_1, x_2, x_3; x]$, i.e. an arrow $f \colon I \to A[x_1, x_2, x_3; x]$ in $\mathcal{V}$, can be represented as the tree

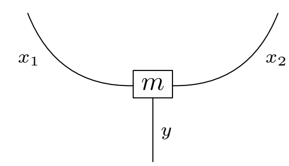


Even if single-sorted operads suffice for many purposes, many-sorted operads are essential for some applications, such as those developed in [**28, 29**]. However, the theory of operads is less developed than that of single-sorted operads. For example, recent monographs on operads [**32, 60**] deal almost exclusively with single-sorted operads. One of the original motivations for this work was to develop further the theory of operads, so as to facilitate applications. As we will see, dropping the restriction of working with single-sorted operads not only does not create problems, but in fact brings to light new mathematical structures.

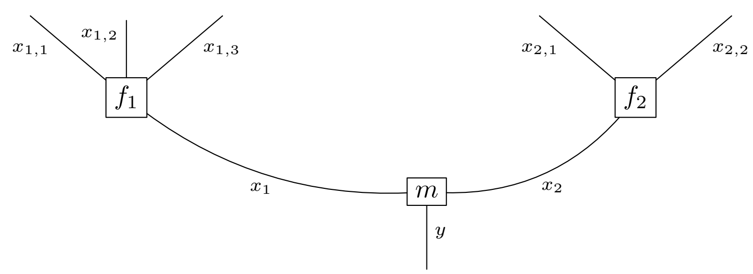
Operads are typically studied for their categories of algebras. For an operad A, an A-algebra consists of a family $T \in \mathcal{V}^{|A|}$ equipped with a left A-action, which can be represented as a composition law for trees of the form



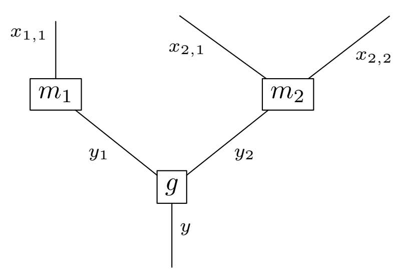
where $f \in A[x_1, x_2, x_3; x]$ and $t_i \in T(x_i)$ for $i = 1, 2, 3$. One can consider algebras not only in $\mathcal{V}$, but also in any symmetric $\mathcal{V}$-rig, i.e. symmetric monoidal closed presentable $\mathcal{V}$-category $\mathcal{R} = (\mathcal{R}, \diamond, e)$. For an operad A, we write $\mathrm{Alg}_{\mathcal{R}}(A)$ for the category of A-algebras in $\mathcal{R}$. The forgeful functor $U \colon \mathrm{Alg}_{\mathcal{R}}(A) \to \mathcal{R}^{|A|}$ has a left adjoint $F \colon \mathcal{R}^{|A|} \to \mathrm{Alg}_{\mathcal{R}}(A)$, the free A-algebra functor. Informally, we may view the category $\mathcal{R}^{|A|}$ as an affine variety, the operad A as a system of equations, and the category $\mathrm{Alg}_{\mathcal{R}}(A)$ as the sub-variety of $\mathcal{R}^{|A|}$ defined by the system of equations A. The left action on an A-algebra The notions of an operad bimodule and operad bimodule map subsume those of algebra and algebra map, as we explain below, and play an important role in the theory of operads [**26, 32, 43, 59, 65**]. Explicitly, for operads A and B, a (B, A)-bimodule in a $\mathcal{V}$-rig $\mathcal{R}$ is a family of objects $M[x_1, \ldots, x_n; y] \in \mathcal{R}$, indexed by sequences $(x_1, \ldots, x_n) \in |A|^n$ and $y \in |B|$, subject to suitable functoriality conditions, equipped with a left B-action and a right A-action that commute with each other. Informally, an element $m \in M[x_1, x_2; y]$ may be represented as a tree of the form



The right A-action can be viewed as composition operation for trees of the form



where $f_1 \in A[x_{1,1}, x_{1,2}, x_{1,3}; x_1]$ and $f_2 \in A[x_{2,1}, x_{2,2}; x_2]$, while the left B-action as a composition operation for trees of the form



where $m_1 \in M[m_{1,1}; y_1]$, $m_2 \in M[x_{2,1}, x_{2,2}; y_2]$ and $g \in B[y_1, y_2; y]$. We write $\mathrm{OpdBim}_{\mathcal{R}}[A, B]$ for the category of (B, A)-bimodules and bimodule maps in $\mathcal{R}$. As we will see, a useful way of understanding operad bimodules is to observe that a (B, A)-bimodule M determines a functor

$$\mathrm{Alg}_{\mathcal{R}}(M) \colon \mathrm{Alg}_{\mathcal{R}}(A) \to \mathrm{Alg}_{\mathcal{R}}(B)\,.$$

which fits into the commutative diagram of the form

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{R}}(A) & \xrightarrow{\mathrm{Alg}_{\mathcal{R}}(M)} & \mathrm{Alg}_{\mathcal{R}}(B) \\ {\scriptstyle F}\uparrow & & \downarrow{\scriptstyle U} \\ \mathcal{R}^{|A|} & \xrightarrow[M]{} & \mathcal{R}^{|B|} \,. \end{array}$$

Here, the vertical arrows are the evident free and forgetful functors and, for $T \in \mathcal{R}^{|A|}$ and $y \in |B|$, we have

$$M(T)(y) =_{\mathrm{def}} \int^{\bar{x} \in S(|A|)} M[\bar{x}; y] \otimes T^{\bar{x}} \,,$$

where $S(|A|)$ is the free symmetric monoidal category on $|A|$, which has tuples of elements of $|A|$ as objects and, for such a tuple $\bar{x} = (x_1, \ldots, x_n)$, we let $T^{\bar{x}} =_{\mathrm{def}} T(x_1) \diamond \ldots \diamond T(x_n)$. We refer to such functors as *analytic functors*, since they are a natural extension the analytic functors originally introduced in [**42**] and their generalization defined in [**30**]. Examples of analytic functors between categories of operad algebras include generalizations of the extension and restriction functors for single-sorted operads studied in [**32**].

One of the upshots of this paper is that for two operads A and B, there are new operads $A \sqcap B$ and B^A such that we have natural equivalences

$$\mathrm{Alg}_{\mathcal{R}}(A \sqcap B) \simeq \mathrm{Alg}_{\mathcal{R}}(A) \times \mathrm{Alg}_{\mathcal{R}}(B) \,, \quad \mathrm{Alg}_{\mathcal{R}}(B^A) \simeq \mathrm{OpdBim}_{\mathcal{R}}[A, B] \,.$$

Thus, $A \sqcap B$ is the operad whose algebras are pairs consisting of an A-algebra and a B-algebra, while B^A is the operad whose algebras are (B, A)-bimodules. We have $|A \sqcap B| = |A| \sqcup |B|$, an instance of a duality phenomenon which pervades this paper. Furthermore, there is an operad $\top$ (the operads whose set of sorts is empty) such that, for any operad A, there are equivalences

$$\mathrm{OpdBim}_{\mathcal{R}}[A, \top] \simeq 1 \,, \quad \mathrm{OpdBim}_{\mathcal{R}}[\top, A] \simeq \mathrm{Alg}_{\mathcal{R}}(A) \,,$$

where 1 is the terminal category. Here, note that the second equivalence shows that operad bimodules subsume operad algebras.

A fundamental notion in our development is that of a bicategory, introduced in [**8**], which generalizes the notion of a monoidal category in the same way in which the notion of a category generalizes that of a monoid. Indeed, it will be very useful for us to regard the notion of an operad as a special case of the general notion of a monad in a bicategory [**8**], which generalizes the notion of a monoid in a monoidal category [**51**]. Taking further ideas in [**4, 30**], we introduce a bicategory, called the bicategory of symmetric sequences and denoted by $\mathrm{Sym}_{\mathcal{V}}$, and show that monads therein are exactly operads. This extends to the many-sorted case of the well-known fact that single-sorted operads can be viewed as monoids in the category of single-sorted symmetric sequences equipped with the substitution monoidal structure [**46, 66**]. Furthermore, the notions of an operad bimodule and an operad bimodule map can be seen as instances of the the general notions of a bimodule and a bimodule map in a bicategory [**71**].

The characterization of operads, operad bimodules and operad bimodule maps as monads, monad bimodules and monad bimodule maps in the bicategory $\mathrm{Sym}_{\mathcal{V}}$, makes it natural to assemble them in a bicategory, called the bicategory of operad bimodules and denoted by $\mathrm{OpdBim}_{\mathcal{V}}$, using the so-called bimodule construction [**71**]. This construction takes what we call a tame bicategory $\mathcal{E}$ (i.e. a bicategory whose hom-categories have reflexive coequalizers and whose composition functors preserve coequalizers in each variable) and returns a new bicategory, called the bicategory of bimodules in $\mathcal{E}$ and denoted by $\mathrm{Bim}(\mathcal{E})$, which has monads, monad bimodules and monad bimodule maps in $\mathcal{E}$ as 0-cells, 1-cells and 2-cells, respectively. The composition operation in $\mathrm{Bim}(\mathcal{E})$ is defined using the assumption that $\mathcal{E}$ is tame, generalizing the definition of the tensor product of ring bimodules. We will prove that the bicategory $\mathrm{Sym}_{\mathcal{V}}$ of symmetric sequences is a tame bicategory (Corollary 4.4.9), a result that allows us to define the bicategory of operad bimodules by letting

$$\mathrm{OpdBim}_{\mathcal{V}} =_{\mathrm{def}} \mathrm{Bim}(\mathrm{Sym}_{\mathcal{V}})\,.$$

Remarkably, the composition of bimodules in $\mathrm{OpdBim}_{\mathcal{V}}$ obtained by specializing to this case the general definition of composition in bicategories of bimodules is a generalization to many-sorted operads of the circle-over construction for single-sorted operads defined in [**65**]. Furthermore, we will show that composition in $\mathrm{OpdBim}_{\mathcal{V}}$ corresponds to composition of analytic functors between categories of operad algebras. The category $\mathrm{Sym}_{\mathcal{V}}$ can be naturally viewed as a sub-bicategory of $\mathrm{OpdBim}_{\mathcal{V}}$ and in between the two there is a further bicategory, called the bicategory of categorical symmetric sequences and denoted by $\mathrm{CatSym}_{\mathcal{V}}$, so that we have inclusions:

$$\mathrm{Sym}_{\mathcal{V}} \subseteq \mathrm{CatSym}_{\mathcal{V}} \subseteq \mathrm{OpdBim}_{\mathcal{V}}\,.$$

When $\mathcal{V} = \mathrm{Set}$, the bicategory $\mathrm{CatSym}_{\mathcal{V}}$ is exactly the bicategory of generalized species of structures defined in [**30**].

Main results. Our main results show that $\mathrm{CatSym}_{\mathcal{V}}$ and $\mathrm{OpdBim}_{\mathcal{V}}$ are cartesian closed bicategories. These facts may be of interest to researchers in mathematical logic and theoretical computer science, since they provide new models of versions of the simply-typed λ-calculus, and contribute to the general program of generalizing domain theory [**19, 39**] and to the study of models of the differential λ-calculus [**27**] inspired by the theory of analytic functors [**31**]. One of our original motivation for showing that $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed was to show that for any two operads A and B, the category of (B, A)-bimodules can be regarded as the category of algebras for an operad. This allows, in particular, to apply to categories of bimodules the known results concerning the existence of Quillen model structures on categories of operad algebras [**11, 12**], although we do not pursue this idea here. As we will explain below, proving that $\mathrm{CatSym}_{\mathcal{V}}$ is cartesian closed can be considered also as a useful step towards show that $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed. It should be noted that, in order for $\mathrm{OpdBim}_{\mathcal{V}}$ to be cartesian closed, it is essential to consider operads and not just single-sorted operads. Indeed, given two single-sorted operads, their exponential in $\mathrm{OpdBim}_{\mathcal{V}}$ is, in general, not single-sorted (see below for more information). We would also like to mention that our results are intended to contribute to an ongoing research programme, which we like to refer to as '2-algebraic geometry', that is concerned with the development of a variant of commutative algebra and algebraic geometry where commutative rings are replaced by symmetric monoidal categories [**4, 22**].

Our first main result (Theorem 3.4.2) is that the bicategory $\mathrm{CatSym}_{\mathcal{V}}$ is cartesian closed. For small $\mathcal{V}$-categories $\mathbb{X}$ and $\mathbb{Y}$, their product $\mathbb{X} \sqcap \mathbb{Y}$ and the exponential $\mathbb{Y}^{\mathbb{X}}$ in $\mathrm{CatSym}_{\mathcal{V}}$ are given by the formulas:

$$\mathbb{X} \sqcap \mathbb{Y} =_{\mathrm{def}} \mathbb{X} \sqcup \mathbb{Y}\,, \quad \mathbb{Y}^{\mathbb{X}} =_{\mathrm{def}} S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}\,,$$

where $\mathbb{X} \sqcup \mathbb{Y}$ denotes the coproduct of $\mathbb{X}$ and $\mathbb{Y}$ in the 2-category of small $\mathcal{V}$-categories. Thus, when $\mathcal{V} = \mathrm{Set}$, we obtain the cartesian closed structure of [**30**]. An interesting aspect of this result is that one obtains a cartesian closed bicategory even when $\mathcal{V}$ is not a cartesian closed category, but has an arbitrary monoidal closed structure. Our approach to the definition of the bicategory $\mathrm{CatSym}_{\mathcal{V}}$ and to the proof that it is cartesian closed differs significantly from the one adopted in [**30**] in the non-enriched case. In particular, its construction and the proof that it has finite products follow immediately from the results that we obtain in the first sections of the paper. There, we develop further the theory of distributors (also known as bimodules or profunctors) [**9, 53**] and introduce and study the notions of a (lax) monoidal distributor and of a symmetric (lax) monoidal distributor, and show how they can be seen as morphisms of appropriate bicategories. These bicategories and the bicategory $\mathrm{CatSym}_{\mathcal{V}}$ are defined using the notion of a Gabriel factorization of a homomorphism (which we introduce in Chapter 1), rather than via the theory of Kleisli bicategories and pseudo-distributive laws [**21, 61, 62**], as done in [**30**]. We prefer this approach since it allows us to avoid the verification of several coherence conditions. Furthermore, our proof that the bicategory $\mathrm{CatSym}_{\mathcal{V}}$ is cartesian closed is broken down and organized into several observations on symmetric monoidal distributors that admit relatively short proofs and does not involve lengthy coend calculations like the proof of the corresponding fact in [**30**].

The second main result in this paper (Theorem 5.4.6) is that the bicategory $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed. Indeed, we have already described above the universal properties that characterize products, written $A \sqcap B$, exponentials, written B^A, and the terminal object, written $\top$, in $\mathrm{OpdBim}_{\mathcal{V}}$. Our proof that $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed relies on two results. The first is that for a tame bicategory $\mathcal{E}$, if $\mathcal{E}$ is cartesian closed, then $\mathrm{Bim}(\mathcal{E})$ is cartesian closed (Theorem 5.1.2). Applying this fact to $\mathrm{CatSym}_{\mathcal{V}}$ (which we show to be tame as well), we obtain that $\mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})$ is cartesian closed. The second is that the inclusion

$$\mathrm{OpdBim}_{\mathcal{V}} = \mathrm{Bim}(\mathrm{Sym}_{\mathcal{V}}) \subseteq \mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})$$

induced by the inclusion $\mathrm{Sym}_{\mathcal{V}} \subseteq \mathrm{CatSym}_{\mathcal{V}}$ is an equivalence of bicategories (Theorem 5.4.5). In order to prove these two auxiliary facts, we develop some aspects of the formal theory of monads (in the sense of [**69**]) in tame bicategories. In particular, we establish a universal property of $\mathrm{Bim}(\mathcal{E})$, namely that of being the Eilenberg-Moore completion of $\mathcal{E}$ as a tame bicategory (a notion that we will define precisely in Section 5.3), which is a special case of a result obtained independently by Richard Garner and Michael Shulman in the context of the theory of categories enriched in a bicategory [**33**].

Organization of the paper. The paper is organized as follows. Chapter 1 provides an overview of the background material used in the paper. We also introduce the notion of a Gabriel factorization of a homomorphism, which we use several times to construct the bicategories of interest.

The rest of the paper is organized in two parts, each leading up to one of our two main results. The first part includes Chapter 2 and Chapter 3. Chapter 2 develops some auxiliary material, needed for Chapter 3. In particular, it introduces the auxiliary notions of a monoidal distributor and of a symmetric monoidal distributor, shows how they can be seen as the morphisms of appropriate bicategories, and establishes some useful facts about these bicategories. Chapter 3 introduces the notion of an S-distributor and the corresponding bicategory $S\text{-}\mathrm{Dist}_{\mathcal{V}}$. We then define $\mathrm{CatSym}_{\mathcal{V}}$ as the opposite of $S\text{-}\mathrm{Dist}_{\mathcal{V}}$. We then establish our first main result (Theorem 3.4.2), asserting that $\mathrm{CatSym}_{\mathcal{V}}$ is cartesian closed.

The second part of the paper comprises Chapter 4 and Chapter 5. Chapter 4 recalls the notions of a monad, bimodule and bimodule map in a bicategory $\mathcal{E}$ and the definition of the bicategory $\mathrm{Bim}(\mathcal{E})$, under the assumption that $\mathcal{E}$ is tame. We then show that $\mathrm{CatSym}_{\mathcal{V}}$ and its subcategory $\mathrm{Sym}_{\mathcal{V}}$ are tame (Corollary 4.4.9), thus allowing us to define the bicategory of operad bimodules using the bimodule construction. Chapter 5 is devoted to the proof of our second main result. First, we show that, for a tame bicategory $\mathcal{E}$, if $\mathcal{E}$ is cartesian closed, then so is $\mathrm{Bim}(\mathcal{E})$ (Theorem 5.1.2). Secondly, we investigate some aspects of the theory of monads in a tame bicategory, leading up to the fact that $\mathrm{Bim}(\mathcal{E})$ can be viewed as the Eilenberg-Moore completion of $\mathcal{E}$ as a tame bicategory (Theorem 5.4.2). A corollary of this fact leads to prove that $\mathrm{OpdBim}_{\mathcal{V}}$ and $\mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})$ are equivalent (Theorem 5.4.5). Combining this fact with earlier results, as described above, we obtain that $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed (Theorem 5.4.6).

Note for expert readers. Since we tried to make the paper as self-contained as possible, some material will be familar to expert readers. In particular, readers who are already acquainted with [**30**] and are willing to assume that the results therein transfer to the enriched setting may skip the first paper of the paper and start reading from Chapter 4. Similarly, readers who are already familiar with the universal property of the bimodule construction from [**33**] may skip the final sections of the paper (Section 5.2, 5.3 and 5.4), apart from Theorem 5.4.5.

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CHAPTER 1

Background

This chapter reviews some notions and results that will be used in the reminder of the paper. In particular, we review the basics of the theory of bicategories (Section 1.1), elements of enriched category theory (Section 1.2) and the definition of the bicategory of distributors (Section 1.3). All the material in this chapter is essentially well-known, with the possible exception of the definition of the Gabriel factorization of a homomorphism, which is the bicategorical counterpart of the factorization of a functor as an essentially surjective functor followed by a fully faithful one. Since Gabriel factorizations are for the definition of several bicategories in the remainder of the paper, we illustrate in some detail how this construction works in the discussion of the bicategory of distributors.

1.1. Review of bicategory theory

For the convenience of the reader, the definitions of bicategory, homomorphism, pseudo-natural transformation and modification are recalled in Appendix A.

For a bicategory $\mathcal{E}$, we write $\mathcal{E}[X, Y]$ for the hom-category between two objects $X, Y \in \mathcal{E}$. A bicategory $\mathcal{E}$ is said to be *locally small* when $\mathcal{E}[X, Y]$ is a small category for every $X, Y \in \mathcal{E}$. A *morphism*, or a 1-*cell* $F\colon X \to Y$ is an object of the category $\mathcal{E}[X, Y]$, and a 2-*cell* $\alpha\colon F \to F'$ is a morphism of the category $\mathcal{E}[X, Y]$. We write $1_X\colon X \to X$ for the identity morphism of an object $X \in \mathcal{E}$. The composition operation of 2-cells, i.e. the the composition operation of the hom-categories of $\mathcal{E}$, is usually referred to as the *vertical composition* in $\mathcal{E}$ and its effect on $\alpha\colon F \to F'$, $\beta\colon F' \to F''$ is written $\beta \cdot \alpha\colon F \to F''$. The identity arrow of an object $F \in \mathcal{E}[X, Y]$ is called an *identity* 2-*cell* of $\mathcal{E}$ and written $1_F\colon F \to F$. We refer to the composition operation

$$(-) \circ (-)\colon \mathcal{E}[Y, Z] \times \mathcal{E}[X, Y] \to \mathcal{E}[X, Z]$$

as the *horizontal composition* of $\mathcal{E}$. The horizontal composite of $F\colon X \to Y$ and $G\colon Y \to Z$ is denoted by $G \circ F\colon X \to Z$. The horizontal composition of $\alpha\colon F \to F'$ with $\beta\colon G \to G'$ is written $\beta \circ \alpha\colon G \circ F \to G' \circ F'$. This 2-cell is the common value of the composites in the following naturality square

$$\begin{array}{ccc} G \circ F & \xrightarrow{G \circ \alpha} & G \circ F' \\ {\scriptstyle \beta \circ F}\downarrow & & \downarrow{\scriptstyle \beta \circ F'} \\ G' \circ F & \xrightarrow[G' \circ \alpha]{} & G' \circ F' . \end{array}$$

We say that a bicategory is *strict* if its composition operation is strictly associative and if the units 1_X are strict. A strict bicategory is the same thing as a 2-category, i.e. a category enriched over the category of locally small categories and functors CAT. Both CAT and the category of small categories Cat have also the structure of a 2-category.

EXAMPLE 1.1.1. A monoidal category $\mathbb{C} = (\mathbb{C}, \otimes, I)$ can be identified with a bicategory, here denoted by $\Sigma(\mathbb{C})$, which has a single object and $\mathbb{C}$ as its hom-category. The horizontal composition of $\Sigma(\mathbb{C})$ is then given by the tensor product of $\mathbb{C}$. Every bicategory with one object is of the form $\Sigma(\mathbb{C})$ for some monoidal category $\mathbb{C}$.

Given two bicategories $\mathcal{E}$ and $\mathcal{F}$, their *cartesian product* $\mathcal{E} \times \mathcal{F}$ is the bicategory with objects

$$\mathrm{Obj}(\mathcal{E} \times \mathcal{F}) =_{\mathrm{def}} \mathrm{Obj}(\mathcal{E}) \times \mathrm{Obj}(\mathcal{F})$$

and hom-categories

$$(\mathcal{E} \times \mathcal{F})[(X,Y),(X',Y')] =_{\mathrm{def}} \mathcal{E}[X,X'] \times \mathcal{F}[Y,Y'] \,,$$

for $X, X' \in \mathcal{E}$ and $Y, Y' \in \mathcal{F}$. Composition is defined in the obvious way. For a bicategory $\mathcal{E}$, we write $\mathcal{E}^{\mathrm{op}}$ for the *opposite bicategory* of $\mathcal{E}$, which is obtained by formally reversing the direction of the morphisms of $\mathcal{E}$, but not that of the 2-cells. For a morphism $F \colon X \to Y$ in $\mathcal{E}$, we write $F^{\mathrm{op}} \colon Y \to X$ for the corresponding morphism in $\mathcal{E}^{\mathrm{op}}$.

Equivalences and adjunctions in a bicategory. A morphism $F \colon X \to Y$ in a bicategory $\mathcal{E}$ is said to be an *equivalence* if there exists a morphism $U \colon Y \to X$ together with invertible 2-cells $\alpha \colon G \circ F \to 1_X$ and $\beta \colon F \circ U \to 1_Y$. We write $X \simeq Y$ to indicate that X and Y are equivalent. An *adjunction* $(F, U, \eta, \varepsilon) \colon X \to Y$ in $\mathcal{E}$ consists of morphisms $F \colon X \to Y$ and $U \colon Y \to X$ and 2-cells $\eta \colon 1_X \to U \circ F$ and $\varepsilon \colon F \circ U \to 1_X$ satisfying the triangular laws, expressed the commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{F \circ \eta} & F \circ U \circ F \\ & \searrow^{1_F} & \downarrow{\scriptstyle \varepsilon \circ F} \\ & & F \,, \end{array} \qquad \begin{array}{ccc} U \circ F \circ U & \xleftarrow{\eta \circ U} & U \\ {\scriptstyle U \circ \varepsilon}\downarrow & \swarrow_{1_U} & \\ U . & & \end{array} \tag{1.1.1}$$

The morphism F is the *left adjoint* and the morphism U is the *right adjoint*; the 2-cell η is the *unit* of the adjunction and the 2-cell ε is the *counit*. Recall that the unit η of an adjunction $(F, U, \eta, \varepsilon)$ determines the counit ε and conversely. More precisely, $\varepsilon \colon F \circ U \to 1_Y$ is the unique 2-cell such that $(U \circ \varepsilon) \cdot (\eta \circ U) = 1_U$, and $\eta \colon 1_X \to U \circ F$ is the unique 2-cell such that $(\varepsilon \circ F) \cdot (F \circ \eta) = 1_F$. We often write $F \dashv U$ to indicate an adjunction $(F, U, \eta, \varepsilon)$. An adjunction is called a *reflection* (resp. *coreflection*) if its counit (resp. unit) is invertible. If $(F, U, \eta, \varepsilon) \colon X \to Y$ is an adjunction in $\mathcal{E}$, then $(U^{\mathrm{op}}, F^{\mathrm{op}}, \eta, \varepsilon) \colon X \to Y$ is an adjunction in the opposite bicategory $\mathcal{E}^{\mathrm{op}}$.

Homomorphisms. For bicategories $\mathcal{E}$ and $\mathcal{F}$, we write $\Phi \colon \mathcal{E} \to \mathcal{F}$ to indicate that Φ is a homomorphism from $\mathcal{E}$ to $\mathcal{F}$. The homomorphisms from $\mathcal{E}$ to $\mathcal{F}$ are the objects of a bicategory $\mathrm{HOM}\big[\mathcal{E}, \mathcal{F}\big]$ whose morphisms are pseudo-natural transformations and 2-cells are modifications. A *contravariant homomorphism* $\Phi \colon \mathcal{E} \to \mathcal{F}$ is defined to be a homomorphism $\Phi \colon \mathcal{E}^{\mathrm{op}} \to \mathcal{F}$. The canonical homomorphism

$$\mathcal{E}[-,-] \colon \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathrm{CAT}$$

takes a pair of objects (X, Y) to the category $\mathcal{E}[X, Y]$. In particular, there is a a covariant homomorphism $\mathcal{E}[K, -] \colon \mathcal{E} \to \mathrm{CAT}$ and a contravariant homomorphism $\mathcal{E}[-, K] \colon \mathcal{E} \to \mathrm{CAT}$ for each object $K \in \mathcal{E}$.

We say that a homomorphism $\Phi \colon \mathcal{E} \to \mathcal{F}$ is *full and faithful* if for every $X, Y \in \mathcal{E}$ the functor

$$\Phi_{X,Y} \colon \mathcal{E}[X, Y] \to \mathcal{F}[\Phi X, \Phi Y]$$

is an equivalence of categories. We say that $\Phi\colon \mathcal{E} \to \mathcal{F}$ is *essentially surjective* if for every object $Y \in \mathcal{F}$ there exists an object $X \in \mathcal{E}$ together with an equivalence $\Phi X \simeq Y$. We say that $\Phi\colon \mathcal{E} \to \mathcal{F}$ is an *equivalence* if it is full and faithful and essentially surjective.

The coherence theorem for bicategories, asserts that every bicategory is equivalent to a 2-category [**52**]. Thanks to this result, it is possible to treat the horizontal composition in a bicategory as if it were strictly associative and unital, which we will often do in the following. We say that a homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ is an *inclusion* if it is injective on objects and full and faithful. In this case, we will often write $\mathcal{E} \subseteq \mathcal{F}$ and treat the action of Φ on objects as if it were the identity.

A homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ takes an adjunction $(F, U, \eta, \varepsilon)\colon X \to Y$ in $\mathcal{E}$ to an adjunction $(\Phi F, \Phi U, \Phi\eta, \Phi\varepsilon)\colon \Phi X \to \Phi Y$ in $\mathcal{F}$. Dually, a contravariant homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ takes an adjunction $(F, U, \eta, \varepsilon)\colon X \to Y$ in $\mathcal{E}$ to an adjunction $(\Phi U, \Phi F, \Phi\eta, \Phi\varepsilon)\colon \Phi Y \to \Phi X$ in $\mathcal{F}$. For example, if K is an object of $\mathcal{E}$, then the homomorphism $\mathcal{E}[K, -]\colon \mathcal{E} \to \mathrm{CAT}$ takes an adjunction $(F, U)\colon X \to Y$ in $\mathcal{E}$ to an adjunction $(\mathcal{E}[K, F], \mathcal{E}[K, U])\colon \mathcal{E}[K, X] \to \mathcal{E}[K, Y]$ in CAT. Dually, the contravariant homomorphism $\mathcal{E}[-, K]\colon \mathcal{E} \to \mathrm{CAT}$ takes an adjunction $(F, U)\colon X \to Y$ in $\mathcal{E}$ to an adjunction $(\mathcal{E}[U, K], \mathcal{E}[F, K])\colon \mathcal{E}[Y, K] \to \mathcal{E}[X, K]$.

Prestacks. By a *prestack* on a (locally small) bicategory $\mathcal{E}$ we mean a contravariant homomorphism $\Phi\colon \mathcal{E} \to \mathrm{Cat}$. The bicategory of prestacks

$$\mathbf{P}(\mathcal{E}) =_{\mathrm{def}} \mathrm{HOM}\big[\mathcal{E}^{\mathrm{op}}, \mathrm{Cat}\big]\,,$$

is a 2-category, since Cat is a 2-category. The Yoneda homomorphism

$$\mathrm{y}_{\mathcal{E}}\colon \mathcal{E} \to \mathbf{P}(\mathcal{E})$$

takes an object $X \in \mathcal{E}$ to the prestack $\mathrm{y}_{\mathcal{E}}(X) =_{\mathrm{def}} \mathcal{E}[-, X]$. For $\Phi \in \mathbf{P}(\mathcal{E})$ and $X \in \mathcal{E}$, there is a functor

$$\mathbf{P}(\mathcal{E})[\mathrm{y}(X), \Phi] \to \Phi(X)$$

which takes a pseudo-natural transformation $\alpha\colon \mathrm{y}_{\mathcal{E}}(X) \to \Phi$ to the object $\alpha_X(1_X) \in \Phi(X)$, and the bicategorical Yoneda lemma asserts that this functor is an equivalence of categories. It follows that the Yoneda homomorphism is full and faithful.

We say that a prestack $\Phi : \mathcal{E} \to \mathrm{Cat}$ on a bicategory $\mathcal{E}$ is *representable* if there exists an object $X \in \mathcal{E}$ together with a pseudo-natural equivalence $\alpha\colon \mathrm{y}(X) \to \Phi$. It follows from Yoneda lemma that α is determined by the object $\alpha_0 =_{\mathrm{def}} \alpha(1_X) \in \Phi(X)$. In this case, we say that Φ is *represented* by the pair (X, α_0). Such a pair (X, α_0) is unique up to equivalence.

Gabriel factorization. In analogy with the way a functor can be factored as a fully faithful functor followed by an essentially surjective one, every homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ admits a factorization of the form

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Phi} & \mathcal{F} \\ & {\scriptstyle\Gamma}\searrow \quad \nearrow{\scriptstyle\Delta} & \\ & \mathcal{G}\,, & \end{array} \tag{1.1.2}$$

where Γ is essentially surjective and Δ is full and faithful. In fact, we may suppose that Γ is the identity on objects, in which case we have a *Gabriel factorization* of Φ. In order to obtain a Gabriel factorization, the bicategory $\mathcal{G}$ is defined as having the same objects as $\mathcal{E}$ and letting, for $X, Y \in \mathcal{E}$,

$$\mathcal{G}[X, Y] =_{\mathrm{def}} \mathcal{F}[\Phi X, \Phi Y]\,.$$

The composition law of $\mathcal{G}$ is defined via the composition law of $\mathcal{F}$ in the evident way. The homomorphism $\Gamma\colon \mathcal{E} \to \mathcal{G}$ is the identity on objects, while $\Gamma_{X,Y}\colon \mathcal{E}[X,Y] \to \mathcal{G}[X,Y]$ is defined to be $\Phi_{X,Y}\colon \mathcal{E}[X,Y] \to \mathcal{F}[\Phi X, \Phi Y]$. The homomorphism $\Delta\colon \mathcal{G} \to \mathcal{F}$ is defined on objects by letting $\Delta(X) =_{\mathrm{def}} \Phi(X)$, for $X \in \mathcal{E}$, while $\Delta_{X,Y}\colon \mathcal{G}[X,Y] \to \mathcal{F}[\Phi X, \Phi Y]$ is the identity functor. These definitions are illustrated in Section 1.3, where we show how the bicategory of distributors arises from a Gabriel factorization.

Example. Let us consider the Gabriel factorization of the Yoneda homomorphism $\mathrm{y}\colon \mathcal{E} \to \mathbf{P}(\mathcal{E})$,

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ \mathrm{y}_{\mathcal{E}}\ } & \mathbf{P}(\mathcal{E}) \\ & {\scriptstyle\Gamma}\searrow \quad \nearrow{\scriptstyle\Delta} & \\ & \mathcal{G}. & \end{array}$$

The bicategory $\mathcal{G}$ is a 2-category, since the bicategory $\mathbf{P}(\mathcal{E})$ is a 2-category, Moreover, the homomorphism Γ is an equivalence of bicategories, since the Yoneda homomorphism is full and faithful. Hence, the bicategory $\mathcal{E}$ is equivalent to a 2-category, giving a proof of the coherence theorem for bicategories [**52**].

There is a slight variation of the definitions given above which arises when we are given not only the homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ but also, for each $X, Y \in \mathcal{E}$, a category $\mathcal{G}[X,Y]$ and an equivalence

$$\Delta_{X,Y}\colon \mathcal{G}[X,Y] \to \mathcal{F}[\Phi X, \Phi Y]\,. \tag{1.1.3}$$

In this case, we obtain again a Gabriel factorization of Φ. The bicategory $\mathcal{G}$ has again the same objects as $\mathcal{E}$ and its hom-categories are given by the given categories $\mathcal{G}[X,Y]$. The composition functors of $\mathcal{G}$ are determined (up to unique isomorphism) by requiring that the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathcal{G}[Y,Z] \times \mathcal{G}[X,Y] & \xrightarrow{\ (-)\circ(-)\ } & \mathcal{G}[X,Z] \\ {\scriptstyle \Delta_{Y,Z}\times\Delta_{X,Y}}\downarrow & & \downarrow{\scriptstyle \Delta_{X,Z}} \\ \mathcal{F}[\Phi X, \Phi Z] \times \mathcal{F}[\Phi X, \Phi Y] & \xrightarrow[\ (-)\circ(-)\ ]{} & \mathcal{F}[\Phi X, \Phi Z]\,. \end{array}$$

Similarly, the identity morphism $1_X\colon X \to X$ on an object $X \in \mathcal{G}$, is determined (up to unique isomorphism) by requiring that there is an isomorphism $\Delta(1_X) \cong 1_{\Phi X}$. These associativity and unit isomorphisms can be defined in a similar way. The definition of the required homomorphism $\Delta\colon \mathcal{G} \to \mathcal{F}$ now follows easily. Its action on objects is given by mapping $X \in \mathcal{G}$ to $\Phi X \in \mathcal{F}$ and its action on hom-categories is given by the equivalences in (1.1.3), so that Δ is full and faithful by construction. The homomorphism $\Delta\colon \mathcal{E} \to \mathcal{G}$ can then be defined as the identity on objects, while its action on hom-categories is essentially determined by requiring that the diagram in (1.1.2) commutes up to pseudo-natural equivalence. We will illustrate this method of constructing bicategories in Section 1.3 and apply it again in Section 2.2 and Section 2.4.

Adjunctions between bicategories. If $\mathcal{E}$ and $\mathcal{F}$ are bicategories, then an *adjunction* (sometimes also referred to as a *biadjunction*) $\theta\colon \Phi \dashv \Psi$ between two homomorphisms $\Phi\colon \mathcal{E} \to \mathcal{F}$ and $\Psi\colon \mathcal{F} \to \mathcal{E}$ is defined to be a pseudo-natural equivalence

$$\theta\colon \mathcal{E}[X, \Psi Y] \simeq \mathcal{F}[\Phi X, Y]\,.$$

The homomorphism Φ is said to be the *left adjoint* and the homomorphism Ψ to be the *right adjoint*. A homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ has a right adjoint if and only if the prestack

$$\mathcal{E}[\Phi(-), Y]\colon \mathcal{E} \to \mathrm{Cat}$$

is representable for every object $Y \in \mathcal{F}$. The *counit* of the adjunction is a pseudo-natural transformation $\varepsilon\colon \Phi \circ \Psi \to \mathrm{Id}_{\mathcal{F}}$ defined by letting $\varepsilon_Y =_{\mathrm{def}} \theta(1_{\Psi Y})$ for $Y \in \mathcal{F}$. The *unit* of the adjunction is a pseudo-natural transformation $\eta\colon \mathrm{Id}_{\mathcal{E}} \to \Psi \circ \Phi$ defined by letting $\eta_X =_{\mathrm{def}} \theta^{-1}(1_{\Phi X})$ for $X \in \mathcal{E}$, where θ^{-1} is a quasi-inverse of θ. Either of the pseudo-natural transformations η and ε determine the adjunction θ.

Cartesian, cocartesian and cartesian closed bicategories. We recall the notion of cartesian bicategory. We say that an object $\top$ in a bicategory $\mathcal{E}$ is *terminal* if the category $\mathcal{E}[C, \top]$ is equivalent to the terminal category for every object $C \in \mathcal{E}$. A terminal object $\top \in \mathcal{E}$ is unique up to equivalence when it exists. Given two objects $X_1, X_2 \in \mathcal{E}$, we say that an object $X \in \mathcal{E}$ equipped with two morphisms $\pi_1\colon X \to X_1$ and $\pi_2\colon X \to X_2$ is the *cartesian product* of X_1 and X_2 if the functor

$$\pi\colon \mathcal{E}[C, X] \to \mathcal{E}[C, X_1] \times \mathcal{E}[C, X_2]\,,$$

defined by letting $\pi(F) =_{\mathrm{def}} (\pi_1 \circ F, \pi_2 \circ F)$ is an equivalence of categories for every object $C \in \mathcal{E}$. The cartesian product of the objects X_1 and X_2 is unique up to equivalence when it exists. In this case, we will denote it by $X_1 \sqcap X_2$ and refer to the morphisms $\pi_k\colon X_1 \sqcap X_2 \to X_k$ $(k = 1, 2)$ as the *projections*. When every pair of objects in $\mathcal{E}$ has a cartesian product, then the diagonal homomorphism $\Delta_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ has a right adjoint,

$$(-) \sqcap (-)\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}\,,$$

which associates to (X_1, X_2) the cartesian product $X_1 \sqcap X_2$. We say that a bicategory $\mathcal{E}$ with a terminal object is *cartesian* if every pair of objects in $\mathcal{E}$ has a cartesian product. Dually, we say that a bicategory $\mathcal{E}$ is *cocartesian* if the opposite bicategory $\mathcal{E}^{\mathrm{op}}$ is cartesian. We write $\perp$ for the initial object and $X_1 \sqcup X_2$ for the coproduct of two objects X_1 and X_2, and refer to the morphisms $\iota_k\colon X_k \to X_1 \sqcup X_2$ $(k = 1, 2)$ as the *inclusions*.

We recall the notion of cartesian closed bicategory. Given objects $X\,, Y$ of a cartesian bicategory $\mathcal{E}$, we will say that an object $E \in \mathcal{E}$ equipped with a morphism $\mathrm{ev}\colon E \sqcap X \to Y$ is the *exponential* of Y by X if the functor

$$\mathcal{E}[K, E] \xrightarrow{(-)\sqcap X} \mathcal{E}[K \sqcap X, E \sqcap X] \xrightarrow{\mathcal{E}[K\sqcap X, \mathrm{ev}]} \mathcal{E}[K \sqcap X, Y]$$

is an equivalence of categories for every object $K \in \mathcal{E}$. This condition means that the prestack

$$\mathcal{E}[(-) \sqcap X, Y]\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat}$$

is represented by the pair (E, ev). The exponential of Y by X is unique up to equivalence when it exists and we denote it by Y^X or $[X, Y]$ and refer to the morphism

$$\mathrm{ev}\colon Y^X \sqcap X \to Y$$

as the *evaluation*. We say that a cartesian bicategory $\mathcal{E}$ is *closed* if the exponential Y^X exists for every $X, Y \in \mathcal{E}$. A cartesian bicategory $\mathcal{E}$ is closed if and only if, for every object $X \in \mathcal{E}$, the homomorphism $(-) \sqcap X\colon \mathcal{E} \to \mathcal{E}$ has a right adjoint $(-)^X\colon \mathcal{E} \to \mathcal{E}$. The resulting homomorphism mapping (X, Y) to Y^X is contravariant in the first variable and covariant in the second.

Monoidal bicategories. A cartesian bicategory is an example of a symmetric monoidal bicategory, a notion that we limit ourselves to review in outline. First, recall that by definition, a monoidal bicategory is a tricategory with one object [**36, 37**]. We will not describe this notion here because of its complexity (see [**20, 36, 37, 68**] for details). It will suffice to say that a monoidal structure on bicategory $\mathcal{E}$ is a 9-tuple

$$(\otimes, \mathbb{I}, \alpha^1, \alpha^2, \lambda^1, \lambda^2, \rho^1, \rho^2, \mu)\,,$$

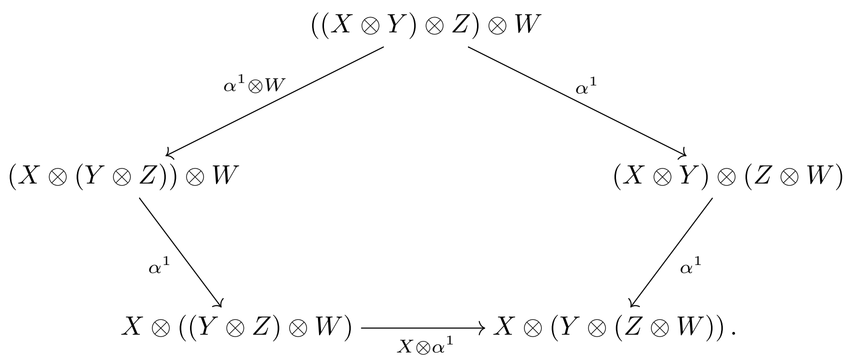
where the tensor product

$$(-) \otimes (-) \colon \mathrm{Bim}(\mathcal{E}) \times \mathrm{Bim}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{E})$$

is a homomorphism, α^1, λ^1 and ρ^1 are pseudo natural (adjoint) equivalences and α^2, λ^2, ρ^2 and μ are invertible modifications. More precisely,

$$\alpha^1(X, Y, Z) \colon (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$$

is the 1-associativity constraint and the 2-associativity constraint $\alpha^2(X, Y, Z, W)$ is a 2-cell fitting in the pentagon



The associativity constraints satisfy coherence conditions that we omit. We also omit the coherence conditions for the unit object $\mathbb{I}$ and its constraints $(\lambda^1, \rho^1, \lambda^2, \rho^2, \mu)$. A *symmetry structure* on a monoidal bicategory as above is a pseudo-natural (adjoint) equivalence

$$\sigma^1_{X,Y} \colon X \otimes Y \simeq Y \otimes X$$

together with certain higher dimensional constraints [**38**].

For further information on the theory of bicategories, we invite the reader to refer to [**17**, Volume I, Chapter 7] and [**8, 49, 70**].

1.2. $\mathcal{V}$-categories and presentable $\mathcal{V}$-categories

Since we will focus on enriched categories and enriched operads, it is convenient to recall some aspects of enriched category theory from [**44**]. Let $\mathcal{V} = (\mathcal{V}, \otimes, I, [-,-])$ be a locally presentable symmetric monoidal closed category, which we shall consider fixed throughout this paper. Examples of such a category include the category of sets, the category of pointed simplicial sets (with the smash product as tensor product), categories of chain complexes of vector spaces over a field, and the category of spectra.

If $\mathbb{X}$ is a small $\mathcal{V}$-category, we write $\mathbb{X}[x, y]$ or simply $[x, y]$ for the hom-object between two objects $x, y \in \mathbb{X}$. We write $\mathrm{Cat}_{\mathcal{V}}$ (resp. $\mathrm{CAT}_{\mathcal{V}}$) for the 2-category of small (resp. locally small)

$\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. The category $\mathrm{Cat}_{\mathcal{V}}$ is complete and cocomplete. In particular, its terminal object is the $\mathcal{V}$-category I defined by letting $\mathrm{Obj}(\mathrm{I}) =_{\mathrm{def}} \{*\}$ and $\mathrm{I}[*,*] =_{\mathrm{def}} \top$, where $\top$ is the terminal object of $\mathcal{V}$. The terminal object of $\mathrm{Cat}_{\mathcal{V}}$ is the $\mathcal{V}$-category 1 defined by letting $\mathrm{Obj}(1) =_{\mathrm{def}} \{*\}$ and $1[*,*] =_{\mathrm{def}} \top$, where $\top$ is the terminal object of $\mathcal{V}$. The category $\mathrm{Cat}_{\mathcal{V}}$ has also a symmetric monoidal closed structure. We write $\mathbb{X} \otimes \mathbb{Y}$ for the tensor product two small $\mathcal{V}$-categories $\mathbb{X}$ and $\mathbb{Y}$. This is defined by letting $\mathrm{Obj}(\mathbb{X} \otimes \mathbb{Y}) =_{\mathrm{def}} \mathrm{Obj}(\mathbb{X}) \times \mathrm{Obj}(\mathbb{Y})$ and

$$(\mathbb{X} \otimes \mathbb{Y})\big[(x,y),(x',y')\big] =_{\mathrm{def}} \mathbb{X}[x,x'] \otimes \mathbb{Y}[y,y'] \, .$$

Sometimes we write $x \otimes y \in \mathbb{X} \otimes \mathbb{Y}$ instead of $(x,y) \in \mathbb{X} \otimes \mathbb{Y}$. The unit object for this monoidal structure is the $\mathcal{V}$-category I defined by letting $\mathrm{Obj}(\mathrm{I}) =_{\mathrm{def}} \{*\}$ and $\mathrm{I}[*,*] =_{\mathrm{def}} I$. The hom-object $[\mathbb{X}, \mathbb{Y}]$ is the $\mathcal{V}$-category of $\mathcal{V}$-functors from $\mathbb{X}$ to $\mathbb{Y}$ and $\mathcal{V}$-natural transformations. For $\mathcal{V}$-categories $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$, a *$\mathcal{V}$-functor of two variables* $F \colon \mathbb{X} \times \mathbb{Y} \to \mathbb{Z}$ is a $\mathcal{V}$-functor $F \colon \mathbb{X} \otimes \mathbb{Y} \to \mathbb{Z}$.

The next definition recalls the notion of a (locally) presentable $\mathcal{V}$-category, with which we will work throughout the paper. The reader is invited to refer to [**45**] for further information about it.

Definition 1.2.1. We say that a $\mathcal{V}$-category $\mathcal{E}$ is (locally) *presentable* if it is $\mathcal{V}$-cocomplete and its underlying ordinary category is (locally) presentable in the usual sense.

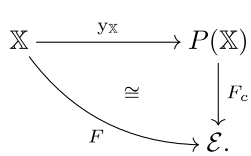
We write $\mathrm{PCAT}_{\mathcal{V}}$ for the 2-category of presentable $\mathcal{V}$-categories, cocontinuous $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. For example, the $\mathcal{V}$-category $P(\mathbb{X}) =_{\mathrm{def}} [\mathbb{X}^{\mathrm{op}}, \mathcal{V}]$ of presheaves on a small $\mathcal{V}$-category $\mathbb{X}$ is presentable. In particular, the terminal $\mathcal{V}$-category $1 \simeq P(0)$ is presentable, where 0 is the $\mathcal{V}$-category with no objects. For a small $\mathcal{V}$-category $\mathbb{X}$, we write $\mathrm{y}_{\mathbb{X}} \colon \mathbb{X} \to P(\mathbb{X})$ for the Yoneda $\mathcal{V}$-functor, which is defined by mapping $x \in \mathbb{X}$ to $\mathrm{y}_{\mathbb{X}}(x) =_{\mathrm{def}} \mathbb{X}[-,x]$. By the enriched version of the Yoneda lemma, there is an isomorphism

$$P(\mathbb{X})[\mathrm{y}_{\mathbb{X}}(x), A] \cong A(x) \, ,$$

for every $A \in P(\mathbb{X})$ and $x \in \mathbb{X}$. It follows that $\mathrm{y}_{\mathbb{X}}$ is full and faithful; we will often regard it as an inclusion by writing x instead of $\mathrm{y}_{\mathbb{X}}(x)$. If $\mathbb{X}$ is a small $\mathcal{V}$-category, then the $\mathcal{V}$-category $P(\mathbb{X})$ is cocomplete and freely generated by $\mathbb{X}$. More precisely, the Yoneda functor $\mathrm{y}_{\mathbb{X}} \colon \mathbb{X} \to P(\mathbb{X})$ exhibits $P(\mathbb{X})$ as the free cocompletion of $\mathbb{X}$. This means that if $\mathcal{E}$ is a cocomplete $\mathcal{V}$-category, and $\mathrm{CCAT}_{\mathcal{V}}[P(\mathbb{X}), \mathcal{E}]$ denotes the (large, locally small) $\mathcal{V}$-category of cocontinuous $\mathcal{V}$-functors from $P(\mathbb{X})$ to $\mathcal{E}$, then the restriction functor

$$\mathrm{y}_{\mathbb{X}}^* \colon \mathrm{CCAT}_{\mathcal{V}}[P(\mathbb{X}), \mathcal{E}] \to [\mathbb{X}, \mathcal{E}] \, , \tag{1.2.1}$$

defined by letting $\mathrm{y}_{\mathbb{X}}^*(F) =_{\mathrm{def}} F \circ \mathrm{y}_{\mathbb{X}}$, is an equivalence of $\mathcal{V}$-categories. In particular, every $\mathcal{V}$-functor $F \colon \mathbb{X} \to \mathcal{E}$ admits a cocontinuous extension $F_c \colon P(\mathbb{X}) \to \mathcal{E}$ which is unique up to a unique $\mathcal{V}$-natural isomorphism,



The $\mathcal{V}$-functor F_c is the left Kan extension of F along $\mathrm{y}_{\mathbb{X}}$ and its action on $A \in P(\mathbb{X})$ is given by the coend formula

$$F_c(A) =_{\mathrm{def}} \int^{x \in \mathbb{X}} F(x) \otimes A(x) \, . \tag{1.2.2}$$

The $\mathcal{V}$-functor F_c is left adjoint to the "singular $\mathcal{V}$-functor" $F^s \colon \mathcal{E} \to P(\mathbb{X})$, given by letting

$$F^s(y)(x) =_{\text{def}} \mathcal{E}[F(x), Y]\,,$$

for $Y \in \mathcal{E}$ and $x \in \mathbb{X}$. We write

$$P \colon \text{Cat}_{\mathcal{V}} \to \text{PCAT}_{\mathcal{V}} \tag{1.2.3}$$

for the homomorphism which takes a small $\mathcal{V}$-category $\mathbb{X}$ to $P(\mathbb{X})$. If $u \colon \mathbb{X} \to \mathbb{Y}$ is a $\mathcal{V}$-functor between small $\mathcal{V}$-categories, we define $P(u) =_{\text{def}} u_! \colon P(\mathbb{X}) \to P(\mathbb{Y})$, i.e. as the cocontinuous extension of $\mathrm{y}_{\mathbb{Y}} \circ u \colon \mathbb{X} \to P(\mathbb{Y})$. Hence, the diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\mathrm{y}_{\mathbb{X}}} & P(\mathbb{X}) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u_!} \\ \mathbb{Y} & \xrightarrow[\mathrm{y}_{\mathbb{Y}}]{} & P(\mathbb{Y}) \end{array}$$

commutes up to a canonical isomorphism and, for every $A \in P(\mathbb{X})$ and $y \in \mathbb{Y}$, we have

$$u_!(A)(y) =_{\text{def}} \int^{x \in \mathbb{X}} \mathbb{X}[y, u(x)] \otimes A(x)\,. \tag{1.2.4}$$

The functor $u_! \colon P(\mathbb{X}) \to P(\mathbb{Y})$ has a right adjoint $u^* \colon P(\mathbb{Y}) \to P(\mathbb{X})$ defined by letting

$$u^*(B)(x) =_{\text{def}} B(u(x))\,, \tag{1.2.5}$$

for $B \in P(\mathbb{Y})$ and $x \in \mathbb{X}$.

If $\mathcal{E}$ is a presentable $\mathcal{V}$-category and $\mathcal{C}$ is a cocomplete $\mathcal{V}$-category, then any cocontinuous $\mathcal{V}$-functor from $\mathcal{E}$ to $\mathcal{C}$ has a right adjoint. Because of this, products and coproducts in $\text{PCAT}_{\mathcal{V}}$ are intimately related, as we now recall. First of all, the cartesian product $\mathcal{E} = \prod_{k \in K} \mathcal{E}_k$ of a family of presentable $\mathcal{V}$-categories $(\mathcal{E}_k \mid k \in K)$ is presentable. Each projection $\pi_k \colon \mathcal{E}_k \to \mathcal{E}$ has a left adjoint $\iota_k \colon \mathcal{E} \to \mathcal{E}_k$ and the family $(\iota_k \mid k \in K)$ is a coproduct diagram in $\text{PCAT}_{\mathcal{V}}$. In particular, the terminal $\mathcal{V}$-category 1 is both initial and terminal in the bicategory $\text{PCAT}_{\mathcal{V}}$.

Lemma 1.2.2. *The homomorphism $P \colon \text{Cat}_{\mathcal{V}} \to \text{PCAT}_{\mathcal{V}}$ preserves coproducts.*

Proof. This follows from the universal property of $P(\mathbb{X})$. Indeed, let us consider a family of small $\mathcal{V}$-categories $(\mathbb{X}_k \mid k \in K)$ and let $\mathbb{X} = \bigsqcup_{k \in K} \mathbb{X}_k$ and let $\iota_k \colon \mathbb{X}_k \to \mathbb{X}$ be the inclusion for $k \in K$. We prove that the family of maps $(\iota_k)_! \colon P(\mathbb{X}_k) \to P(\mathbb{X})$ $(k \in K)$ is a coproduct diagram in $\text{PCAT}_{\mathcal{V}}$. For this, it suffices to show for every presentable $\mathcal{V}$-category $\mathcal{E}$, the family of functors

$$\text{PCAT}_{\mathcal{V}}[(\iota_k)_!, \mathcal{E}] \colon \text{PCAT}_{\mathcal{V}}[P(\mathbb{X}), \mathcal{E}] \to \text{PCAT}_{\mathcal{V}}[P(\mathbb{X}_k), \mathcal{E}] \qquad (k \in K)$$

is a product diagram in the 2-category CAT. But this family is equivalent to the family of functors

$$[\iota_k, \mathcal{E}] \colon [\mathbb{X}, \mathcal{E}] \to [\mathbb{X}_k, \mathcal{E}] \qquad (k \in K)$$

by the equivalence in (1.2.1). This proves the result, since the family $\iota_k \colon \mathbb{X}_k \to \mathbb{X}$, for $k \in K$, is a coproduct diagram also in the 2-category of locally small $\mathcal{V}$-categories. $\square$

Remark 1.2.3. If $\mathcal{E}$ and $\mathcal{F}$ are presentable $\mathcal{V}$-categories, then so is the $\mathcal{V}$-category $\text{CCAT}_{\mathcal{V}}[\mathcal{E}, \mathcal{F}]$ of cocontinuous $\mathcal{V}$-functors from $\mathcal{E}$ to $\mathcal{F}$. This defines the hom-object of a symmetric monoidal closed structure on the 2-category $\text{PCAT}_{\mathcal{V}}$. By definition, the *completed tensor product* $\mathcal{E} \mathbin{\widehat{\otimes}} \mathcal{F}$ of two presentable $\mathcal{V}$-categories $\mathcal{E}$ and $\mathcal{F}$ is a presentable $\mathcal{V}$-category equipped with a $\mathcal{V}$-functor in two variables from $\mathcal{E} \times \mathcal{F}$ to $\mathcal{E} \mathbin{\widehat{\otimes}} \mathcal{F}$ that is $\mathcal{V}$-cocontinuous in each variable and universal with respect to that property. The unit object for the completed tensor product is $\mathcal{V}$. If we consider the

2-categories $\mathrm{Cat}_{\mathcal{V}}$ and $\mathrm{PCAT}_{\mathcal{V}}$ as equipped with the symmetric monoidal structures, the homomorphism $P \colon \mathrm{Cat}_{\mathcal{V}} \to \mathrm{PCAT}_{\mathcal{V}}$ is symmetric monoidal. Indeed, for $\mathbb{X}\,, \mathbb{Y} \in \mathrm{Cat}_{\mathcal{V}}$, we have a $\mathcal{V}$-functor of two variables

$$\phi_{\mathbb{X},\mathbb{Y}} \colon P(\mathbb{X}) \times P(\mathbb{Y}) \to P(\mathbb{X} \otimes \mathbb{Y})$$

defined by letting $\phi_{\mathbb{X},\mathbb{Y}}(F, G)(x \otimes y) =_{\mathrm{def}} F(x) \otimes G(y)$. This $\mathcal{V}$-functor exhibits $P(\mathbb{X} \otimes \mathbb{Y})$ as the completed tensor product of $P(\mathbb{X})$ and $P(\mathbb{Y})$ and so we have an equivalence

$$P(\mathbb{X}) \mathbin{\widehat{\otimes}} P(\mathbb{Y}) \simeq P(\mathbb{X} \otimes \mathbb{Y})\,.$$

We conclude this section with a straightforward observation, which we state explicitly for future reference. Recall that if $\mathbb{X}$ is a small $\mathcal{V}$-category and $\mathcal{E}$ is a locally small $\mathcal{V}$-category, then the $\mathcal{V}$-category $[\mathbb{X}, \mathcal{E}]$ of $\mathcal{V}$-functors from $\mathbb{X}$ to $\mathcal{E}$ is locally small.

Proposition 1.2.4. *Let $\mathbb{X}, \mathbb{Y}$ be a small $\mathcal{V}$-categories and $\mathcal{E}$ be a locally small $\mathcal{V}$-category. The $\mathcal{V}$-functors*

$$\lambda^{\mathbb{Y}} \colon [\mathbb{X} \otimes \mathbb{Y}, \mathcal{E}] \to [\mathbb{X}, [\mathbb{Y}, \mathcal{E}]]\,, \quad \lambda^{\mathbb{X}} \colon [\mathbb{X} \otimes \mathbb{Y}, \mathcal{E}] \to [\mathbb{Y}, [\mathbb{X}, \mathcal{E}]]$$

defined by letting

$$(\lambda^{\mathbb{Y}} F)(x)(y) =_{\mathrm{def}} F(x, y)\,, \quad (\lambda^{\mathbb{X}} F)(y)(x) =_{\mathrm{def}} F(x, y)\,,$$

for $F \colon \mathbb{X} \otimes \mathbb{Y} \to \mathcal{E}$, $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, are equivalences of $\mathcal{V}$-categories. □

1.3. Distributors

Let us recall the notion of a distributor (sometimes called bimodule or profunctor in the literature) [**9, 53**] and some basic facts about it. In particular, we show how the bicategory of distributors fits into a Gabriel factorization.

Definition 1.3.1. Let $\mathbb{X}, \mathbb{Y} \in \mathrm{Cat}_{\mathcal{V}}$. A *distributor* $F \colon \mathbb{X} \to \mathbb{Y}$ is a $\mathcal{V}$-functor $F \colon \mathbb{Y}^{\mathrm{op}} \otimes \mathbb{X} \to \mathcal{V}$.

For a distributor $F \colon \mathbb{X} \to \mathbb{Y}$, we write $F[y, x]$ for the result of applying F to $(y, x) \in \mathbb{Y}^{\mathrm{op}} \otimes \mathbb{X}$. Small $\mathcal{V}$-categories, distributors and $\mathcal{V}$-transformations form a bicategory, called the bicategory of distributors and denoted by $\mathrm{Dist}_{\mathcal{V}}$, in which the hom-category of morphisms between two small $\mathcal{V}$-categories $\mathbb{X}$ and $\mathbb{Y}$ is defined by letting

$$\mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{Y}] =_{\mathrm{def}} [\mathbb{Y}^{\mathrm{op}} \otimes \mathbb{X}, \mathcal{V}]\,.$$

The bicategory $\mathrm{Dist}_{\mathcal{V}}$ fits into a Gabriel factorization of the form

$$\begin{array}{ccc} \mathrm{Cat}_{\mathcal{V}} & \xrightarrow{\quad P \quad} & \mathrm{PCAT}_{\mathcal{V}} \\ & {\scriptstyle (-)_{\bullet}} \searrow \quad \nearrow {\scriptstyle (-)^{\dagger}} & \\ & \mathrm{Dist}_{\mathcal{V}}. & \end{array} \tag{1.3.1}$$

The Gabriel factorization essentially determines the composition operation and the unit morphisms of the bicategory $\mathrm{Dist}_{\mathcal{V}}$ and provides a proof that they satisfy the appropriate coherence conditions. We illustrate this fact since we will use the same method to define other bicategories in Chapter 2 and Chapter 3. First of all, for small $\mathcal{V}$-categories $\mathbb{X}, \mathbb{Y}$, we define

$$(-)^{\dagger} \colon \mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{Y}] \to \mathrm{PCAT}_{\mathcal{V}}[P(\mathbb{X}), P(\mathbb{Y})] \tag{1.3.2}$$

to be the composite of first equivalence in Proposition 1.2.4 and the quasi-inverse of the equivalence in (1.2.1):

$$\mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{Y}] \xrightarrow{\ \lambda\ } \mathrm{CAT}_{\mathcal{V}}[\mathbb{X}, P(\mathbb{Y})] \xrightarrow{(-)_{\mathrm{c}}} \mathrm{PCAT}_{\mathcal{V}}[P(\mathbb{X}), P(\mathbb{Y})]\,.$$

It is convenient to express the effect of this functor with the following notation:

$$\cfrac{\cfrac{F\colon \mathbb{X} \to \mathbb{Y}}{\lambda F\colon \mathbb{X} \to P(\mathbb{Y})}}{F^{\dagger}\colon P(\mathbb{X}) \to P(\mathbb{Y})\,.}$$

Here and in the rest of the paper, a horizontal line means that the data above it allows us to define the data data below it. Explicitly, for $A \in P(\mathbb{X})$, we have

$$\begin{aligned} F^{\dagger}(A)(y) &= (\lambda F)_c(A)(y) \\ &= \int^{x \in \mathbb{X}} (\lambda F)(x)(y) \otimes A(x) \\ &= \int^{x \in \mathbb{X}} F[y,x] \otimes A(x)\,. \end{aligned}$$

The functor in (1.3.2) is an equivalence of categories, since it is the composite of equivalences. Because of this, the composite $G \circ F$ of two distributors $F\colon \mathbb{X} \to \mathbb{Y}$ and $G\colon \mathbb{Y} \to \mathbb{Z}$ is determined up to unique isomorphism by the requirement that there is an isomorphism

$$\varphi_{F.G}\colon (G \circ F)^{\dagger} \to G^{\dagger} \circ F^{\dagger}\,. \tag{1.3.3}$$

Thus, $G \circ F\colon \mathbb{X} \to \mathbb{Z}$ can be defined by letting

$$(G \circ F)[z,x] =_{\mathrm{def}} \int^{y \in \mathbb{Y}} G[z,y] \otimes F[y,x]\,. \tag{1.3.4}$$

Similarly, the identity distributor $\mathrm{Id}_{\mathbb{X}}\colon \mathbb{X} \to \mathbb{X}$ is determined up to unique isomorphism by the requirement that there is an isomorphism

$$\varphi_{\mathbb{X}}\colon (\mathrm{Id}_{\mathbb{X}})^{\dagger} \to 1_{P(\mathbb{X})}\,. \tag{1.3.5}$$

Thus, $\mathrm{Id}_{\mathbb{X}}\colon \mathbb{X} \to \mathbb{X}$ can be defined by letting

$$\mathrm{Id}_{\mathbb{X}}[x,y] =_{\mathrm{def}} \mathbb{X}[x,y]\,.$$

Using the same reasoning, it is possible to show that horizontal composition of distributors is functorial and associative up to coherent isomorphism, and that the identity morphisms provide two-sided units for this operation up to coherent isomorphism. For example, for distributors $F\colon \mathbb{X} \to \mathbb{Y}$, $G\colon \mathbb{Y} \to \mathbb{Z}$, $H\colon \mathbb{Z} \to \mathbb{W}$, the associativity isomorphism

$$\alpha_{F,G,H}\colon (H \circ G) \circ F \to H \circ (G \circ F)$$

can be defined as the unique 2-cell such that the following diagram commutes (where we omit subscripts of the 2-cells to improve readability):

$$\begin{array}{ccc} \big((H \circ G) \circ F\big)^{\dagger} & \xrightarrow{\alpha^{\dagger}} & \big(H \circ (G \circ F)\big)^{\dagger} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (H \circ G)^{\dagger} \circ F^{\dagger} & & H^{\dagger} \circ (G \circ F)^{\dagger} \\ {\scriptstyle \varphi \circ F^{\dagger}}\downarrow & & \downarrow{\scriptstyle H^{\dagger} \circ \varphi} \\ (H^{\dagger} \circ G^{\dagger}) \circ F^{\dagger} & = & H^{\dagger} \circ (G^{\dagger} \circ F^{\dagger})\,. \end{array} \tag{1.3.6}$$

It follows that we can define a homomorphism $(-)^\dagger \colon \mathrm{Dist}_\mathcal{V} \to \mathrm{PCAT}_\mathcal{V}$ by letting

$$\mathbb{X}^\dagger =_{\mathrm{def}} P(\mathbb{X})$$

and taking its action on morphisms and 2-cells be defined by the functor in (1.3.2). The required isomorphisms expressing pseudo-functoriality are then given by the 2-cells in (1.3.3) and (1.3.5), which satisfy the required coherence conditions by the definition of the associativity and unit isomorphisms in $\mathrm{Dist}_\mathcal{V}$, as done above. For example, the diagram in (1.3.6) states exactly one the coherence conditions. Furthermore, by construction, the homomorphism $(-)^\dagger \colon \mathrm{Dist}_\mathcal{V} \to \mathrm{PCAT}_\mathcal{V}$ is full and faithful, as required from the second part of a Gabriel factorization.

We now define the homomorphism $(-)_\bullet \colon \mathrm{Cat}_\mathcal{V} \to \mathrm{Dist}_\mathcal{V}$ which provides the first part of the Gabriel factorization in (1.3.1). Its action on objects is the identity. Furthermore, the requirement that the diagram in (1.3.1) commutes up to pseudo-natural isomorphism forces us to send a $\mathcal{V}$-functor $u \colon \mathbb{X} \to \mathbb{Y}$ to a distributor $u_\bullet \colon \mathbb{X} \to \mathbb{Y}$ such that $u_! \cong (u_\bullet)^\dagger$. Such a distributor, which is unique up to unique isomorphism, is defined by letting, for $x \in \mathbb{X}$, $y \in \mathbb{Y}$,

$$u_\bullet[y, x] =_{\mathrm{def}} \mathbb{Y}[y, u(x)]\,.$$

Indeed, for $A \in P(\mathbb{X})$ and $y \in \mathbb{Y}$, we have

$$\begin{aligned} (u_\bullet)^\dagger(A)(y) &= \int^{x\in\mathbb{X}} \mathbb{Y}[y, u(x)] \otimes A(x) \\ &= u_!(A)(y)\,. \end{aligned}$$

The distributor $u_\bullet \colon \mathbb{X} \to \mathbb{Y}$ has a right adjoint $u^\bullet \colon \mathbb{Y} \to \mathbb{X}$, which is defined by letting for $x \in \mathbb{X}$, $y \in \mathbb{Y}$,

$$u^\bullet[x, y] =_{\mathrm{def}} \mathbb{Y}[u(x), y]\,.$$

Indeed, we have that $u^* \cong (u^\bullet)^\dagger$, since for $B \in P(\mathbb{Y})$ and $x \in \mathbb{X}$, we have

$$\begin{aligned} (u^\bullet)^\dagger(B)(x) &= \int^{y\in\mathbb{Y}} \mathbb{Y}[u(x), y] \otimes B(y) \\ &\cong B(u(x)) \\ &= u^*(B)(x)\,. \end{aligned}$$

Since there is an adjunction $u_! \dashv u^*$, we also have an adjunction $u_\bullet \dashv u^\bullet$. The components of its unit $\eta \colon \mathrm{Id}_\mathbb{X} \to u^\bullet \circ u_\bullet$ are the maps $u_{x,x'} \colon \mathbb{X}[x, x'] \to \mathbb{Y}[u(x), u(x')]$ given by u. The components of the counit $\varepsilon \colon u_\bullet \circ u^\bullet \to \mathrm{Id}_\mathbb{Y}$ are the canonical maps

$$\varepsilon_{y,y'} \colon \int^{x\in\mathbb{X}} \mathbb{Y}[y, u(x)] \otimes \mathbb{Y}[u(x), y'] \to \mathbb{Y}[y, y']\,.$$

For $u \colon \mathbb{X} \to \mathbb{Y}\,, v \colon \mathbb{X} \to \mathbb{Y}$ we have $(v \circ u)_! \cong v_! \circ u_!$, and for $\mathbb{X} \in \mathrm{Cat}_\mathcal{V}$, we have $(1_\mathbb{X})_! \cong 1_{P(\mathbb{X})}$. Therefore, there are canonical isomorphisms

$$(v \circ u)_\bullet \cong v_\bullet \circ u_\bullet\,, \quad (1_\mathbb{X})_\bullet \cong \mathrm{Id}_\mathbb{X}\,,$$

which necessarily satisfy the coherence conditions for a homomorphism $(-)_\bullet \colon \mathrm{Cat}_\mathcal{V} \to \mathrm{Dist}_\mathcal{V}$.

Part (i) of the next lemma will be used to prove Theorem 3.1.1, while part (ii) will be used in the proof of Proposition 3.2.6

LEMMA 1.3.2. *Let $F\colon \mathbb{X} \to \mathbb{Y}$ be a distributor.*

(i) For all $\mathcal{V}$-functors $u : \mathbb{X}' \to \mathbb{X}$, $\lambda(F \circ u_\bullet) \cong \lambda(F) \circ u$.
(ii) For all $\mathcal{V}$-functors $u : \mathbb{X}' \to \mathbb{X}$ and $v : \mathbb{Y}' \to \mathbb{Y}$, $(v^\bullet \circ F \circ u_\bullet)[y', x'] \cong F[v(y'), u(x')]$.

PROOF. For (i), let $x' \in \mathbb{X}$. Then

$$\begin{aligned} \lambda(F \circ u_\bullet)(x')(y) &= (F \circ u_\bullet)[y, x'] \\ &= \int^{x \in \mathbb{X}} F[y, x] \otimes u_\bullet[x, x'] \\ &= \int^{x \in \mathbb{X}} F[y, x] \otimes \mathbb{X}'[x, u(x')] \\ &\cong F[y, u(x')] \\ &= (\lambda(F) \circ u)(x')(y)\,. \end{aligned}$$

□

For (ii), let $x' \in \mathbb{X}'$ and $y' \in \mathbb{Y}'$. Then

$$\begin{aligned} (v^\bullet \circ F \circ u_\bullet)[y', x'] &\cong \int^{x \in \mathbb{X}} \int^{y \in \mathbb{Y}} v^\bullet[y', y] \otimes F[y, x] \otimes u_\bullet[x, x'] \\ &= \int^{x \in \mathbb{X}} \int^{y \in \mathbb{Y}} \mathbb{Y}[v(y'), y] \otimes F(y, x) \otimes \mathbb{X}[x, u(x')] \\ &\cong F[v(y'), u(x')]\,. \end{aligned}$$

PROPOSITION 1.3.3. *The bicategory* $\mathrm{Dist}_\mathcal{V}$ *has coproducts and the homomorphisms*

$$(-)_\bullet : \mathrm{Cat}_\mathcal{V} \to \mathrm{Dist}_\mathcal{V}\,, \quad (-)^\dagger : \mathrm{Dist}_\mathcal{V} \to \mathrm{PCAT}_\mathcal{V}$$

preserve coproducts.

PROOF. This follows from Lemma 1.2.2 and the fact that $\mathrm{Dist}_\mathcal{V}$ fits into a Gabriel factorization. □

REMARK 1.3.4. Although we will not need it in the following, let us recall that the symmetric monoidal structure on $\mathrm{Cat}_\mathcal{V}$ extends to a symmetric monoidal structure on $\mathrm{Dist}_\mathcal{V}$, defined in the same way on objects. The tensor product $F_1 \otimes F_2 \colon \mathbb{X}_1 \otimes \mathbb{X}_2 \to \mathbb{Y}_1 \otimes \mathbb{Y}_2$ of two distributors $F_1 \colon \mathbb{X}_1 \to \mathbb{Y}_1$ and $F_2 \colon \mathbb{X}_2 \to \mathbb{Y}_2$ is defined by letting

$$(F_1 \otimes F_2)[(y_1, y_2), (x_1, x_2)] =_{\mathrm{def}} F_1[y_1, x_1] \otimes F_2[y_2, x_2]\,,$$

for $x_1 \in \mathbb{X}_1$, $x_2 \in \mathbb{X}_2$, $y_1 \in \mathbb{Y}_1$ and $y_2 \in \mathbb{Y}_2$. This defines a symmetric monoidal structure on the bicategory $\mathrm{Dist}_\mathcal{V}$. The homomorphisms involved in the Gabriel factorization in (1.3.1) are symmetric monoidal. Let us also remark that the symmetric monoidal bicategory $\mathrm{Dist}_\mathcal{V}$ is compact [**68**] (also called rigid): the dual of a small $\mathcal{V}$-category $\mathbb{X}$ is the opposite $\mathcal{V}$-category $\mathbb{X}^{\mathrm{op}}$. The counit $\varepsilon\colon \mathbb{X}^{\mathrm{op}} \otimes \mathbb{X} \to \mathrm{I}$ is given by the hom-functor $\mathrm{I}^{\mathrm{op}} \otimes \mathbb{X}^{\mathrm{op}} \otimes \mathbb{X} = \mathbb{X}^{\mathrm{op}} \otimes \mathbb{X} \to \mathcal{V}$ and similarly for the unit $\eta\colon \mathrm{I} \to \mathbb{X} \otimes \mathbb{X}^{\mathrm{op}}$. Here, I is the $\mathcal{V}$-category giving the unit for the tensor product on $\mathrm{Cat}_\mathcal{V}$, as defined in Section 1.2.

Remark 1.3.5 (The bicategory of matrices). For our goals in Chapter 3, it will be useful for to give an explicit description of the sub-bicategory of $\mathrm{Dist}_{\mathcal{V}}$ spanned by sets, viewed as discrete $\mathcal{V}$-categories. In order to do so, we need to recall the definition and some basic properties of the functor mapping an ordinary category to the free $\mathcal{V}$-category on it. If I is the unit object of the monoidal category $\mathcal{V}$, then the functor $\mathcal{V}(I, -) \colon \mathcal{V} \to \mathrm{Set}$ has a left adjoint $(-) \cdot I \colon \mathrm{Set} \to \mathcal{V}$ which associates to a set S the coproduct $S \cdot I = \bigsqcup_S I$ of S copies of I. This left adjoint functor is symmetric (strong) monoidal. Hence, for any pair of sets S and T, we have an isomorphism

$$(S \times T) \cdot I \cong (S \cdot I) \otimes (T \cdot I)\,.$$

A similar situation occurs for the functor $(-) \cdot \mathrm{I} \colon \mathrm{Set} \to \mathrm{Cat}_{\mathcal{V}}$ which takes a set S to the $\mathcal{V}$-category $S \cdot \mathrm{I} = \bigsqcup_S \mathrm{I}$. The functor $\mathrm{Und} \colon \mathrm{Cat}_{\mathcal{V}} \to \mathrm{Cat}$ mapping a $\mathcal{V}$-category to its underlying category has also a left adjoint $(-) \cdot \mathrm{I} \colon \mathrm{Cat} \to \mathrm{Cat}_{\mathcal{V}}$ which associates to a category $\mathbb{C}$ the $\mathcal{V}$-category $\mathbb{C} \cdot \mathrm{I}$ defined by letting $\mathrm{Obj}(\mathbb{C} \cdot \mathrm{I}) =_{\mathrm{def}} \mathrm{Obj}(\mathbb{C})$ and $(\mathbb{C} \cdot \mathrm{I})[x, y] =_{\mathrm{def}} \mathbb{C}[x, y] \cdot I$. This left adjoint is symmetric (strong) monoidal. Hence, for every $\mathbb{C}, \mathbb{D} \in \mathrm{Cat}$, we have an isomorphism

$$(\mathbb{C} \times \mathbb{D}) \cdot \mathrm{I} \cong (\mathbb{C} \cdot \mathrm{I}) \otimes (\mathbb{D} \cdot \mathrm{I})\,.$$

Recall that, for sets X and Y, a *matrix* $F \colon X \to Y$ is a functor $F \colon Y \times X \to \mathcal{V}$, i.e. a family of sets $F(y, x)$, for $(y, x) \in Y \times X$. Sets, matrices and natural transformations form a bicategory, called the bicategory of matrices and denoted by $\mathrm{Mat}_{\mathcal{V}}$, which can be identified with the full sub-bicategory of $\mathrm{Dist}_{\mathcal{V}}$ spanned by discrete $\mathcal{V}$-categories. Indeed, for a set X, the discrete $\mathcal{V}$-category with set of objects X is the same thing as the free $\mathcal{V}$-category on the discrete category with set of objects X, which is denoted here by $X \cdot \mathrm{I}$. Furthermore, for every pair of sets X and Y, we have isomorphisms of categories

$$\begin{aligned}
\mathrm{Mat}_{\mathcal{V}}[X, Y] &= \mathrm{CAT}[Y \times X, \mathcal{V}] \\
&\cong \mathrm{CAT}_{\mathcal{V}}[(Y \times X) \cdot \mathrm{I}, \mathcal{V}] \\
&\cong \mathrm{CAT}_{\mathcal{V}}[(Y \cdot \mathrm{I}) \otimes (X \cdot \mathrm{I}), \mathcal{V}] \\
&\cong \mathrm{Dist}_{\mathcal{V}}[X \cdot \mathrm{I}, Y \cdot \mathrm{I}]\,.
\end{aligned}$$

The composition and identity morphisms in $\mathrm{Mat}_{\mathcal{V}}$ can be then defined so that we have a full and faithful homomorphism from $\mathrm{Mat}_{\mathcal{V}}$ to $\mathrm{Dist}_{\mathcal{V}}$. Given two matrices $F \colon X \to Y$ and $G \colon Y \to Z$, their composite $G \circ F \colon X \to Z$ is defined so that there is an isomorphism $(G \circ F) \cdot \mathrm{I} \cong (G \cdot \mathrm{I}) \circ (F \cdot \mathrm{I})$. It follows that, for $x \in X$ and $z \in Z$, we can define

$$(G \circ F)[z, x] =_{\mathrm{def}} \bigsqcup_{y \in Y} G[z, y] \otimes F[y, x]\,.$$

For a set X, the identity matrix $\mathrm{Id}_X : X \to X$ is defined so that $\mathrm{Id}_X \cdot \mathrm{I} \cong \mathrm{Id}_{X \cdot \mathrm{I}}$. Hence, for $x, y \in X$, we can define

$$\mathrm{Id}_X[x, y] =_{\mathrm{def}} \begin{cases} I & \text{if } x = y\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where I and 0 denote the unit object and the initial object of $\mathcal{V}$, respectively. These definitions determine an inclusion $\mathrm{Mat}_{\mathcal{V}} \subseteq \mathrm{Dist}_{\mathcal{V}}$.

We conclude this section by defining the operation of composition of a distributor with a $\mathcal{V}$-functor, which will be useful in the discussion of composition of analytic functors in Section 3.3. If $\mathcal{E}$ is a presentable $\mathcal{V}$-category, then we define the *composite* of a $\mathcal{V}$-distributor $F \colon \mathbb{X} \to \mathbb{Y}$ with a

$\mathcal{V}$-functor $T \colon \mathbb{Y} \to \mathcal{E}$ as the $\mathcal{V}$-functor $T \circ F \colon \mathbb{X} \to \mathcal{E}$ defined by mapping $x \in \mathbb{X}$ to

$$(T \circ F)(x) =_{\text{def}} \int^{y \in \mathbb{Y}} T(y) \otimes F(y, x) \,. \tag{1.3.7}$$

Lemma 1.3.6. *Let $\mathbb{X}, \mathbb{Y}$ be small $\mathcal{V}$-categories and $\mathcal{E}$ a presentable $\mathcal{V}$-category. Let $F \colon \mathbb{X} \to \mathbb{Y}$ be a distributor and $T \colon \mathbb{Y} \to \mathcal{E}$ be a $\mathcal{V}$-functor. There is an isomorphism*

$$(T \circ F)_c \cong T_c \circ F^\dagger \,,$$

where $T_c \colon P(\mathbb{Y}) \to \mathcal{E}$ is the cocontinuous extension of $T \colon \mathbb{Y} \to \mathcal{E}$.

Proof. The functor $T_c \circ F^\dagger \colon P(\mathbb{X}) \to \mathcal{E}$ is cocontinuous and for every $x \in \mathbb{X}$ we have

$$\begin{aligned} (T_c \circ F^\dagger)(\mathrm{y}_{\mathbb{X}}(x)) &\cong T_c(\lambda F)(x) \\ &= \int^{y \in \mathbb{Y}} T(y) \otimes \lambda(F)(x)(y) \\ &= \int^{y \in \mathbb{Y}} T(y) \otimes F(y, x) \\ &= (T \circ F)(x) \,. \end{aligned}$$

Thus, $(T \circ F)_c \cong T_c \circ F^\dagger$ by the uniqueness up to unique isomorphism of the concontinous extension of a functor. □

Proposition 1.3.7. *Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in \mathrm{Cat}_{\mathcal{V}}$ and $\mathcal{E} \in \mathrm{PCAT}_{\mathcal{V}}$.*

(i) For all distributors $F \colon \mathbb{X} \to \mathbb{Y}$, $G \colon \mathbb{Y} \to \mathbb{Z}$ and $\mathcal{V}$-functors $T \colon \mathbb{Z} \to \mathcal{E}$, $(T \circ G) \circ F \cong T \circ (G \circ F)$.

(ii) For all $T \colon \mathbb{Z} \to \mathcal{E}$, $T \circ \mathrm{Id}_{\mathbb{Z}} \cong T$.

Proof. For part (i), it suffices to show that we have $((T \circ G) \circ F)_c \cong T \circ (G \circ F)_c$. By Lemma 1.3.6 and the isomorphism in (1.3.3) we have

$$\begin{aligned} ((T \circ G) \circ F)_c &\cong (T \circ G)_c \circ F^\dagger \\ &\cong (T_c \circ G^\dagger) \circ F^\dagger \\ &= T_c \circ (G^\dagger \circ F^\dagger) \\ &\cong T_c \circ (G \circ F)^\dagger \\ &= T \circ (G \circ F) \,. \end{aligned}$$

Part (ii) follows by a similar reasoning. □

See [**17**, Volume I, Chapter 7] and [**9, 10, 53**] for further information and [**19**] for applications of distributors in theoretical computer science.

CHAPTER 2

Monoidal distributors

This chapter introduces several auxiliary bicategories of distributors, with the ultimate goal of defining the bicategory of symmetric monoidal distributors, denoted by $\mathrm{SMonDist}_{\mathcal{V}}$. This bicategory will be used in Chapter 3 to define bicategories of symmetic sequences. The chapter is organized in two parts: the first consists of Section 2.1 and Section 2.2, and the second of Section 2.3 and Section 2.4. The two parts have a parallel development: we begin by defining a suitable homomorphisms and then define bicategories of interest by considering their Gabriel factorization. In particular, Section 2.1 and Section 2.2 lead up to the definition of the bicategories of lax monoidal distributors (Theorem 2.2.2) and of monoidal distributors (Theorem 2.2.2), while Sections 2.3 and Section 2.4 lead to the definition of the bicategories of symmetric lax monoidal distributors (Theorem 2.4.2) and of symmetric monoidal distributors (Theorem 2.4.4).

2.1. Monoidal $\mathcal{V}$-categories and $\mathcal{V}$-rigs

We suppose that the reader familiar with the notions of monoidal $\mathcal{V}$-category, lax monoidal $\mathcal{V}$-functor, and monoidal $\mathcal{V}$-natural transformation. We limit ourselves to recalling that, for monoidal $\mathcal{V}$-categories $(\mathbb{M}, \otimes, e)$ and $(\mathbb{N}, \otimes, e)$, a *lax monoidal $\mathcal{V}$-functor* $F \colon \mathbb{M} \to \mathbb{N}$ is equipped with multiplication and a unit

$$\mu(x,y) \colon F(x) \otimes F(y) \to F(x \otimes y)\,, \quad \eta \colon e \to F(e)\,.$$

We say that F is a *monoidal $\mathcal{V}$-functor* if μ and η are invertible. Recall also that a $\mathcal{V}$-natural transformation between lax monoidal $\mathcal{V}$-functors is *monoidal* if it respects the multiplication and unit. We write $\mathrm{MonCat}_{\mathcal{V}}^{\mathrm{lax}}$ (resp. $\mathrm{MonCat}_{\mathcal{V}}$) for the 2-category of small monoidal $\mathcal{V}$-categories, lax monoidal (resp. monoidal) $\mathcal{V}$-functors and monoidal $\mathcal{V}$-natural transformations. If $\mathbb{M}$ and $\mathbb{N}$ are monoidal $\mathcal{V}$-categories, then so is the $\mathcal{V}$-category $\mathbb{M} \otimes \mathbb{N}$. This defines a symmetric monoidal structure on the categories $\mathrm{MonCat}_{\mathcal{V}}^{\mathrm{lax}}$ and $\mathrm{MonCat}_{\mathcal{V}}$. The unit object is the $\mathcal{V}$-category I that is the unit for the tensor product on $\mathrm{Cat}_{\mathcal{V}}$, defined in Section 1.2. It is easy to verify that I is initial in the bicategory $\mathrm{MonCat}_{\mathcal{V}}$, in the sense that for every $\mathbb{M} \in \mathrm{MonCat}_{\mathcal{V}}$ we have an equivalence of categories $\mathrm{MonCat}_{\mathcal{V}}[\mathrm{I}, \mathbb{M}] \simeq 1$, where 1 is the terminal category.

Definition 2.1.1 below introduces the notion of a $\mathcal{V}$-rig. The comparison between this notion and notions already existing in in the literature is simpler to describe in the symmetric case, so we postpone it until Definition 2.3.2, where we introduce the notion of a symmetric $\mathcal{V}$-rig.

Definition 2.1.1. A *$\mathcal{V}$-rig* is a monoidal closed presentable $\mathcal{V}$-category.

A $\mathcal{V}$-rig can be defined equivalently as a monoid (in an appropriately weak sense) in the monoidal bicategory $(\mathrm{PCAT}_{\mathcal{V}}, \widehat{\otimes}, \mathcal{V})$. If $\mathcal{R}$ and $\mathcal{S}$ are $\mathcal{V}$-rigs, we say that a cocontinuous $\mathcal{V}$-functor $F \colon \mathcal{R} \to \mathcal{S}$ is a *lax homomorphism* (resp. homomorphism) of $\mathcal{V}$-rigs if it is a lax monoidal (resp. monoidal) functor. We write $\mathrm{Rig}_{\mathcal{V}}^{\mathrm{lax}}$ (resp. $\mathrm{Rig}_{\mathcal{V}}$) for the 2-category of $\mathcal{V}$-rigs, lax homomorphisms (resp. homomorphism) and monoidal natural transformations.

We need to recall some basic facts about Day's convolution tensor product [**24, 25, 40**]. For a small monoidal $\mathcal{V}$-category $\mathbb{M} = (\mathbb{M}, \oplus, 0)$ and a $\mathcal{V}$-rig $\mathcal{R} = (\mathcal{R}, \diamond, e)$, the $\mathcal{V}$-category $[\mathbb{M}, \mathcal{R}]$ can be equipped with a monoidal structure, called the *convolution tensor product*, making it into a $\mathcal{V}$-rig. By definition, the convolution product $A_1 * A_2$ of two $\mathcal{V}$-functors $A_1, A_2 \colon \mathbb{M} \to \mathcal{R}$ is defined by letting, for $x \in \mathbb{M}$,

$$(A_1 * A_2)(x) =_{\mathrm{def}} \int^{x_1 \in \mathbb{M}} \int^{x_2 \in \mathbb{M}} A_1(x_1) \diamond A_2(x_2) \otimes \mathbb{M}[x_1 \oplus x_2, x] \, . \tag{2.1.1}$$

The unit object for the convolution product is the functor $E =_{\mathrm{def}} \mathbb{M}(0, -) \otimes e$. An important case of the convolution tensor product is given by considering $\mathcal{V}$-rigs of the form $P(\mathbb{M}) = [\mathbb{M}^{\mathrm{op}}, \mathcal{V}]$, where $\mathbb{M} = (\mathbb{M}, \oplus, 0)$ is a small monoidal $\mathcal{V}$-category. In this case, for $A_1, A_2 \in P(\mathbb{M})$, $x \in \mathbb{M}$, we have

$$(A_1 * A_2)(x) = \int^{x_1 \in \mathbb{M}} \int^{x_2 \in \mathbb{M}} A_1(x_1) \otimes A_2(x_2) \otimes \mathbb{M}[x, x_1 \oplus x_2] \, .$$

The function mapping a small monoidal $\mathcal{V}$-category $\mathbb{M}$ to the $\mathcal{V}$-rig $P(\mathbb{M})$ extends to a homomorphism

$$P \colon \mathrm{MonCat}_{\mathcal{V}}^{\mathrm{lax}} \to \mathrm{Rig}_{\mathcal{V}}^{\mathrm{lax}} \, . \tag{2.1.2}$$

Indeed, for every small monoidal $\mathcal{V}$-category $\mathbb{M}$ the Yoneda embedding $\mathrm{y}_{\mathbb{M}} \colon \mathbb{M} \to P(\mathbb{M})$ becomes a monoidal functor and it exhibits $P(\mathbb{M})$ as the free $\mathcal{V}$-rig on $\mathbb{M}$. More precisely, the restriction functor

$$\mathrm{y}_{\mathbb{M}}^* \colon \mathrm{Rig}_{\mathcal{V}}^{\mathrm{lax}}[P(\mathbb{M}), \mathcal{R}] \to \mathrm{MONCAT}_{\mathcal{V}}^{\mathrm{lax}}[\mathbb{M}, \mathcal{R}] \tag{2.1.3}$$

along $\mathrm{y}_{\mathbb{M}} \colon \mathbb{M} \to P(\mathbb{M})$ is an equivalence of categories for any $\mathcal{V}$-rig $\mathcal{R}$. The inverse equivalence takes a lax monoidal $\mathcal{V}$-functor $F \colon \mathbb{M} \to \mathcal{R}$ to its cocontinuous extension $F_c \colon P(\mathbb{M}) \to \mathcal{R}$, defined as in (1.2.4), which can be equipped with a lax monoidal structure. Thus, the homomorphism in (2.1.2) takes a lax monoidal $\mathcal{V}$-functor $u \colon \mathbb{M} \to \mathbb{N}$ to the lax homomorphism of rigs $P(u) =_{\mathrm{def}} u_! \colon P(\mathbb{M}) \to P(\mathbb{N})$. All of the above restricts in an evident way to the 2-category $\mathrm{MonCat}_{\mathcal{V}}$ so as to give also a homomorphism

$$P \colon \mathrm{MonCat}_{\mathcal{V}} \to \mathrm{Rig}_{\mathcal{V}} \, . \tag{2.1.4}$$

Remark 2.1.2. If $\mathcal{R}$ and $\mathcal{S}$ are $\mathcal{V}$-rigs, then so is the presentable $\mathcal{V}$-category $\mathcal{R} \mathbin{\widehat{\otimes}} \mathcal{S}$ discussed in Remark 1.2.3. This defines the tensor product of a symmetric monoidal closed structure on the 2-categories $\mathrm{Rig}_{\mathcal{V}}^{\mathrm{lax}}$ and $\mathrm{Rig}_{\mathcal{V}}$, with unit the category $\mathcal{V}$. Furthermore, $\mathcal{V}$ is initial in the 2-category $\mathrm{Rig}_{\mathcal{V}}$, in the sense that for every $\mathcal{R} \in \mathrm{Rig}_{\mathcal{V}}$ we have an equivalence $\mathrm{Rig}_{\mathcal{V}}[\mathcal{V}, \mathcal{R}] \simeq 1$. If we consider the 2-categories $\mathrm{MonCat}_{\mathcal{V}}^{\mathrm{lax}}$ and $\mathrm{Rig}_{\mathcal{V}}^{\mathrm{lax}}$ (resp. $\mathrm{MonCat}_{\mathcal{V}}$ and $\mathrm{Rig}_{\mathcal{V}}$) as equipped with their symmetric monoidal structures, the homomorphism $P \colon \mathrm{MonCat}_{\mathcal{V}}^{\mathrm{lax}} \to \mathrm{Rig}_{\mathcal{V}}^{\mathrm{lax}}$ (resp. $P \colon \mathrm{MonCat}_{\mathcal{V}} \to \mathrm{Rig}_{\mathcal{V}}$) is symmetric monoidal. Indeed, if $\mathbb{M}$ and $\mathbb{N}$ are small monoidal $\mathcal{V}$-categories, then the equivalence of presentable categories

$$\phi_{\mathbb{M},\mathbb{N}} \colon P(\mathbb{M}) \mathbin{\widehat{\otimes}} P(\mathbb{N}) \to P(\mathbb{M} \otimes \mathbb{N})$$

of Remark 1.2.3 is an equivalence of $\mathcal{V}$-rigs.

The homomorphisms in (2.1.2) and (2.1.4) will be used in Section 2.2 to define the bicategories of lax monoidal distributors and of monoidal distributors, respectively, via a Gabriel factorization. In order to do this, we establish some auxiliary results.

LEMMA 2.1.3. *Let* $\mathbb{M}\,,\mathbb{N}$ *be small monoidal* $\mathcal{V}$*-categories, and* $\mathcal{R}$ *be a* $\mathcal{V}$*-rig. The equivalences*

$$\lambda^{\mathbb{M}} \colon [\mathbb{M}\otimes\mathbb{N},\mathcal{R}] \to [\mathbb{N},[\mathbb{M},\mathcal{R}]]\,, \quad \lambda^{\mathbb{N}} \colon [\mathbb{M}\otimes\mathbb{N},\mathcal{R}] \to [\mathbb{M},[\mathbb{N},\mathcal{R}]]$$

are monoidal.

PROOF. Let $\lambda = \lambda^{\mathbb{M}}$. For $\mathcal{V}$-functors $A_1\,,A_2 \colon \mathbb{M}\otimes\mathbb{N} \to \mathcal{R}$, we construct a natural isomorphism

$$\lambda(A_1) * \lambda(A_2) \cong \lambda(A_1 * A_2)\,.$$

By definition, for $y \in \mathbb{N}$, we have

$$\big(\lambda(A_1) * \lambda^{\mathbb{M}}(A_2)\big)(y) = \int^{y_1\in\mathbb{N}}\int^{y_2\in\mathbb{N}} \lambda(A_1)(y_1) * \lambda(A_2)(y_2) \otimes \mathbb{N}[y_1\oplus y_2, y]\,.$$

Hence, for $x \in \mathbb{M}$ and $y \in \mathbb{N}$, we have

$$\big(\lambda(A_1) * \lambda^{\mathbb{M}}(A_2)\big)(y)(x) = \int^{y_1\in\mathbb{N}}\int^{y_2\in\mathbb{N}} \big(\lambda(A_1)(y_1) * \lambda(A_2)(y_2)\big)(x) \otimes \mathbb{N}[y_1\oplus y_2, y]\,. \tag{2.1.5}$$

But, for $x \in \mathbb{M}$ and $y_1, y_2 \in \mathbb{N}$, we have

$$\begin{aligned}\big(\lambda(A_1)(y_1) * \lambda(A_2)(y_2)\big)(x) &= \int^{x_1\in\mathbb{M}}\int^{x_2\in\mathbb{M}} \lambda(A_1)(y_1)(x_1) \diamond \lambda(A_2)(y_2)(x_2) \otimes \mathbb{M}[x_1\oplus x_2, x] \\ &\cong \int^{x_1\in\mathbb{M}}\int^{x_2\in\mathbb{M}} A_1(x_1,y_1) \diamond A_2(x_2,y_2) \otimes \mathbb{M}[x_1\oplus x_2, x]\,.\end{aligned} \tag{2.1.6}$$

By substituting the right-hand side of (2.1.6) in the right-hand side of (2.1.5), it follows that

$$\begin{aligned}&\big(\lambda(A_1) * \lambda^{\mathbb{M}}(A_2)\big)(y)(x) \\ &\quad\cong \int^{y_1\in\mathbb{N}}\int^{y_2\in\mathbb{N}}\int^{x_1\in\mathbb{M}}\int^{x_2\in\mathbb{M}} A_1(x_1,y_1) \diamond A_2(x_2,y_2) \otimes \mathbb{M}[x_1\oplus x_2, x] \otimes \mathbb{N}[y_1\oplus y_2, y] \\ &\quad\cong \int^{(x_1,y_1)\in\mathbb{M}\otimes\mathbb{N}}\int^{(x_2,y_2)\in\mathbb{M}\otimes\mathbb{N}} A_1(x_1,y_1) \diamond A_2(x_2,y_2) \otimes (\mathbb{M}\otimes\mathbb{N})[(x_1,y_1)\oplus(x_2,y_2),(x,y)] \\ &\quad= (A_1 * A_2)(x,y) \\ &\quad= \lambda^{\mathbb{M}}(A_1 * A_2)(y)(x)\,,\end{aligned}$$

as required. □

Let $\mathbb{M} = (\mathbb{M},\oplus,0)$ be a small monoidal $\mathcal{V}$-category and $\mathcal{R} = (\mathcal{R},\diamond,e)$ be a $\mathcal{V}$-rig. By the definition of the convolution product of $A_1, A_2 \in [\mathbb{M},\mathcal{R}]$, as given in (2.1.1), there is a canonical map

$$can \colon A_1(x_1) \diamond A_2(x_2) \to (A_1 * A_2)(x_1\oplus x_2)\,. \tag{2.1.7}$$

If $A = (A,\mu,\eta)$ is a monoid object in $[\mathbb{M},\mathcal{R}]$, then the composite

$$A(x_1) \diamond A(x_2) \xrightarrow{can} (A * A)(x_1\oplus x_2) \xrightarrow{\mu(x_1\oplus x_2)} A(x_1\oplus x_2)$$

is a lax monoidal structure on A with components $\mu(x_1,x_2) \colon A(x_1)\diamond A(x_2) \to A(x_1\oplus x_2)$ on A. This defines a functor $\rho \colon \mathrm{Mon}[\mathbb{M},\mathcal{R}] \to \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M},\mathcal{R}]$, where $\mathrm{Mon}[\mathbb{M},\mathcal{R}]$ denotes the category of monoids in $\mathrm{CAT}_{\mathcal{V}}[\mathbb{M},\mathcal{R}]$ and $\mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M},\mathcal{R}]$ denotes the category of lax monoidal $\mathcal{V}$-functors from $\mathbb{M}$ to $\mathcal{R}$. The next lemma is essentially as in [**24**, Example 3.2.2] and [**58**, Proposition 22.1].

LEMMA 2.1.4. *The functor* $\rho \colon \mathrm{Mon}[\mathbb{M},\mathcal{R}] \to \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M},\mathcal{R}]$ *is an equivalence of categories.*

Proof. Let us describe an inverse to ρ. Let $\mu(x_1, x_2)\colon A(x_1) \diamond A(x_2) \to A(x_1 \oplus x_2)$ be a lax monoidal structure on a functor $A\colon \mathbb{M} \to \mathcal{R}$. We have

$$(A * A)(x) = \int^{x_1 \in \mathbb{M}} \int^{x_2 \in \mathbb{M}} \mathbb{M}(x_1 \oplus x_2, x) \otimes A(x_1) \diamond A(x_2)$$

and the natural transformation $\mu(x_1, x_2)\colon A(x_1) \diamond A(x_2) \to A(x_1 \oplus x_2)$ induces a map

$$(A * A)(x) \to \int^{x_1 \in \mathbb{M}} \int^{x_2 \in \mathbb{M}} \mathbb{M}(x_1 \oplus x_2, x) \otimes A(x_1 \oplus x_2) \to A(x)$$

which defines the multiplication $\mu\colon A * A \to A$ of a monoid object (A, μ, η) in $[\mathbb{M}, \mathcal{R}]$. It is easy to verify that this is an inverse to ρ. □

We can now extend Proposition 1.2.4 to categories of monoidal functors.

Proposition 2.1.5. *The equivalences of categories*

$$\lambda^{\mathbb{M}}\colon [\mathbb{M} \otimes \mathbb{N}, \mathcal{R}] \to [\mathbb{N}, [\mathbb{M}, \mathcal{R}]]\,, \quad \lambda^{\mathbb{N}}\colon [\mathbb{M} \otimes \mathbb{N}, \mathcal{R}] \to \mathrm{CAT}_{\mathcal{V}}[\mathbb{M}, [\mathbb{N}, \mathcal{R}]]$$

restrict to equivalences of categories

$$\lambda^{\mathbb{M}}\colon \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M} \otimes \mathbb{N}, \mathcal{R}] \to \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{N}, [\mathbb{M}, \mathcal{R}]]\,,$$
$$\lambda^{\mathbb{N}}\colon \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M} \otimes \mathbb{N}, \mathcal{R}] \to \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M}, [\mathbb{N}, \mathcal{R}]]\,.$$

Proof. We consider only $\lambda^{\mathbb{M}}$, since the argument for $\lambda^{\mathbb{N}}$ is analogous. First of all, observe that $\lambda^{\mathbb{M}}$ is monoidal by Lemma 2.1.3. It thus induces an equivalence between the category of monoids in the monoidal category $[\mathbb{M} \otimes \mathbb{N}, \mathcal{R}]$ and the category of monoids in the monoidal category $[\mathbb{N}, [\mathbb{M}, \mathcal{R}]]$. The claim then follows by Lemma 2.1.4. □

Corollary 2.1.6. *Let $\mathbb{M}$, $\mathbb{N}$ be small monoidal $\mathcal{V}$-categories. For every distributor $F\colon \mathbb{M} \to \mathbb{N}$, there is a bijective correspondence between the lax monoidal structures on the $\mathcal{V}$-functors*

$$F\colon \mathbb{N}^{\mathrm{op}} \otimes \mathbb{M} \to \mathcal{V}\,, \quad \lambda F\colon \mathbb{M} \to P(\mathbb{N})\,, \quad F^{\dagger}\colon P(\mathbb{M}) \to P(\mathbb{N})\,.$$

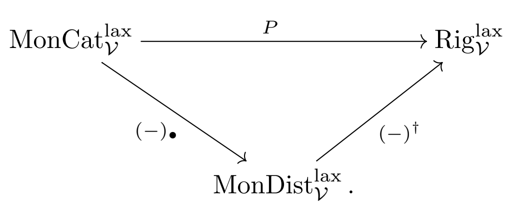
Proof. By Proposition 2.1.5 and the equivalence in (2.1.3). □

2.2. Monoidal distributors

Definition 2.2.1. Let $\mathbb{M}$, $\mathbb{N}$ be small monoidal $\mathcal{V}$-categories. A *lax monoidal distributor* $F\colon \mathbb{M} \to \mathbb{N}$ is a lax monoidal functor $F\colon \mathbb{N}^{\mathrm{op}} \otimes \mathbb{M} \to \mathcal{V}$.

The next theorem introduces the bicategory of lax monoidal distributors.

Theorem 2.2.2. *Small monoidal $\mathcal{V}$-categories, lax monoidal distributors and monoidal $\mathcal{V}$-transformations form a bicategory, called the bicategory of lax monoidal distributors and denoted by $\mathrm{MonDist}^{\mathrm{lax}}_{\mathcal{V}}$, which fits into a Gabriel factorization diagram*



PROOF. For small monoidal $\mathcal{V}$-categories $\mathbb{M}$ and $\mathbb{N}$, we define the hom-category of lax monoidal distributors from $\mathbb{M}$ to $\mathbb{N}$ by letting

$$\mathrm{MonDist}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M}, \mathbb{N}] =_{\mathrm{def}} \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{N}^{\mathrm{op}} \otimes \mathbb{M}, \mathcal{V}] \, .$$

Then, we define the functor

$$(-)^{\dagger} \colon \mathrm{MonDist}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M}, \mathbb{N}] \to \mathrm{Rig}^{\mathrm{lax}}_{\mathcal{V}}[P(\mathbb{M}), P(\mathbb{N})]$$

as the composite

$$\mathrm{MonDist}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M}, \mathbb{N}] \xrightarrow{\ \lambda\ } \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M}, P(\mathbb{N})] \xrightarrow{(-)_c} \mathrm{Rig}^{\mathrm{lax}}_{\mathcal{V}}[P(\mathbb{M}), P(\mathbb{N})] \, .$$

The functor λ is an equivalence by Proposition 2.1.5. Since $(-)_c$ is also an equivalence (being a quasi-inverse to composition with $\mathrm{y}_{\mathbb{M}}$), it follows that $(-)^{\dagger}$ is an equivalence, as required. The rest of the data necessary to have a bicategory is determined by the requirement to have a Gabriel factorization. In particular, the second part of the Gabriel factorization is then defined by mapping a small monoidal $\mathcal{V}$-category to $\mathbb{M}^{\dagger} =_{\mathrm{def}} P(\mathbb{M})$, viewed as a $\mathcal{V}$-rig with the convolution monoidal structure. For the first part of the factorization, we need to show that if $u \colon \mathbb{M} \to \mathbb{N}$ is a lax monoidal $\mathcal{V}$-functor, then the distributor $u_{\bullet} \colon \mathbb{M} \to \mathbb{N}$ is lax monoidal. But the functor $u_! \colon P(\mathbb{M}) \to P(\mathbb{N})$ is lax monoidal, since the functor $u \colon \mathbb{M} \to \mathbb{N}$ is lax monoidal. Corollary 2.1.6 implies that the distributor $u_{\bullet}$ is lax monoidal, since we have $(u_{\bullet})^{\dagger} \cong u_!$. □

The composition operation of $\mathrm{MonDist}^{\mathrm{lax}}_{\mathcal{V}}$ is obtained as a restriction of the composition operation of $\mathrm{Dist}_{\mathcal{V}}$. Indeed, for lax monoidal distributors $F \colon \mathbb{M} \to \mathbb{N}$ and $G \colon \mathbb{N} \to \mathbb{P}$, the composite distributor $G \circ F \colon \mathbb{M} \to \mathbb{P}$ is lax monoidal, since there is an isomorphism

$$(G \circ F)^{\dagger} \cong G^{\dagger} \circ F^{\dagger}$$

and lax monoidal $\mathcal{V}$-functors are closed under composition. Similarly, the identity morphism on a small monoidal $\mathcal{V}$-category is the identity distributor $\mathrm{Id}_{\mathbb{M}} \colon \mathbb{M} \to \mathbb{M}$, which is lax monoidal since the hom-functor of $\mathbb{M}$ is a lax monoidal $\mathcal{V}$-functor. All of the above admits a restriction to the case of monoidal, rather than lax monoidal, $\mathcal{V}$-functors. In order to make the theory work out smoothly, however, it is appropriate to define the notion of a monoidal distributor as follows.

DEFINITION 2.2.3. Let $\mathbb{M}\,,\mathbb{N}$ be small monoidal $\mathcal{V}$-categories. A *monoidal distributor* $F \colon \mathbb{M} \to \mathbb{N}$ is a lax monoidal distributor such that the lax monoidal functor $\lambda F \colon \mathbb{N} \to P(\mathbb{M})$ is monoidal.

Let us point out that requiring a lax monoidal distributor $F \colon \mathbb{M} \to \mathbb{N}$ to be monoidal is not equivalent to requiring the lax monoidal $\mathcal{V}$-functor $F \colon \mathbb{N}^{\mathrm{op}} \otimes \mathbb{M} \to \mathcal{V}$ to be monoidal. For example, consider the identity distributor $\mathrm{Id}_{\mathbb{M}} \colon \mathbb{M} \to \mathbb{M}$, which is given by the hom-functor $\mathbb{M}(-,-) \colon \mathbb{M}^{\mathrm{op}} \otimes \mathbb{M} \to \mathcal{V}$. This $\mathcal{V}$-functor is lax monoidal, but not monoidal. However, $\mathrm{Id}_{\mathbb{M}} \colon \mathbb{M} \to \mathbb{M}$ is a monoidal distributor since $\lambda(\mathrm{Id}_{\mathbb{M}}) \colon \mathbb{M} \to P(\mathbb{M})$ is the Yoneda embedding $\mathrm{y}_{\mathbb{M}} \colon \mathbb{M} \to P(\mathbb{M})$, which is monoidal. Note that, by Corollary 2.1.6, a lax monoidal distributor $F \colon \mathbb{M} \to \mathbb{N}$ is monoidal if and only if the lax monoidal functor $F^{\dagger} \colon P(\mathbb{N}) \to P(\mathbb{M})$ is monoidal.

The next theorem defines the bicategory of monoidal distributors.

THEOREM 2.2.4. *Small monoidal $\mathcal{V}$-categories, monoidal distributors and monoidal $\mathcal{V}$-transformations form a bicategory, called the bicategory of monoidal distributors and denoted by* $\mathrm{MonDist}_{\mathcal{V}}$, *which fits in a Gabriel factorization diagram*

$$\begin{array}{ccc} \mathrm{MonCat}_{\mathcal{V}} & \xrightarrow{\quad P \quad} & \mathrm{Rig}_{\mathcal{V}} \\ & {\scriptstyle (-)_\bullet} \searrow \qquad \nearrow {\scriptstyle (-)^\dagger} & \\ & \mathrm{MonDist}_{\mathcal{V}}\,. & \end{array}$$

Proof. For small monoidal $\mathcal{V}$-categories $\mathbb{M}$ and $\mathbb{N}$, we define the category $\mathrm{MonDist}_{\mathcal{V}}[\mathbb{M},\mathbb{N}]$ as the full sub-category of $\mathrm{MonDist}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{M},\mathbb{N}]$ spanned by monoidal distributors. The rest of the proof follows the pattern of the one of Theorem 2.2.2. In particular, the functor λ used in the proof of Theorem 2.2.2 restricts to an equivalence

$$\lambda \colon \mathrm{MonDist}_{\mathcal{V}}[\mathbb{M},\mathbb{N}] \to \mathrm{Rig}_{\mathcal{V}}[\mathbb{M}, P(\mathbb{N})]$$

by the very definition of the notion of a monoidal distributor. ☐

Remark 2.2.5. The tensor product $F_1 \otimes F_2 \colon \mathbb{M}_1 \otimes \mathbb{M}_2 \to \mathbb{N}_1 \otimes \mathbb{N}_2$ of lax monoidal (resp. monoidal) distributors $F_1 \colon \mathbb{M}_1 \to \mathbb{N}_1$ and $F_2 \colon \mathbb{M}_2 \to \mathbb{N}_2$ is lax monoidal (resp. monoidal). The operation defines a symmetric monoidal structure on the bicategories $\mathrm{MonDist}^{\mathrm{lax}}_{\mathcal{V}}$ and $\mathrm{MonDist}_{\mathcal{V}}$. Moreover, the homomorphisms in the Gabriel factorizations of Theorem 2.2.2 and Theorem 2.2.4 are symmetric monoidal. Let us also point out that the symmetric monoidal bicategory $\mathrm{MonDist}^{\mathrm{lax}}_{\mathcal{V}}$ is compact: the dual of a monoidal $\mathcal{V}$-category $\mathbb{M}$ is the opposite $\mathcal{V}$-category $\mathbb{M}^{\mathrm{op}}$. The counit $\varepsilon \colon \mathbb{M}^{\mathrm{op}} \otimes \mathbb{M} \to \mathrm{I}$ is given by the hom-functor $\mathrm{I}^{\mathrm{op}} \otimes \mathbb{M}^{\mathrm{op}} \otimes \mathbb{M} = \mathbb{M}^{\mathrm{op}} \otimes \mathbb{M} \to \mathcal{V}$ and similarly for the unit $\eta \colon \mathrm{I} \to \mathbb{M} \otimes \mathbb{M}^{\mathrm{op}}$. In contrast, the symmetric monoidal bicategory $\mathrm{MonDist}_{\mathcal{V}}$ is *not* compact.

We conclude this section by restricting to the monoidal case the operation of composition of a distributor with a functor, defined in (1.3.7).

Proposition 2.2.6. *Let $\mathcal{R}$ be a $\mathcal{V}$-rig. Then the composite of a monoidal distributor $F \colon \mathbb{M} \to \mathbb{N}$ with a monoidal functor $T \colon \mathbb{N} \to \mathcal{R}$ is a monoidal functor $T \circ F \colon \mathbb{M} \to \mathcal{R}$.*

Proof. It suffices to show that $(T \circ F)_c \colon P(\mathbb{M}) \to \mathcal{R}$ is monoidal. The functor $F^\dagger \colon P(\mathbb{M}) \to P(\mathbb{N})$ is monoidal, since the distributor $F \colon \mathbb{M} \to \mathbb{N}$ is monoidal, and the functor $T_c \colon P(\mathbb{N}) \to \mathcal{R}$ is monoidal, since T is monoidal. Hence, the functor $T_c \circ F^\dagger \colon P(\mathbb{M}) \to \mathcal{R}$ is monoidal. This proves the result, since we have $(T \circ F)_c \cong T_c \circ F^\dagger$ by Lemma 1.3.6. ☐

2.3. Symmetric monoidal $\mathcal{V}$-categories and symmetric $\mathcal{V}$-rigs

The aim of this section is to develop the counterpart for symmetric monoidal $\mathcal{V}$-categories of the material in Section 2.1. Let us recall that, for symmetric monoidal $\mathcal{V}$-categories $\mathbb{A}$ and $\mathbb{B}$, a lax monoidal $\mathcal{V}$-functor $F \colon \mathbb{A} \to \mathbb{B}$ is said to be *symmetric* if, for any pair of objects $x \in \mathbb{A}$ and $y \in \mathbb{B}$, the following square commutes

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{\ \mu(x,y)\ } & F(x \otimes y) \\ {\scriptstyle \sigma} \downarrow & & \downarrow {\scriptstyle F(\sigma)} \\ F(y) \otimes F(x) & \xrightarrow[\ \mu(y,x)\ ]{} & F(y \otimes x)\,, \end{array}$$

where we use σ to denote the symmetry isomorphism of both $\mathbb{A}$ and $\mathbb{B}$. We write $\mathrm{SMonCat}_{\mathcal{V}}^{\mathrm{lax}}$ (resp. $\mathrm{SMonCat}_{\mathcal{V}}$) for the 2-category of small symmetric monoidal $\mathcal{V}$-categories, symmetric lax monoidal (resp. symmetric monoidal) $\mathcal{V}$-functors and monoidal $\mathcal{V}$-natural transformations. If $\mathbb{A}$ and $\mathbb{B}$ are symmetric monoidal $\mathcal{V}$-categories, then so is the $\mathcal{V}$-category $\mathbb{A} \otimes \mathbb{B}$. This operation defines a symmetric monoidal structure on the categories $\mathrm{SMonCat}_{\mathcal{V}}^{\mathrm{lax}}$ and $\mathrm{SMonCat}_{\mathcal{V}}$. The unit object is the $\mathcal{V}$-category I giving the unit for the tensor product of $\mathrm{Cat}_{\mathcal{V}}$, as defined in Section 1.2. It is easy to verify that I is initial in the 2-category $\mathrm{SMonCat}_{\mathcal{V}}$. If $\mathbb{A} = (\mathbb{A}, \oplus, 0, \sigma)$ is a symmetric monoidal category, then the interchange law

$$(x_1 \oplus x_2) \oplus (y_1 \oplus y_2) \cong (x_1 \oplus y_1) \oplus (x_2 \oplus y_2)$$

is a natural (symmetric) monoidal structure on the tensor product functor of $\mathbb{A}$.

Lemma 2.3.1. *The* 2*-category* $\mathrm{SMonCat}_{\mathcal{V}}$ *has finite coproducts. In particular, the coproduct of two small symmetric monoidal* $\mathcal{V}$*-categories* $\mathbb{A}_1$ *and* $\mathbb{A}_2$ *is their tensor product* $\mathbb{A}_1 \otimes \mathbb{A}_2$*.*

Proof. Let $\mathbb{A}_1 = (\mathbb{A}_1, \oplus, 0)\,, \mathbb{A}_2 = (\mathbb{A}_2, \oplus, 0)$ be two symmetric monoidal categories. We define the functors $\iota_1 \colon \mathbb{A}_1 \to \mathbb{A}_1 \otimes \mathbb{A}_2$ and $\iota_2 \colon \mathbb{A}_2 \to \mathbb{A}_1 \otimes \mathbb{A}_2$ by letting $\iota_1(x) =_{\mathrm{def}} (x, 0)$ and $\iota_2(x) =_{\mathrm{def}} (0, x)$. We now have to show that the functor

$$\pi \colon \mathrm{SMonCat}_{\mathcal{V}}[\mathbb{A}_1 \otimes \mathbb{A}_2, \mathbb{B}] \to \mathrm{SMonCat}_{\mathcal{V}}[\mathbb{A}_1, \mathbb{B}] \times \mathrm{SMonCat}_{\mathcal{V}}[\mathbb{A}_2, \mathbb{B}]\,,$$

defined by letting $\pi(F) =_{\mathrm{def}} (F \circ \iota_1, F \circ \iota_2)$, is an equivalence of categories for any symmetric monoidal $\mathcal{V}$-category $\mathbb{B}$. The tensor product functor $\mu \colon \mathbb{B} \otimes \mathbb{B} \to \mathbb{B}$ is symmetric monoidal, since the monoidal category $\mathbb{B}$ is symmetric. Thus, if $F_1 \colon \mathbb{A}_1 \to \mathbb{B}$ and $F_2 \colon \mathbb{A}_2 \to \mathbb{B}$ are symmetric monoidal functors, then the functor $\mu \circ (F_1 \otimes F_2) \colon \mathbb{A}_1 \otimes \mathbb{A}_2 \to \mathbb{B}$ is symmetric monoidal. Hence, we obtain a functor

$$\rho \colon \mathrm{SMonCat}_{\mathcal{V}}[\mathbb{A}, \mathbb{B}] \times \mathrm{SMonCat}_{\mathcal{V}}[\mathbb{A}_2, \mathbb{B}] \to \mathrm{SMonCat}_{\mathcal{V}}[\mathbb{A}_1 \otimes \mathbb{A}_2, \mathbb{B}]$$

defined by letting $\rho(F_1, F_2) =_{\mathrm{def}} \mu \circ (F_1 \otimes F_2)$. The verification that the functors π and ρ are mutually pseudo-inverse is left to the reader. $\square$

The notion of a symmetric $\mathcal{V}$-rig, introduced in Definition 2.3.2 below, generalizes to the enriched case the notion of a 2-ring introduced in [**22**] (where it is defined as a symmetric closed monoidal presentable category), which is a special case of the notion of a 2-rig introduced in [**4**] (where it is defined as a symmetric monoidal cocomplete category in which the tensor product preserves colimits in both variables).

Definition 2.3.2. A *symmetric* $\mathcal{V}$*-rig* is a symmetric monoidal closed presentable $\mathcal{V}$-category.

If $\mathcal{R}$ and $\mathcal{S}$ are symmetric $\mathcal{V}$-rigs, we say that a lax homomorphism (resp. homomorphism) of $\mathcal{V}$-rigs $F \colon \mathcal{R} \to \mathcal{S}$ is a *symmetric* if it is symmetric as a lax monoidal (resp. monoidal) $\mathcal{V}$-functor. We write $\mathrm{SRig}_{\mathcal{V}}^{\mathrm{lax}}$ (resp. $\mathrm{SRig}_{\mathcal{V}}$) for the 2-category of symmetric $\mathcal{V}$-rigs, symmetric lax homomorphisms (resp. symmetric homomorphisms) and monoidal $\mathcal{V}$-natural transformations.

The convolution tensor product extends in a natural way to the symmetric case [**25, 40**]. Indeed, if $\mathbb{A} = (\mathbb{A}, \oplus, 0, \sigma_{\mathbb{A}})$ is a small symmetric monoidal $\mathcal{V}$-category and $\mathcal{R} = (\mathcal{R}, \diamond, e, \sigma_{\mathcal{R}})$ is symmetric $\mathcal{V}$-rig then the $\mathcal{V}$-rig $[\mathbb{A}, \mathcal{R}]$ is symmetric. By definition, for $F, G \colon \mathbb{A} \to \mathcal{R}$, the value at $x \in \mathbb{A}$ of the symmetry isomorphism $\sigma \colon F * G \to G * F$ is the coend over $x_1, x_2 \in \mathbb{A}$ of the maps

$$\sigma_{\mathcal{R}} \otimes \mathbb{A}[\sigma_{\mathbb{A}}, x] \colon F(x_1) \diamond G(x_2) \otimes \mathbb{A}[x_1 \oplus x_2, x] \to G(x_2) \diamond F(x_1) \otimes \mathbb{A}[x_2 \oplus x_1, x]\,.$$

If $\mathbb{A}$ is a small symmetric monoidal $\mathcal{V}$-category, then $P(\mathbb{A}) = [\mathbb{A}^{\mathrm{op}}, \mathcal{V}]$ is a symmetric $\mathcal{V}$-rig. The Yoneda functor $\mathrm{y}_{\mathbb{A}} \colon \mathbb{A} \to P(\mathbb{A})$ is symmetric monoidal and exhibits $P(\mathbb{A})$ as the free symmetric $\mathcal{V}$-rig on $\mathbb{A}$. More precisely, this means that the restriction functor

$$\mathrm{y}_{\mathbb{A}}^{*} \colon \mathrm{SRig}_{\mathcal{V}}^{\mathrm{lax}}[P(\mathbb{A}), \mathcal{R}] \to \mathrm{SMONCAT}_{\mathcal{V}}^{\mathrm{lax}}[\mathbb{A}, \mathcal{R}] \tag{2.3.1}$$

along the Yoneda functor $\mathrm{y}_{\mathbb{A}} \colon \mathbb{A} \to P(\mathbb{A})$ is an equivalence of categories for any symmetric $\mathcal{V}$-rig $\mathcal{R}$. The inverse equivalence takes a symmetric lax monoidal $\mathcal{V}$-functor $F \colon \mathbb{A} \to \mathcal{R}$ to the functor $F_c \colon P(\mathbb{A}) \to \mathcal{R}$, which can be equipped with the structure of a lax homomorphism of $\mathcal{V}$-rigs. We write

$$P \colon \mathrm{SMonCat}_{\mathcal{V}}^{\mathrm{lax}} \to \mathrm{SRig}_{\mathcal{V}}^{\mathrm{lax}}$$

for the homomorphism of bicategories which takes a symmetric lax monoidal functor $u \colon \mathbb{A} \to \mathbb{B}$ to the lax symmetric homomorphism of symmetric $\mathcal{V}$-rigs $P(u) =_{\mathrm{def}} u_! \colon P(\mathbb{A}) \to P(\mathbb{B})$. All of the above restricts to symmetric monoidal $\mathcal{V}$-functors and symmetric homomorphisms of $\mathcal{V}$-rigs and so we obtain a homomorphism

$$P \colon \mathrm{SMonCat}_{\mathcal{V}} \to \mathrm{SRig}_{\mathcal{V}}\,.$$

Remark 2.3.3. If $\mathcal{R}$ and $\mathcal{S}$ are symmetric $\mathcal{V}$-rigs, then so is their completed tensor product $\mathcal{R} \mathbin{\widehat{\otimes}} \mathcal{S}$ (cf. Remark 1.2.3) This defines a symmetric monoidal structure on the bicategories $\mathrm{SRig}_{\mathcal{V}}^{\mathrm{lax}}$ and $\mathrm{SRig}_{\mathcal{V}}$. The unit object is the category $\mathcal{V}$. If $\mathbb{A}$ and $\mathbb{B}$ are symmetric monoidal $\mathcal{V}$-categories, then the canonical functor

$$P(\mathbb{A}) \mathbin{\widehat{\otimes}} P(\mathbb{B}) \to P(\mathbb{A} \otimes \mathbb{B})$$

is an equivalence of symmetric $\mathcal{V}$-rigs. This witnesses the fact that the homomorphisms of bicategories $P \colon \mathrm{SMonCat}_{\mathcal{V}} \to \mathrm{SRig}_{\mathcal{V}}$ and $P \colon \mathrm{SMonCat}_{\mathcal{V}}^{\mathrm{lax}} \to \mathrm{SRig}_{\mathcal{V}}^{\mathrm{lax}}$ are symmetric monoidal.

We now proceed to extend Proposition 2.1.5 to functor categories of symmetric lax monoidal $\mathcal{V}$-functors. The first step is the following lemma, which is a counterpart of Lemma 2.1.3 for symmetric monoidal $\mathcal{V}$-categories.

Lemma 2.3.4. *Let $\mathbb{A}\,, \mathbb{B}$ be small symmetric monoidal $\mathcal{V}$-categories and $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig. Then, the monoidal equivalences*

$$\lambda^{\mathbb{A}} \colon [\mathbb{A} \otimes \mathbb{B}, \mathcal{R}] \to [\mathbb{B}, [\mathbb{A}, \mathcal{R}]]\,, \quad \lambda^{\mathbb{B}} \colon [\mathbb{A} \otimes \mathbb{B}, \mathcal{R}] \to [\mathbb{A}, [\mathbb{B}, \mathcal{R}]]$$

are symmetric.

Proof. Similar to the proof of Lemma 2.1.3. □

Let $\mathbb{A} = (\mathbb{A}, \oplus, 0, \sigma_{\mathbb{A}})$ be a small symmetric monoidal $\mathcal{V}$-category and $\mathcal{R} = (\mathcal{R}, \diamond, e, \sigma_{\mathcal{R}})$ be a symmetric $\mathcal{V}$-rig. As we have just seen, the $\mathcal{V}$-rig $[\mathbb{A}, \mathcal{R}]$ is symmetric. Now, for $F, G \colon \mathbb{A} \to \mathcal{R}$, if

$$\sigma \colon F * G \to G * F$$

is the symmetry isomorphism, then the following diagram commutes for every $x_1, x_2 \in \mathbb{A}$,

$$\begin{array}{ccc} F(x_1) \diamond G(x_2) & \xrightarrow{\ can\ } & (F * G)(x_1 \oplus x_2) \\ {\scriptstyle \sigma_{\mathcal{R}}} \downarrow & & \downarrow {\scriptstyle \sigma(\sigma_{\mathbb{A}})} \\ G(x_2) \diamond F(x_1) & \xrightarrow[\ can\ ]{} & (G * F)(x_2 \oplus x_1)\,, \end{array} \tag{2.3.2}$$

where the maps labelled *can* are as in (2.1.7). For a small symmetric monoidal $\mathcal{V}$-category $\mathbb{A}$ and a symmetric $\mathcal{V}$-rig $\mathcal{R}$, we write $\mathrm{CMon}[\mathbb{A}, \mathcal{R}]$ for the category of commutative monoid objects in the symmetric $\mathcal{V}$-rig $[\mathbb{A}, \mathcal{R}]$, which is a full subcategory of the category $\mathrm{Mon}[\mathbb{A}, \mathcal{R}]$ of monoid objects in $[\mathbb{A}, \mathcal{R}]$. The next lemma is a counterpart of Lemma 2.1.4 for symmetric monoidal $\mathcal{V}$-categories.

LEMMA 2.3.5. *Let $\mathbb{A}$ be a small symmetric monoidal $\mathcal{V}$-category and $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig. Then, the equivalence of categories*

$$\rho\colon \mathrm{Mon}[\mathbb{A}, \mathcal{R}] \to \mathrm{MONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{A}, \mathcal{R}]$$

restricts to an equivalence of categories

$$\mathrm{CMon}[\mathbb{A}, \mathcal{R}] \to \mathrm{SMONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{A}, \mathcal{R}]\,.$$

PROOF. Let us show that the functor ρ takes a commutative monoid object $F = (F, \mu, \eta)$ in $[\mathbb{A}, \mathcal{R}]$ to a symmetric lax monoidal functor. By definition, the lax monoidal structure

$$\mu(x_1, x_2)\colon F(x_1) \diamond F(x_2) \to F(x_1 \oplus x_2)$$

on the functor $F\colon \mathbb{A} \to \mathcal{V}$ is obtained by composing the maps

$$F(x_1) \diamond F(x_2) \xrightarrow{can} (F * F)(x_1 \oplus x_2) \xrightarrow{\mu(x_1 \oplus x_2)} F(x_1 \oplus x_2). \tag{2.3.3}$$

Let us now consider the following diagram:

$$\begin{array}{ccc}
(F * F)(x_1 \oplus x_2) & \xrightarrow{\mu(x_1 \oplus x_2)} & F(x_1 \oplus x_2) \\
{\scriptstyle \sigma(x_1 \oplus x_2)} \downarrow & & \| \\
(F * F)(x_1 \oplus x_2) & \xrightarrow{\mu(x_1 \oplus x_2)} & F(x_1 \oplus x_2) \\
{\scriptstyle (F*F)(\sigma_{\mathbb{A}})} \downarrow & & \downarrow {\scriptstyle F(\sigma_{\mathbb{A}})} \\
(F * F)(x_2 \oplus x_1) & \xrightarrow[\mu(x_2 \oplus x_1)]{} & F(x_2 \oplus x_1)\,.
\end{array}$$

Its top square commutes since the product $\mu\colon F * F \to F$ is commutative, while the bottom square commutes by naturality. It follows by composing that following square commutes,

$$\begin{array}{ccc}
(F * F)(x_1 \oplus x_2) & \xrightarrow{\mu(x_1 \oplus x_2)} & F(x_1 \oplus x_2) \\
{\scriptstyle \sigma(\sigma_{\mathbb{A}})} \downarrow & & \downarrow {\scriptstyle F(\sigma_{\mathbb{A}})} \\
(F * F)(x_2 \oplus x_1) & \xrightarrow[\mu(x_2 \oplus x_1)]{} & F(x_2 \oplus x_1)\,.
\end{array} \tag{2.3.4}$$

But the following square commutes, being an instance of the diagram in (2.3.2)

$$\begin{array}{ccc}
F(x_1) \diamond F(x_2) & \xrightarrow{can} & (F * F)(x_1 \oplus x_2) \\
{\scriptstyle \sigma_{\mathcal{R}}} \downarrow & & \downarrow {\scriptstyle \sigma(\sigma_{\mathbb{A}})} \\
F(x_2) \diamond F(x_1) & \xrightarrow[can]{} & (F * F)(x_2 \oplus x_1).
\end{array} \tag{2.3.5}$$

If we compose horizontally the squares in (2.3.4) and (2.3.5), we obtain the following commutative square

$$\begin{array}{ccc} F(x_1) \diamond F(x_2) & \xrightarrow{\mu(x_1,x_2)} & F(x_1 \oplus x_2) \\ {\scriptstyle \sigma_{\mathcal{R}}} \downarrow & & \downarrow {\scriptstyle F(\sigma_{\mathbb{A}})} \\ F(x_2) \diamond F(x_1) & \xrightarrow[\mu(x_2,x_1)]{} & F(x_2 \oplus x_1)\,. \end{array}$$

This shows that the lax monoidal structure in (2.3.3) is symmetric. □

It is now possible to extend Proposition 2.1.5 to the symmetric case.

Proposition 2.3.6. *The equivalences of categories*

$$\lambda^{\mathbb{A}} \colon [\mathbb{A} \otimes \mathbb{B}, \mathcal{R}] \to [\mathbb{B}, [\mathbb{A}, \mathcal{R}]]\,, \quad \lambda^{\mathbb{B}} \colon [\mathbb{A} \otimes \mathbb{B}, \mathcal{R}] \to [\mathbb{A}, [\mathbb{B}, \mathcal{R}]]$$

restrict to equivalences of categories

$$\mathrm{SMONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{A} \otimes \mathbb{B}, \mathcal{R}] \simeq \mathrm{SMONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{B}, [\mathbb{A}, \mathcal{R}]]\,,$$
$$\mathrm{SMONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{A} \otimes \mathbb{B}, \mathcal{R}] \simeq \mathrm{SMONCAT}^{\mathrm{lax}}_{\mathcal{V}}[\mathbb{A}, [\mathbb{B}, \mathcal{R}]]\,.$$

Proof. This follows from Lemma 2.3.4 and Lemma 2.3.5. □

The next corollary is the counterpart of Corollary 2.1.6 in the symmetric monoidal case.

Corollary 2.3.7. *Let $\mathbb{A}\,, \mathbb{B}$ be small symmetric $\mathcal{V}$-categories. For every distributor $F \colon \mathbb{A} \to \mathbb{B}$, the symmetric lax monoidal structures on the $\mathcal{V}$-functors*

$$F \colon \mathbb{B}^{\mathrm{op}} \otimes \mathbb{A} \to \mathcal{V}\,, \quad \lambda F \colon \mathbb{A} \to P(\mathbb{B})\,, \quad F^{\dagger} \colon P(\mathbb{A}) \to P(\mathbb{B})$$

are in bijective correspondence.

Proof. The claim follows by Proposition 2.3.6 and the equivalence in (2.3.1). □

2.4. Symmetric monoidal distributors

Definition 2.4.1. Let $\mathbb{A}$, $\mathbb{B}$ be small symmetric monoidal $\mathcal{V}$-categories. A *symmetric lax monoidal distributor* $F \colon \mathbb{A} \to \mathbb{B}$ is a symmetric lax monoidal functor $F \colon \mathbb{B}^{\mathrm{op}} \otimes \mathbb{A} \to \mathcal{V}$.

The next theorem introduces the bicategory of symmetric lax monoidal distributors.

Theorem 2.4.2. *Small symmetric monoidal $\mathcal{V}$-categories, symmetric lax monoidal distributors and monoidal $\mathcal{V}$-natural transformations form a bicategory, called the bicategory of symmetric lax monoidal distributors and denoted by $\mathrm{SMonDist}^{\mathrm{lax}}_{\mathcal{V}}$, which fits in a Gabriel factorization*

$$\begin{array}{ccc} \mathrm{SMonCat}^{\mathrm{lax}}_{\mathcal{V}} & \xrightarrow{\quad P \quad} & \mathrm{SRig}^{\mathrm{lax}}_{\mathcal{V}} \\ & {\scriptstyle (-)_{\bullet}} \searrow \quad \nearrow {\scriptstyle (-)^{\dagger}} & \\ & \mathrm{SMonDist}^{\mathrm{lax}}_{\mathcal{V}}\,. & \end{array}$$

Proof. For small symmetric monoidal $\mathcal{V}$-categories $\mathbb{A}$ and $\mathbb{B}$ we define the hom-category of symmetric lax monoidal distributors from $\mathbb{A}$ to $\mathbb{B}$ by letting

$$\mathrm{SMonDist}_{\mathcal{V}}^{\mathrm{lax}}[\mathbb{A}, \mathbb{B}] =_{\mathrm{def}} \mathrm{SMONCAT}_{\mathcal{V}}^{\mathrm{lax}}[\mathbb{B}^{\mathrm{op}} \otimes \mathbb{A}, \mathcal{V}]\,.$$

With this definition, we have an equivalence of categories

$$(-)^{\dagger} \colon \mathrm{SMonDist}_{\mathcal{V}}^{\mathrm{lax}}[\mathbb{A}, \mathbb{B}] \to \mathrm{SRig}_{\mathcal{V}}^{\mathrm{lax}}[P(\mathbb{A}), P(\mathbb{B})]\,,$$

which is defined as the composite of the following two equivalences:

$$\mathrm{SMonDist}_{\mathcal{V}}^{\mathrm{lax}}[\mathbb{A}, \mathbb{B}] \xrightarrow{\ \lambda\ } \mathrm{SMONCAT}_{\mathcal{V}}^{\mathrm{lax}}[\mathbb{A}, P(\mathbb{B})] \xrightarrow{\ (-)_c\ } \mathrm{SRig}_{\mathcal{V}}^{\mathrm{lax}}[P(\mathbb{A}), P(\mathbb{B})]\,.$$

The rest of the data is determined by the requirement of having a Gabriel factorization. In particular, the action of $(-)^{\dagger} \colon \mathrm{SMonDist}_{\mathcal{V}}^{\mathrm{lax}} \to \mathrm{SRig}_{\mathcal{V}}^{\mathrm{lax}}$ on objects by letting $\mathbb{A}^{\dagger} =_{\mathrm{def}} P(\mathbb{A})$. If $u \colon \mathbb{A} \to \mathbb{B}$ is a symmetric lax monoidal $\mathcal{V}$-functor between symmetric monoidal $\mathcal{V}$-categories, then the distributor $u_{\bullet} \colon \mathbb{A} \to \mathbb{B}$ is symmetric lax monoidal. By Corollary 2.3.7 the lax monoidal functor $u_! \colon P(\mathbb{A}) \to P(\mathbb{B})$ is symmetric, since the lax monoidal functor $u \colon \mathbb{A} \to \mathbb{B}$ is symmetric. Hence the lax monoidal distributor $u_{\bullet}$ is symmetric, since we have $(u_{\bullet})^{\dagger} \cong u_!$. □

Note that the composition law and the identity morphisms are defined as in $\mathrm{Dist}_{\mathcal{V}}$. Indeed, for symmetric lax monoidal distributors $F \colon \mathbb{A} \to \mathbb{B}$ and $G \colon \mathbb{B} \to \mathbb{C}$, the distributor $G \circ F \colon \mathbb{A} \to \mathbb{C}$ is symmetric lax monoidal, since $(G \circ F)^{\dagger} \cong G^{\dagger} \circ F^{\dagger}$ and symmetric lax monoidal functors are closed under composition. Furthermore, the identity distributor $\mathrm{Id}_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}$ is symmetric lax monoidal, since the hom-functor from $\mathbb{A}^{\mathrm{op}} \otimes \mathbb{A}$ to $\mathcal{V}$ is symmetric lax monoidal.

Definition 2.4.3. If $\mathbb{A}$ and $\mathbb{B}$ are small symmetric monoidal $\mathcal{V}$-categories, we say that a monoidal distributor $F \colon \mathbb{A} \to \mathbb{B}$ is *symmetric* if the monoidal functor $\lambda F \colon \mathbb{A} \to P(\mathbb{B})$ is symmetric.

The next theorem introduces the bicategory of symmetric monoidal distributors.

Theorem 2.4.4. *Small symmetric monoidal $\mathcal{V}$-categories, symmetric monoidal distributors and monoidal $\mathcal{V}$-transformations form a bicategory* $\mathrm{SMonDist}_{\mathcal{V}}$ *which fits in a Gabriel factorization*

$$\begin{array}{ccc} \mathrm{SMonCat}_{\mathcal{V}} & \xrightarrow{\quad P \quad} & \mathrm{SRig}_{\mathcal{V}} \\ & {\scriptstyle (-)_{\bullet}} \searrow \quad \nearrow {\scriptstyle (-)^{\dagger}} & \\ & \mathrm{SMonDist}_{\mathcal{V}}\,. & \end{array}$$

Proof. For small symmetric monoidal $\mathcal{V}$-categories $\mathbb{A}$ and $\mathbb{B}$, we define $\mathrm{SMonDist}_{\mathcal{V}}[\mathbb{A}, \mathbb{B}]$ as the full sub-category of $\mathrm{SMonDist}_{\mathcal{V}}^{\mathrm{lax}}[\mathbb{A}, \mathbb{B}]$ spanned by symmetric monoidal distributors. In this way, the functor λ in the proof of Theorem 2.4.2 restricts to an equivalence

$$\lambda \colon \mathrm{SMonDist}_{\mathcal{V}}[\mathbb{A}, \mathbb{B}] \to \mathrm{SRig}_{\mathcal{V}}[\mathbb{A}, P(\mathbb{B})]\,.$$

The rest of the proof follows the usual pattern. □

Once again, the composition law and the identity morphisms of $\mathrm{SMonDist}_{\mathcal{V}}$ are defined as in $\mathrm{Dist}_{\mathcal{V}}$.

Remark 2.4.5. The tensor product $F_1 \otimes F_2 \colon \mathbb{A}_1 \otimes \mathbb{A}_2 \to \mathbb{B}_1 \otimes \mathbb{B}_2$ of symmetric lax monoidal (resp. symmetric monoidal) distributors $F_1 \colon \mathbb{A}_1 \to \mathbb{B}_1$ and $F_2 \colon \mathbb{A}_2 \to \mathbb{B}_2$ is symmetric lax monoidal (resp. symmetric monoidal). The operation defines a symmetric monoidal structure on the bicategories $\mathrm{SMonDist}_{\mathcal{V}}^{\mathrm{lax}}$ and $\mathrm{SMonDist}_{\mathcal{V}}$. Moreover, the homomorphisms involved in the Gabriel factorizations of Theorem 2.4.2 and Theorem 2.4.4 are symmetric monoidal. The symmetric monoidal bicategory $\mathrm{SMonDist}_{\mathcal{V}}^{\mathrm{lax}}$ is compact: the dual of a symmetric monoidal $\mathcal{V}$-category $\mathbb{A}$ is the opposite $\mathcal{V}$-category $\mathbb{A}^{\mathrm{op}}$. The counit $\varepsilon \colon \mathbb{A}^{\mathrm{op}} \otimes \mathbb{A} \to \mathrm{I}$ is given by the hom-functor $\mathrm{I}^{\mathrm{op}} \otimes \mathbb{A}^{\mathrm{op}} \otimes \mathbb{A} = \mathbb{A}^{\mathrm{op}} \otimes \mathbb{A} \to \mathcal{V}$ and similarly for the unit $\eta \colon \mathrm{I} \to \mathbb{A} \otimes \mathbb{A}^{\mathrm{op}}$. In contrast, the symmetric monoidal bicategory $\mathrm{SMonDist}_{\mathcal{V}}$ is *not* compact.

We conclude the section by extending Proposition 2.2.6 to the symmetric case.

Proposition 2.4.6. *Let $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig. Then the composite of a symmetric monoidal distributor $F \colon \mathbb{A} \to \mathbb{B}$ with a symmetric monoidal functor $T \colon \mathbb{B} \to \mathcal{R}$ is a symmetric monoidal functor $T \circ F \colon \mathbb{A} \to \mathcal{R}$.*

Proof. Similar to the proof of Proposition 2.2.6. □

CHAPTER 3

Symmetric sequences

This chapter introduces the bicategory of categorical symmetric sequences, which is denoted by $\mathrm{CatSym}_{\mathcal{V}}$, and presents our first main result, asserting that $\mathrm{CatSym}_{\mathcal{V}}$ is cartesian closed (Theorem 3.4.2). This generalizes the main result in [**30**] to the enriched setting. In order to define the bicategory $\mathrm{CatSym}_{\mathcal{V}}$, we construct an auxiliary bicategory, called the bicategory of S-distributors and denoted by $S\text{-}\mathrm{Dist}_{\mathcal{V}}$. The definition of this bicategory is based on that of the bicategory of symmetric monoidal distributors (presented in Section 2.4), some basic facts about the construction of free symmetric monoidal $\mathcal{V}$-categories, and the notion of a Gabriel factorization. The special case of this bicategory obtained by considering $\mathcal{V} = \mathrm{Set}$ was introduced in [**30**] using the theory of Kleisli bicategories and pseudo-distributive laws. Although that approach is likely to carry over to the enriched case and lead to an equivalent definition to the one given here, we prefer to use the idea of a Gabriel factorization, since it allows us to avoid almost entirely the discussion of coherence conditions. Furthermore, when $\mathcal{V} = \mathrm{Set}$, the monoidal categories $\mathrm{Sym}_{\mathcal{V}}[X, X]$ (with tensor product given by composition in $\mathrm{Sym}_{\mathcal{V}}$) are exactly those defined in [**4**] in order to characterize operads as monoids.

We begin in Section 3.1 by reviewing some material on free symmetric monoidal categories. Section 3.2 introduces the bicategory of S-distributors and establishes some basic facts about it. In particular, we prove that it has coproducts. In Section 3.3 we then define the bicategory of categorical symmetric sequences by duality, letting

$$\mathrm{CatSym}_{\mathcal{V}} =_{\mathrm{def}} (S\text{-}\mathrm{Dist}_{\mathcal{V}})^{\mathrm{op}} \,.$$

We also define the analytic functor associated to a categorical symmetric sequence and present the sub-category $\mathrm{Sym}_{\mathcal{V}}$ of $\mathrm{CatSym}_{\mathcal{V}}$ spanned by sets. The morphisms in that bicategory are called symmetric sequences and they can be thought of as the many-sorted generalization of the single-sorted symmetric sequences used in connection with single-sorted operads. Indeed, we will use $\mathrm{Sym}_{\mathcal{V}}$ in Chapter 4 to define the bicategory of operad bimodules. The proof that $\mathrm{CatSym}_{\mathcal{V}}$ is cartesian closed is given in Section 3.4, which concludes the chapter.

3.1. Free symmetric monoidal $\mathcal{V}$-categories

The forgetful 2-functor $U \colon \mathrm{SMonCat}_{\mathcal{V}} \to \mathrm{Cat}_{\mathcal{V}}$ has a left adjoint $S \colon \mathrm{Cat}_{\mathcal{V}} \to \mathrm{SMonCat}_{\mathcal{V}}$ which associates to a small $\mathcal{V}$-category $\mathbb{X}$ the symmetric monoidal $\mathcal{V}$-category $S(\mathbb{X})$ freely generated by $\mathbb{X}$ [**15**]. This $\mathcal{V}$-category is defined by letting

$$S(\mathbb{X}) =_{\mathrm{def}} \bigsqcup_{n \in \mathbb{N}} S^n(\mathbb{X}) \,, \tag{3.1.1}$$

where $S^n(\mathbb{X})$ is the *symmetric n-power* of $\mathbb{X}$. More explicitly, observe that the n-th symmetric group Σ_n acts naturally on the $\mathcal{V}$-category $\mathbb{X}^n$ with the right action defined by letting

$$\bar{x} \cdot \sigma =_{\mathrm{def}} (x_{\sigma 1}, \ldots, x_{\sigma n}) \,,$$

for $\bar{x} = (x_1, \dots, x_n) \in \mathbb{X}^n$ and $\sigma \in \Sigma_n$. If we apply the Grothendieck construction to this right action, we obtain the symmetric n-power of $\mathbb{X}$,

$$S^n(\mathbb{X}) =_{\mathrm{def}} \Sigma_n \int \mathbb{X}^n \,.$$

Explicitly, an object of $S^n(\mathbb{X})$ is a sequence $\bar{x} = (x_1, \dots, x_n)$ of objects of $\mathbb{X}$, and the hom-object between $\bar{x}, \bar{y} \in S^n(\mathbb{X})$ is defined by letting

$$S^n(\mathbb{X})\big[\bar{x}, \bar{y}\big] =_{\mathrm{def}} \bigsqcup_{\sigma \in \Sigma_n} \mathbb{X}[x_1, y_{\sigma(1)}] \otimes \cdots \otimes \mathbb{X}[x_n, y_{\sigma(n)}] \,,$$

where the coproduct on the right-hand side is taken in $\mathcal{V}$. The tensor product of $\bar{x} \in S^m(\mathbb{X})$ and $\bar{y} \in S^n(\mathbb{X})$ is the concatenation

$$\bar{x} \oplus \bar{y} =_{\mathrm{def}} (x_1, \dots, x_m, y_1, \dots, y_n) \,.$$

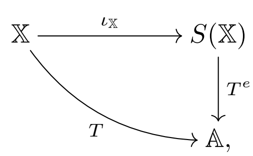
The symmetry $\sigma_{\bar{x},\bar{y}} \colon \bar{x} \oplus \bar{y} \to \bar{y} \oplus \bar{x}$ is the shuffle permutation swapping the first m-elements of the first sequence with the last m-elements of the second. The unit is the empty sequence e. The inclusion $\mathcal{V}$-functor

$$\iota_{\mathbb{X}} \colon \mathbb{X} \to S(\mathbb{X}) \,,$$

which takes $x \in \mathbb{X}$ to the one-element sequence $(x) \in S^1(\mathbb{X})$, exhibits $S(\mathbb{X})$ as the free symmetric monoidal $\mathcal{V}$-category on $\mathbb{X}$. More precisely, for every symmetric monoidal $\mathcal{V}$-category $\mathbb{A} = (\mathbb{A}, \oplus, 0, \sigma)$ the restriction functor

$$\iota_{\mathbb{X}}^* \colon \mathrm{SMonCat}_{\mathcal{V}}[S(\mathbb{X}), \mathbb{A}] \to \mathrm{Cat}_{\mathcal{V}}[\mathbb{X}, \mathbb{A}] \,,$$

defined by letting $\iota_{\mathbb{X}}^*(v) =_{\mathrm{def}} v \circ \iota_{\mathbb{X}}$, is an equivalence of categories. It follows that every $\mathcal{V}$-functor $T \colon \mathbb{X} \to \mathbb{A}$ admits a symmetric monoidal extension $T^e \colon S(\mathbb{X}) \to \mathbb{A}$ fitting in the diagram,



and that T^e is unique up to unique isomorphism of symmetric monoidal $\mathcal{V}$-functors. Explicitly, for $\bar{x} = (x_1, \dots, x_n) \in S(\mathbb{X})$, we have

$$T^e(\bar{x}) =_{\mathrm{def}} T(x_1) \oplus \ldots \oplus T(x_n) \,.$$

Our next theorem shows how the adjunction between $\mathrm{Cat}_{\mathcal{V}}$ and $\mathrm{SMonCat}_{\mathcal{V}}$ extends to an adjunction between $\mathrm{Dist}_{\mathcal{V}}$ and $\mathrm{SMonDist}_{\mathcal{V}}$.

THEOREM 3.1.1. *The forgetful homomorphism* $U \colon \mathrm{SMonDist}_{\mathcal{V}} \to \mathrm{Dist}_{\mathcal{V}}$ *has a left adjoint*

$$S \colon \mathrm{Dist}_{\mathcal{V}} \to \mathrm{SMonDist}_{\mathcal{V}} \,.$$

PROOF. Let $\iota = \iota_{\mathbb{X}} \colon \mathbb{X} \to S(\mathbb{X})$, so that we have a distributor $\iota_\bullet \colon \mathbb{X} \to S(\mathbb{X})$. We need to show that the restriction functor

$$\iota_\bullet^* \colon \mathrm{SMonDist}_{\mathcal{V}}[S(\mathbb{X}), \mathbb{A}] \to \mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{A}]$$

defined by letting $\iota_\bullet^*(F) =_{\mathrm{def}} F \circ \iota_\bullet$ is an equivalence of categories for any symmetric monoidal $\mathcal{V}$-category $\mathbb{A}$. The following diagram commutes up to isomorphism by part (i) of Lemma 1.3.2

$$\begin{array}{ccc} \mathrm{SMonDist}_{\mathcal{V}}[S(\mathbb{X}), \mathbb{A}] & \xrightarrow{(-)\circ \iota_\bullet} & \mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{A}] \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \mathrm{SMonCat}_{\mathcal{V}}[S(\mathbb{X}), P(\mathbb{A})] & \xrightarrow[(-)\circ \iota]{} & \mathrm{CAT}_{\mathcal{V}}\big[\mathbb{X}, P(\mathbb{A})\big] \, . \end{array}$$

The bottom side of this diagram is an equivalence of categories. Hence also the top side, since the vertical sides are equivalences. □

We give an explicit formula for the symmetric monoidal extension $F^e \colon S(\mathbb{X}) \to \mathbb{A}$ of a distributor $F \colon \mathbb{X} \to \mathbb{A}$, which fits in the diagram of S-distributors

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{(\iota_{\mathbb{X}})_\bullet} & S(\mathbb{X}), \\ & {\scriptstyle F} \searrow & \downarrow{\scriptstyle F^e} \\ & & \mathbb{A} \end{array}$$

and is the unique such distributor up to a unique isomorphism of symmetric monoidal $\mathcal{V}$-distributors. By construction, the functor $\lambda(F^e) \colon S(\mathbb{X}) \to P(\mathbb{A})$ is the symmetric monoidal extension of the $\mathcal{V}$-functor $\lambda F \colon \mathbb{X} \to P(\mathbb{A})$. Thus, $\lambda(F^e) = (\lambda F)^e$ and it follows that

$$F^e[y; \bar{x}] =_{\mathrm{def}} \int^{y_1 \in \mathbb{A}} \ldots \int^{y_n \in \mathbb{A}} F[y_1, x_1] \otimes \cdots \otimes F[y_n, x_n] \otimes \mathbb{A}[y, y_1 \oplus \cdots \oplus y_n] \, , \qquad (3.1.2)$$

for $\bar{x} = (x_1, \ldots, x_n) \in S^n(\mathbb{X})$ and $y \in \mathbb{A}$. A special case of these definitions that will be of importance for our development arises by considering $\mathbb{A} = S(\mathbb{Y})$, where $\mathbb{Y}$ is a small $\mathcal{V}$-category. In this case, the symmetric monoidal extension $F^e \colon S(\mathbb{X}) \to S(\mathbb{Y})$ of a distributor $F \colon \mathbb{X} \to S(\mathbb{Y})$ is defined by letting

$$F^e[\bar{y}; \bar{x}] =_{\mathrm{def}} \int^{\bar{y}_1 \in S(\mathbb{Y})} \cdots \int^{\bar{y}_n \in S(\mathbb{Y})} F[\bar{y}_1; x_1] \otimes \cdots \otimes F[\bar{y}_n; x_n] \otimes S(\mathbb{Y})[\bar{y}, \bar{y}_1 \oplus \cdots \oplus \bar{y}_n] \, , \qquad (3.1.3)$$

for $\bar{x} = (x_1, \ldots, x_n) \in S^n(\mathbb{X})$ and $\bar{y} \in S(\mathbb{Y})$.

It will be useful to describe the action of the homomorphism $S \colon \mathrm{Dist}_{\mathcal{V}} \to \mathrm{SMonDist}_{\mathcal{V}}$ on morphisms. For a distributor $F \colon \mathbb{X} \to \mathbb{Y}$, the symmetric monoidal distributor $S(F) \colon S(\mathbb{X}) \to S(\mathbb{Y})$ is defined by letting $S(F) =_{\mathrm{def}} (\iota_\bullet \circ F)^e$ and therefore makes following diagram commute up to a canonical isomorphism

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\iota_\bullet} & S(\mathbb{X}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle S(F)} \\ \mathbb{Y} & \xrightarrow[\iota_\bullet]{} & S(\mathbb{Y}). \end{array}$$

Explicitly, for $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_m)$, we have

$$S(F)[\bar{y}, \bar{x}] =_{\mathrm{def}} \begin{cases} \bigsqcup_{\sigma \in \Sigma_n} F[y_1, x_{\sigma(1)}] \otimes \cdots \otimes F[y_n, x_{\sigma(n)}] \, , & \text{if } m = n \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

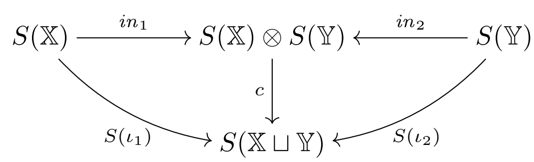
We conclude this section by stating an important property of the interaction between tensor products and coproducts of $\mathcal{V}$-categories. For $\mathbb{X}, \mathbb{Y} \in \mathrm{Cat}_\mathcal{V}$, define the $\mathcal{V}$-functor

$$c_{\mathbb{X},\mathbb{Y}} \colon S(\mathbb{X}) \otimes S(\mathbb{Y}) \to S(\mathbb{X} \sqcup \mathbb{Y}), \tag{3.1.4}$$

by letting $c_{\mathbb{X},\mathbb{Y}}(\bar{x} \otimes \bar{y}) =_{\mathrm{def}} \bar{x} \oplus \bar{y}$. The next proposition will be used in the proofs of Proposition 3.2.6 and Theorem 3.4.2.

PROPOSITION 3.1.2. *For every* $\mathbb{X}, \mathbb{Y} \in \mathrm{Cat}_\mathcal{V}$*, the* $\mathcal{V}$*-functor* $c_{\mathbb{X},\mathbb{Y}} \colon S(\mathbb{X}) \otimes S(\mathbb{Y}) \to S(\mathbb{X} \sqcup \mathbb{Y})$ *is a symmetric monoidal equivalence.*

PROOF. Let us write c for $c_{\mathbb{X},\mathbb{Y}}$. It is easy to verify that $c \colon S(\mathbb{X}) \otimes S(\mathbb{Y}) \to S(\mathbb{X} \sqcup \mathbb{Y})$ is symmetric monoidal. Let us show that it is an equivalence. The 2-functor $S \colon \mathrm{Cat}_\mathcal{V} \to \mathrm{SMonCat}_\mathcal{V}$ preserves coproducts, since it is a left adjoint. Thus, if $\iota_1 \colon \mathbb{X} \to \mathbb{X} \sqcup \mathbb{Y}$ and $\iota_2 \colon \mathbb{Y} \to \mathbb{X} \sqcup \mathbb{Y}$ are the inclusions, then the functors $S(\iota_1)$ and $S(\iota_2)$ exhibit $S(\mathbb{X} \sqcup \mathbb{Y})$ as the coproduct of $S(\mathbb{X})$ and $S(\mathbb{Y})$. But the functors $in_1 \colon S(\mathbb{X}) \to S(\mathbb{X}) \otimes S(\mathbb{Y})$ and $in_2 \colon S(\mathbb{Y}) \to S(\mathbb{X}) \otimes S(\mathbb{Y})$ defined by letting $in_1(\bar{x}) =_{\mathrm{def}} \bar{x} \otimes e$ and $in_2(\bar{y}) =_{\mathrm{def}} e \otimes \bar{y}$, where we write e for the empty sequence in both $S(\mathbb{X})$ and $S(\mathbb{Y})$, exhibit $S(\mathbb{X}) \otimes S(\mathbb{Y})$ as the coproduct of $S(\mathbb{X})$ and $S(\mathbb{Y})$ by Lemma 2.3.1 It follows that the $\mathcal{V}$-functor $c \colon S(\mathbb{X}) \otimes S(\mathbb{Y}) \to S(\mathbb{X} \sqcup \mathbb{Y})$ is an equivalence, since the diagram



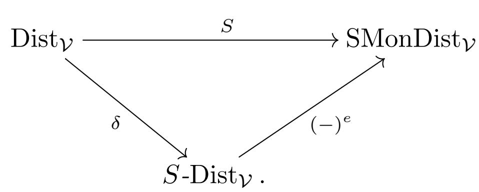
commutes. □

3.2. S-distributors

DEFINITION 3.2.1. Let $\mathbb{X}, \mathbb{Y}$ be small $\mathcal{V}$-categories. An *S-distributor* $F \colon \mathbb{X} \to \mathbb{Y}$ is a distributor $F \colon \mathbb{X} \to S(\mathbb{Y})$, i.e. a $\mathcal{V}$-functor $F \colon S(\mathbb{Y})^{\mathrm{op}} \otimes \mathbb{X} \to \mathcal{V}$.

We write $F[\bar{y}; x]$ for the result of applying an S-distributor $F \colon \mathbb{X} \to \mathbb{Y}$ to $(\bar{y}, x) \in S(\mathbb{Y})^{\mathrm{op}} \otimes \mathbb{X}$. The next theorem introduces the bicategory of S-distributors. Once again, its proof uses a Gabriel factorization.

THEOREM 3.2.2. *Small* $\mathcal{V}$*-categories,* S*-distributors and* $\mathcal{V}$*-transformations form a bicategory, called the bicategory of* S*-distributors and denoted by* $S\text{-}\mathrm{Dist}_\mathcal{V}$*, which fits into a Gabriel factorization diagram*



PROOF. Let $\mathbb{X}$, $\mathbb{Y}$ be small $\mathcal{V}$-categories. We define the hom-category of S-distributors from $\mathbb{X}$ to $\mathbb{Y}$ by letting $S\text{-}\mathrm{Dist}_\mathcal{V}[\mathbb{X}, \mathbb{Y}] =_{\mathrm{def}} \mathrm{Dist}_\mathcal{V}[\mathbb{X}, S(\mathbb{Y})]$. We then define a functor

$$(-)^e \colon S\text{-}\mathrm{Dist}_\mathcal{V}[\mathbb{X}, \mathbb{Y}] \to \mathrm{SMonDist}_\mathcal{V}[S(\mathbb{X}), S(\mathbb{Y})] \tag{3.2.1}$$

by mapping an S-distributor $F\colon \mathbb{X} \to \mathbb{Y}$, given by a distributor $F\colon \mathbb{X} \to S(\mathbb{Y})$, to its symmetric monoidal extension $F^e\colon S(\mathbb{X}) \to S(\mathbb{Y})$, defined as in (3.1.3). We represent the action of this functor by the derivation

$$\frac{F\colon \mathbb{X} \to \mathbb{Y}}{F^e\colon S(\mathbb{X}) \to S(\mathbb{Y}).}$$

Given two S-distributors $F\colon \mathbb{X} \to \mathbb{Y}$ and $G\colon \mathbb{Y} \to \mathbb{Z}$, we define their composite $G \circ F\colon \mathbb{X} \to \mathbb{Z}$ so as to have an isomorphism $(G \circ F)^e \cong G^e \circ F^e$. For $\bar{x} \in S(\mathbb{X})$ and $\bar{z} \in S(\mathbb{Z})$, we have

$$(G^e \circ F^e)[\bar{z};\bar{x}] = \int^{\bar{y} \in S(\mathbb{Y})} G^e[\bar{z};\bar{y}] \otimes F^e[\bar{y};\bar{x}]\,,$$

and therefore, for $x \in \mathbb{X}$ and $\bar{z} \in S(\mathbb{Z})$, we let

$$(G \circ F)[\bar{z};x] =_{\mathrm{def}} \int^{\bar{y} \in S(\mathbb{Y})} G^e[\bar{z};\bar{y}] \otimes F[\bar{y};x]\,. \tag{3.2.2}$$

Here, the distributor $G^e\colon S(\mathbb{Y}) \to S(\mathbb{Z})$ is defined via the formula in (3.1.3). The horizontal composition of S-distributors is functorial, coherently associative since the horizontal composition of the bicategory $\mathrm{SMonDist}_{\mathcal{V}}$ is so. The identity S-distributor $\mathrm{Id}_{\mathbb{X}}\colon \mathbb{X} \to \mathbb{X}$ is determined in an analogous way as the composition operation; explicitly, we define it as the distributor $(\iota_{\mathbb{X}})_\bullet\colon \mathbb{X} \to S(\mathbb{X})$, where $\iota_{\mathbb{X}}\colon \mathbb{X} \to S(\mathbb{X})$ is the inclusion $\mathcal{V}$-functor. Thus,

$$\mathrm{Id}_{\mathbb{X}}[x_1, \ldots, x_n; x] =_{\mathrm{def}} S(\mathbb{X})\big[(x_1, \ldots, x_n), (x)\big] = \begin{cases} \mathbb{X}[x_1, x] & \text{if } n = 1\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.3}$$

This completes the definition of the bicategory $S\text{-}\mathrm{Dist}_{\mathcal{V}}$. The homomorphism

$$(-)^e\colon S\text{-}\mathrm{Dist}_{\mathcal{V}} \to \mathrm{SMonDist}_{\mathcal{V}}$$

is then defined as follows. Its action on objects is defined by letting $\mathbb{X}^e =_{\mathrm{def}} S(\mathbb{X})$, while its action on hom-categories is given by the functors in (3.2.1). We complete the definition of the required Gabriel factorization by defining the homomorphism $\delta\colon \mathrm{Dist}_{\mathcal{V}} \to S\text{-}\mathrm{Dist}_{\mathcal{V}}$. Its action on object is the identity. Given a distributor $F\colon \mathbb{X} \to \mathbb{Y}$ we define the S-distributor $\delta(F)\colon \mathbb{X} \to \mathbb{Y}$ by letting $\delta(F) =_{\mathrm{def}} \iota_\bullet \circ F$, where $\iota\colon \mathbb{Y} \to S(\mathbb{Y})$ is the inclusion $\mathcal{V}$-functor. In this way, we have

$$S(F) \cong (\iota_\bullet \circ F)^e = \delta(F)^e\ . \tag{3.2.4}$$

Explicitly, we have

$$\delta(F)[\bar{y}, x] =_{\mathrm{def}} \begin{cases} F[y, x] & \text{if } \bar{y} = (y) \in S^1(\mathbb{Y})\,, \\ 0 & \text{otherwise.} \end{cases}$$

For distributors $F\colon \mathbb{X} \to \mathbb{Y}$ and $G\colon \mathbb{Y} \to \mathbb{Z}$, the pseudo-functoriality isomorphism

$$\delta(G) \circ \delta(F) \cong \delta(G \circ F)$$

is obtained combining the fact that the functor in (3.2.1) is an equivalence with the existence of an isomorphism

$$\delta(G \circ F)^e \cong S(G \circ F) \cong S(G) \circ S(F) \cong \delta(G)^e \circ \delta(F)^e\,,$$

where we used the isomorphism in (3.2.4) twice. In a similar way one obtains an isomorphism $\delta(\mathrm{Id}_{\mathbb{X}}) \cong \mathrm{Id}_{\mathbb{X}}$ for every small $\mathcal{V}$-category $\mathbb{X}$. □

REMARK 3.2.3. For every pair of small $\mathcal{V}$-categories $\mathbb{X}$, $\mathbb{Y}$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{Y}] & \xrightarrow{\lambda} & \mathrm{Cat}_{\mathcal{V}}[\mathbb{X}, PS(\mathbb{Y})] \\ {\scriptstyle (-)^e}\downarrow & & \downarrow{\scriptstyle (-)^e} \\ \mathrm{SMonDist}_{\mathcal{V}}[S(\mathbb{X}), S(\mathbb{Y})] & \xrightarrow[\lambda]{} & \mathrm{SMonCat}_{\mathcal{V}}[S(\mathbb{X}), PS(\mathbb{Y})]\,. \end{array}$$

Thus, for a distributor $F\colon \mathbb{X} \to \mathbb{Y}$, we write $\lambda F^e\colon S(\mathbb{X}) \to PS(\mathbb{Y})$ for the common value of the composite functors. Note that the functor $(\lambda F^e)_c\colon PS(\mathbb{X}) \to PS(\mathbb{Y})$ is a homomorphism of symmetric $\mathcal{V}$-rigs. Symbolically,

$$\frac{\dfrac{\dfrac{F\colon \mathbb{X} \to \mathbb{Y} \text{ in } S\text{-}\mathrm{Dist}_{\mathcal{V}}}{F^e\colon S(\mathbb{X}) \to S(\mathbb{Y}) \text{ in } \mathrm{SMonDist}_{\mathcal{V}}}}{\lambda F^e\colon S(\mathbb{X}) \to PS(\mathbb{Y}) \text{ in } \mathrm{SMonCat}_{\mathcal{V}}}}{(\lambda F^e)_c\colon PS(\mathbb{X}) \to PS(\mathbb{Y}) \text{ in } \mathrm{SRig}_{\mathcal{V}}\,.}$$

LEMMA 3.2.4. *If $\mathbb{X}_1$ and $\mathbb{X}_2$ are small $\mathcal{V}$-categories and $\iota_k\colon \mathbb{X}_k \to \mathbb{X}_1 \sqcup \mathbb{X}_2$ is the k-th inclusion, then the symmetric monoidal distributors $S(\iota_k)_\bullet\colon S(\mathbb{X}_k) \to S(\mathbb{X}_1 \sqcup \mathbb{X}_2)$ for $k = 1, 2$ exhibit $S(\mathbb{X}_1 \sqcup \mathbb{X}_2)$ as the coproduct of $S(\mathbb{X}_1)$ and $S(\mathbb{X}_2)$ in the bicategory* $\mathrm{SMonDist}_{\mathcal{V}}$.

PROOF. The distributors $(\iota_k)_\bullet\colon \mathbb{X}_k \to \mathbb{X}_1 \sqcup \mathbb{X}_2$ (for $k = 1, 2$) exhibit $\mathbb{X}_1 \sqcup \mathbb{X}_2$ as the coproduct of $\mathbb{X}_1$ and $\mathbb{X}_2$ in the bicategory $\mathrm{Dist}_{\mathcal{V}}$ by Proposition 1.3.3. Hence the symmetric monoidal distributors $S(\iota_k)_\bullet = S((\iota_k)_\bullet)\colon S(\mathbb{X}_k) \to S(\mathbb{X}_1 \sqcup \mathbb{X}_2)$ exhibits the coproduct of $S(\mathbb{X}_1)$ and $S(\mathbb{X}_2)$ in the bicategory $\mathrm{SMonDist}_{\mathcal{V}}$, since the homomorphism $S\colon \mathrm{Dist}_{\mathcal{V}} \to \mathrm{SMonDist}_{\mathcal{V}}$ is a left adjoint and hence preserves finite coproducts. □

PROPOSITION 3.2.5. *The bicategory S-$\mathrm{Dist}_{\mathcal{V}}$ has finite coproducts and the homomorphisms*

$$\delta\colon \mathrm{Dist}_{\mathcal{V}} \to S\text{-}\mathrm{Dist}_{\mathcal{V}}\,, \quad (-)^e\colon S\text{-}\mathrm{Dist}_{\mathcal{V}} \to \mathrm{SMonDist}_{\mathcal{V}}$$

preserve finite coproducts.

PROOF. This follows from Lemma 3.2.4, since the homomorphism $S\colon \mathrm{Dist}_{\mathcal{V}} \to \mathrm{SMonDist}_{\mathcal{V}}$ preserves finite coproducts. □

We establish an explicit formula for the coproduct homomorphism on S-distributors, which will be used in the proof of Theorem 3.4.2. By definition, the coproduct of two S-distributors $F_1\colon \mathbb{X}_1 \to \mathbb{Y}_1$ and $F_2\colon \mathbb{X}_2 \to \mathbb{Y}_2$ is a S-distributor $F_1 \sqcup_S F_2\colon \mathbb{X}_1 \sqcup \mathbb{X}_2 \to \mathbb{Y}_1 \sqcup \mathbb{Y}_2$ fitting in a commutative diagram of S-distributors,

$$\begin{array}{ccc} \mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{Y}_1 \\ {\scriptstyle \delta(i^1_\bullet)}\downarrow & & \downarrow{\scriptstyle \delta(j^1_\bullet)} \\ \mathbb{X}_1 \sqcup \mathbb{X}_2 & \xrightarrow{F_1 \sqcup_S F_2} & \mathbb{Y}_1 \sqcup \mathbb{Y}_2 \\ {\scriptstyle \delta(i^2_\bullet)}\uparrow & & \uparrow{\scriptstyle \delta(j^2_\bullet)} \\ \mathbb{X}_2 & \xrightarrow[F_2]{} & \mathbb{Y}_2\,, \end{array} \tag{3.2.5}$$

where $i^k\colon \mathbb{X}_k \to \mathbb{X}_1 \sqcup \mathbb{X}_2$ and $j^k\colon \mathbb{Y}_k \to \mathbb{Y}_1 \sqcup \mathbb{Y}_2$ are the inclusions.

PROPOSITION 3.2.6. *Given S-distributors $F_1 \colon \mathbb{X}_1 \to \mathbb{Y}_1$ and $F_2 \colon \mathbb{X}_2 \to \mathbb{Y}_2$, then*

$$(F_1 \sqcup_S F_2)^e[\bar{y}_1 \oplus \bar{y}_2, \bar{x}_1 \oplus \bar{x}_2] \cong F_1^e[\bar{y}_1, \bar{x}_1] \otimes F_2^e[\bar{y}_2, \bar{x}_2]$$

for $\bar{x}_1 \in S(\mathbb{X}_1)$, $\bar{x}_2 \in S(\mathbb{X}_2)$, $\bar{y}_1 \in S(\mathbb{Y}_1)$ and $\bar{y}_2 \in S(\mathbb{Y}_2)$.

PROOF. The image of the diagram in (3.2.5) by the homomorphism $(-)^e \colon S\text{-Dist}_{\mathcal{V}} \to \text{SMonDist}_{\mathcal{V}}$ is a commutative diagram of symmetric monoidal distributors,

$$\begin{array}{ccc}
S(\mathbb{X}_1) & \xrightarrow{\quad F_1^e \quad} & S(\mathbb{Y}_1) \\
{\scriptstyle S(i_1)_\bullet}\downarrow & & \downarrow{\scriptstyle S(j_1)_\bullet} \\
S(\mathbb{X}_1 \sqcup \mathbb{X}_2) & \xrightarrow{(F_1 \oplus F_2)^e} & S(\mathbb{Y}_1 \sqcup \mathbb{Y}_2) \\
{\scriptstyle S(i_2)_\bullet}\uparrow & & \uparrow{\scriptstyle S(j_2)_\bullet} \\
S(\mathbb{X}_2) & \xrightarrow[\quad F_2^e \quad]{} & S(\mathbb{Y}_2)\,,
\end{array}$$

from which we obtain the following commutative square of symmetric monoidal distributors,

$$\begin{array}{ccc}
S(\mathbb{X}_1) \otimes S(\mathbb{X}_2) & \xrightarrow{F_1^e \otimes F_2^e} & S(\mathbb{Y}_1) \otimes S(\mathbb{Y}_2) \\
{\scriptstyle c_\bullet}\downarrow & & \downarrow{\scriptstyle c_\bullet} \\
S(\mathbb{X}_1 \sqcup \mathbb{X}_2) & \xrightarrow[(F_1 \oplus F_2)^e]{} & S(\mathbb{Y}_1 \sqcup \mathbb{Y}_2)\,,
\end{array}$$

where $c \colon S(\mathbb{X}_1) \otimes S(\mathbb{X}_2) \to S(\mathbb{X}_1 \sqcup \mathbb{X}_2)$ is the $\mathcal{V}$-functor of (3.1.4), which in this case is defined by letting $c(\bar{x}_1 \otimes \bar{x}_2) =_{\text{def}} \bar{x}_1 \oplus \bar{x}_2$. We have $c^\bullet \circ c_\bullet \cong \text{Id}_{S(\mathbb{Y}_1) \otimes S(Y_2)}$, since c is an equivalence by Proposition 3.1.2. Hence the following diagram of distributors commutes

$$\begin{array}{ccc}
S(\mathbb{X}_1) \otimes S(\mathbb{X}_2) & \xrightarrow{F_1^e \otimes F_2^e} & S(\mathbb{Y}_1) \otimes S(\mathbb{Y}_2) \\
{\scriptstyle c_\bullet}\downarrow & & \uparrow{\scriptstyle c^\bullet} \\
S(\mathbb{X}_1 \sqcup \mathbb{X}_2) & \xrightarrow[(F_1 \oplus F_2)^e]{} & S(\mathbb{Y}_1 \sqcup \mathbb{Y}_2)\,.
\end{array}$$

This proves the result, since

$$(F_1^e \otimes F_2^e)[\bar{y}_1 \otimes \bar{y}_2, \bar{x}_1 \otimes \bar{x}_2] =_{\text{def}} F_1^e[\bar{y}_1, \bar{x}_1] \otimes F_2^e[\bar{y}_2, \bar{x}_2]$$

and, by part (ii) of Lemma 1.3.2, we have

$$(c^\bullet \circ (F_1 \oplus F_2)^e \circ c_\bullet)[\bar{y}_1 \otimes \bar{y}_2, \bar{x}_1 \otimes \bar{x}_2] \cong (F_1 \oplus F_2)^e[\bar{y}_1 \oplus \bar{y}_2, \bar{x}_1 \oplus \bar{x}_2]\,. \qquad \square$$

REMARK 3.2.7 (The bicategory of S-matrices). We give an explicit description of the full the sub-bicategory of $S\text{-Dist}_{\mathcal{V}}$ spanned by sets, in analogy with what we did in Remark 1.3.5, where we showed how the bicategory of matrices $\text{Mat}_{\mathcal{V}}$ can be seen as the full sub-bicategory of the bicategory of distributors spanned by sets. If $\mathbb{M}$ is a symmetric monoidal category, then $\mathbb{M} \cdot \text{I}$ has the structure of a symmetric monoidal $\mathcal{V}$-category and the functor mapping $\mathbb{M}$ to $\mathbb{M} \cdot \text{I}$ is left adjoint to the functor $\text{Und} \colon \text{SMonCat}_{\mathcal{V}} \to \text{SMonCat}$ mapping a symmetric monoidal $\mathcal{V}$-category to its underlying symmetric monoidal category. We also have a natural isomorphism,

$$S(\mathbb{C}) \cdot \text{I} \cong S(\mathbb{C} \cdot \text{I})$$

since the diagram

$$\begin{array}{ccc} \mathrm{SMonCat}_{\mathcal{V}} & \xrightarrow{\mathrm{Und}} & \mathrm{SMonCat} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Cat}_{\mathcal{V}} & \xrightarrow[\mathrm{Und}]{} & \mathrm{Cat} \end{array}$$

commutes and a composite of left adjoints is left adjoint to the composite. As a special case of the definitions in Section 3.1, the free symmetric monoidal category $S(X)$ on a set X admits the following direct description. For $n \in \mathbb{N}$, let $S^n(X)$ be the category whose objects are sequences $\bar{x} = (x_1, \ldots, x_n)$ of elements of X and whose morphisms $\sigma \colon (x_1, \ldots, x_n) \to (x'_1, \ldots, x'_n)$ are permutations $\sigma \in \Sigma_n$ such that $x'_i = x_{\sigma(i)}$ for $1 \leq i \leq n$. We then let $S(X)$ be the coproduct of the categories $S_n(X)$, for $n \in \mathbb{N}$,

$$S(X) =_{\mathrm{def}} \bigsqcup_{n \in \mathbb{N}} S^n(X)\,.$$

For sets X and Y, we define an *S-matrix* $F \colon X \to Y$ to be a functor $F \colon S(Y)^{\mathrm{op}} \times X \to \mathcal{V}$. Sets, S-matrices and natural transformations form a bicategory $S\text{-}\mathrm{Mat}_{\mathcal{V}}$ which can be identified with the full sub-bicategory of the bicategory $S\text{-}\mathrm{Dist}_{\mathcal{V}}$ of S-distributors spanned by discrete $\mathcal{V}$-categories. Indeed, for sets X and Y, we have the following chain of isomorphisms:

$$\begin{aligned} S\text{-}\mathrm{Mat}_{\mathcal{V}}[X, Y] &= \mathrm{Cat}[S(Y)^{\mathrm{op}} \times X\,, \mathcal{V}] \\ &\cong \mathrm{Cat}_{\mathcal{V}}\big[\big(S(Y)^{\mathrm{op}} \times X\big) \cdot \mathrm{I}, \mathcal{V}\big] \\ &\cong \mathrm{Cat}_{\mathcal{V}}\big[\big(S(Y)^{\mathrm{op}} \cdot \mathrm{I}\big) \otimes \big(X \cdot \mathrm{I}\big), \mathcal{V}\big] \\ &\cong \mathrm{Cat}_{\mathcal{V}}\big[\big(S(Y) \cdot \mathrm{I}\big)^{\mathrm{op}} \otimes \big(X \cdot \mathrm{I}\big), \mathcal{V}\big] \\ &\cong S\text{-}\mathrm{Dist}_{\mathcal{V}}[X \cdot \mathrm{I}\,, Y \cdot \mathrm{I}]\,. \end{aligned}$$

The composition operation and the identity morphisms of $S\text{-}\mathrm{Mat}_{\mathcal{V}}$ are determined by those of $S\text{-}\mathrm{Dist}_{\mathcal{V}}$ analogously to the way in which composition operation and the identity morphisms of $\mathrm{Mat}_{\mathcal{V}}$ are determined by those of $\mathrm{Dist}_{\mathcal{V}}$. We do not unfold these definitions, since in Section 4.4 we will describe explicitly its opposite bicategory. Note that we obtain the following diagram of inclusions:

$$\begin{array}{ccc} \mathrm{Mat}_{\mathcal{V}} & \longrightarrow & S\text{-}\mathrm{Mat}_{\mathcal{V}} \\ \downarrow & & \downarrow \\ \mathrm{Dist}_{\mathcal{V}} & \longrightarrow & S\text{-}\mathrm{Dist}_{\mathcal{V}}\,. \end{array}$$

We conclude this section by defining the operation of compositon of an S-distributor with a functor, in analogy with the definition of composition of a distributor with a functor, given in (1.3.7). Let $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig. Then the *composite* of an S-distributor $F \colon \mathbb{X} \to \mathbb{Y}$ with a $\mathcal{V}$-functor $T \colon \mathbb{Y} \to \mathcal{R}$ is the $\mathcal{V}$-functor $T \circ F \colon \mathbb{X} \to \mathcal{R}$ defined by letting

$$(T \circ F)(x) =_{\mathrm{def}} \int^{\bar{y} \in S(\mathbb{Y})} T^e(\bar{y}) \otimes F[\bar{y}; x]\,, \tag{3.2.6}$$

for $x \in \mathbb{X}$.

Lemma 3.2.8. *Let $\mathbb{X}, \mathbb{Y}$ be small $\mathcal{V}$-categories and $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig. For every S-distributor $F \colon \mathbb{X} \to \mathbb{Y}$ and $\mathcal{V}$-functor $T \colon \mathbb{Y} \to \mathcal{R}$, we have*

$$(T \circ F)^e \cong T^e \circ F^e\,,$$

where

$$(T^e \circ F^e)(\bar{x}) = \int^{\bar{y} \in S(\mathbb{Y})} T^e(\bar{y}) \otimes F^e[\bar{y}; \bar{x}] \, .$$

Proof. The functor $T^e \circ F^e \colon S(\mathbb{X}) \to \mathcal{R}$ is symmetric monoidal by Proposition 2.4.6, since the distributor $F^e \colon S(\mathbb{X}) \to S(\mathbb{Y})$ is symmetric monoidal and the $\mathcal{V}$-functor $T^e \colon S(\mathbb{Y}) \to \mathcal{R}$ is symmetric monoidal. Moreover, if $\bar{x} = (x)$ with $x \in \mathbb{X}$, then

$$(T^e \circ F^e)((x)) = \int^{\bar{y} \in S(\mathbb{Y})} T^e(\bar{y}) \otimes F^e[\bar{y}; (x)] = \int^{\bar{y} \in S(\mathbb{Y})} T^e(\bar{y}) \otimes F[\bar{y}; x] = (T \circ F)(x) \, .$$

Hence the symmetric monoidal functor $T^e \circ F^e \colon S(\mathbb{X}) \to \mathcal{R}$ is an extension of $T \circ F \colon \mathbb{X} \to \mathcal{R}$. This shows that $(T \circ F)^e \cong T^e \circ F^e$ by the uniqueness up to unique isomorphism of the extension. ☐

Proposition 3.2.9. *Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ be small $\mathcal{V}$-categories and $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig.*

(i) For all distributors $F \colon \mathbb{X} \to \mathbb{Y}$, $G \colon \mathbb{Y} \to \mathbb{Z}$ and $\mathcal{V}$-functors $T \colon \mathbb{Z} \to \mathcal{R}$, $(T \circ G) \circ F \cong T \circ (G \circ F)$.

(ii) For every $\mathcal{V}$-functor $T \colon \mathbb{X} \to \mathcal{R}$, $T \circ \mathrm{Id}_{\mathbb{X}} \cong T$.

Proof. For part (i), it suffices to show that $((T \circ G) \circ F)^e \cong (T \circ (G \circ F))^e$. But, by Proposition 1.3.7 and Lemma 3.2.8, we have

$$\begin{aligned} ((T \circ G) \circ F)^e &\cong (T \circ G)^e \circ F^e \\ &\cong (T^e \circ G^e) \circ F^e \\ &\cong T^e \circ (G^e \circ F^e) \\ &\cong T^e \circ (G \circ F)^e \, . \end{aligned}$$

The proof of part (ii) is analogous. ☐

3.3. Symmetric sequences and analytic functors

Definition 3.3.1. Let $\mathbb{X}, \mathbb{Y}$ be small $\mathcal{V}$-categories. A *categorical symmetric sequence* $F \colon \mathbb{X} \to \mathbb{Y}$ is an S-distributor from $\mathbb{Y}$ to $\mathbb{X}$, i.e. a $\mathcal{V}$-functor $F \colon S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y} \to \mathcal{V}$.

If $F \colon \mathbb{X} \to \mathbb{Y}$ is a categorical symmetric sequence, then for a symmetric $\mathcal{V}$-rig $\mathcal{R} = (\mathcal{R}, \diamond, e)$ we define its associated analytic functor $F \colon \mathcal{R}^{\mathbb{X}} \to \mathcal{R}^{\mathbb{Y}}$ by letting, for $T \in \mathcal{R}^{\mathbb{X}}$ and $y \in \mathbb{Y}$,

$$F(T)(y) =_{\mathrm{def}} \int^{\bar{x} \in S(\mathbb{X})} F[\bar{x}; y] \otimes T^{\bar{x}} \, ,$$

where, for $\bar{x} = (x_1, \ldots, x_n) \in S(\mathbb{X})$, $T^{\bar{x}} =_{\mathrm{def}} T^e(\bar{x}) = T(x_1) \diamond \ldots \diamond T(x_n)$. We represent the correspondence between categorical symmetric sequences and analytic functors as follows:

$$\frac{F \colon \mathbb{X} \to \mathbb{Y}}{F \colon \mathcal{R}^{\mathbb{X}} \to \mathcal{R}^{\mathbb{Y}} \, .}$$

Example 3.3.2. For $\mathcal{V} = \mathcal{R} = \mathrm{Set}$ and $\mathbb{X} = \mathbb{Y} = 1$, where 1 is the terminal category, we obtain exactly the notion of an analytic functor introduced in [**42**]. Indeed, a symmetric sequence $F \colon 1 \to 1$ is the same thing as a functor $F \colon \mathrm{S} \to \mathrm{Set}$. Here, $\mathrm{S} = S(1)$ is the category of natural numbers and permutations. The analytic functor $F \colon \mathrm{Set} \to \mathrm{Set}$ associated to such a symmetric sequence has the following form:

$$F(T) =_{\mathrm{def}} \int^{n \in \mathrm{S}} F[n] \otimes T^n \, .$$

See [**3, 13, 41**] for applications of the theory of analytic functors to combinatorics and [**2**] for recent work on categorical aspects of the theory.

EXAMPLE 3.3.3. For $\mathcal{V} = \mathcal{R} = \mathrm{Set}$, we obtain the notion of analytic functor between categories of covariant presheaves[1] considered in [**30**]. In that context, the analytic functor $F \colon \mathrm{Set}^{\mathbb{X}} \to \mathrm{Set}^{\mathbb{Y}}$ of a symmatric sequence $F \colon \mathbb{X} \to \mathbb{Y}$, i.e. a functor $F \colon S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y} \to \mathrm{Set}$, is obtained as a left Kan extension, fitting in the diagram

$$\begin{array}{ccc} S(\mathbb{X})^{\mathrm{op}} & \xrightarrow{\sigma_{\mathbb{X}}} & \mathrm{Set}^{\mathbb{X}} \\ & \searrow^{\lambda F} & \downarrow{\scriptstyle F} \\ & & \mathrm{Set}^{\mathbb{Y}} \, , \end{array}$$

where $\sigma_{\mathbb{X}} \colon S(\mathbb{X})^{\mathrm{op}} \to \mathrm{Set}^{\mathbb{X}}$ is the functor defined by letting

$$\sigma_{\mathbb{X}}(x_1, \ldots, x_n) =_{\mathrm{def}} \bigsqcup_{1 \leq k \leq n} \mathbb{X}[x_k, -] \, .$$

Hence, for $T \in \mathrm{Set}^{\mathbb{X}}$ and $y \in \mathbb{Y}$, we have

$$\begin{aligned} F(T)(y) &= \int^{\bar{x} \in S(\mathbb{X})} \lambda F(\bar{x})(y) \otimes \big[\sigma_{\mathbb{X}}(\bar{x}), T\big] \\ &= \bigsqcup_{n \in \mathbb{N}} \int^{\bar{x} \in S^n(\mathbb{X})} F[\bar{x}; y] \otimes \Big[\bigsqcup_{1 \leq k \leq n} \mathbb{X}[x_k, -], T \Big] \\ &= \bigsqcup_{n \in \mathbb{N}} \int^{\bar{x} \in S^n(\mathbb{X})} F[\bar{x}; y] \otimes \prod_{1 \leq k \leq n} P(\mathbb{X})\big[\mathbb{X}[x_k, -], T\big] \\ &= \bigsqcup_{n \in \mathbb{N}} \int^{\bar{x} \in S^n(\mathbb{X})} F[\bar{x}; y] \otimes \prod_{1 \leq k \leq n} T(x_k) \\ &= \int^{\bar{x} \in S(\mathbb{X})} F[\bar{x}; y] \otimes T^{\bar{x}} \, , \end{aligned}$$

where for $\bar{x} = (x_1, \ldots, x_n)$, we have $T^{\bar{x}} = T(x_1) \times \ldots \times T(x_n)$. Note how this construction of analytic functors as a left Kan extension does not carry over to the enriched setting.

Small $\mathcal{V}$-categories, categorical symmetric sequences and $\mathcal{V}$-natural transformations form a bicategory, called the bicategory of categorical symmetric sequences and denoted by $\mathrm{CatSym}_{\mathcal{V}}$. This bicategory is defined as the opposite of the bicategory $S\text{-}\mathrm{Dist}_{\mathcal{V}}$ of S-distributors:

$$\mathrm{CatSym}_{\mathcal{V}} =_{\mathrm{def}} (S\text{-}\mathrm{Dist}_{\mathcal{V}})^{\mathrm{op}} \, .$$

In particular, for small $\mathcal{V}$-categories $\mathbb{X}$ and $\mathbb{Y}$, we have

$$\mathrm{CatSym}_{\mathcal{V}}[\mathbb{X}, \mathbb{Y}] = S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, \mathbb{X}] = \mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, S(\mathbb{X})] = \mathrm{CAT}_{\mathcal{V}}[S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}, \mathcal{V}] \, .$$

[1]The analytic functors studied in [**30**] were between categories of presheaves, but we prefer to consider covariant preshaves to match our earlier definitions.

REMARK 3.3.4 (The bicategory of symmetric sequences). We write $\mathrm{Sym}_{\mathcal{V}}$ for the full sub-bicategory of $\mathrm{CatSym}_{\mathcal{V}}$ spanned by sets, viewed as discrete $\mathcal{V}$-categories. This bicategory can be defined simply as

$$\mathrm{Sym}_{\mathcal{V}} =_{\mathrm{def}} (S\text{-}\mathrm{Mat}_{\mathcal{V}})^{\mathrm{op}}\,,$$

where S-$\mathrm{Mat}_{\mathcal{V}}$ is full sub-bicategory of S-$\mathrm{Dist}_{\mathcal{V}}$ spanned by sets, as defined in Remark 3.2.7. We unfold some definitions since they will be useful in Section 4.4. The objects of $\mathrm{Sym}_{\mathcal{V}}$ are sets and the hom-category between two sets X and Y is defined by

$$\mathrm{Sym}_{\mathcal{V}}[X, Y] =_{\mathrm{def}} S\text{-}\mathrm{Mat}_{\mathcal{V}}[Y, X] = \mathrm{Cat}[S(X)^{\mathrm{op}} \times Y, \mathcal{V}]\,.$$

For sets X and Y, we define a *symmetric sequence* $F \colon X \to Y$ to be a functor $F \colon S(X)^{\mathrm{op}} \times Y \to \mathcal{V}$. For a symmetric $\mathcal{V}$-rig $\mathcal{R}$, the extension $F \colon \mathcal{R}^X \to \mathcal{R}^Y$ of such a symmetric sequence is given by

$$F(T)(y) =_{\mathrm{def}} \bigsqcup_{n \in \mathbb{N}} \int^{(x_1,\ldots,x_n)\in S^n(X)} F[x_1, \ldots, x_n; y] \otimes T(x_1) \otimes \ldots \otimes T(x_n)\,. \tag{3.3.1}$$

It will be useful to have also an explicit description of the composition operation in $\mathrm{Sym}_{\mathcal{V}}$. For sets X, Y, Z and symmetric sequences $F \colon X \to Y$, $G \colon Y \to Z$, their composite $G \circ F \colon X \to Z$ is given by

$$\begin{aligned}(G \circ F)(\bar{x}; z) =_{\mathrm{def}} \bigsqcup_{m\in\mathbb{N}} \int^{(y_1,\ldots,y_m)\in S^m(Y)} & G[y_1, \ldots, y_m; z] \otimes \\ \int^{\bar{x}_1 \in S(X)} \cdots \int^{\bar{x}_m \in S(X)} & [\bar{x}, \bar{x}_1 \oplus \ldots \oplus \bar{x}_m] \otimes F[\bar{x}_1; y_1] \otimes \cdots \otimes F[\bar{x}_m; y_m]\,. \end{aligned} \tag{3.3.2}$$

For a set X, the identity symmetric sequence $\mathrm{Id}_X \colon X \to X$ is defined by letting

$$\mathrm{Id}_X[\bar{x}; x] = \begin{cases} I & \text{if } \bar{x} = (x)\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.3}$$

EXAMPLE 3.3.5. Let $\mathrm{S} = S(1)$, the category of natural numbers and permutations. The monoidal structure on $\mathrm{Sym}_{\mathcal{V}}[1, 1] \cong [\mathrm{S}^{\mathrm{op}}, \mathcal{V}]$ given by the horizontal composition in $\mathrm{Sym}_{\mathcal{V}}$ as defined in (3.3.2) is exactly the substitution monoidal structure discussed in the introduction, which characterizes the notion of a single-sorted operad, in the sense that monoids in $[\mathrm{S}^{\mathrm{op}}, \mathcal{V}]$ with respect to this monoidal structure are exactly single-sorted operads [**46**]. Indeed, as a special case of the formula in (3.3.2), we get

$$(G \circ F)[n] = \int^{m\in\mathrm{S}} G[m] \otimes \int^{n_1\in\mathrm{S}} \cdots \int^{n_m\in\mathrm{S}} [n, n_1 + \ldots n_m] \otimes F[n_1] \otimes \cdots \otimes F[n_m]\,.$$

Similarly, the unit J for the substitution tensor product is given by a special case of the formula in (3.3.3):

$$J[n] = \begin{cases} I & \text{if } n = 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

We will see in Section 4.1 that the horizontal composition of $\mathrm{Sym}_{\mathcal{V}}$ can be used to characterize the notion of an operad.

We conclude this section by relating the composition of categorical symmetric sequences with the composition of analytic functors. In particular, it shows how the analytic functor associated to the composite of two categorical symmetric sequences is isomorphic to the composites of the analytic functors associated to the categorical symmetric sequences. We also show that the analytic functor associated to the identity categorical symmetric sequence is naturally isomorphic to the identity functor. This generalizes Theorem 3.2 in [**30**] to the enriched setting.

THEOREM 3.3.6. *Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ be small $\mathcal{V}$-categories and $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig.*

(i) For every pair of categorical symmetric sequences $F\colon \mathbb{X} \to \mathbb{Y}$ and $G\colon \mathbb{Y} \to \mathbb{Z}$, there is a natural isomorphism with components

$$(G \circ F)(T) \cong G(F(T)) \, .$$

(ii) There is a natural isomorphism with components

$$\mathrm{Id}_{\mathbb{X}}(T) \cong T \, .$$

PROOF. For this proof, it is convenient to have some auxiliary notation: for a categorical symmetric sequence $F\colon \mathbb{X} \to \mathbb{Y}$, we write $F^{\mathrm{op}}\colon \mathbb{Y} \to \mathbb{X}$ for the corresponding S-distributor. For part (i), let $T\colon \mathbb{X} \to \mathcal{R}$. We begin by showing that $F(T) = T \circ F^{\mathrm{op}}$, where $T \circ F^{\mathrm{op}}$ denotes the composite of the S-distributor $F^{\mathrm{op}}\colon \mathbb{Y} \to \mathbb{X}$ with the $\mathcal{V}$-functor $T\colon \mathbb{X} \to \mathcal{R}$, defined in (3.2.6). For $y \in \mathbb{Y}$, we have

$$\begin{aligned} F(T)(y) &= \int^{\bar{x} \in S(\mathbb{X})} F[\bar{x}; y] \otimes T^{\bar{x}} \\ &= \int^{\bar{x} \in S(\mathbb{X})} F[\bar{x}; y] \otimes T^{e}(\bar{x}) \\ &= (T \circ F^{\mathrm{op}})(y) \, . \end{aligned}$$

It then follows by part (i) of Proposition 3.2.9 that we have

$$\begin{aligned} (G \circ F)(T) &= T \circ (G \circ F)^{\mathrm{op}} \\ &= T \circ (F^{\mathrm{op}} \circ G^{\mathrm{op}}) \\ &\cong (T \circ F^{\mathrm{op}}) \circ G^{\mathrm{op}} \\ &= F(T) \circ G^{\mathrm{op}} \\ &= G(F(T)) \, . \end{aligned}$$

For part (ii), if $T\colon \mathbb{X} \to \mathcal{R}$, then by part (ii) of Proposition 3.2.9 we have

$$\begin{aligned} \mathrm{Id}_{\mathbb{X}}(T) &= T \circ \mathrm{Id}_{\mathbb{X}}^{\mathrm{op}} \\ &\cong T \, . \end{aligned}$$

□

3.4. Cartesian closure of categorical symmetric sequences

We conclude this chapter with our first main result, asserting that that the bicategory of categorical symmetric sequences is cartesian closed. This generalizes to the enriched setting the main result of [**30**]. The existence of products in $\mathrm{CatSym}_{\mathcal{V}}$ follows easily by our earlier results, but we state the result explicitly for emphasis.

PROPOSITION 3.4.1. *The bicategory $\mathrm{CatSym}_{\mathcal{V}}$ has finite products. In particular, the product of two small $\mathcal{V}$-categories $\mathbb{X}$ and $\mathbb{Y}$ in $\mathrm{CatSym}_{\mathcal{V}}$ is given by their coproduct $\mathbb{X} \sqcup \mathbb{Y}$ in $\mathrm{Cat}_{\mathcal{V}}$.*

PROOF. This follows by Proposition 3.2.5 by duality, since the bicategory S-$\mathrm{Dist}_{\mathcal{V}}$ has finite coproducts and these are given by coproducts in $\mathrm{Cat}_{\mathcal{V}}$. □

As done in [**30**] in the case $\mathcal{V} = \mathrm{Set}$, the fact that coproducts in $\mathrm{CatSym}_{\mathcal{V}}$ are given by coproducts in $\mathrm{Cat}_{\mathcal{V}}$ can be seen intuitively by the following chain of equivalences:

$$\begin{aligned} \mathrm{CatSym}_{\mathcal{V}}[\mathbb{Z}, \mathbb{X}] \times \mathrm{CatSym}_{\mathcal{V}}[\mathbb{Z}, \mathbb{Y}] &= S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{Z}] \times S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, \mathbb{Z}] \\ &= \mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, S(\mathbb{Z})] \times \mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, S(\mathbb{Z})] \\ &\simeq \mathrm{Dist}_{\mathcal{V}}[\mathbb{X} \sqcup \mathbb{Y}, S(\mathbb{Z})] \\ &= S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathbb{X} \sqcup \mathbb{Y}, \mathbb{Z}] \\ &= \mathrm{CatSym}_{\mathcal{V}}[\mathbb{Z}, \mathbb{X} \sqcup \mathbb{Y}] \,. \end{aligned}$$

We now consider the definition of exponentials in $\mathrm{CatSym}_{\mathcal{V}}$. For small $\mathcal{V}$-categories $\mathbb{X}$ and $\mathbb{Y}$, let us define

$$[\mathbb{X}, \mathbb{Y}] =_{\mathrm{def}} S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y} \,.$$

Then for every a small $\mathcal{V}$-category $\mathbb{Z}$, we have

$$\begin{aligned} \mathrm{CatSym}_{\mathcal{V}}[\mathbb{Z}, [\mathbb{X}, \mathbb{Y}]] &= S\text{-}\mathrm{Dist}_{\mathcal{V}}[S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}, \mathbb{Z}] \\ &= \mathrm{Dist}_{\mathcal{V}}[S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}, S(\mathbb{Z})] \\ &= \mathrm{Cat}_{\mathcal{V}}[S(\mathbb{Z})^{\mathrm{op}} \otimes S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}, \mathcal{V}] \\ &= \mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, S(\mathbb{Z}) \otimes S(\mathbb{X})] \\ &\simeq \mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, S(\mathbb{Z} \sqcup \mathbb{X})] \\ &= S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, \mathbb{Z} \sqcup \mathbb{X}] \\ &= \mathrm{CatSym}_{\mathcal{V}}[\mathbb{Z} \sqcap \mathbb{X}, \mathbb{Y}] \,. \end{aligned}$$

Let us consider the effect of this chain of equivalences on a categorical symmetric sequence $F \colon \mathbb{Z} \to [\mathbb{X}, \mathbb{Y}]$, which is is a distributor $F \colon \mathbb{Y} \to S(\mathbb{Z}) \otimes S(\mathbb{X})$. Let $c = c_{\mathbb{Z},\mathbb{X}} \colon S(\mathbb{Z}) \otimes S(\mathbb{X}) \to S(\mathbb{Z} \sqcup \mathbb{X})$ be the functor in (3.1.4), which in this case is given by $c(\bar{z} \otimes \bar{x}) =_{\mathrm{def}} \bar{z} \oplus \bar{x}$. The distributor $c_\bullet \colon S(\mathbb{Z}) \otimes S(\mathbb{X}) \to S(\mathbb{Z} \sqcup \mathbb{X})$ is an equivalence, since the functor c is an equivalence, as stated in Proposition 3.1.2. We then have

$$\frac{\dfrac{\dfrac{\dfrac{F \colon \mathbb{Z} \to [\mathbb{X}, \mathbb{Y}] \text{ in } \mathrm{CatSym}_{\mathcal{V}}}{F \colon \mathbb{Y} \to S(\mathbb{Z}) \otimes S(\mathbb{X}) \text{ in } \mathrm{Dist}_{\mathcal{V}}}}{c_\bullet \circ F \colon Y \to S(\mathbb{Z} \sqcup \mathbb{X}) \text{ in } \mathrm{Dist}_{\mathcal{V}}}}{c_\bullet \circ F \colon Y \to \mathbb{Z} \sqcup \mathbb{X} \text{ in } S\text{-}\mathrm{Dist}_{\mathcal{V}}}}{c_\bullet \circ F \colon \mathbb{Z} \sqcap \mathbb{X} \to \mathbb{Y} \text{ in } \mathrm{CatSym}_{\mathcal{V}} \,.}$$

By considering the particular case of $\mathbb{Z} = [\mathbb{X}, \mathbb{Y}]$ we define the categorical symmetric sequence

$$\mathrm{ev} \colon [\mathbb{X}, \mathbb{Y}] \sqcap \mathbb{X} \to \mathbb{Y} \,,$$

by letting $\mathrm{ev} =_{\mathrm{def}} c_\bullet \circ \mathrm{Id}$, where $\mathrm{Id} \colon [\mathbb{X}, \mathbb{Y}] \to [\mathbb{X}, \mathbb{Y}]$ is the identity categorical symmetric sequence on $[\mathbb{X}, \mathbb{Y}]$. By definition, we have

$$\mathrm{ev} = (c_\bullet \circ \mathrm{Id}) \colon \mathbb{Y} \to \big(S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}\big) \sqcup \mathbb{X}$$

where $\mathrm{Id} \colon S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y} \to S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}$ is the identity S-distributor, as in (3.2.3), which in this case is given by the distributor $\mathrm{Id} \colon S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y} \to S(S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y})$ defined by

$$\mathrm{Id}(\bar{v}, \bar{x}^{\mathrm{op}} \otimes y) = \big[\bar{v}, \bar{x}^{\mathrm{op}} \otimes y\big] =_{\mathrm{def}} S(S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y})[\bar{v}, \bar{x}^{\mathrm{op}} \otimes y] \,,$$

for $\bar{v} \in S(S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y})$, $\bar{x} \in S(\mathbb{X})$ and $y \in \mathbb{Y}$. Hence,

$$\mathrm{ev}[\overline{w}; y] = \int^{\bar{v} \in S(S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y})} \int^{\bar{x} \in S(\mathbb{X})} \big[\overline{w}, \bar{v} \oplus \bar{x}\big] \otimes \big[\bar{v}, \bar{x}^{\mathrm{op}} \otimes y\big] = \int^{\bar{x} \in S(\mathbb{X})} \big[\overline{w}, (\bar{x}^{\mathrm{op}} \otimes y) \oplus \bar{x}\big]\,, \quad (3.4.1)$$

for $\overline{w} \in S(S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y} \sqcup \mathbb{X})$ and $y \in \mathbb{Y}$.

We can now state and prove our first main result, which generalizes to the enriched case the main result in [**30**].

Theorem 3.4.2. *The bicategory* $\mathrm{CatSym}_{\mathcal{V}}$ *is cartesian closed. More precisely, the analytic functor* $\mathrm{ev} \colon [\mathbb{X}, \mathbb{Y}] \sqcap \mathbb{X} \to \mathbb{Y}$ *exhibits the exponential of* $\mathbb{Y}$ *by* $\mathbb{X}$.

Proof. We have to show that the functor

$$\varepsilon \colon \mathrm{CatSym}_{\mathcal{V}}[\mathbb{Z}, [\mathbb{X}, \mathbb{Y}]] \to \mathrm{CatSym}_{\mathcal{V}}[\mathbb{Z} \sqcap \mathbb{X}, \mathbb{Y}]$$

defined by letting $\varepsilon(F) =_{\mathrm{def}} \mathrm{ev} \circ (F \sqcap \mathbb{X})$ is an equivalence of categories for every small $\mathcal{V}$-category $\mathbb{Z}$. By duality, this amounts to showing that the functor

$$\varepsilon \colon S\text{-}\mathrm{Dist}_{\mathcal{V}}[S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}, \mathbb{Z}] \to S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, \mathbb{Z} \sqcup \mathbb{X}]$$

defined by letting $\varepsilon(F) =_{\mathrm{def}} (F \sqcup \mathbb{X}) \circ \mathrm{ev}$ is an equivalence of categories for every small $\mathcal{V}$-category $\mathbb{Z}$. Notice that ε has the form

$$\varepsilon \colon \mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, S(\mathbb{Z}) \otimes S(\mathbb{X})] \to \mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, S(\mathbb{Z} \sqcup \mathbb{X})]$$

since

$$S\text{-}\mathrm{Dist}_{\mathcal{V}}[S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}, \mathbb{Z}] = \mathrm{Cat}_{\mathcal{V}}[S(\mathbb{Z})^{\mathrm{op}} \otimes S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}, \mathcal{V}] = \mathrm{Dist}_{\mathcal{V}}[\mathbb{Y}, S(\mathbb{Z}) \otimes S(\mathbb{X})]\,.$$

If $c \colon S(\mathbb{Z}) \otimes S(\mathbb{X}) \to S(\mathbb{Z} \sqcup \mathbb{X})$ is the functor in (3.1.4), which in this case is defined by letting $c(\bar{z} \otimes \bar{x}) =_{\mathrm{def}} \bar{z} \oplus \bar{x}$, let us show that we have $\varepsilon(F) = c_{\bullet} \circ F$

$$\begin{array}{ccc} \mathbb{Y} & \xrightarrow{\;F\;} & S(\mathbb{Z}) \otimes S(\mathbb{X}) \\ & \searrow^{\varepsilon(F)} & \downarrow{\scriptstyle c_{\bullet}} \\ & & S(\mathbb{Z} \sqcup \mathbb{X})\,. \end{array}$$

Observe that we have $c_{\bullet} \circ c^{\bullet} \cong \mathrm{Id}_{S(\mathbb{Z}) \otimes S(\mathbb{X})}$, since the functor c is an equivalence of categories by Proposition 3.1.2. Hence it suffices to show that we have $c^{\bullet} \circ \varepsilon(F) = F$, as in the following diagram of distibutors:

$$\begin{array}{ccc} \mathbb{Y} & \xrightarrow{\;F\;} & S(\mathbb{Z}) \otimes S(\mathbb{X}) \\ & \searrow^{\varepsilon(F)} & \uparrow{\scriptstyle c^{\bullet}} \\ & & S(\mathbb{Z} \sqcup \mathbb{X})\,. \end{array}$$

In other words, it suffice to show that for $\bar{z} \in S(\mathbb{Z})$, $\bar{x} \in S(\mathbb{X})$ and $y \in \mathbb{Y}$ we have

$$\varepsilon(F)[\bar{z} \oplus \bar{x}; y] = F[\bar{z} \otimes \bar{x}; y]\,.$$

By definition, $\varepsilon(F)$ is a composite of S-distributors:

$$\begin{array}{ccc} \mathbb{Y} & \xrightarrow{\;\mathrm{ev}\;} & \big(S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y}\big) \sqcup \mathbb{X} \\ & \searrow^{\varepsilon(F)} & \downarrow{\scriptstyle F \sqcup \mathbb{X}} \\ & & \mathbb{Z} \sqcup \mathbb{X}\,. \end{array}$$

Thus,

$$\varepsilon(F)[\bar{z} \oplus \bar{x}; y] = \int^{\overline{w} \in S(S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y} \sqcup \mathbb{X})} (F \sqcup \mathbb{X})^e[\bar{z} \oplus \bar{x}, \overline{w}] \otimes \mathrm{ev}(\overline{w}; y)\,.$$

It then follows from (3.4.1) that

$$\begin{aligned} \varepsilon(F)(\bar{z} \oplus \bar{x}; y) &= \int^{\overline{w} \in S(\mathbb{Z}^{\mathbb{Y}} \sqcup \mathbb{Y})} \int^{\bar{v} \in S(\mathbb{Y})} (F \sqcup \mathbb{Y})^e[\bar{z} \oplus \bar{x}, \overline{w}] \otimes \big[\overline{w}, (\bar{v}^{\mathrm{op}} \otimes y) \oplus \bar{v}\big] \\ &= \int^{\bar{v} \in S(\mathbb{Y})} (F \sqcup \mathbb{Y})^e[\bar{z} \oplus \bar{x}, (\bar{v}^{\mathrm{op}} \otimes y) \oplus \bar{v}]\,. \end{aligned}$$

But we have $(F \sqcup \mathbb{Y})^e[\bar{z} \oplus \bar{x}, (\bar{v}^{\mathrm{op}} \otimes y) \oplus \bar{v}] = F^e[\bar{z}, \bar{v}^{\mathrm{op}} \otimes y] \otimes [\bar{x}, \bar{v}]$ by Proposition 3.2.6. Thus,

$$\begin{aligned} \varepsilon(F)(\bar{z} \oplus \bar{x}; y) &= \int^{\bar{v} \in S(\mathbb{Y})} F^e[\bar{z}, \bar{v}^{\mathrm{op}} \otimes y] \otimes [\bar{x}, \bar{v}] \\ &= F^e[\bar{z}, \bar{x}^{\mathrm{op}} \otimes y] \\ &= F^e[\bar{z} \otimes \bar{x}, y] \\ &= F(\bar{z} \otimes \bar{x}, y)\,. \end{aligned}$$

This proves that $\varepsilon(F) = c_\bullet \circ F$. Hence the functor mapping F to $\varepsilon(F)$ is an equivalence, as required, since the distributor $c_\bullet$ is an equivalence. □

CHAPTER 4

The bicategory of operad bimodules

The aim of this chapter is to define the bicategory of operad bimodules, which we denote by $\mathrm{OpdBim}_{\mathcal{V}}$. The first step to do this is to identify operads, operad bimodules and operad bimodules maps as monads, monad bimodules and bimodule maps in the bicategory of symmetric sequences $\mathrm{Sym}_{\mathcal{V}}$ introduced in Chapter 3. The second step is to apply to $\mathrm{Sym}_{\mathcal{V}}$ the so-called bimodule construction [**71**], which takes a bicategory $\mathcal{E}$ satisying appropriate assumptions and produces a bicategory $\mathrm{Bim}(\mathcal{E})$ with monads, monad bimodules and bimodule maps in $\mathcal{E}$ as 0-cells, 1-cells and 2-cells, respectively. The appropriate assumptions on $\mathcal{E}$ to perform this construction are that its hom-categories have reflexive coequalizers and the composition functors preserve coequalizers in each variable. Although there are important examples of such bicategories, they do not seem to have been isolated with specific terminology; we will call them *tame*. Thus, we are led to establish that $\mathrm{Sym}_{\mathcal{V}}$ is tame, a fact that allows us to define the bicategory of operad bimodules by letting

$$\mathrm{OpdBim}_{\mathcal{V}} =_{\mathrm{def}} \mathrm{Bim}(\mathrm{Sym}_{\mathcal{V}})\,.$$

We establish that $\mathrm{Sym}_{\mathcal{V}}$ is tame as a consequence of the fact that the bicategory of $\mathrm{CatSym}_{\mathcal{V}}$ is tame, a result that will be useful also to prove that $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed in Chapter 5

This chapter is organized as follows. Section 4.1 recalls the notions of a monad, monad bimodule and bimodule map in a bicategory, and shows how operads, operad bimodules and operad bimodule maps are instances of these notions. Section 4.2 recalls the bimodule construction and establishes some elementary facts about it. In Section 4.3 relates, for a tame bicategory $\mathcal{E}$, the bicategory of bimodules $\mathrm{Bim}(\mathcal{E})$ with the bicategory of monads $\mathrm{Mnd}(\mathcal{E})$, defined as in [**69**]. The results established there are used in Section 4.5 to provide examples of analytic functors and in Chapter 5 to prove that $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed. Section 4.4 is devoted to proving that $\mathrm{Sym}_{\mathcal{V}}$ is tame. We end the chapter in Section 4.5 by introducing the analytic functor associated to an operad bimodule and providing examples thereof.

4.1. Monads, modules and bimodules

For this section, let $\mathcal{E}$ be a fixed bicategory.

DEFINITION 4.1.1. Let $X \in \mathcal{E}$.

(i) A *monad* on X is a triple (A, μ, η) consisting of a morphism $A \colon X \to X$, a 2-cell $\mu \colon A \circ A \to A$, called the *multiplication* of the monad, and a 2-cell $\eta \colon 1_X \to A$, called the *unit* of the monad, such that the following diagrams (expressing associativity and unit axioms) commute:

$$\begin{array}{ccc} A \circ A \circ A & \xrightarrow{A \circ \mu} & A \circ A \\ {\scriptstyle \mu \circ A}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \circ A & \xrightarrow[\mu]{} & A, \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{A \circ \eta} & A \circ A & \xleftarrow{\eta \circ A} & A \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_A} & \\ & & A. & & \end{array}$$

(ii) Let $A = (A, \mu_A, \eta_A)$ and $B = (B, \mu_B, \eta_B)$ be monads on X. A *monad map* from A to B is a 2-cell $\pi \colon A \to B$ such that the following diagrams commute:

$$\begin{array}{ccc} A \circ A & \xrightarrow{\pi \circ \pi} & B \circ B \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ A & \xrightarrow[\pi]{} & B \end{array} \qquad \begin{array}{ccc} 1_X & \xrightarrow{\eta_A} & A \\ & {\scriptstyle \eta_B}\searrow & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

For $X \in \mathcal{E}$, we write $\mathrm{Mon}(X)$ for the category of monads on X and monad maps. Sometimes we will use μ and η for the multiplication and the unit of different monads, whenever the context does not lead to confusion. Note that the notion of a monad is self-dual, in the sense that a monad in $\mathcal{E}$ is the same thing as a monad in $\mathcal{E}^{\mathrm{op}}$. The category $\mathrm{Mon}(X)$ can be defined equivalently as the category of monoids and monoid morphisms in the category $\mathcal{E}[X, X]$, considered as a monoidal category with composition as tensor product and the identity morphism $1_X \colon X \to X$ as unit. Hence, a homomorphism $\Phi \colon \mathcal{E} \to \mathcal{F}$ sends a monad $A \colon X \to X$ to a monad $\Phi(A) \colon \Phi(X) \to \Phi(X)$, since it induces a monoidal functor $\Phi_{X,X} \colon \mathcal{E}[X, X] \to \mathcal{F}[\Phi(X), \Phi(X)]$. Clearly, monads on a small $\mathcal{V}$-category in the 2-category $\mathrm{Cat}_{\mathcal{V}}$ are $\mathcal{V}$-monads in the usual sense. We give some further examples below.

Example 4.1.2 (Monoids as monads). For a monoidal category $\mathbb{C}$, monads in the bicategory $\Sigma(\mathbb{C})$ are monoids in $\mathbb{C}$ [**8**, Section 5.4.1]. In particular, monads in $\Sigma(\mathrm{Ab})$ are rings [**51**, Section VII.3].

Example 4.1.3 (Categories as monads). We recall from [**8**] that a monad on a set X in the bicategory of matrices $\mathrm{Mat}_{\mathcal{V}}$ of Section 1.3 is the same thing as a $\mathcal{V}$-category with X as its set of objects. Indeed, if $A \colon X \to X$ is a monad, we can define a $\mathcal{V}$-category $\mathbb{X}$ with $\mathrm{Obj}(\mathbb{X}) = X$ by letting $\mathbb{X}[x, y] =_{\mathrm{def}} A[x, y]$, for $x, y \in X$, since the matrix $A \colon X \to X$ is a function $A \colon X \times X \to \mathcal{V}$. The composition operation and the identity morphisms of $\mathbb{X}$ are given by the multiplication and unit of the monad, since they have components of the form

$$\mu_{x,z} \colon \bigsqcup_{y \in X} \mathbb{X}[y, z] \times \mathbb{X}[x, y] \to \mathbb{X}[x, z] \,, \quad \eta_x \colon I \to \mathbb{X}[x, x] \,.$$

The associativity and unit axioms for a monad, as stated in Definition 4.1.1, then reduce to the associativity and unit axioms for the composition operation in a $\mathcal{V}$-category.

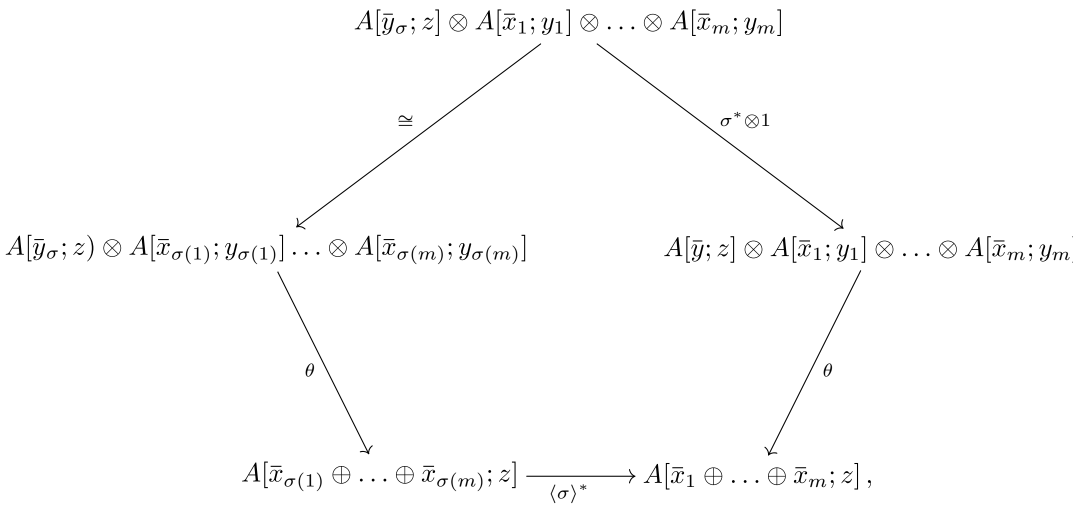
Example 4.1.4 (Operads as monads). A monad on a set X in $\mathrm{Sym}_{\mathcal{V}}$ is the same thing as an operad (by which we mean a symmetric many-sorted $\mathcal{V}$-operad, which is the same thing as a symmetric $\mathcal{V}$-multicategory), with X as its set of sorts (or set of objects). For $\mathcal{V} = \mathrm{Set}$, this was shown in [**4**, Proposition 9]. We give an outline of the proof for a general $\mathcal{V}$, which follows from an an immediate generalization of a result by Kelly [**46**] recalled in Remark 3.3.5. Let $A \colon X \to X$ be a monad in $\mathrm{Sym}_{\mathcal{V}}$, i.e. a symmetric sequence $A \colon X \to X$, given by a functor $A \colon S(X)^{\mathrm{op}} \times X \to \mathcal{V}$, equipped with a multiplication and a unit. We define an operad with set of objects X as follows. First of all, for $x_1, \ldots, x_n, x \in X$, the object of operations with inputs of sorts $x_1, \ldots, x_n$ and output of sort x to be $A[x_1, \ldots, x_n; x]$. The symmetric group actions required to have an operad follow from the functoriality of A, since we have a morphism

$$\sigma^* \colon \mathbb{X}[x_{\sigma(1)}, \ldots, x_{\sigma(n)}; x] \to \mathbb{X}[x_1, \ldots, x_n; x]$$

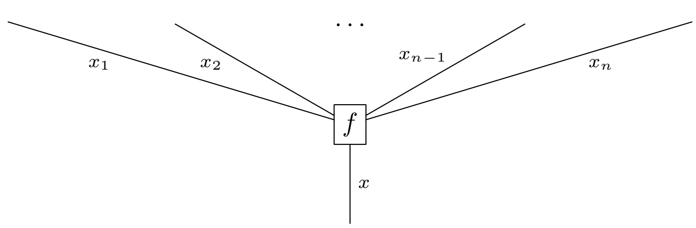
for each each permutation $\sigma \in \Sigma_n$ and $(x_1, \ldots, x_n, x) \in S_n(X)^{\mathrm{op}} \times X$. By the definition of the composition operation in $\mathrm{Sym}_{\mathcal{V}}$, as given in (3.3.2), the multiplication $\mu \colon A \circ A \to A$ amounts to having a family of morphisms

$$\theta_{\bar{x}_1,\ldots,\bar{x}_m,\bar{x},x} \colon A[\bar{x};x] \otimes A[\bar{x}_1;x_1] \otimes \ldots \otimes A[\bar{x}_m;x_m] \to A[\bar{x}_1 \oplus \ldots \oplus \bar{x}_m;x]\,,$$

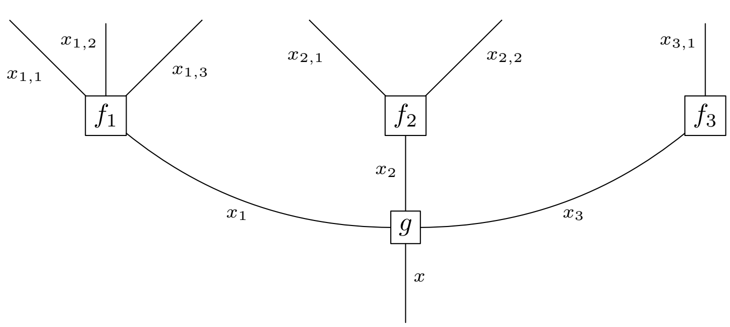
where $\bar{x} = (x_1, \ldots, x_m)$, which is natural in $\bar{x}, \bar{x}_1, \ldots, \bar{x}_m \in S(X)^{\mathrm{op}}$ and satisfies the equivariance condition expressed by the commutativity of the following diagram:



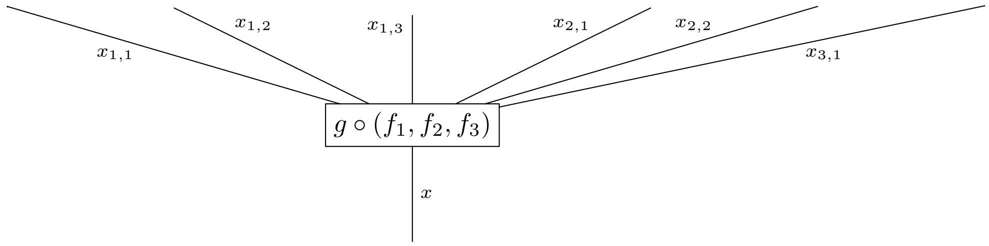
where $\langle\sigma\rangle \colon \bar{x}_{\sigma(1)} \oplus \ldots \oplus \bar{x}_{\sigma(m)} \to \bar{x}_1 \oplus \ldots \oplus \bar{x}_m$ is the evident morphism in $S(X)^{\mathrm{op}}$ induced by σ. It is common to represent maps $f \colon I \to A[x_1, \ldots, x_n;x]$ as corollas of the form



With this notation, the composition operation may be represented diagrammatically as a grafting operation. For example, the composite represented by the following grafting diagram



is represented by



where $g \circ (f_1, f_2, f_3) = \theta(g, f_1, f_2, f_3)$. By the definition of the identity symmetric sequence in (3.3.3), the unit $\eta \colon \mathrm{Id}_X \to A$ then amounts to having a morphism $1_x \colon I \to A[(x); x]$ for each $x \in X$. These give the identity operations of the operad. The associativity and unit axioms for a monad then correspond to the associativity and unit axioms for operads.

An operad (X, A) determines, for every symmetric $\mathcal{V}$-rig $\mathcal{R}$, a monad $A \colon \mathcal{R}^X \to \mathcal{R}^X$, whose underlying functor is the analytic functor associated to the symmetric sequence $A \colon X \to X$, which is given by the formula

$$A(T)(x) =_{\mathrm{def}} \int^{\bar{x} \in S(X)} A[\bar{x}, x] \otimes T^{\bar{x}} \, .$$

We write $\mathrm{Alg}_{\mathcal{R}}(A)$ for the category of algebras and algebra morphisms for this monad, which is related to the category $\mathcal{V}^X$ by a monadic adjunction

$$\mathrm{Alg}_{\mathcal{R}}(A) \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{R}^X \, .$$

It should be mentioned here that characterizations of several kinds of operads as monads in appropriate bicategories are also given in [**54**]. Our setting is different to that considered in [**54**], since we work with bicategories of distributors rather than bicategories of spans (see [**23**] for a discussion of the two settings).

The next definition recalls the standard notion of a left module for a monad, i.e. what is often called a generalized algebra [**49**].

Definition 4.1.5. Let $A \colon X \to X$ be a monad in $\mathcal{E}$. Let $K \in \mathcal{E}$.

(i) A *left A-module with domain K* is a morphism $M \colon K \to X$ equipped with a left A-action, i.e. a 2-cell $\lambda \colon A \circ M \to M$ such that the following diagrams commute:

$$\begin{array}{ccc} A \circ A \circ M & \xrightarrow{A \circ \lambda} & A \circ M \\ {\scriptstyle \mu \circ M}\downarrow & & \downarrow{\scriptstyle \lambda} \\ A \circ M & \xrightarrow[\lambda]{} & M, \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\eta \circ M} & A \circ M \\ & {\scriptstyle 1_M} \searrow & \downarrow{\scriptstyle \lambda} \\ & & M \, . \end{array}$$

(ii) If M and M' are left A-modules with domain K, then a *left A-module map* from M to M' is a 2-cell $f\colon M \to M'$ such that the following diagram commutes:

$$\begin{array}{ccc} A \circ M & \xrightarrow{A \circ f} & A \circ M' \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda'} \\ M & \xrightarrow[f]{} & M' \,. \end{array}$$

We write $\mathcal{E}[K,X]^A$ for the category of left A-modules with domain K and left A-module maps.

EXAMPLE 4.1.6 (Left modules in a monoidal category). For a monoidal category $\mathbb{C} = (\mathbb{C}, \otimes, I)$, a left module over a monoid A, viewed as a monad in $\Sigma(\mathbb{C})$, is the same thing as an object $M \in \mathbb{C}$ equipped with a left A-action, i.e. a morphism $\lambda\colon A \otimes M \to M$ satisfying associativity and unit axioms. In particular, we obtain the familiar notion of left modules over a ring when $\mathbb{C} = \mathrm{Ab}$.

EXAMPLE 4.1.7 (Left modules for categories). For a small $\mathcal{V}$-category $\mathbb{X}$, viewed as a monad on $X =_{\mathrm{def}} \mathrm{Obj}(\mathbb{X})$ in $\mathrm{Mat}_{\mathcal{V}}$, left $\mathbb{X}$-modules are families of presheaves on $\mathbb{X}$, i.e. controvariant $\mathcal{V}$-functors from $\mathbb{X}$ to $\mathcal{V}$. Indeed, a left $\mathbb{X}$-module M with domain K is a matrix $M\colon K \to \mathrm{Obj}(\mathbb{X})$, i.e. a functor $M\colon \mathrm{Obj}(\mathbb{X}) \times K \to \mathcal{V}$, equipped with a natural transformation with components

$$\lambda_{x,k}\colon \bigsqcup_{x' \in \mathrm{Obj}(\mathbb{X})} \mathbb{X}[x,x'] \otimes M[x',k] \to M[x,k] \,,$$

satisfying associativity and unit axioms. It is immediate to see that this is the same thing as a family of $\mathcal{V}$-functors $M_k\colon \mathbb{X}^{\mathrm{op}} \to \mathcal{V}$, for $k \in K$.

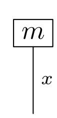
EXAMPLE 4.1.8 (Left modules for operads). Let us consider an operad, given as a monad (X, A) in $\mathrm{Sym}_{\mathcal{V}}$. A left A-module with domain K consists of an symmetric sequence $M\colon K \to X$, i.e. a functor $M\colon S(K)^{\mathrm{op}} \times X \to \mathcal{V}$ equipped with a left A-action $\lambda\colon A \circ M \to M$. By the definition of the composition operation in $\mathrm{Sym}_{\mathcal{V}}$, as given in (3.3.2), such a left action amounts to having maps in $\mathcal{V}$ of the form

$$M[\bar{k}_1; x_1] \otimes \ldots \otimes M[\bar{k}_m; x_m] \otimes A[x_1, \ldots, x_m; x] \to M[\bar{k}_1 \oplus \ldots \oplus \bar{k}_m; x] \,,$$

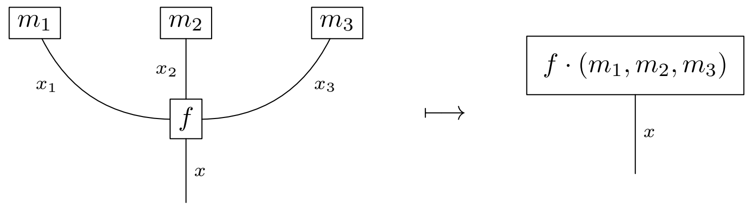
which satisfy associativity and unit axioms and an equivariance condition. When $K = \emptyset$, we have $S(K) \cong 1$ and therefore a left A-module with domain $\emptyset$ is a family of objects $M(x) \in \mathcal{V}$, for $x \in X$, equipped with maps in $\mathcal{V}$ of the form

$$M(x_1) \otimes \ldots \otimes M(x_m) \otimes A[x_1, \ldots, x_m; x] \to M(x) \,,$$

satisfying the associativity and unit axioms and an equivariance condition. Such left modules and their left module maps are exactly algebras and algebra morphisms for A in $\mathcal{V}$, in the sense of Example 4.1.4, and so we have $\mathrm{Sym}_{\mathcal{V}}[\emptyset, X]^A = \mathrm{Alg}_{\mathcal{V}}(A)$. Diagrammatically, if we represent a map $m\colon I \to M(x)$ as



then the left A-action can be seen, for example as acting as follows:



REMARK 4.1.9. Let $\mathbb{K} \in \mathcal{E}$. The homomorphism $\mathcal{E}[K, -] \colon \mathcal{E} \to \mathrm{Cat}$ sends a monad $A \colon X \to X$ to a monad $\mathcal{E}[K, A] \colon \mathcal{E}[K, X] \to \mathcal{E}[K, X]$. Then, the category of left A-modules with domain K and left A-module maps is exactly the category of algebras and algebra morphisms for the monad $\mathcal{E}[K, A]$. Therefore, we have an adjunction

$$\mathcal{E}[K, X]^A \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{E}[K, X],$$

where the right adjoint is the forgetful functor and the left adjoint takes a morphism $M \colon K \to X$ to the free left A-module on it, $A \circ M \colon K \to X$.

Right modules and right module maps are defined in a dual way to left modules and left module maps. We state the explicit definition below.

DEFINITION 4.1.10. Let $A \colon X \to X$ be a monad in $\mathcal{E}$. Let $K \in \mathcal{E}$.

(i) A *right A-module with codomain K* is a morphism $M \colon X \to K$ equipped with a right A-action $\rho \colon M \circ A \to M$ such that the following diagrams commute:

$$\begin{array}{ccc} M \circ A \circ A & \xrightarrow{\rho \circ A} & M \circ A \\ {\scriptstyle M \circ \mu}\downarrow & & \downarrow{\scriptstyle \rho} \\ M \circ A & \xrightarrow[\rho]{} & M\,, \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{M \circ \eta} & M \circ A \\ & {\scriptstyle 1_M}\searrow & \downarrow{\scriptstyle \rho} \\ & & M\,. \end{array}$$

(ii) If M and M' are two right A-modules with codomain K, then a *right A-module map* from M to M' is a 2-cell $f \colon M \to M'$ such that the following diagram commutes:

$$\begin{array}{ccc} M \circ A & \xrightarrow{f \circ A} & M' \circ A \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho'} \\ M & \xrightarrow[f]{} & M'. \end{array}$$

We write $\mathcal{E}[X, K]_A$ for the category of right A-module with codomain K and right A-module maps.

EXAMPLE 4.1.11 (Right modules in a monoidal category). For a monoidal category $\mathbb{C} = (\mathbb{C}, \otimes, I)$, a right module over a monoid A, viewed as a monad in $\Sigma(\mathbb{C})$, is the same thing as an object $M \in \mathbb{C}$ equipped with a right A-action $\rho \colon M \otimes A \to M$. In particular, right modules in Ab are the same thing as right modules over a ring in the usual sense.

EXAMPLE 4.1.12 (Right modules for categories). For a small $\mathcal{V}$-category $\mathbb{X}$, viewed as a monad (X, A) in $\mathrm{Mat}_{\mathcal{V}}$, right A-modules are families of covariant $\mathcal{V}$-functors from $\mathbb{X}$ to $\mathcal{V}$. Indeed, a right

A-module with codomain K is a matrix $M\colon X \to K$, i.e. a functor $M\colon K \times X \to \mathcal{V}$, equipped with a natural transformation with components

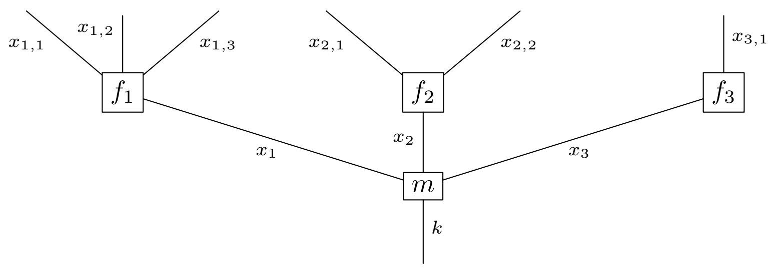
$$\rho_{k,x}\colon \bigsqcup_{x' \in X} M[k, x'] \otimes A[x', x] \to M[k, x]\,,$$

satisfying associativity and unit axioms. It is immediate to see that this is the same thing as a family of $\mathcal{V}$-functors $M_k\colon \mathbb{X} \to \mathcal{V}$, for $k \in K$.

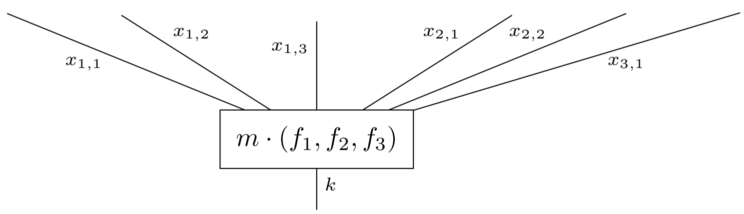
EXAMPLE 4.1.13 (Right modules for operads). For a small $\mathcal{V}$-operad, viewed as a monad (X, A) in $\mathrm{Sym}_{\mathcal{V}}$, a right A-module with codomain K consists of a symmetric sequence $M\colon X \to K$, i.e. a functor $M\colon S(X)^{\mathrm{op}} \times K \to \mathcal{V}$ equipped with a right $\mathbb{X}$-action $\rho\colon M \circ A \to M$. By the definition of composition in $\mathrm{Sym}_{\mathcal{V}}$, as given in (3.3.2), such an action amounts to having maps in $\mathcal{V}$ of the form

$$A[\bar{x}_1; x_1] \otimes \ldots \otimes A[\bar{x}_m; x_m] \otimes M[x_1, \ldots, x_m; k] \to M[\bar{x}_1 \oplus \ldots \oplus \bar{x}_m; k]$$

which satisfy associativity and unit axioms, as well as an equivariance condition. Diagrammatically, using notation analogous to the one adopted above, we have, for example, that



is mapped to



When $X = 1$, these maps have the form

$$A[n_1] \otimes \ldots \otimes A[n_m] \otimes M_k(m) \to M_k(n_1 + \ldots + n_m)\,,$$

where we write $M_k(n)$ for $M[n; k]$. These are K-indexed families of right A-modules for the operad, as usually defined in the literature (see, for example, [**32, 43**]).

REMARK 4.1.14. Let $\mathbb{K} \in \mathcal{E}$. The homomorphism $\mathcal{E}[K, -]\colon \mathcal{E} \to \mathrm{Cat}$ sends a monad $A\colon X \to X$ to a monad $\mathcal{E}[K, A]\colon \mathcal{E}[X, K] \to \mathcal{E}[X, K]$. Right A-modules with domain K and right A-module maps are the algebras and the algebra morphisms for the monad $\mathcal{E}[K, A]$. Hence, we have an adjunction

$$\mathcal{E}[K, X]_A \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{E}[K, X],$$

where the right adjoint is the forgetful functor and the left adjoint takes a morphism $M\colon X \to K$ to the right A-module $M \circ A\colon K \to X$.

Next, we define the notions of a bimodule and bimodule map, which will play a fundamental role throughout the rest of the paper. In particular, in Section 4.2 we will recall how (under appropriate assumptions on $\mathcal{E}$) monads, bimodules and bimodule maps form a bicategory.

DEFINITION 4.1.15. Let $A\colon X \to X$ and $B\colon X \to X$ be monads in $\mathcal{E}$.

(i) A (B, A)-*bimodule* is a morphism $M\colon X \to Y$ equipped with a left B-action $\lambda\colon B \circ M \to M$ and a right A-action $\rho\colon M \circ A \to M$ which commute with each other, in the sense that the following diagram commutes:

$$\begin{array}{ccc} B \circ M \circ A & \xrightarrow{\lambda \circ A} & M \circ A \\ {\scriptstyle B\circ\rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ B \circ M & \xrightarrow[\lambda]{} & M\,. \end{array} \tag{4.1.1}$$

(ii) If $M, M\colon X \to Y'$ are (B, A)-bimodules, then a *bimodule map* from M to M' is a 2-cell $f\colon M \to M'$ that is a map of left B-modules and of right A-modules.

We write $\mathcal{E}[X, Y]^B_A$ for the category of (B, A)-bimodules and bimodule maps.

EXAMPLE 4.1.16 (Bimodules in a monoidal category). For a monoidal category $\mathbb{C} = (\mathbb{C}, \otimes, I)$, bimodules in a $\Sigma(\mathbb{C})$ are the same thing as objects of $\mathbb{C}$ equipped with a right action and a left action by a monoid which distribute over each other. In particular, bimodules in $\Sigma(\mathrm{Ab})$ in the sense of the previous definition are the same thing as bimodules over a ring in the standard algebraic sense.

EXAMPLE 4.1.17 (Bimodules for categories). As is well-known, bimodules in the bicategory $\mathrm{Mat}_{\mathcal{V}}$ are exactly distributors, in the sense of Definition 1.3.1. Indeed, for a small $\mathcal{V}$-category $\mathbb{X}$ with set of objects X and a small $\mathcal{V}$-category $\mathbb{Y}$ with set of objects Y, a $(\mathbb{Y}, \mathbb{X})$-bimodule is a function $M\colon Y \times X \to \mathcal{V}$ equipped with natural transformations with components

$$\rho_{y,x}\colon \bigsqcup_{x' \in X} M[y, x'] \otimes \mathbb{X}[x', x] \to M[y, x]\,, \quad \lambda_{x,y}\colon \bigsqcup_{y' \in X} \mathbb{Y}[y, y'] \otimes M[y', y] \to M[y, x]\,,$$

satisfying associativity and unit axioms. It is immediate to see that this is the same thing as a functor $M\colon \mathbb{Y}^{\mathrm{op}} \otimes \mathbb{X} \to \mathcal{V}$, i.e. a distributor $M\colon \mathbb{X} \to \mathbb{Y}$.

EXAMPLE 4.1.18 (Operad bimodules). Bimodules in $\mathrm{Sym}_{\mathcal{V}}$ are operad bimodules [**59, 65**], which we define explicitly below. Let us consider two operads $\mathbb{X} = (X, A)$ and $\mathbb{Y} = (Y, B)$. Then, an (B, A)-bimodule consists of a symmetric sequence $M\colon X \to Y$, i.e. a functor $M\colon S(X)^{\mathrm{op}} \times Y \to \mathcal{V}$, equipped with a right A-action $\rho\colon M \circ A \to M$ and a left B-action $\lambda\colon B \circ M \to M$ satisfying the compatibility condition in (4.1.1). Explicitly, the right A-action amounts to having maps of the form

$$A[\bar{x}_1; x_1] \otimes \ldots \otimes A[\bar{x}_m; x_m] \otimes M[x_1, \ldots, x_m; y] \to M[\bar{x}_1 \oplus \ldots \oplus \bar{x}_m; y]\,,$$

while the left B-action amounts to having maps of the form

$$M[\bar{x}_1; y_1] \otimes \ldots \otimes M[\bar{x}_m; y_m] \otimes B[y_1, \ldots, y_m; y] \to M[\bar{x}_1 \oplus \ldots \oplus \bar{x}_m; y]\,,$$

all satisfying associativity, unit, compatibility and equivariance conditions. Bimodules for non-symmetric operads were defined in [**54**, Definition 2.36].

PROPOSITION 4.1.19. *The forgetful functor* $U\colon \mathcal{E}[X, Y]^B_A \to \mathcal{E}[X, Y]$ *is monadic.*

PROOF. Observe that the endofunctor $\mathcal{E}[A, B] \colon \mathcal{E}[X, Y] \to \mathcal{E}[X, Y]$ has the structure of a monad with multiplication $\mu = \mathcal{E}[\mu_A, \mu_B]$ and unit $\eta = \mathcal{E}[\eta_A, \eta_B]$. A (B, A)-bimodule is the same thing as an $\mathcal{E}[A, B]$-algebra, which is a morphism $M \colon X \to Y$, equipped with a 2-cell $\alpha \colon B \circ M \circ A \to M$ such that the following diagrams commute:

$$
\begin{array}{ccc}
B \circ B \circ M \circ A \circ A & \xrightarrow{B \circ \alpha \circ A} & B \circ M \circ A \\
{\scriptstyle \mu_B \circ M \circ \mu_A} \downarrow & & \downarrow {\scriptstyle \alpha} \\
B \circ M \circ A & \xrightarrow[\alpha]{} & M \,,
\end{array}
\qquad
\begin{array}{ccc}
M & \xrightarrow{\eta_B \circ M \circ \eta_A} & B \circ M \circ A \\
& {\scriptstyle 1_M} \searrow & \downarrow {\scriptstyle \alpha} \\
& & M .
\end{array}
$$

From the 2-cell α we obtain two actions $\lambda = \alpha \cdot (B \circ M \circ \eta_A)$ and $\rho = \alpha \cdot (\eta_B \circ M \circ A)$ which commute with each other. Conversely, from a commuting pair of actions (λ, ρ) we obtain a 2-cell α which makes the required diagrams commute by letting $\alpha =_{\text{def}} \rho \cdot (\lambda \circ A) = \lambda \cdot (B \circ \rho)$, i.e. the common value of the composites in (4.1.1). $\square$

REMARK 4.1.20. The category $\mathcal{E}[X, Y]_A^B$ is related to the categories $\mathcal{E}[X, Y]^B$ and $\mathcal{E}[X, Y]_A$ by the following commutative squares of monadic forgetful functors (all written U) and left adjoints (all written F):

$$
\begin{array}{ccc}
\mathcal{E}[X,Y]_A^B & \xrightarrow{U} & \mathcal{E}[X,Y]^B \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
\mathcal{E}[X,Y]_A & \xrightarrow[U]{} & \mathcal{E}[X,Y] \,,
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{E}[X,Y]_A^B & \xleftarrow{F} & \mathcal{E}[X,Y]^B \\
{\scriptstyle F}\uparrow & & \uparrow{\scriptstyle F} \\
\mathcal{E}[X,Y]_A & \xleftarrow[F]{} & \mathcal{E}[X,Y] \,,
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{E}[X,Y]_A^B & \xleftarrow{F} & \mathcal{E}[X,Y]^B \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
\mathcal{E}[X,Y]_A & \xleftarrow[F]{} & \mathcal{E}[X,Y] \,,
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{E}[X,Y]_A^B & \xrightarrow{U} & \mathcal{E}[X,Y]^B \\
{\scriptstyle F}\uparrow & & \uparrow{\scriptstyle F} \\
\mathcal{E}[X,Y]_A & \xrightarrow[U]{} & \mathcal{E}[X,Y] \,.
\end{array}
$$

4.2. Tame bicategories and bicategories of bimodules

We review the bimodule construction, which assembles monads, bimodules and bimodule maps in a bicategory $\mathcal{E}$ satisfying appropriate assumptions into a new bicategory $\mathrm{Bim}(\mathcal{E})$. For more information on the bimodule construction, see [**14, 18, 33, 48, 71**].

DEFINITION 4.2.1.

(i) We say that $\mathcal{E}$ is *tame* if for every $X, Y \in \mathcal{E}$ the category $\mathcal{E}[X, Y]$ has reflexive coequalizers and the horizontal composition functor of $\mathcal{E}$ preserves coequalizers in each variable.

(ii) If $\mathcal{E}$ and $\mathcal{F}$ are tame bicategories, we say that a homomorphism $\Phi \colon \mathcal{E} \to \mathcal{F}$ is *tame* if for every $X, Y \in \mathcal{E}$, the functor $\Phi_{X,Y} \colon \mathcal{E}[X, Y] \to \mathcal{F}[\Phi X, \Phi Y]$ preserves reflexive coequalizers.

As we will see below, the condition that a bicategory is tame is used in order to define the composition of bimodules. If $\mathcal{E}$ and $\mathcal{F}$ are tame bicategories, we write $\mathrm{REG}[\mathcal{E}, \mathcal{F}]$ for the full sub-bicategory of $\mathrm{HOM}[\mathcal{E}, \mathcal{F}]$ whose objects are tame homomorphisms from $\mathcal{E}$ to $\mathcal{F}$.

In order to make evident how different components of the structure of a bimodule play different roles in the definition of the composition functors, we discuss ways of combining a bimodule with a (right or left) module. From now until the end of the section, let $\mathcal{E}$ be a fixed tame bicategory. Let $A \colon X \to X$, $B \colon Y \to Y$ be monads in $\mathcal{E}$. For a left A-module $M \colon K \to X$ and a (B, A)-bimodule

$F\colon X \to Y$, we define a left B-module $F \circ_A M \colon K \to Y$ as follows. Its underlying morphism is defined by following reflexive coequalizer diagram:

$$F \circ A \circ M \underset{F \circ \lambda}{\overset{\rho \circ M}{\rightrightarrows}} F \circ M \xrightarrow{\ q\ } F \circ_A M \,. \tag{4.2.1}$$

The left B-action is determined by the universal property of coequalizers, as follows:

$$\begin{array}{ccccc}
B \circ F \circ A \circ M & \underset{B \circ F \circ \lambda}{\overset{B \circ \rho \circ M}{\rightrightarrows}} & B \circ F \circ M & \xrightarrow{\ B \circ q\ } & B \circ F \circ_A M \\
\Big\downarrow{\scriptstyle \lambda \circ A \circ M} & & \Big\downarrow{\scriptstyle \lambda \circ M} & & \Big\downarrow{\scriptstyle \lambda} \\
F \circ A \circ M & \underset{F \circ \lambda}{\overset{\rho \circ M}{\rightrightarrows}} & F \circ M & \xrightarrow[\ q\ ]{} & F \circ_A M \,,
\end{array}$$

Here, the top row is a coequalizer diagram by the assumption that $\mathcal{E}$ is tame, being obtained from the diagram in (4.2.1) by composition with B. The verification of the axioms for a left A-action uses the fact that the actions of A and B commute with each other and it is essentially straightforward. Furthermore, this definition can easily be shown to extend to a functor

$$(-) \circ_A (-) \colon \mathcal{E}[X,Y]^B_A \times \mathcal{E}[K,X]^A \to \mathcal{E}[K,Y]^B \,.$$

Dually, for a (B,A)-bimodule $F\colon X \to Y$ and a right B-module $M\colon Y \to K$, we define a right A-module $M \circ_B F \colon X \to K$ as follows. Its underlying morphism is defined by following reflexive coequalizer diagram:

$$M \circ B \circ F \underset{\rho \circ F}{\overset{M \circ \lambda}{\rightrightarrows}} M \circ F \xrightarrow{\ q\ } M \circ_B F \,. \tag{4.2.2}$$

The left B-action is determined by the universal property of coequalizers, as follows:

$$\begin{array}{ccccc}
M \circ B \circ F \circ A & \underset{\rho \circ F \circ A}{\overset{M \circ \lambda \circ A}{\rightrightarrows}} & M \circ F \circ A & \xrightarrow{\ q \circ A\ } & M \circ_B F \circ A \\
\Big\downarrow{\scriptstyle M \circ B \circ \rho} & & \Big\downarrow{\scriptstyle M \circ \rho} & & \Big\downarrow{\scriptstyle \rho} \\
M \circ B \circ F & \underset{\rho \circ F}{\overset{M \circ \lambda}{\rightrightarrows}} & M \circ F & \xrightarrow[\ q\ ]{} & M \circ_B F \,.
\end{array} \tag{4.2.3}$$

In this way, we obtain a functor

$$(-) \circ_B (-) \colon \mathcal{E}[X,Y]^B_A \times \mathcal{E}[Y,K]_B \to \mathcal{E}[X,K]_A \,.$$

Let us now assume we have monads $A\colon X \to X$, $B\colon Y \to Y$ and $C\colon Z \to Z$, a (B,A)-bimodule $F\colon X \to Y$ and a (C,B)-bimodule $G\colon Y \to Z$. If we consider F as a left B-module and G as a (C,B)-bimodule, the morphism given by the formula in (4.2.1) coincides with the morphism given by the formula in (4.2.2), applied considering F as a (B,A)-bimodule and G as a right B-module. Hence, this morphism $G \circ_B F \colon \mathbb{X} \to \mathbb{Z}$ is equipped with both a left C-action and a right A-action, which can be easily seen to commute with each other, thus giving us a (C,A)-bimodule $G \circ_B F \colon X \to Z$. Explicitly, the morphism $G \circ_B F$ is defined by the reflexive coequalizer

$$G \circ B \circ F \underset{G \circ \lambda}{\overset{\rho \circ F}{\rightrightarrows}} G \circ F \xrightarrow{\ q\ } G \circ_B F \,.$$

Its right A-action is determined by the diagram

$$\begin{array}{ccccc}
G \circ B \circ F \circ A & \overset{\rho \circ F \circ A}{\underset{G \circ \lambda \circ A}{\rightrightarrows}} & G \circ F \circ A & \xrightarrow{q \circ A} & (G \circ_B F) \circ A \\
{\scriptstyle G \circ B \circ \rho}\downarrow & & \downarrow{\scriptstyle G \circ \rho} & & \downarrow{\scriptstyle \rho} \\
G \circ B \circ F & \overset{\rho \circ F}{\underset{G \circ \lambda}{\rightrightarrows}} & G \circ F & \xrightarrow[q]{} & G \circ_B F \,,
\end{array}$$

while its left C-action is given by the diagram

$$\begin{array}{ccccc}
C \circ G \circ B \circ F & \overset{C \circ G \circ \lambda}{\underset{C \circ \rho \circ F}{\rightrightarrows}} & C \circ G \circ F & \xrightarrow{C \circ q} & C \circ (G \circ_B F) \\
{\scriptstyle \lambda \circ B \circ F}\downarrow & & \downarrow{\scriptstyle \lambda \circ F} & & \downarrow{\scriptstyle \lambda} \\
G \circ B \circ F & \overset{G \circ \lambda}{\underset{\rho \circ F}{\rightrightarrows}} & G \circ F & \xrightarrow[q]{} & G \circ_B F \,.
\end{array}$$

In this way, we obtain a functor

$$(-) \circ_B (-) \colon \mathcal{E}[Y,Z]^C_B \times \mathcal{E}[X,Y]^B_A \to \mathcal{E}[X,Z]^C_A \,, \tag{4.2.4}$$

which we sometimes call *relative composition*. This operation generalizes the circle over construction defined by Rezk [**65**, Section 2.3.10], which is called the relative composition product in [**32**, Section 5.1.5].

We can now define the bicategory $\mathrm{Bim}(\mathcal{E})$. The objects of $\mathrm{Bim}(\mathcal{E})$ are pairs of the form (X, A), where $X \in \mathcal{E}$ and $A \colon X \to X$ is a monad. In order to simplify the notation, if we consider a pair (X, A) as an object of $\mathrm{Bim}(\mathcal{E})$ we write it as X/A and sometimes refer to it simply as a monad. For a pair of monads X/A and Y/B, we then define

$$\mathrm{Bim}(\mathcal{E})[X/A, Y/B] =_{\mathrm{def}} \mathcal{E}[X,Y]^B_A \,.$$

Hence, a morphism in $\mathrm{Bim}(\mathcal{E})$ from X/A to Y/B is a (B, A)-bimodule $M \colon X/A \to Y/B$ and a 2-cell $f \colon M \to N$ in $\mathrm{Bim}(\mathcal{E})$ is a bimodule map. The composition operation of $\mathrm{Bim}(\mathcal{E})$ is then given by the relative composition of bimodules in (4.2.4). For an object $X/A \in \mathrm{Bim}(\mathcal{E})$, the identity bimodule $1_{X/A} \colon X/A \to X/A$ is given by the morphism $A \colon X \to X$, viewed as an (A, A)-bimodule by taking the monad multiplication $\mu \colon A \circ A \to A$ as both the left and the right A-action. In order to complete the definition of the data of the bicategory $\mathrm{Bim}(\mathcal{E})$, it remains to exhibit the associativity and unit isomorphisms. For the associativity isomorphisms, let us define the joint composition of three bimodules

$$(V, D) \xrightarrow{L} (X, A) \xrightarrow{M} (Y, B) \xrightarrow{N} (Z, C)$$

as the colimit $N \circ_B M \circ_A L$ of a double (reflexive) graph

$$\begin{array}{ccc}
N \circ B \circ M \circ A \circ L & \rightrightarrows & N \circ B \circ M \circ L \\
\downarrow\downarrow & & \downarrow\downarrow \\
N \circ M \circ A \circ L & \rightrightarrows & N \circ M \circ L \,.
\end{array}$$

If the colimit is calculated horizontally and then vertically, we obtain $N \circ_B (M \circ_A L)$. If the colimit is calculated vertically and then horizontally, instead, we obtain $(N \circ_B M) \circ_A L$. Thus, we have the required isomorphism $a_{L,M,N} \colon N \circ_B (M \circ_A L) \to (N \circ_B M) \circ_A L$. This isomorphism is the unique 2-cell a fitting in the commutative diagram of canonical maps,

$$\begin{array}{ccc} N \circ (M \circ L) & = & (N \circ M) \circ L \\ \downarrow & & \downarrow \\ N \circ_B (M \circ_A L) & \xrightarrow[a]{} & (N \circ_B M) \circ_A L\,. \end{array}$$

For the unit isomorphisms, observe that for $X/A, Y/B \in \mathrm{Bim}(\mathcal{E})$ and $M \colon X/A \to Y/B$, we have an isomorphism $\ell_M \colon B \circ_B M \to M$ which fits in the diagram

$$\begin{array}{ccccc} B \circ B \circ M & \overset{M \circ \mu}{\underset{\rho \circ A}{\rightrightarrows}} & B \circ M & \xrightarrow{q} & B \circ_B M \\ & & & \searrow^{\rho} & \downarrow \ell_M \\ & & & & M \end{array} \tag{4.2.5}$$

since

$$B \circ B \circ M \overset{\mu \circ M}{\underset{B \circ \lambda}{\rightrightarrows}} B \circ M \xrightarrow{\lambda} M, \qquad \eta \circ B \circ M \colon B \circ M \to B \circ B \circ M, \quad \eta \circ M \colon M \to B \circ M$$

is a split fork. Dually, we have also an isomorphism $r_M \colon M \circ_A A \to M$ making the following diagram commute

$$\begin{array}{ccccc} M \circ A \circ A & \overset{M \circ \mu}{\underset{\rho \circ A}{\rightrightarrows}} & M \circ A & \xrightarrow{q} & M \circ_A A \\ & & & \searrow^{\rho} & \downarrow r_M \\ & & & & M \end{array} \tag{4.2.6}$$

since

$$M \circ A \circ A \overset{M \circ \mu}{\underset{\rho \circ A}{\rightrightarrows}} M \circ A \xrightarrow{\rho} M, \qquad M \circ A \circ \eta \colon M \circ A \to M \circ A \circ A, \quad M \circ \eta \colon M \to M \circ A$$

is a split fork. The verification of the coherence axioms for a bicategory is a straightforward diagram-chasing argument.

Remark 4.2.2. For every tame bicategory $\mathcal{E}$, there is a homomorphism

$$\mathrm{J}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$$

which maps an object $X \in \mathcal{E}$ to the identity monad $X/1_X$. The action of $\mathrm{J}_{\mathcal{E}}$ on morphisms and 2-cells is evident and hence we do not spell it out.

Remark 4.2.3. If the monad $B \colon Y \to Y$ is the identity $1_Y \colon Y \to Y$, then we have a canonical isomorphism $G \circ_{1_Y} F \cong G \circ F$, which we will consider as an equality for simplicity.

Example 4.2.4 (Bimodules in a monoidal category). For a monoidal category $\mathbb{C}$ with reflexive coequalizers in which the tensor product preserves reflexive coequalizers, the bicategory $\mathrm{Bim}(\mathbb{C})$ has monoids in $\mathbb{C}$ as objects, bimodules as morphisms and bimodule maps as 2-cells (see also [**5**] for a discussion of this example). In particular, $\mathrm{Bim}(\mathrm{Ab})$ is the bicategory of rings, ring bimodules and bimodule maps. Given a (B, A)-bimodule M and a (C, B)-bimodule N, where A, B and C are rings, their tensor product $N \otimes_B M$ fits in the following reflexive coequalizer in Ab:

$$N \otimes B \otimes M \underset{N\otimes\lambda}{\overset{\rho\otimes M}{\rightrightarrows}} N \otimes M \xrightarrow{q} N \otimes_B M\,.$$

Example 4.2.5 (Distributors). The bicategory of bimodules in the bicategory of matrices $\mathrm{Mat}_{\mathcal{V}}$ is exactly the bicategory of distributors:

$$\mathrm{Dist}_{\mathcal{V}} = \mathrm{Bim}(\mathrm{Mat}_{\mathcal{V}})\,.$$

Indeed, we have seen that small $\mathcal{V}$-categories, which are the objects of $\mathrm{Dist}_{\mathcal{V}}$, are monads in $\mathrm{Mat}_{\mathcal{V}}$, distributors $F\colon \mathbb{X} \to \mathbb{Y}$ are the same thing as $(\mathbb{Y}, \mathbb{X})$-bimodules, and $\mathcal{V}$-natural transformations between distributors are exactly a bimodule maps. Furthermore, composition and identities in $\mathrm{Dist}_{\mathcal{V}}$ arise as special cases of the general definition of composition and identities in bicategories of bimodules, as direct calculations show.

Remark 4.2.6. For every tame bicategory $\mathcal{E}$, there is an isomorphism

$$\mathrm{Bim}(\mathcal{E})^{\mathrm{op}} \cong \mathrm{Bim}(\mathcal{E}^{\mathrm{op}})\,.$$

Recall that we write $F^{\mathrm{op}}\colon \mathbb{Y} \to \mathbb{X}$ for the morphism in $\mathcal{E}^{\mathrm{op}}$ associated to a morphism $F\colon \mathbb{X} \to \mathbb{Y}$ in $\mathcal{E}$. The required isomorphism sends an object $X/A \in \mathrm{Bim}(\mathcal{E})^{\mathrm{op}}$ to the object $X/A^{\mathrm{op}} \in \mathrm{Bim}(\mathcal{E}^{\mathrm{op}})$. Given $X/A\,, Y/B \in \mathrm{Bim}(\mathcal{E})$, if $M\colon X \to Y$ has a (B, A)-bimodule structure in $\mathcal{E}$, then $M^{\mathrm{op}}\colon Y \to X$ has an $(A^{\mathrm{op}}, B^{\mathrm{op}})$-bimodule structure in $\mathcal{E}^{\mathrm{op}}$ and so we have an isomorphism

$$\mathrm{Bim}(\mathcal{E}^{\mathrm{op}})[Y/B^{\mathrm{op}}, X/A^{\mathrm{op}}] \cong \mathrm{Bim}(\mathcal{E})[X/A, Y/B] = (\mathrm{Bim}(\mathcal{E}))^{\mathrm{op}}[Y/B, X/A]\,.$$

Moreover, if $N\colon Y \to Z$ has a (C, B)-bimodule structure, we have $(N \circ_B M)^{\mathrm{op}} = M^{\mathrm{op}} \circ_{B^{\mathrm{op}}} N^{\mathrm{op}}$.

Remark 4.2.7. For every tame homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$, there is a homomorphism

$$\mathrm{Bim}(\Phi)\colon \mathrm{Bim}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{F})$$

defined by letting $\mathrm{Bim}(\Phi)(X/A) =_{\mathrm{def}} \Phi(X)/\Phi(A)$ for $X/A \in \mathrm{Bim}(\mathcal{E})$, $\mathrm{Bim}(\Phi)(M) =_{\mathrm{def}} \Phi(M)$ for $M\colon X/A \to X/B$ and $\mathrm{Bim}(\Phi)(\alpha) =_{\mathrm{def}} \Phi(\alpha)$ for $\alpha\colon M \to N$. The condition that Φ is tame ensures that $\mathrm{Bim}(\Phi)\colon \mathrm{Bim}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{F})$ preserves composition and identity morphisms up to coherent natural isomorphism. In this way, an inclusion $\mathcal{E} \subseteq \mathcal{F}$ determines an inclusion $\mathrm{Bim}(\mathcal{E}) \subseteq \mathrm{Bim}(\mathcal{F})$.

4.3. Monad morphisms and bimodules

The aim of this section is to relate the bicategory of monads, bimodules and bimodule maps introduced in Section 4.2 to the bicategory of monads, monad morphisms and maps of monad morphisms defined as in [**69**]. We begin by recalling some definitions from [**69**], using a slightly different terminology.

Definition 4.3.1. Let $A\colon X \to X\,, B\colon Y \to Y$ be monads in $\mathcal{E}$.

(i) A *lax monad morphism* $(F, \phi)\colon (X, A) \to (Y, B)$ consists of a morphism $F\colon X \to Y$ in $\mathcal{E}$ and a 2-cell $\phi\colon B \circ F \to F \circ A$ such that the following diagrams commute:

$$\begin{array}{ccccc} B \circ B \circ F & \xrightarrow{B\circ\phi} & B \circ F \circ A & \xrightarrow{\phi\circ A} & F \circ A \circ A \\ {\scriptstyle \mu_B\circ F}\downarrow & & & & \downarrow{\scriptstyle F\circ\mu_A} \\ B \circ F & & \xrightarrow{\qquad\phi\qquad} & & F \circ A\,, \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\eta_B\circ F} & B \circ F \\ & {\scriptstyle F\circ\eta_A}\searrow & \downarrow{\scriptstyle\phi} \\ & & F \circ A\,. \end{array}$$

(ii) A *map of lax monad morphisms* $f\colon (F,\phi)\to(F',\phi')$ is a 2-cell $f\colon F\to F'$ in $\mathcal{E}$ such that the following diagram commutes:

$$\begin{array}{ccc} B\circ F & \xrightarrow{\phi} & F\circ A \\ {\scriptstyle B\circ f}\downarrow & & \downarrow{\scriptstyle f\circ A} \\ B\circ F' & \xrightarrow[\phi']{} & F'\circ A\,. \end{array}$$

Following [**69**], we write $\mathrm{Mnd}(\mathcal{E})$ for the bicategory whose objects are pairs (X,A), where $X\in\mathcal{E}$ and $A\colon X\to X$ is a monad, morphisms are lax monad morphisms and 2-cells are maps of lax monad morphisms. The composition operation of morphisms in $\mathrm{Mnd}(\mathcal{E})$ is defined in the following way: for monad morphisms $(F,\phi)\colon(X,A)\to(Y,B)$ and $(G,\psi)\colon(Y,B)\to(Z,C)$, their composite is given by the morphism $G\circ F\colon X\to Z$ equipped with the 2-cell

$$C\circ G\circ F\xrightarrow{\psi\circ F}G\circ B\circ F\xrightarrow{G\circ\phi}G\circ F\circ A\,.$$

Note that a lax monad morphism $(F,\phi)\colon(X,A)\to(Y,B)$ is an equivalence in $\mathrm{Mnd}(\mathcal{E})$ if and only if $F\colon X\to Y$ is an equivalence in $\mathcal{E}$ and ϕ is invertible. Also note that a homomorphism $\Phi\colon\mathcal{E}\to\mathcal{F}$ induces a homomorphism $\mathrm{Mnd}(\Phi)\colon\mathrm{Mnd}(\mathcal{E})\to\mathrm{Mnd}(\mathcal{F})$, defined in the evident way. As we show in Lemma 4.3.2 below, every lax monad morphism gives rise to a bimodule and every map of lax monad morphisms gives rise to a bimodule map.

Lemma 4.3.2.

(i) If $(F,\phi)\colon(X,A)\to(Y,B)$ is a lax monad morphism, then the morphism $F\circ A\colon X\to Y$ has the structure of a (B,A)-bimodule.

(ii) If $f\colon(F,\phi)\to(F',\phi')$ is a map of lax monad morphisms, then $f\circ A\colon F\circ A\to F'\circ A$ is a bimodule map.

Proof. The morphism $F\circ A\colon X\to Y$ has the structure of a free right A-module with the right action $\rho=F\circ\mu_A\colon(F\circ A)\circ A\to F\circ A$. The left action $\lambda\colon B\circ(F\circ A)\to F\circ A$ of the monad B is defined to be the composite

$$B\circ F\circ A\xrightarrow{\phi\circ A}F\circ A\circ A\xrightarrow{F\circ\mu_A}F\circ A\,.$$

The proof the required properties is a straightforward diagram-chasing argument. □

Lemma 4.3.2 allows us to define a homomorphism $R\colon\mathrm{Mnd}(\mathcal{E})\to\mathrm{Bim}(\mathcal{E})$ as follows. Its action on objects is the identity. For a monad morphism $(F,\phi)\colon(X,A)\to(Y,B)$, we define $R(F,\phi)\colon X/A\to Y/B$ to be the (B,A)-bimodule with underlying morphism $F\circ A\colon X\to Y$, as in the proof of Lemma 4.3.2, Given a monad 2-cell $f\colon(F,\phi)\to(F',\phi')$, we define $R(f)\colon F\circ A\to F'\circ A$ to be $f\circ A\colon F\circ A\to F'\circ A$, which is a bimodule map by part (ii) of Lemma 4.3.2. The remaining data necessary to define a homomorphism can be derived easily. In particular, for lax monad morphisms $(F,\phi)\colon(X,A)\to(Y,B)$ and $(G,\psi)\colon(Y,B)\to(Z,C)$, we have an isomorphism

$$R(G,\psi)\circ_B R(F,\phi)\cong R(G\circ F,(G\circ\phi)\cdot(\psi\circ F))\,,$$

since

$$R(G,\psi)\circ_B R(F,\phi)=(G\circ B)\circ_B(F\circ A)\cong G\circ F\circ A=R(G\circ F)\,.$$

Recall that for a bicategory $\mathcal{E}$, we write $\mathcal{E}^{\mathrm{op}}$ for the bicategory obtained from $\mathcal{E}$ by formally reversing the direction of morphisms, but not that of 2-cells. Since $(\mathrm{Bim}(\mathcal{E}^{\mathrm{op}}))^{\mathrm{op}}=\mathrm{Bim}(\mathcal{E})$, Lemma 4.3.2 admits a dual, which we state explicitly below since it will be useful in the following. We begin by recalling from [**69**] an explicit description of the morphisms and 2-cells in $(\mathrm{Mnd}(\mathcal{E}^{\mathrm{op}}))^{\mathrm{op}}$.

DEFINITION 4.3.3. Let $A\colon X \to X$, $B\colon Y \to Y$ be monads in $\mathcal{E}$.

(i) An *oplax monad morphism* $(F,\phi)\colon (X,A) \to (Y,B)$ consists of a morphism $F\colon X \to Y$ in $\mathcal{E}$ and a 2-cell $\psi\colon F \circ A \to B \circ F$ such that the following diagrams commute:

$$\begin{array}{ccccc} F\circ A\circ A & \xrightarrow{\psi\circ A} & B\circ F\circ A & \xrightarrow{B\circ\psi} & B\circ B\circ F \\ {\scriptstyle F\circ\mu_A}\downarrow & & & & \downarrow{\scriptstyle \mu_B\circ F} \\ F\circ A & & \xrightarrow[\psi]{} & & B\circ F\,, \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{F\circ\eta_A} & F\circ A \\ & {\scriptstyle \eta_B\circ F}\searrow & \downarrow{\scriptstyle\psi} \\ & & B\circ F\,. \end{array}$$

(ii) A *map of oplax monad morphisms* $f\colon (F,\psi) \to (F',\psi')$ is a 2-cell $f\colon F \to F'$ in $\mathcal{E}$ such that the following diagram commutes:

$$\begin{array}{ccc} F\circ A & \xrightarrow{\psi} & B\circ F \\ {\scriptstyle f\circ A}\downarrow & & \downarrow{\scriptstyle B\circ f} \\ F'\circ A & \xrightarrow[\psi']{} & B\circ F'\,. \end{array}$$

The bicategory $\mathrm{Mnd}(\mathcal{E}^{\mathrm{op}})^{\mathrm{op}}$ has the same objects as $\mathrm{Mnd}(\mathcal{E})$, oplax monad morphisms as morphisms and maps of oplax monad morphisms as 2-cells. We now state the dual of Lemma 4.3.2.

LEMMA 4.3.4.

(i) If $(F,\psi)\colon (X,A) \to (Y,B)$ is an oplax monad morphism in a bicategory $\mathcal{E}$, then the morphism $B \circ F\colon X \to Y$ has the structure of a (B,A)-bimodule.

(ii) If $f\colon (F,\psi) \to (F',\phi')$ is a map of oplax monad morphisms, then $B \circ f\colon B \circ F \to B \circ F'$ is a bimodule map.

PROOF. The morphism $B \circ F : X \to Y$ has the structure of a free left B-module with the left action $\mu \circ F\colon B \circ B \circ F \to B \circ B$. The right A-action is defined to be the composite

$$B\circ F\circ A \xrightarrow{B\circ\psi} F\circ A\circ A \xrightarrow{\mu_B\circ A} B\circ F\,.$$

As for Lemma 4.3.2, we omit the details of the verification. □

By Lemma 4.3.4, it is possible to define a homomorphism $L\colon (\mathrm{Mnd}(\mathcal{E}^{\mathrm{op}}))^{\mathrm{op}} \to \mathrm{Bim}(\mathcal{E})$, in complete analogy with the way in which we defined the homomorphism $R\colon \mathrm{Mnd}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{E})$ using Lemma 4.3.2. We omit the details, which are straightforward. The following remark will be useful in the proofs of Proposition 5.1.1 and Theorem 5.1.2.

REMARK 4.3.5. Let $\Phi, \Psi\colon \mathcal{E} \to \mathcal{F}$ be homomorphisms and let $F\colon \Phi \to \Psi$ be a pseudo-natural transformation. If $A\colon X \to X$ is a monad in $\mathcal{E}$, we have monads $\Phi(A)\colon \Phi(X) \to \Phi(X)$ and $\Psi(A)\colon \Phi(X) \to \Psi(X)$ in $\mathcal{F}$. Then, the morphism $F_X\colon \Phi(X) \to \Psi(X)$ and the pseudo-naturality 2-cell

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{F_X} & \Psi(X) \\ {\scriptstyle\Phi(A)}\downarrow & \Downarrow f_A & \downarrow{\scriptstyle\Psi(A)} \\ \Phi(X) & \xrightarrow[F_X]{} & \Psi(X) \end{array}$$

give us a lax monad morphism

$$(F_X, f_A)\colon (\Phi(X), \Phi(A)) \to (\Psi(X), \Psi(A))\,.$$

Since the 2-cell f_A is invertible, we also have an oplax monad morphism

$$(F_X, f_A^{-1})\colon (\Phi(X), \Phi(A)) \to (\Psi(X), \Psi(A))\,.$$

We now show how, under appropriate assumptions, for monads $A\colon X \to X$ and $B\colon Y \to Y$ in $\mathcal{E}$, an adjunction $(F, G, \eta, \varepsilon)\colon X \to Y$ induces an adjunction $(F', G', \eta', \varepsilon')\colon (X, A) \to (Y, B)$ in $\mathrm{Bim}(\mathcal{E})$. In order to do this, we will exploit Lemma 4.3.2 and Lemma 4.3.4. Let us begin by observing that if $(F, G)\colon X \to Y$ is an adjunction in $\mathcal{E}$, then a monad $B\colon Y \to Y$ induces a monad $B'\colon X \to X$, where $B' =_{\mathrm{def}} G \circ B \circ F$, with multiplication defined as the composite

$$G \circ B \circ F \circ G \circ B \circ F \xrightarrow{G \circ B \circ \varepsilon \circ B \circ F} G \circ B \circ B \circ F \xrightarrow{G \circ \mu_B \circ F} G \circ B \circ F\,,$$

and unit $\eta'\colon 1_X \to G \circ B \circ F$ defined as the composite

$$1_X \xrightarrow{\eta} G \circ F \xrightarrow{G \circ \eta_B \circ F} G \circ B \circ F\,.$$

Theorem 4.3.6 below allows us, under appropriate hypotheses, to construct adjunctions in $\mathrm{Bim}(\mathcal{E})$ from adjunctions in $\mathcal{E}$.

THEOREM 4.3.6. *Let $\mathcal{E}$ be a tame bicategory. Let $A\colon X \to X$, $B\colon Y \to Y$ be monads in $\mathcal{E}$ and let $(F, G, \eta, \varepsilon)\colon X \to Y$ be an adjunction in $\mathcal{E}$. Then, a monad map $\xi\colon A \to G \circ B \circ F$ determines an adjunction $(F', G', \eta', \varepsilon')\colon (X, A) \to (Y, B)$ in* $\mathrm{Bim}(\mathcal{E})$.

PROOF. Given a monad map (in the sense of Definition 4.1.1) $\xi\colon A \to G \circ B \circ F$, we define the 2-cell $\psi\colon F \circ A \to B \circ F$ as the composite

$$F \circ A \xrightarrow{F \circ \xi} F \circ G \circ B \circ F \circ \xrightarrow{\varepsilon \circ B \circ F} B \circ F\,.$$

A standard diagram-chasing argument shows that $(F, \psi)\colon (X, A) \to (Y, B)$ is an oplax monad morphism. Similarly, we define the 2-cell $\phi\colon A \circ G \to G \circ B$ as the composite

$$A \circ G \xrightarrow{\xi \circ G} G \circ B \circ F \circ G \xrightarrow{G \circ B \circ \varepsilon} G \circ B$$

and obtain a lax monad morphism $(G, \phi)\colon (Y, B) \to (X, A)$. We define $F'\colon (X, A) \to (Y, B)$ to be the (B, A)-bimodule associated to the oplax monad morphism $(F, \psi)\colon (X, A) \to (Y, B)$, as in Lemma 4.3.4. Explicitly, the morphism $F' =_{\mathrm{def}} B \circ F$ is equipped with the left B-action $\lambda\colon B \circ F' \to F'$ given by $\mu \circ F\colon B \circ B \circ F \to B \circ F$ and the right A-action $\rho\colon F' \circ A \to F'$ given by the composite

$$(\mu_B \circ F) \cdot (B \circ \psi)\colon B \circ F \circ A \to B \circ F\,.$$

Similarly, the right adjoint $G'\colon (Y, B) \to (X, A)$ is the (A, B)-bimodule associated to the lax monad morphism $(G, \phi)\colon (Y, B) \to (X, A)$, as in Lemma 4.3.2. Explicitly, $G' =_{\mathrm{def}} G \circ B$ is equipped with the left A-action $\lambda\colon A \circ G' \to G'$ given by the composite

$$(G \circ \mu_B) \cdot (\phi \circ B)\colon A \circ G \circ B \to G \circ B$$

and the right A-action $\rho\colon G' \circ A \to G'$ given by $G \circ \mu_B\colon G \circ B \circ B \to G \circ B$.

In order to define the unit of the adjunction $\eta'\colon 1_{(X,A)} \to G' \circ_B F'$, observe that by the definition of the relative composition we have

$$G' \circ_B F' \cong G \circ B \circ F\,.$$

Hence, we define η' to be the monoid map $\xi \colon A \to G \circ B \circ F$. The counit $\varepsilon' \colon F' \circ_A G' \to 1_{(Y,B)}$ is obtained via the universal property of coqualizers, via the diagram

$$\begin{array}{ccccc} F' \circ A \circ G' & \rightrightarrows & F' \circ G' & \longrightarrow & F' \circ_A G' \\ & & & \searrow^{\sigma} & \downarrow{\scriptstyle \varepsilon'} \\ & & & & B \, , \end{array}$$

where σ is the composite

$$B \circ F \circ G \circ B \xrightarrow{B \varepsilon B} B \circ B \xrightarrow{\mu} B \, .$$

Unfolding the relevant definitions, the triangular laws amount to the commutativity of the diagrams

$$\begin{array}{ccc} B \circ F \circ_A A & \xrightarrow{B \circ F \circ_A \xi} & B \circ F \circ_A G \circ B \circ F \\ & \searrow^{\ell_{B \circ F}} & \downarrow{\scriptstyle \varepsilon' \circ F} \\ & & B \circ F \, , \end{array} \qquad \begin{array}{ccc} A \circ_A G \circ B & \xrightarrow{\xi \circ_A G \circ B} & G \circ B \circ F \circ_A G \circ B \\ & \searrow^{r_{G \circ B}} & \downarrow{\scriptstyle G \circ \varepsilon'} \\ & & G \circ B \, , \end{array}$$

where $\ell_{B \circ F}$ and $r_{G \circ B}$ are the unit isomorphisms of $\mathrm{Bim}(\mathcal{E})$, defined as in (4.2.5) and (4.2.6). In both cases, the required commutativity follows by the universal property defining the relative composition operation and in particular the definition of the unit isomorphisms. $\square$

Let us remark that the adjunction $(F', G', \eta', \varepsilon')$ in $\mathrm{Bim}(\mathcal{E})$ constructed in the proof of Theorem 4.3.6 is not obtained by applying the homomorphism $\mathrm{J}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$ to the adjunction $(F, G, \eta, \varepsilon)$ in $\mathcal{E}$. Indeed, applying $\mathrm{J}_{\mathcal{E}}$ to the latter would give us an adjunction in $\mathrm{Bim}(\mathcal{E})$ between the identity monads $(X, 1_X)$ and $(Y, 1_Y)$.

4.4. Tameness of bicategories of symmetric sequences

The aim of this section is to prove that the bicategory $\mathrm{Sym}_{\mathcal{V}}$ of symmetric sequences, defined in Section 3.3, is tame. This generalizes the corresponding fact for the category of single-sorted symmetric sequences, equipped with the substitution monoidal structure, proved in [**65**]. Showing that $\mathrm{Sym}_{\mathcal{V}}$ is tame allows us to organize operads, operad bimodules and operad bimodule maps into a bicategory, called the bicategory of operad bimodules and denoted by $\mathrm{OpdBim}_{\mathcal{V}}$, using the bimodule construction of Section 4.2. More precisely, we define

$$\mathrm{OpdBim}_{\mathcal{V}} =_{\mathrm{def}} \mathrm{Bim}(\mathrm{Sym}_{\mathcal{V}}) \, . \tag{4.4.1}$$

In particular, for operads (X, A) and (Y, B), we have

$$\mathrm{OpdBim}_{\mathcal{V}}[(X, A), (Y, B)] = \mathrm{Sym}_{\mathcal{V}}[X, Y]^B_A \, .$$

In order to prove that $\mathrm{Sym}_{\mathcal{V}}$ is tame, we will show that the bicategory $S\text{-}\mathrm{Dist}_{\mathcal{V}}$ defined of Section 3.2 is tame. Since, for small $\mathcal{V}$-categories $\mathbb{X}$ and $\mathbb{Y}$, we have

$$S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{Y}] = \mathrm{CAT}_{\mathcal{V}}[S(\mathbb{Y})^{\mathrm{op}} \otimes \mathbb{X}, \mathcal{V}] \, ,$$

the existence of reflexive coequalizers in the hom-categories of $S\text{-}\mathrm{Dist}_{\mathcal{V}}$ is clear. Thus, it remains to show that the composition functors of $S\text{-}\mathrm{Dist}_{\mathcal{V}}$ preserve reflexive coequalizers in both variables. We will actually show something stronger, namely that the composition functors in $S\text{-}\mathrm{Dist}_{\mathcal{V}}$ preserve sifted colimits in the first variable and are cocontinuous in the second variable, which implies that composition functors preserve reflexive coequalizers, since reflexive coequalizers are sifted colimits. For the convenience of the reader, we begin by recalling the notion of a sifted category and recall some basic facts about it. For further information on this notion, see [**1**, Chapter 3].

Definition 4.4.1. We say that a small category $\mathbb{K}$ is *sifted* if the colimit functor

$$\mathrm{colim}_{\mathbb{K}} \colon \mathrm{Set}^{\mathbb{K}} \to \mathrm{Set}$$

preserves finite products. A category $\mathbb{K}$ is said to be *cosifted* if the opposite category $\mathbb{K}^{\mathrm{op}}$ is sifted.

A category $\mathbb{K}$ is sifted if and only if it is non-empty and the diagonal functor $d_{\mathbb{K}} \colon \mathbb{K} \to \mathbb{K} \times \mathbb{K}$ is cofinal. Dually, $\mathbb{K}$ is cosifted if and only if it is non-empty and the diagonal functor $d_{\mathbb{K}} \colon \mathbb{K} \to \mathbb{K} \times \mathbb{K}$ is coinitial. Let us say that a presheaf $X \colon \mathbb{K}^{\mathrm{op}} \to \mathrm{Set}$ is *connected* if $\mathrm{colim}_{\mathbb{K}} X = 1$. Then a category $\mathbb{K}$ is cosifted if and only if it is non-empty and the cartesian product $\mathbb{K}(-, j) \times \mathbb{K}(-, k)$ of representable presheaves is connected. A sifted (or cosifted) category is connected. In the statement of the next lemma, we write Δ for the usual category of finite ordinals and monotone maps, and $\Delta_{|1}$ for its full subcategory spanned by the objects $[0]$ and $[1]$.

Lemma 4.4.2. *The categories Δ and $\Delta_{|1}$ are cosifted.*

Proof. The colimit of a simplicial set $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is the set $\pi_0(X)$ of its connected components. It is well known that the canonical map $\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$ is bijective for any pair of simplicial sets X and Y. Moreover, $\pi_0(\Delta[0]) = 1$. Similarly, a presheaf X on $\Delta_{|1}$ is a reflexive graph and its colimit is the set $\pi_0(X)$ of its connected components. It is easy to verify that the canonical map $\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$ is bijective for any pair of reflexive graphs X and Y. Moreover, $\pi_0(\Delta[0]) = 1$. ☐

Remark 4.4.3. Since a reflexive graph in $\mathcal{E}$ is exactly a contravariant functor $X \colon \Delta_{|1}^{\mathrm{op}} \to \mathcal{E}$, reflexive coequalizers are sifted colimits.

We now establish some auxiliary facts which will allow us to establish that S-$\mathrm{Dist}_{\mathcal{V}}$ is tame. If $\mathbb{K}$ is a small category and $\mathcal{R}$ is a symmetric $\mathcal{V}$-rig, then category $\mathcal{R}^{\mathbb{K}}$ of $\mathbb{K}$-indexed diagrams in $\mathcal{R}$ has a symmetric monoidal (closed) structure with the pointwise tensor product:

$$(A \otimes B)(k) =_{\mathrm{def}} A(k) \otimes B(k)\,.$$

The unit object for the pointwise tensor product is the constant diagram $cI \colon \mathbb{K} \to \mathcal{R}$ with value the unit object $I \in \mathcal{R}$. If $A, B \in \mathcal{R}^{\mathbb{K}}$ then the canonical map

$$\mathrm{colim}_{(i,j) \in \mathbb{K} \times \mathbb{K}} A(i) \otimes B(j) \to \mathrm{colim}_{i \in \mathbb{K}} A(i) \otimes \mathrm{colim}_{j \in \mathbb{K}} B(j) \tag{4.4.2}$$

is an isomorphism, since the tensor product functor of $\mathcal{R}$ is cocontinuous in each variable. If $\mathbb{K}$ is sifted, then the canonical map

$$\mathrm{colim}_{i \in \mathbb{K}} A(i) \otimes B(i) \to \mathrm{colim}_{(i,j) \in \mathbb{K} \times \mathbb{K}} A(i) \otimes B(j)$$

is an isomorphism, since the diagonal $d_{\mathbb{K}} \colon \mathbb{K} \to \mathbb{K} \times \mathbb{K}$ is cofinal.

Proposition 4.4.4. *Let $\mathbb{K}$ be a small category and $\mathcal{R}$ a symmetric $\mathcal{V}$-rig. If $\mathbb{K}$ is sifted, then $\mathrm{colim}_{\mathbb{K}} \colon \mathcal{R}^{\mathbb{K}} \to \mathcal{R}$ is a symmetric monoidal functor.*

Proof. We saw above that the map in (4.4.2) is an isomorphism for any pair of diagrams $A, B \colon \mathbb{K} \to \mathcal{R}$. Moreover, the canonical map $\mathrm{colim}_{\mathbb{K}}\, cI \to I$ is an isomorphism, since $\mathbb{K}$ is connected. ☐

We define the n-fold tensor product functor $T^n \colon \mathcal{R}^n \to \mathcal{R}$ by letting

$$T^n(X_1, \ldots, X_n) =_{\text{def}} X_1 \otimes \ldots \otimes X_n \, .$$

COROLLARY 4.4.5. *The n-fold tensor product functor $T^n \colon \mathcal{R}^n \to \mathcal{R}$ preserves sifted colimits for every $n \geq 0$.*

PROOF. If $\mathbb{K}$ is a sifted category, then the colimit functor $\text{colim}_{\mathbb{K}} \colon \mathcal{R}^{\mathbb{K}} \to \mathcal{R}$ is monoidal by Proposition 4.4.4. It thus preserves n-fold tensor products, from which the claim follows. □

Let $\mathbb{W}$ be a small $\mathcal{V}$-category and $[\mathbb{W}, \mathcal{R}]$ be the $\mathcal{V}$-category of $\mathcal{V}$-functors from $\mathbb{W}$ to $\mathcal{R}$. If $w \in \mathbb{W}$, then the evaluation functor $ev_w \colon [\mathbb{W}, \mathcal{R}] \to [\mathbb{W}, \mathcal{R}]$ defined by letting $ev_w(F) = F(w)$ is cocontinuous. If $\overline{w} = (w_1, \ldots, w_n) \in \mathbb{W}^n$, let us put

$$ev_{\overline{w}}(F) = F(w_1) \otimes \ldots \otimes F(w_n) \, .$$

LEMMA 4.4.6. *The functor $ev_{\overline{w}} \colon [\mathbb{W}, \mathcal{R}] \to \mathcal{R}$ preserves sifted colimits for every $\overline{w} \in \mathbb{W}^n$.*

PROOF. The functor $ev_{\overline{w}}$ is the composite of the functor $\rho_{\overline{w}} \colon [\mathbb{W}, \mathcal{R}] \to \mathcal{R}^n$ defined by letting $\rho_{\overline{w}}(F) =_{\text{def}} (F(w_1), \ldots, F(w_n))$ followed by the n-fold tensor product functor $T^n \colon \mathcal{R}^n \to \mathcal{R}$. The first functor is cocontinuous, while the second preserves sifted colimits by Corollary 4.4.5. □

Recall from Section 3.2 that to an S-distributor $F \colon \mathbb{X} \to \mathbb{Y}$, i.e. a distributor $F \colon \mathbb{X} \to S(\mathbb{Y})$, we associate a distributor $F^e \colon S(X) \to S(\mathbb{Y})$ defined as in (3.1.3).

LEMMA 4.4.7. *For every pair of small $\mathcal{V}$-categories $\mathbb{X}, \mathbb{Y}$ the functor*

$$(-)^e \colon \text{Dist}_{\mathcal{V}}[\mathbb{X}, S(\mathbb{Y})] \to \text{Dist}_{\mathcal{V}}[S(\mathbb{X}), S(\mathbb{Y})]$$

preserves sifted colimits.

PROOF. The claim follows if we show that the functor

$$(-)^e \colon \text{CAT}_{\mathcal{V}}[\mathbb{X}, PS(\mathbb{Y})] \to \text{CAT}[S(\mathbb{X}), PS(\mathbb{Y})]$$

preserves sifted colimits. But this is a consequence of Lemma 4.4.6 since for a $\mathcal{V}$-functor $F \colon \mathbb{X} \to PS(\mathbb{Y})$, we have $F^e(\bar{x}) = ev_{\bar{x}}(F)$. □

THEOREM 4.4.8. *The bicategory S-$\text{Dist}_{\mathcal{V}}$ is tame. In particular, the horizontal composition functors of S-$\text{Dist}_{\mathcal{V}}$ preserve sifted colimits in the first variable and are cocontinuous in the second variable.*

PROOF. The hom-categories of S-$\text{Dist}_{\mathcal{V}}$ clearly have reflexive coequalizers. Recall from (3.2.2) that for $F \in S$-$\text{Dist}_{\mathcal{V}}[X, Y]$ and $G \in S$-$\text{Dist}_{\mathcal{V}}[Y, Z]$ we have

$$\begin{aligned}(G \circ F)[\bar{z}; x] &= \int^{\bar{y} \in S(\mathbb{Y})} G^e[\bar{z}; \bar{y}] \otimes F[\bar{y}; x] \\ &= (G^e \circ F)[\bar{z}; x] \, .\end{aligned}$$

But the composition functors in $\text{Dist}_{\mathcal{V}}$ are cocontinuous in each variable. Hence, the functor $F \mapsto G \circ F$ is cocontinuous, while the functor $G \mapsto G^e \circ F$ preserves sifted colimits by Lemma 4.4.7. □

COROLLARY 4.4.9. *The bicategories $\text{CatSym}_{\mathcal{V}}$ and $\text{Sym}_{\mathcal{V}}$ are tame.*

PROOF. Since a bicategory is tame if and only if its opposite is so, the claim that the bicategory $\mathrm{CatSym}_{\mathcal{V}}$ is tame follows from Theorem 4.4.8, since $\mathrm{CatSym}_{\mathcal{V}}$ is the opposite of $S\text{-Dist}_{\mathcal{V}}$, which is tame by Theorem 4.4.8. The bicategory $\mathrm{Sym}_{\mathcal{V}}$ is tame since it is a full sub-bicategory of the tame bicategory $\mathrm{CatSym}_{\mathcal{V}}$. □

Since we defined the bicategory of categorical symmetric sequences as the opposite of the bicategory of $S\text{-Dist}_{\mathcal{V}}$, by letting $\mathrm{CatSym}_{\mathcal{V}} =_{\mathrm{def}} S\text{-Dist}_{\mathcal{V}}^{\mathrm{op}}$ and the bicategory of operad bimodules by letting $\mathrm{OpdBim}_{\mathcal{V}} =_{\mathrm{def}} \mathrm{Bim}(S\text{-Mat}_{\mathcal{V}}^{\mathrm{op}})$, the inclusion $S\text{-Mat}_{\mathcal{V}}^{\mathrm{op}} \subseteq S\text{-Dist}_{\mathcal{V}}^{\mathrm{op}}$ induces an inclusion

$$\mathrm{OpdBim}_{\mathcal{V}} \subseteq \mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})\,. \tag{4.4.3}$$

As we will see in Theorem 5.4.5, this inclusion is actually an equivalence.

4.5. Analytic functors

We define the analytic functors associated to an operad bimodule. Recall that for an operad (X, A), we write $\mathrm{Alg}_{\mathcal{R}}(A)$ for its category of algebras and algebra morphisms in a symmetric $\mathcal{V}$-ring $\mathcal{R}$. Let (X, A) and (Y, B) be operads. Given an operad bimodule $F\colon (X, A) \to (Y, B)$, for a symmetric $\mathcal{V}$-rig $\mathcal{R}$ we define the *analytic functor*

$$\mathrm{Alg}_{\mathcal{R}}(F)\colon \mathrm{Alg}_{\mathcal{R}}(A) \to \mathrm{Alg}_{\mathcal{R}}(B)$$

associated to F as follows. For an A-algebra M, we let

$$\mathrm{Alg}_{\mathcal{R}}(F)(M) =_{\mathrm{def}} F \circ_A M\,, \tag{4.5.1}$$

where $F \circ_A M$ is given by the following reflexive coequalizer diagram

$$F \circ A \circ M \underset{F\circ\lambda}{\overset{\rho\circ M}{\rightrightarrows}} F \circ M \longrightarrow F \circ_A M\,.$$

This object has a B-algebra structure given as in (4.2.3). This functor fits in the following commutative diagram

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{R}}(A) & \xrightarrow{\mathrm{Alg}_{\mathcal{R}}(F)} & \mathrm{Alg}_{\mathcal{R}}(B) \\ \uparrow & & \downarrow \\ \mathcal{R}^X & \xrightarrow[F]{} & \mathcal{R}^Y\,, \end{array}$$

where the vertical arrows are the evident free algebra and forgetful functors, and $F\colon \mathcal{R}^X \to \mathcal{R}^Y$ is the analytic functor associated to the symmetric sequence $F\colon X \to Y$, defined in (3.3.1).

We now show how the *restriction functor* and *extension functor* (see, e.g., [**32**] for these functors in the case of single-sorted operads) associated to an operad morphism are analytic functors. This will follow by an application of Theorem 4.3.6. Let us begin by recalling some definitions. Let $\mathbb{X} = (X, A)$, $\mathbb{Y} = (Y, B)$ be operads, viewed as monads in the bicategory $\mathrm{Sym}_{\mathcal{V}}$. Thus, $X\,, Y$ are sets and $A\colon X \to X$, $B\colon Y \to Y$ are symmetric sequences equipped with multiplication and unit. Let us also fix an operad morphism $(u, \xi)\colon X \to \mathbb{Y}$, which consists of a function $u\colon X \to Y$ and a monoid morphism $\xi\colon A \to B'$, where $B'\colon X \to X$ is the monad whose underlying symmetric sequence is defined by letting

$$B'[x_1, \ldots, x_n; x] =_{\mathrm{def}} B[ux_1, \ldots, ux_n; ux]\,. \tag{4.5.2}$$

It will be useful to define the symmetric sequence $u^\circ \colon X \to Y$ and $u_\circ \colon Y \to X$ by letting

$$u^\circ[x_1, \ldots, x_n; y] =_{\mathrm{def}} B[ux_1, \ldots, ux_n; y]\,, \quad u_\circ[y_1, \ldots, y_n; x] =_{\mathrm{def}} B[y_1, \ldots, y_n; ux]\,. \tag{4.5.3}$$

We wish to show that $u_\circ$ and u° form an adjunction in $\mathrm{Sym}_{\mathcal{V}}$. We make some preliminary observations. As a special case of the corresponding facts for $\mathcal{V}$-functors and distributors, recalled in Section 1.3, the function $u \colon X \to Y$ determines an adjunction $(u_\bullet, u^\bullet) \colon X \to Y$ in $\mathrm{Mat}_{\mathcal{V}}$. The homomorphism $\delta \colon \mathrm{Mat}_{\mathcal{V}} \to S\text{-}\mathrm{Mat}_{\mathcal{V}}$ takes this adjunction to an adjunction $(\delta(u_\bullet), \delta(u^\bullet)) \colon X \to Y$ in $S\text{-}\mathrm{Mat}_{\mathcal{V}}$, which gives us an adjunction $(\delta(u^\bullet), \delta(u_\bullet)) \colon X \to Y$ in $\mathrm{Sym}_{\mathcal{V}}$ by duality. Explicitly, the symmetric sequence $\delta(u^\bullet) \colon X \to Y$ and $\delta(u_\bullet) \colon Y \to X$ are defined by letting

$$\delta(u^\bullet)[\bar{x}; y] =_{\mathrm{def}} \begin{cases} I & \text{if } \bar{x} = (x) \text{ and } ux = y\,, \\ 0 & \text{otherwise,} \end{cases} \qquad \delta(u_\bullet)[\bar{y}; x] =_{\mathrm{def}} \begin{cases} I & \text{if } \bar{y} = (ux)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Here, I and 0 denote the unit and the initial object of $\mathcal{V}$, respectively. We now state and prove a lemma which relates these symmetric sequences with those defined in (4.5.3).

Lemma 4.5.1. *There are isomorphisms*

(i) $u^\circ \cong B \circ \delta(u^\bullet)$,

(ii) $u_\circ \cong \delta(u_\bullet) \circ B$.

Proof. For the proof, we write $u\bar{x}$ for $(ux_1, \ldots, ux_n)$, where $\bar{x} = (x_1, \ldots, x_n)$. For (i), observe that by the definition of composition in $\mathrm{Sym}_{\mathcal{V}}$, we have

$$(B \circ \delta(u^\bullet))[\bar{x}; y] = \bigsqcup_{m \in \mathbb{N}} \int^{(y_1, \ldots, y_m) \in S^n(Y)} \int^{\bar{x}_1 \in S(X)} \cdots \int^{\bar{x}_m \in S(X)} \delta(u^\bullet)[\bar{x}_1; y_1] \otimes \ldots$$
$$\ldots \otimes \delta(u^\bullet)[\bar{x}_m; y_m] \otimes S(X)[\bar{x}, \bar{x}_1 \oplus \ldots \oplus \bar{x}_m] \otimes B[y_1, \ldots y_m; y]\,.$$

By the definition of $\delta(u^\bullet)$, the right-hand side is isomorphic to

$$\bigsqcup_{m \in \mathbb{N}} \int^{x_1 \in X} \cdots \int^{x_m \in X} S(X)[\bar{x}, (x_1, \ldots, x_m)] \otimes B[ux_1, \ldots ux_m; y]\,.$$

This, in turn, is isomorphic to $B[u\bar{x}; y]$, as required. The proof of (ii) is similar. □

The next lemma gives an alternative description of the monad $B' \colon X \to X$ defined in (4.5.2).

Lemma 4.5.2. *There is an isomorphism* $B' \cong \delta(u_\bullet) \circ B \circ \delta(u^\bullet)$.

Proof. By part (i) of Lemma 4.5.1, it is sufficient to exhibit an isomorphism $B' \cong \delta(u_\bullet) \circ u^\circ$. We have

$$(\delta(u_\bullet) \circ u^\circ)[\bar{x}; x] = \bigsqcup_{n \in \mathbb{N}} \int^{(y_1, \ldots, y_n) \in S^n(Y)} \int^{\bar{x}_1 \in S(X)} \cdots \int^{\bar{x}_n \in S(X)} u^\circ[\bar{x}_1; y_1] \otimes \ldots$$
$$\ldots \otimes u^\circ[\bar{x}_n; y_n] \otimes S(X)[\bar{x}, \bar{x}_1 \oplus \ldots \oplus \bar{x}_n] \otimes \delta(u_\bullet)[y_1, \ldots, y_n; x]\,.$$

By the definition of $\delta(u_\bullet)$, the left-hand side is isomorphic to

$$\int^{\bar{x}' \in S(X)} u^\circ[\bar{x}'; ux] \otimes S(X)[\bar{x}, \bar{x}']\,,$$

which is isomorphic to $u^\circ[\bar{x}; ux]$. But, by definition of u° and B', we have $u^\circ[\bar{x}; ux] = B'[\bar{x}; x]$. □

LEMMA 4.5.3. *The symmetric sequence* $u^\circ\colon X \to Y$ *has the structure of a* (B, A)*-bimodule, the symmetric sequence* $u_\circ\colon Y \to X$ *has the structure of an* (A, B)*-bimodule and the resulting operad bimodules form an adjunction* $(u^\circ, u_\circ)\colon (X, A) \to (Y, B)$ *in the bicategory* $\mathrm{OpdBim}_{\mathcal{V}}$*.*

PROOF. First of all, observe that we have an adjunction $(\delta(u^\bullet), \delta(u_\bullet))\colon X \to Y$ in the bicategory $\mathrm{Sym}_{\mathcal{V}}$. Secondly, the monoid morphism $\xi\colon A \to B'$ determines, by Lemma 4.5.2, a monoid morphism $\xi'\colon A \to \delta(u_\bullet) \circ B \circ \delta(u^\bullet)$. By Proposition 4.3.6, it follows that we have an adjunction

$$(B \circ \delta(u^\bullet), \delta(u_\bullet) \circ B)\colon (X, A) \to (Y, B)$$

in $\mathrm{OpdBim}_{\mathcal{V}}$. But, by Lemma 4.5.1, the symmetric sequences $B \circ \delta(u^\bullet)$ and $\delta(u_\bullet) \circ B$ are isomorphic to the symmetric sequences u° and $u_\circ$, which therefore inherit a bimodule structure so as to give us the required adjunction. $\square$

We can apply Lemma 4.5.3 to give a general version of the restriction and extension functors between categories of algebras for operads. We continue to consider a fixed operad morphism $(u, \xi)\colon (X, A) \to (Y, B)$. For a set K, we define the restriction functor $u^*\colon [K, X]^A \to [K, Y]^B$ as follows. For a left B-module $N\colon K \to Y$, we define the left A-module $u^*(N)\colon K \to X$ by letting

$$u^*(N)[k_1, \ldots, k_n; x] =_{\mathrm{def}} N[k_1, \ldots, k_n; u(x)]\,.$$

THEOREM 4.5.4. *For every operad morphism* $(u, \xi)\colon X \to Y$*, the functor*

$$u^*\colon [K, X]^A \to [K, Y]^B$$

is an analytic functor and it has a left adjoint

$$u_!\colon [K, Y]^B \to [K, X]^A\,,$$

which is also an analytic functor.

PROOF. We show that u^* is the analytic functor associated to the operad bimodule $u_\circ\colon (Y, B) \to (X, A)$. By the formula in (4.5.1), this amounts to showing that we have a natural isomorphism

$$u^*(N) \cong u_\circ \circ_B N\,, \tag{4.5.4}$$

for every left B-module $N\colon K \to Y$. By part (ii) of Lemma 4.5.1 and the unit isomorphism of $\mathrm{OpdBim}_{\mathcal{V}}$, we have $u_\circ \circ_B N \cong \delta(u_\bullet) \circ B \circ_B N \cong \delta(u_\bullet) \circ N$. Hence, it suffices to exhibit an isomorphism $u^*(N) \cong \delta(u_\bullet) \circ N$, which can be done using calculations similar to those in the proofs of Lemma 4.5.1 and Lemma 4.5.2. By the isomorphism in (4.5.4) and Lemma 4.5.3, it follows that we can define the left adjoint $u_!\colon [K, Y]^B \to [K, X]^A$ as the analytic functor associated to the operad bimodule $u^\circ : (X, A) \to (Y, B)$. Explicitly, for a left A-module $M\colon K \to X$, we have

$$u_!(M) =_{\mathrm{def}} u^\circ \circ_A M\,.$$

The required adjointness $u_! \dashv u^*$ now follows immediately from the adjointness $u^\circ \dashv u_\circ$ proved in Lemma 4.5.3. $\square$

CHAPTER 5

Cartesian closure of operad bimodules

The goal of this chapter is to prove that the bicategory of operad bimodules $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed, which is our second main result. The proof of this fact uses two auxiliary results. The first is that, for a tame bicategory $\mathcal{E}$, if $\mathcal{E}$ is cartesian closed, then so is $\mathrm{Bim}(\mathcal{E})$. We establish this in Section 5.1. The second auxiliary result is that the inclusion $\mathrm{OpdBim}_{\mathcal{V}} \subseteq \mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})$ determined by the inclusion $\mathrm{Sym}_{\mathcal{V}} \subseteq \mathrm{CatSym}_{\mathcal{V}}$ is an equivalence. We establish this in Section 5.4, as a consequence of the development of some aspects of monad theory within tame bicategories and bicategories of bimodules, which is given in Section 5.2 and Section 5.3. In particular, we show how, for a tame bicategory $\mathcal{E}$, the bicategory $\mathrm{Bim}(\mathcal{E})$ can be seen as the Eilenberg-Moore completion of $\mathcal{E}$ as a tame bicategory, which is a special case of a general result obtained independently in [**33**]. This is proved in Appendix B.

5.1. Cartesian closed bicategories of bimodules

We show that if a tame bicategory $\mathcal{E}$ is cartesian closed, then so is the bicategory $\mathrm{Bim}(\mathcal{E})$. We begin by considering the cartesian structure.

PROPOSITION 5.1.1. *Let $\mathcal{E}$ be a tame bicategory. If $\mathcal{E}$ is cartesian, then so is* $\mathrm{Bim}(\mathcal{E})$*.*

PROOF. Let us first verify that a terminal object $\top$ in $\mathcal{E}$ remains a terminal object in $\mathrm{Bim}(\mathcal{E})$. For this, we have to show that the category $\mathcal{E}[X/A, \top]$ is equivalent to the terminal category for every object $X/A \in \mathrm{Bim}(\mathcal{E})$. By definition, $\mathcal{E}[X/A, \top] = \mathcal{E}[X, \top]_A$ is the category of algebras of the monad $\mathcal{E}[A, \top]$ acting on the category $\mathcal{E}[X, \top]$. But the monad $\mathcal{E}[A, \top]$ is isomorphic to the identity monad, since every morphism in $\mathcal{E}[X, \top]$ is invertible. It follows that the category $\mathcal{E}[X, \top]_A$ is equivalent to the category $\mathcal{E}[X, \top]$. Hence the category $\mathcal{E}[X/A, \top]$ is equivalent to the terminal category.

Let us now show that the category $\mathrm{Bim}(\mathcal{E})$ admits binary cartesian products. The cartesian product homomorphism $(-) \times (-) \colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ takes a monad $(B_1, B_2) \colon (Y_1, Y_2) \to (Y_1, Y_2)$ in $\mathcal{E} \times \mathcal{E}$ to a monad $B_1 \times B_2 \colon Y_1 \times Y_2 \to Y_1 \times Y_2$ in $\mathcal{E}$. We will prove that

$$Y_1/B_1 \times Y_2/B_2 = (Y_1 \times Y_2)/(B_1 \times B_2)\,,$$

i.e. that the product of Y_1/B_1 and Y_2/B_2 in $\mathrm{Bim}(\mathcal{E})$ is given by $(Y_1 \times Y_2)/(B_1 \times B_2)$. The projections $\pi_1 \colon Y_1 \times Y_2 \to Y_1$ and $\pi_2 \colon Y_1 \times Y_2 \to Y_2$ are components of pseudo-natural transformations. Hence the left hand square in the following diagrams commute up to a canonical isomorphism $\sigma_1 \colon B_1 \circ \pi_1 \cong \pi_1 \circ (B_1 \times B_2)$ and the right hand square up to a canonical isomorphism $\sigma_2 \colon B_2 \circ \pi_2 \cong \pi_2 \circ (B_1 \times B_2)$:

$$\begin{array}{ccccc} Y_1 & \xleftarrow{\ \pi_1\ } & Y_1 \times Y_2 & \xrightarrow{\ \pi_2\ } & Y_2 \\ {\scriptstyle B_1}\downarrow & & {\scriptstyle B_1 \times B_2}\downarrow & & \downarrow{\scriptstyle B_2} \\ Y_1 & \xleftarrow[\ \pi_1\ ]{} & Y_1 \times Y_2 & \xrightarrow[\ \pi_2\ ]{} & Y_2\,. \end{array} \tag{5.1.1}$$

Moreover, $(\pi_1, \sigma_1)\colon (Y_1 \times Y_2, B_1 \times B_2) \to (Y_1, B_1)$ and $(\pi_2, \sigma_2)\colon (Y_1 \times Y_2, B_1 \times B_2) \to (Y_2, B_2)$ are lax monad morphisms by Remark 4.3.5. It follows by Lemma 4.3.2 that the morphism

$$\tilde{\pi}_1 =_{\mathrm{def}} \pi_1 \circ (B_1 \times B_2)\colon Y \to Y_1$$

has the structure of a $(B_1, B_1 \times B_2)$-bimodule and that the morphism

$$\tilde{\pi}_2 =_{\mathrm{def}} \pi_2 \circ (B_1 \times B_2)\colon Y \to Y_2$$

has the structure of a $(B_2, B_1 \times B_2)$-bimodule. Let us put $Y =_{\mathrm{def}} Y_1 \times Y_2$ and $B =_{\mathrm{def}} B_1 \times B_2$ and show that the object $Y/B \in \mathrm{Bim}(\mathcal{E})$ equipped with the morphisms $\tilde{\pi}_1\colon Y/B \to Y_1/B_1$ and $\tilde{\pi}_2\colon Y/B \to Y_2/B_2$ is the cartesian product of the objects Y_1/B_1 and Y_2/B_2. For this we have to show that the functor

$$(\tilde{\pi}_1, \tilde{\pi}_2) \circ_B (-)\colon \mathcal{E}[X, Y]_A^{B_1 \times B_2} \longrightarrow \mathcal{E}[X, Y_1]_A^{B_1} \times \mathcal{E}[X, Y_2]_A^{B_2} \tag{5.1.2}$$

defined by letting $(\tilde{\pi}_1, \tilde{\pi}_2) \circ_B M = (\tilde{\pi}_1 \circ_B M, \tilde{\pi}_2 \circ_B M)$ is an equivalence of categories for every object $X/A \in \mathrm{Bim}(\mathcal{E})$. The equivalence of categories

$$(\pi_1, \pi_2) \circ (-) = (\mathcal{E}[X, \pi_1], \mathcal{E}[X, \pi_2])\colon \mathcal{E}[X, Y] \to \mathcal{E}[X, Y_1] \times \mathcal{E}[X, Y_2]$$

is the component associated to the triple $(X, Y_1, Y_2) \in \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \times \mathcal{E}$ of a pseudo-natural transformation. By Remark 4.3.5, it induces a lax monad morphism

$$\big(\mathcal{E}[X, Y_1 \times Y_2], \mathcal{E}[A, B_1 \times B_2]\big) \to \big(\mathcal{E}[X, Y_1], \mathcal{E}[A, B_1]\big) \times \big(\mathcal{E}[X, Y_2], \mathcal{E}[A, B_2]\big)$$

in Cat. This is an equivalence in the 2-category Mnd(Cat) since the functor $(\mathcal{E}[X, \pi_1], \mathcal{E}[X, \pi_2])$ is an equivalence. Hence the induced functor

$$\mathcal{E}[X, Y]_A^{B_1 \times B_2} \longrightarrow \mathcal{E}[X, Y_1]_A^{B_1} \times \mathcal{E}[X, Y_2]_A^{B_2}\,, \tag{5.1.3}$$

which takes a $M \in \mathcal{E}[X, Y]_A^B$ to $(\pi_1 \circ M, \pi_2 \circ M)$, is an equivalence of categories. But we have

$$\tilde{\pi}_1 \circ_B M \cong \pi_1 \circ B \circ_B \circ M \cong \pi_1 \circ M \quad \text{and} \quad \tilde{\pi}_2 \circ_B M \cong \pi_2 \circ B \circ_B \circ M \cong \pi_2 \circ M\,.$$

Thus, $(\tilde{\pi}_1 \circ_B M, \tilde{\pi}_2 \circ_B M) \cong (\pi_1 \circ M, \pi_2 \circ M)$. This shows that the functor in (5.1.2) is an equivalence of categories. ☐

THEOREM 5.1.2. *Let $\mathcal{E}$ be a tame bicategory. If $\mathcal{E}$ is cartesian closed, then so is* $\mathrm{Bim}(\mathcal{E})$.

PROOF. The internal hom homomorphism $(-)^{(-)}\colon \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{E}$ takes a monad (B, C) on the object $(Y, Z) \in \mathcal{E}^{\mathrm{op}} \times \mathcal{E}$ to a monad C^B on the object $Z^Y \in \mathcal{E}$. We will prove that

$$(Z/C)^{Y/B} = Z^Y / C^B\,, \tag{5.1.4}$$

i.e. that the exponential of (Z/C) by Y/B in the bicategory $\mathrm{Bim}(\mathcal{E})$ is given by Z^Y/C^B. If $\mathrm{ev}\colon Y^Z \times Y \to Z$ is the evaluation then the adjunction

$$\theta\colon \mathcal{E}[X, Z^Y] \to \mathcal{E}[X \times Y, Z]$$

is defined by letting $\theta(M) =_{\mathrm{def}} \mathrm{ev} \circ (M \times Y)$ for every $X \in \mathcal{E}$ and $M\colon X \to Z^Y$. If (A, B, C) is a monad on the object $(X, Y, Z) \in \mathcal{E}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} \times \mathcal{E}$, then by Remark 4.3.5, θ induces a lax monad morphism

$$\big(\mathcal{E}[X, Z^Y], \mathcal{E}[A, C^B]\big) \to \big(\mathcal{E}[X \times Y, Z], \mathcal{E}[A \times B, C]\big)\,,$$

since it is a component of a pseudo-natural transformation. This lax monad morphism induces an equivalence between the categories of algebras over these monads

$$\tilde{\theta}\colon \mathcal{E}[X, Z^Y]_A^{C^B} \to \mathcal{E}[X \times Y, Z]_{A \times B}^C$$

since θ is an equivalence. By definition, we have a commutative square

$$\begin{array}{ccc} \mathcal{E}[X, Z^Y]_A^{C^B} & \xrightarrow{\tilde{\theta}} & \mathcal{E}[X \times Y, Z]_{A \times B}^{C} \\ \downarrow & & \downarrow \\ \mathcal{E}[X, Z^Y] & \xrightarrow[\theta]{} & \mathcal{E}[X \times Y, Z]\,, \end{array}$$

in which the vertical arrows are forgetful functors. Note that

$$\mathcal{E}[X, Z^Y]_A^{C^B} = \mathrm{Bim}(\mathcal{E})[X/A,\ Z^Y/C^B]$$

and

$$\mathcal{E}[X \times Y, Z]_{A \times B}^{C} = \mathrm{Bim}(\mathcal{E})[X/A \times Y/B,\ Z/C]\,.$$

Let us show that the equivalence

$$\tilde{\theta} \colon \mathrm{Bim}(\mathcal{E})[X/A,\ Z^Y/C^B] \to \mathrm{Bim}(\mathcal{E})[X/A \times Y/B,\ Z/C]$$

is natural in $X/A \in \mathrm{Bim}(\mathcal{E})$. But $\tilde{\theta}$ is natural if and only if it is of the form

$$\tilde{\theta}(M) = \widetilde{\mathrm{ev}} \circ_{C^B \times B} (M \times Y/B)$$

for some morphism $\widetilde{\mathrm{ev}} \colon Z^Y/C^B \times Y/B \to Z/C$ in the bicategory $\mathrm{Bim}(\mathcal{E})$. If we apply this formula to the case $M = 1_{Z^Y} = C^B$, we obtain that

$$\widetilde{\mathrm{ev}} = \tilde{\theta}(C^B \times B) = \mathrm{ev} \circ (C^B \times B)\,.$$

Conversely, let us define $\widetilde{\mathrm{ev}} \colon Z^Y \times Y \to Z$ by letting

$$\widetilde{\mathrm{ev}} =_{\mathrm{def}} \mathrm{ev} \circ (C^B \times B)\,.$$

Let us show that the morphism $\widetilde{\mathrm{ev}}$ so defined has the structure of a $(C, C^B \times B)$-bimodule. Note that $C^B \times B = (C^Y \times Y) \circ (Z^B \times B)$ and that the monad $C^Y \times Y$ commutes with the monad $Z^B \times B$, since we have

$$C^Y \circ Z^B = C^B = Z^B \circ C^Y$$

by functoriality. The morphism $\mathrm{ev} \colon Z^Y \times Y \to Z$ is a component of a pseudo-natural transformation. By Remark 4.3.5 it defines a lax monad morphism $(\mathrm{ev}, \alpha) \colon (Z^Y \times Y, C^Y \times Y) \to (Z, C)$ and it follows by Lemma 4.3.2 that the morphism $\mathrm{ev} \circ (C^Y \times Y)$ has the structure of a $(C, C^Y \times Y)$-bimodule. Hence the morphism

$$\widetilde{\mathrm{ev}} =_{\mathrm{def}} \mathrm{ev} \circ (C^B \times B) = \mathrm{ev} \circ (C^Y \times Y) \circ (Z^B \times B)$$

has the structure of a $(C, C^B \times B)$-bimodule, since the monad $Z^B \times B$ commutes with the monad $C^Y \times Y$. For every $M \in \mathcal{E}[X, Z^Y]_A^{C^B}$ we have

$$\tilde{\theta}(M) = \mathrm{ev} \circ (M \times Y) = \mathrm{ev} \circ (C^B \times B) \circ_{(C^B \times B)} (M \times Y) = \widetilde{\mathrm{ev}} \circ_{(C^B \times B)} (M \times Y)\,.$$

This shows that the equivalence $\tilde{\theta}$ is natural and hence that $(Z/C)^{Y/B} = Z^Y/C^B$. □

Recall from Theorem 3.4.2 that the tame bicategory $\mathrm{CatSym}_{\mathcal{V}}$ is cartesian closed. Hence, by Theorem 5.1.2, the bicategory $\mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})$ is also cartesian closed. In order to prove that $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed, recall that $\mathrm{OpdBim}_{\mathcal{V}} = \mathrm{Bim}(\mathrm{Sym}_{\mathcal{V}})$ and that have an inclusion $\mathrm{Sym}_{\mathcal{V}} \subseteq \mathrm{CatSym}_{\mathcal{V}}$. Thus, we have an induced inclusion

$$\mathrm{OpdBim}_{\mathcal{V}} \subseteq \mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})$$

Thus, in order to prove that $\mathrm{OpdBim}_{\mathcal{V}}$ is cartesian closed, it is sufficient to show that this inclusion is an equivalence. The next two, final, sections of this paper lead to a proof of this fact.

5.2. Monad theory in tame bicategories

The aim of this section is to develop some aspects of the formal theory of monads in the setting of a tame bicategory. We begin by reviewing some notions and results from [**69**], adapting them from the setting of 2-category to that of a bicategory, and then focus on the particular aspects that arise in tame bicategories. Let $\mathcal{E}$ be a fixed bicategory. Recall from Section 4.1 that, for a monad $A\colon X \to X$ in $\mathcal{E}$ and $K \in \mathcal{E}$, we write $\mathcal{E}[K, X]^A$ for the category of left A-modules with domain K and left A-module maps. The category $\mathcal{E}[K, X]^A$ depends pseudo-functorially on K and so we obtain a prestack $\mathcal{E}[-, X]^A \colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat}$.

Definition 5.2.1. An *Eilenberg-Moore object* for a monad $A\colon X \to X$ in $\mathcal{E}$ is a representing object for the prestack $\mathcal{E}[-, X]^A \colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat}$.

Concretely, an Eilenberg-Moore object for a monad $A\colon X \to X$ consists of an object $X^A \in \mathcal{E}$ and a morphism $U\colon X^A \to X$ equipped with a left A-action $\lambda\colon A \circ U \to U$, which is universal in the following sense: the functor

$$\mathcal{E}[K, X^A] \to \mathcal{E}[K, X]^A \tag{5.2.1}$$

which takes a morphism $N\colon K \to X^A$ to the morphism $U \circ N\colon K \to X$ equipped with the left action $\lambda \circ N\colon A \circ U \circ N \to U \circ N$ is an equivalence of categories for every object $K \in \mathcal{E}$. In particular, for any left A-module $M\colon K \to X$ there exists a morphism $N\colon K \to X^A$ together with an invertible 2-cell $\alpha\colon M \to U \circ N$ such that the following square commutes,

$$\begin{array}{ccc} A \circ M & \xrightarrow{A\circ\alpha} & A \circ U \circ N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda\circ N} \\ M & \xrightarrow[\alpha]{} & U \circ N. \end{array}$$

In the following, we will adopt a slight abuse of language and refer to either the object X^A or the morphism $U\colon X^A \to X$ as the Eilenberg-Moore object for a monad $A\colon X \to X$. Note that the universal property characterizing an Eilenberg-Moore object considered here is weaker than introduced in [**69**], even in a 2-category, since we require the functor in (5.2.1) to be an equivalence rather than an isomorphism. In particular, an Eilenberg-Moore object for a monad, as defined here, is unique up to equivalence rather than up to isomorphism as in [**69**].

Proposition 5.2.2. *If $U\colon X^A \to X$ is an Eilenberg-Moore object for a monad $A\colon X \to X$, then the functor $\mathcal{E}[K, U]\colon \mathcal{E}[K, X^A] \to \mathcal{E}[K, X]$ is monadic for every $K \in \mathcal{E}$.*

Proof. We have a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{E}[K, X^A] & \xrightarrow{\simeq} & \mathcal{E}[K, X]^A\,, \\ & {\scriptstyle\mathcal{E}[K,U]}\searrow & \downarrow \\ & & \mathcal{E}[K, X]\,, \end{array}$$

where the vertical arrow is the evident forgetful functor, which is monadic by construction. Hence also the functor $\mathcal{E}[K, U]$, since it is the composite of a monadic functor and an equivalence of categories. ☐

The next proposition is a version of [**69**, Theorem 2] in the context of bicategories. We omit the proof.

Proposition 5.2.3. *Every Eilenberg-Moore object* $U\colon X^A \to X$ *for a monad* $A\colon X \to X$ *has a left adjoint* $F\colon X \to X^A$ *and the monad map* $\pi\colon A \to U \circ F$ *given by the composite*

$$A \xrightarrow{A\circ\eta} A \circ U \circ F \xrightarrow{\lambda\circ F} U \circ F$$

is invertible. □

Let $A\colon X \to X$ be a monad in $\mathcal{E}$. Recall from Section 4.1 that we write $\mathcal{E}[K, X]_A$ for the category of right A-modules with domain K and right A-module maps between them. The category $\mathcal{E}[K, X]_A$ depends pseudo-functorially on the object K and so we obtain a prestack $\mathcal{E}[X, -]_A\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat}$.

Definition 5.2.4. A *Kleisli object* for a monad $A\colon X \to X$ in a bicategory $\mathcal{E}$ is a representing object for the prestack $\mathcal{E}[X, -]_A\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat}$.

Concretely, a Kleisli object for a monad $A\colon X \to X$ consists of an object $X_A \in \mathcal{E}$ and a morphism $F\colon X \to X_A$ equipped with a right A-action $\rho\colon F \circ A \to F$ which is universal in the following sense: the functor

$$\mathcal{E}[F, K]\colon \mathcal{E}[X_A, K] \to \mathcal{E}[X, K]_A$$

which takes a morphism $N\colon X_A \to K$ to the morphism $N \circ F_A\colon X \to K$ equipped with the right action $N \circ \rho\colon N \circ F \circ A \to N \circ F$ is an equivalence of categories for every $K \in \mathcal{E}$. In particular, for any right A-module $M\colon X \to K$ there exists a morphism $N\colon X_A \to K$ together with an invertible 2-cell $\alpha\colon M \to N \circ F$ such that the following square commutes:

$$\begin{array}{ccc} M \circ A & \xrightarrow{\alpha\circ A} & N \circ F \circ A \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle N\circ\rho} \\ M & \xrightarrow[\alpha]{} & N \circ F\,. \end{array}$$

Observe that a right A-module $F\colon X \to X_A$ is a Kleisli object for a monad $A\colon X \to X$ in $\mathcal{E}$ if and only if the left A^{op}-module $F^{\mathrm{op}}\colon X_A \to X$ in $\mathcal{E}^{\mathrm{op}}$ is an Eilenberg-Moore object for the monad $A^{\mathrm{op}}\colon X \to X$ in $\mathcal{E}^{\mathrm{op}}$.

Proposition 5.2.5. *If* $F\colon X \to X_A$ *is a Kleisli object for a monad* $A\colon X \to X$*, then the functor* $\mathcal{E}[F, K]\colon \mathcal{E}[X_A, K] \to \mathcal{E}[X, K]$ *is monadic for every object* $K \in \mathcal{E}$*.*

Proof. This follows by duality from Proposition 5.2.2. □

Proposition 5.2.6. *Every Kleisli object* $F\colon X_A \to X$ *for a monad* $A\colon X \to X$ *has a right adjoint* $U\colon X \to X_A$ *and the monad map* $\pi\colon A \to U \circ F$ *given by the composite*

$$A \xrightarrow{\eta\circ A} U \circ F \circ A \xrightarrow{U\circ\rho} U \circ F$$

is invertible.

Proof. This follows by duality from Proposition 5.2.3. □

The next definition introduces the notions of an opmonadic and bimonadic adjunction. Opmonadic adjunctions should not be confused with the comonadic adjunctions, which involve comonads rather than monads.

Definition 5.2.7. We say that an adjunction $(F, U, \eta, \varepsilon)\colon X \to Y$ in $\mathcal{E}$ is

(i) *monadic* if the morphism $U\colon Y \to X$, equipped with the left action by the monad $U \circ F$, is an Eilenberg-Moore object for the monad $U \circ F\colon X \to X$,
(ii) *opmonadic* if the morphism $F\colon X \to Y$, equipped with the right action of the monad $U \circ F$, is a Kleisli object for the monad $U \circ F\colon X \to X$,
(iii) *bimonadic* if it is both monadic and opmonadic.

In the 2-category Cat, an adjunction is monadic in the sense of Definition 5.2.7 if and only if it is monadic in the usual sense [**6**, Section 3.3]. An adjunction $(F, U)\colon X \to Y$ in a bicategory $\mathcal{E}$ is monadic if and only if the adjunction $(\mathcal{E}[K, F], \mathcal{E}[K, G])\colon \mathcal{E}[K, X] \to \mathcal{E}[K, Y]$ is monadic in Cat for every $K \in \mathcal{E}$. By Proposition 5.2.3, the adjunction $(F, U)\colon X \to X^A$ associated to an Eilenberg-Moore object is monadic. Dually, by Proposition 5.2.6, the adjunction $(F, U)\colon X \to X_A$ associated to a Kleisli object is opmonadic. Observe that an adjunction $(F, U, \eta, \varepsilon)$ is opmonadic in $\mathcal{E}$ if and only if the opposite adjunction $(U^{\mathrm{op}}, F^{\mathrm{op}}, \eta, \varepsilon)$ is monadic in $\mathcal{E}^{\mathrm{op}}$. Consequently, the notion of a bimonadic adjunction is self-dual, in the sense that an adjunction $(F, U, \eta, \varepsilon)$ in $\mathcal{E}$ is bimonadic if and only if the opposite adjunction $(U^{\mathrm{op}}, F^{\mathrm{op}}, \eta, \varepsilon)$ in $\mathcal{E}^{\mathrm{op}}$ is bimonadic.

Definition 5.2.8. We will say that an adjunction $(F, U, \eta, \varepsilon)\colon X \to Y$ in $\mathcal{E}$ is *effective* if the fork

$$F \circ U \circ F \circ U \underset{F \circ U \circ \varepsilon}{\overset{\varepsilon \circ F \circ U}{\rightrightarrows}} F \circ U \xrightarrow{\ \varepsilon\ } 1_Y$$

is a coequalizer diagram.

The notion of an effective adjunction is self-dual, in the sense that an adjunction $(F, U, \eta, \varepsilon)$ is effective in $\mathcal{E}$ if and only if the opposite adjunction $(U^{\mathrm{op}}, F^{\mathrm{op}}, \eta, \varepsilon)$ is effective in $\mathcal{E}^{\mathrm{op}}$. Effective adjunctions have been studied extensively in category theory (see [**47**] and references therein for further information).

Proposition 5.2.9.
(i) A monadic adjunction is effective.
(ii) An opmonadic adjunction is effective.

Proof. For part (i), let us show that a monadic adjunction $(F, U, \eta, \varepsilon)\colon X \to Y$ is effective. The adjunction

$$\mathcal{E}[K, X] \underset{\mathcal{E}[K, U]}{\overset{\mathcal{E}[K, F]}{\rightleftarrows}} \mathcal{E}[K, Y] \quad (\bot)$$

is monadic for every $K \in \mathcal{E}$, since $U\colon Y \to X$ is an Eilenberg-Moore object for the monad $U \circ F\colon X \to X$. This is true in particular in the case where $K = Y$. Let us now show that the fork

$$F \circ U \circ F \circ U \underset{F \circ U \circ \varepsilon}{\overset{\varepsilon \circ F \circ U}{\rightrightarrows}} F \circ U \xrightarrow{\ \varepsilon\ } 1_Y \tag{5.2.2}$$

is a coequalizer diagram. But the image of the fork in (5.2.2) by the functor $U \circ (-)$ is a split coequalizer:

$$U \circ F \circ U \circ F \circ U \underset{U \circ F \circ U \circ \varepsilon}{\overset{U \circ \varepsilon \circ F \circ U}{\rightrightarrows}} U \circ F \circ U \xrightarrow{\ U \circ \varepsilon\ } U.$$

with splittings $\eta \circ U \circ F \circ U\colon U \circ F \circ U \to U \circ F \circ U \circ F \circ U$ and $\eta \circ U\colon U \to U \circ F \circ U$.

By Beck's monadicity theorem, which we can apply because the adjunction $\mathcal{E}[K,F] \dashv \mathcal{E}[K,U]$ is monadic, the fork in (5.2.2) is a coequalizer diagram. Part (ii) follows by duality. □

Theorem 5.2.10 below shows that in a tame bicategory $\mathcal{E}$ also the converse of each of the implications of Proposition 5.2.9 also holds and therefore monadic, opmonadic and effective adjunctions coincide. Its proof makes use of the Crude Monadicity Theorem, according to which an adjunction $(F,U)\colon X \to Y$ in the 2-category Cat, where Y has reflexive coequalizers, is monadic if the functor U preserves reflexive coequalizers and is conservative [**6**, Section 3.5].

THEOREM 5.2.10. *For an adjunction $(F,U,\eta,\varepsilon)\colon X \to Y$ in a tame bicategory $\mathcal{E}$, the conditions of being effective, monadic, opmonadic and bimonadic are equivalent.*

PROOF. We already saw in Proposition 5.2.9 that every monadic adjunction is effective. Conversely, let us show that if $\mathcal{E}$ is tame, then every effective adjunction $(F,U,\eta,\varepsilon)\colon X \to Y$ in $\mathcal{E}$ is monadic. For this, we show that the adjunction

$$\mathcal{E}[K,X] \underset{\mathcal{E}[K,U]}{\overset{\mathcal{E}[K,F]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{E}[K,Y]$$

is monadic for every $K \in \mathcal{E}$. We apply the Crude Monadicity Theorem. The category $\mathcal{E}[K,Y]$ has reflexive coequalizers and the functor $\mathcal{E}[K,U]\colon \mathcal{E}[K,Y] \to \mathcal{E}[K,X]$ preserves them by the hypothesis on $\mathcal{E}$. Hence it remains to show that the functor $\mathcal{E}[K,U]$ is conservative. Let $f\colon M \to N$ be a 2-cell in $\mathcal{E}[K,Y]$ and suppose that the 2-cell $U \circ f\colon U \circ M \to U \circ N$ is invertible. Let us show that f is invertible. Consider the following commutative diagram,

$$\begin{array}{ccccc}
F\circ U\circ F\circ U\circ M & \underset{F\circ U\circ\varepsilon\circ M}{\overset{\varepsilon\circ F\circ U\circ M}{\rightrightarrows}} & F\circ U\circ M & \xrightarrow{\varepsilon\circ M} & M \\
\big\downarrow{\scriptstyle F\circ U\circ F\circ U\circ f} & & \big\downarrow{\scriptstyle F\circ U\circ f} & & \big\downarrow{\scriptstyle f} \\
F\circ U\circ F\circ U\circ N & \underset{F\circ U\circ\varepsilon\circ N}{\overset{\varepsilon\circ F\circ U\circ N}{\rightrightarrows}} & F\circ U\circ N & \xrightarrow[\varepsilon\circ N]{} & N\,.
\end{array}$$

The top fork of the diagram is a coequalizer diagram, since the the adjunction $F \dashv U$ is effective and the functor $\mathcal{E}[M,Y]\colon \mathcal{E}[Y,Y] \to \mathcal{E}[K,Y]$ preserves reflexive coequalizers. Similarly, the bottom fork of the diagram is a coequalizer diagram. But the left and middle vertical cells of the diagram are invertible, since the 2-cell $U \circ f$ is invertible. It follows that f is invertible. We have proved that the adjunction $F \dashv U$ in $\mathcal{E}$ is monadic. It follows by duality that the adjunction $U^{\mathrm{op}} \dashv F^{\mathrm{op}}$ in $\mathcal{E}^{\mathrm{op}}$ is monadic if and only if it is effective. But the adjunction $U^{\mathrm{op}} \dashv F^{\mathrm{op}}$ is monadic in $\mathcal{E}^{\mathrm{op}}$ if and only if the adjunction $F \dashv U$ is opmonadic in $\mathcal{E}$. Moreover, the adjunction $U^{\mathrm{op}} \dashv F^{\mathrm{op}}$ is effective in $\mathcal{E}^{\mathrm{op}}$ if and only if the adjunction $F \dashv U$ is effective. This proves that the adjunction $F \dashv U$ is opmonadic if and only if it is effective. □

COROLLARY 5.2.11. *Let $\mathcal{E}$ be a tame bicategory and $(F,U)\colon X \to Y$ be an adjunction in $\mathcal{E}$. The following conditions are equivalent:*

(i) the adjunction $(F,U)\colon X \to Y$ is effective,
(ii) the morphism $U\colon Y \to X$ is an Eilenberg-Moore object for the monad $U \circ F\colon X \to X$,
(iii) the morphism $F\colon X \to Y$ is a Kleisli object for the monad $U \circ F\colon X \to X$. □

COROLLARY 5.2.12. *Let $\mathcal{E}$ a tame bicategory. If $U\colon X^A \to X$ is an Eilenberg-Moore object for a monad $A\colon X \to X$, then its left adjoint $F\colon X \to X^A$ is a Kleisli object for A. Conversely, if*

$F\colon X \to X_A$ is a Kleisli object for A, then its right adjoint $U\colon X_A \to X$ is an Eilenberg-Moore object for A. □

Note that Corollary 5.2.12 implies that Kleisli objects and Eilenberg-Moore objects coincide in a tame bicategory. The next proposition involves the notion of a tame homomorphism between tame bicategories introduced in Definition 4.2.1.

PROPOSITION 5.2.13. *Let $\mathcal{E}$ and $\mathcal{F}$ be tame bicategories. A tame homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ preserves monadic (and hence also opmonadic, bimonadic and effective) adjunctions. It thus preserves Eilenberg-Moore (and hence also Kleisli) objects.*

PROOF. Let us show that Φ preserves effective adjunctions. If $(F, U, \eta, \varepsilon)\colon X \to Y$ is an effective adjunction in $\mathcal{E}$, let us show that the adjunction $(\Phi F, \Phi U, \Phi\eta, \Phi\varepsilon)\colon \Phi X \to \Phi X$ is effective. The fork

$$F \circ U \circ F \circ U \underset{F\circ U\circ\varepsilon}{\overset{\varepsilon\circ F\circ U}{\rightrightarrows}} F \circ U \xrightarrow{\varepsilon} 1_Y$$

is a reflexive coequalizer diagram in $\mathcal{E}[Y, Y]$, since the adjunction $(F, U, \eta, \varepsilon)$ is effective. Hence also the fork

$$\Phi F \circ \Phi U \circ \Phi F \circ \Phi U \underset{\Phi F\circ\Phi U\circ\Phi\varepsilon}{\overset{\Phi\varepsilon\circ\Phi F\circ\Phi U}{\rightrightarrows}} \Phi F \circ \Phi U \xrightarrow{\Phi\varepsilon} 1_{\Phi Y}$$

is a reflexive coequalizer, since the functor $\Phi_{X,Y}\colon \mathcal{E}[Y, Y] \to \mathcal{F}[\Phi Y, \Phi Y]$ preserves reflexive coequalizers for every $Y \in \mathcal{E}$. This proves that Φ preserves effective adjunctions. It then follows from Theorem 5.2.10 that Φ preserves monadic, opmonadic and bimonadic adjunctions. It thus preserves Eilenberg-Moore and Kleisli objects by Corollary 5.2.11. □

5.3. Monad theory in bicategories of bimodules

Recall from Remark 4.2.2 that for every tame bicategory $\mathcal{E}$ there is a homomorphism

$$\mathrm{J}_{\mathcal{E}}\colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$$

which maps an object $X \in \mathcal{E}$ to the identity monad $X/1_X$ We introduce the appropriate notion of morphism between tame bicategories and make an observation about the functorial character of the bimodule construction.

PROPOSITION 5.3.1. *Let $\mathcal{E}$ be a tame bicategory. The bicategory* $\mathrm{Bim}(\mathcal{E})$ *is tame and the homomorphism* $\mathrm{J}_{\mathcal{E}}\colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$ *is full and faithful and proper.*

PROOF. Let us show that $\mathrm{J}_{\mathcal{E}}\colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$ is fully faithful. A morphism $M\colon X \to Y$ in $\mathcal{E}[X, Y]$ has a unique structure of $(1_Y, 1_X)$-bimodule, and every 2-cell $f\colon M \to N$ in $\mathcal{E}[X, Y]$ is a map of $(1_Y, 1_X)$-bimodules. This shows that the functor

$$\mathrm{J}_{X,Y}\colon \mathcal{E}[X, Y] \to \mathrm{Bim}(\mathcal{E})[X/1_X, Y/1_Y]$$

is an isomorphism of categories for every pair of objects $X, Y \in \mathcal{E}$.

Let us now show that the bicategory $\mathrm{Bim}(\mathcal{E})$ is tame. First of all, recall that the category $\mathcal{E}[X, Y]$ has reflexive coequalizers by the assumption that $\mathcal{E}$ is tame. Moreover, the monad $B \circ (-) \circ A\colon \mathcal{E}[X, Y] \to \mathcal{E}[X, Y]$ preserves reflexive coequalizers since it is defined by composition. By Proposition 4.1.19 the category $\mathcal{E}[X, Y]^B_A$ has reflexive coequalizers and the forgetful functor $\mathcal{E}[X, Y]^B_A \to \mathcal{E}[X, Y]$ preserves and reflects reflexive coequalizers. Let us now show that the horizontal composition functors of $\mathrm{Bim}(\mathcal{E})$ preserve coequalizers on the left. For this we have to show that the functor $N \circ_B (-) \colon [X/A, Y/B] \to [X/A, Z/C]$ preserves reflexive coequalizers for every morphism $N \colon Y/B \to Z/C$ in $\mathrm{Bim}(\mathcal{E})$. The following square commutes

$$\begin{array}{ccc} \mathcal{E}[X,Y]^B_A & \xrightarrow{N \circ_B (-)} & \mathcal{E}[X,Z]^C_A \\ {\scriptstyle U_1}\downarrow & & \downarrow{\scriptstyle U_2} \\ \mathcal{E}[X,Y]^B & \xrightarrow[N \circ_B (-)]{} & \mathcal{E}[X,Z] \end{array}$$

by definition of the functor $N \circ_B (-)$, where U_1 and U_2 are the forgetful functors. Moreover, the functors U_1 and U_2 preserve and reflect reflexive coequalizers. Hence, it suffices to show that the composite

$$U_2(N \circ_B (-)) \colon \mathcal{E}[X,Y]^B_A \to \mathcal{E}[X,Y]^C_A \to \mathcal{E}[X,Z]$$

preserves reflexive coequalizers. But for this it suffices to show that the functor

$$N \circ_B (-) \colon \mathcal{E}[X,Y]^B \to \mathcal{E}[X,Z]$$

preserves reflexive coequalizers, since the square commutes and the functor U_1 preserves reflexive coequalizers. For every $M \in \mathcal{E}[X,Y]^B$ we have a coequalizer diagram

$$N \circ B \circ M \underset{N \circ \lambda}{\overset{\rho \circ M}{\rightrightarrows}} N \circ M \xrightarrow{q} N \circ_B M \tag{5.3.1}$$

in the category $\mathcal{E}[X,Z]$, where ρ_N is the right action of B on N and λ_M is the left action of B on M. The functors

$$N \circ B \circ (-) \colon [X,Y] \to [X,Z]\,, \quad N \circ (-) \colon [X,Y] \to [X,Z]$$

preserve reflexive coequalizers, since the bicategory $\mathcal{E}$ is tame. Hence also their composite with the forgetful functor $[X,Y]^B \to [X,Y]$. This shows that the functor $N \circ_B (-)$ is a colimit of functors preserving reflexive coequalizers. It follows that the functor $N \circ_B (-)$ preserves reflexive coequalizers, since colimits commute with colimits. We have proved that the horizontal composition functors of $\mathrm{Bim}(\mathcal{E})$ preserve coequalizers on the left. It follows by the duality of Remark 4.2.6 that the horizontal composition functors of $\mathrm{Bim}(\mathcal{E})$ preserve coequalizers also on the right, and hence $\mathcal{E}$ is tame.

It remains to show that the homomorphism $\mathrm{J}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$ is proper. But $\mathrm{J}_{\mathcal{E}}$ preserves local reflexive coequalizers, since the functor $\mathrm{J}_{X,Y} \colon \mathcal{E}[X,Y] \to \mathrm{Bim}(\mathcal{E})[X/1_X, Y/1_Y]$ is an equivalence of categories for every pair of objects $X, Y \in \mathcal{E}$. □

Proposition 5.3.1 implies that $\mathrm{J}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$ can be regarded as an inclusion $\mathcal{E} \subseteq \mathrm{Bim}(\mathcal{E})$. Because of this, in the following we will identify an object $X \in \mathcal{E}$ with the object $X/1_X$ of $\mathrm{Bim}(\mathcal{E})$ and the category $\mathcal{E}[X,Y]$ with the category $\mathrm{Bim}(\mathcal{E})[X/1_X, Y/1_X]$ for every pair $X, Y \in \mathcal{E}$. Our next goal is to establish that $\mathrm{Bim}(\mathcal{E})$ is Eilenberg-Moore complete. We begin with two observations about the relationship between monads in $\mathcal{E}$ and in $\mathrm{Bim}(\mathcal{E})$.

Proposition 5.3.2. *Let $\mathcal{E}$ a tame bicategory.*

(i) An adjunction in $\mathcal{E}$ is monadic in $\mathcal{E}$ if and only if it is monadic in $\mathrm{Bim}(\mathcal{E})$.
(ii) A morphism $U \colon Y \to X$ in $\mathcal{E}$ is an Eilenberg-Moore object for a monad $A \colon X \to X$ if and only if it is an Eilenberg-Moore object for $A \colon X/1_X \to X/1_X$ in $\mathrm{Bim}(\mathcal{E})$.

Proof. Let us show that an adjunction $\mathcal{E}$ is monadic if and only if it is monadic in $\mathrm{Bim}(\mathcal{E})$. But an adjunction in $\mathcal{E}$ is effective if and only if it is effective in $\mathrm{Bim}(\mathcal{E})$, since the functor

$$\mathrm{J}_{X,Y}\colon \mathcal{E}[Y,Y] \to \mathrm{Bim}(\mathcal{E})[Y,Y]$$

is an equivalence of categories (it is actually an isomorphism of categories). The result then follows from Theorem 5.2.10, since $\mathcal{E}$ and $\mathrm{Bim}(\mathcal{E})$ are tame. Corollary 5.2.11 implies that a morphism $U\colon Y \to X$ in $\mathcal{E}$ is an Eilenberg-Moore object for a monad $A\colon X \to X$ if and only if it is an Eilenberg-Moore object for $A\colon X/1_X \to X/1_X$ in $\mathrm{Bim}(\mathcal{E})$. □

Let $A\colon X \to X$ be a monad in a tame bicategory $\mathcal{E}$. Then the morphism $A\colon X \to X$ has the structure of a left A-module $F\colon X \to X/A$ and of a right A-module $U\colon X/A \to X$. We wish to show that we have an adjunction $F \dashv U$ in $\mathrm{Bim}(\mathcal{E})$. Since $U \circ_A F = A \circ_A A \cong A$ and $F \circ U = A \circ A$, we define the unit $\eta\colon 1_X \to U \circ_A F$ to be $\eta_A\colon 1_X \to A$ and the counit $\varepsilon\colon F \circ U \to 1_{X/A}$ to be $\mu_A\colon A \circ A \to A$. To prove that we have an adjunction, we need to show that the triangular identities hold. This amounts to proving that

$$(A \circ_A \mu_A) \cdot (\eta_A \circ A) = 1_A\,, \quad (\mu_A \circ_A A) \cdot (A \circ \eta_A) = 1_A\,.$$

But we have $A \circ_A \mu_A = \mu_A$, since the 2-cell $\mu_A\colon A \circ A \to A$ is a map of left A-modules. Thus,

$$(A \circ_A \mu_A) \cdot (\eta_A \circ A) = \mu_A \cdot (\eta_A \circ A) = 1_A\,,$$

since η_A is a unit for the multiplication μ_A. Dually, we have $\mu_A \circ_A A = \mu_A$, since the 2-cell $\mu_A\colon A \circ A \to A$ is a map of right A-modules. Thus,

$$(\mu_A \circ_A A) \cdot (A \circ \eta) = \mu_A \cdot (A \circ \eta_A) = 1_A\,,$$

since η_A is a unit for the multiplication μ_A.

Lemma 5.3.3. *Let $\mathcal{E}$ be a tame bicategory. Let $A\colon X \to X$ be a monad in $\mathrm{Bim}(\mathcal{E})$. Then the adjunction $(F,U)\colon X \to X/A$ in $\mathrm{Bim}(\mathcal{E})$ described above is monadic and the monad $U \circ_A F\colon X \to X$ is isomorphic to $A\colon X \to X$. Hence, $U\colon X/A \to X$ is an Eilenberg-Moore object for A.*

Proof. Let us begin by verifying that the monad $U \circ_A F$ is isomorphic to A. Obviously, $U \circ_A F = A \circ_A A = A$. The unit of the monad $U \circ_A F$ is defined to be the unit η of the adjunction $F \dashv U$. But we have $\eta = \eta_A$ by definition. The multiplication of the monad $U \circ_A F$ is defined to be the 2-cell $U \circ_A \varepsilon \circ_A F$. But we have

$$U \circ_A \varepsilon \circ_A F = A \circ_A \mu_A \circ_A A = \mu_A\,,$$

since $\mu_A\colon A \circ A \to A$ is a map of (A,A)-bimodules. Let us now show that the adjunction (F,U,η,ε) is monadic. By Proposition 5.3.1 the bicategory $\mathrm{Bim}(\mathcal{E})$ is tame and therefore, by Theorem 5.2.10, it suffices to show that the adjunction is effective. For this we have to show that the fork

$$F \circ U \circ_A F \circ U \underset{F \circ U \circ_A\, \varepsilon}{\overset{\varepsilon \circ_A F \circ U}{\rightrightarrows}} F \circ U \xrightarrow{\ \varepsilon\ } 1_{X/A}$$

is a coequalizer diagram in the category of (A,A)-bimodules. But this fork is isomorphic to

$$A \circ A \circ A \underset{A \circ \mu_A}{\overset{\mu_A \circ A}{\rightrightarrows}} A \circ A \xrightarrow{\ \mu_A\ } A \tag{5.3.2}$$

since $F = U = A$, $A \circ_A A \cong A$ and $\varepsilon = \mu_A$. Let us show that the fork in (5.3.2) is a coequalizer diagram. But its image by the forgetful functor $U \colon \mathcal{E}[X,X]^A_A \to \mathcal{E}[X,X]$ splits in the category $\mathcal{E}[X,X]$, as we have the following diagram

$$A \circ A \circ A \underset{A \circ \mu_A}{\overset{\mu_A \circ A}{\rightrightarrows}} A \circ A \xrightarrow{\mu_A} A \, , \qquad \eta_A \circ A \circ A \colon A \circ A \to A \circ A \circ A \, , \quad \eta_A \circ A \colon A \to A \circ A \, .$$

This shows that the fork in (5.3.2) is a coequalizer diagram, since the functor U is monadic, as observed in Proposition 4.1.19. □

Remark 5.3.4. Let $\mathrm{Und}_\mathcal{E} \colon \mathrm{Mnd}(\mathcal{E}) \to \mathcal{E}$ be the homomorphism mapping a monad (X, A) to its underlying object X. For a monad (X, A) in $\mathcal{E}$, the bimodule $U \colon X/A \to X$ of Lemma 5.3.3 can be viewed as a morphism

$$U^A \colon R(X, A) \to (\mathrm{J}_\mathcal{E} \circ \mathrm{Und}_\mathcal{E})(X, A) \, ,$$

in $\mathrm{Bim}(\mathcal{E})$, where $R_\mathcal{E} \colon \mathrm{Mnd}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{E})$ is the homomorphism defined via Lemma 4.3.2. The family of morphisms $U^A \colon X/A \to X$, for $(X, A) \in \mathrm{Mnd}(\mathcal{E})$, can then be seen as the components of a pseudo-natural transformation fitting in the diagram

$$\begin{array}{ccc} \mathrm{Mnd}(\mathcal{E}) & \xrightarrow{\mathrm{Und}_\mathcal{E}} & \mathcal{E} \\ & {\scriptstyle R_\mathcal{E}} \searrow & \downarrow {\scriptstyle \mathrm{J}_\mathcal{E}} \\ & & \mathrm{Bim}(\mathcal{E}) \, . \end{array}$$

For a monad morphism $(F, \phi) \colon (X, A) \to (Y, B)$, the required pseudo-naturality 2-cell, which should fit in the diagram

$$\begin{array}{ccc} X/A & \xrightarrow{U^A} & X/1_X \\ {\scriptstyle R(F)} \downarrow & \Downarrow u_F & \downarrow {\scriptstyle F} \\ Y/B & \xrightarrow[U^B]{} & Y/1_Y \, , \end{array}$$

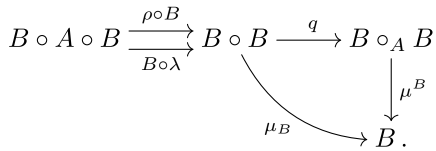
is given by the following chain of isomorphisms and equalities:

$$F \circ U^A = F \circ A \cong B \circ_B F \circ A = B \circ_B R(F) = U^B \circ_B R(F) \, .$$

Lemma 5.3.3 shows that every monad in $\mathcal{E}$ admits an Eilenberg-Moore object in $\mathrm{Bim}(\mathcal{E})$. Below, we show that in fact every monad in $\mathrm{Bim}(\mathcal{E})$ has an Eilenberg-Moore object in $\mathrm{Bim}(\mathcal{E})$. In order to do this, we need some preliminary observations. Let (A, μ_A, η_A) be a monad on $X \in \mathcal{E}$. Then, for a monad (B, μ_B, η_B) on X and a map of monads $\pi \colon A \to B$, the morphism $B \colon X \to X$ has the structure of an (A, A)-bimodule, which we denote by $B_A \colon X/A \to X/A$. The left action $\lambda \colon A \circ B \to B$ and the right action $\rho \colon B \circ A \to B$ are defined by the following diagrams

$$\begin{array}{ccccc} A \circ B & \xrightarrow{\pi \circ B} & B \circ B & \xleftarrow{B \circ \pi} & B \circ A \\ & {\scriptstyle \lambda} \searrow & \downarrow {\scriptstyle \mu_B} & \swarrow {\scriptstyle \rho} & \\ & & B \, . & & \end{array}$$

Furthermore, the bimodule $B_A\colon X/A \to X/A$ has the structure of a monad in $\mathrm{Bim}(\mathcal{E})$ with the multiplication $\mu^B\colon B \circ_A B \to B$ defined by the following commutative diagram, determined by the universal property of $B \circ_A B$, as follows:



The unit of the monad $B_A\colon X/A \to X/A$ is the 2-cell $\pi\colon A \to B$. This defines a functor $(-)_A\colon A\backslash\mathrm{Mon}_{\mathcal{E}}(X) \to \mathrm{Mon}_{\mathrm{Bim}(\mathcal{E})}(X/A)$.

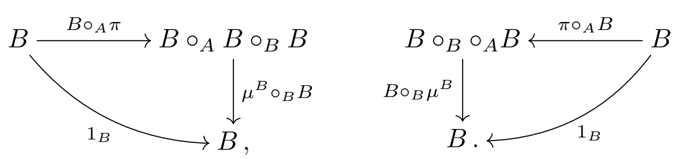
LEMMA 5.3.5. *For every monad (A, μ_A, η_A) in a tame category $\mathcal{E}$, the functor*

$$(-)_A\colon A\backslash\mathrm{Mon}_{\mathcal{E}}(X) \to \mathrm{Mon}_{\mathrm{Bim}(\mathcal{E})}(X/A)$$

is essentially surjective.

PROOF. Let (E, μ, η) be a monad on X/A in $\mathrm{Bim}(\mathcal{E})$. Thus, we have a morphism $E\colon X \to X$ equipped with the structure of an (A, A)-bimodule together with bimodule maps $\mu\colon E \circ_A E \to E$ and $\eta\colon A \to E$ satisfying the monad axioms. We define a monad (B, μ_B, η_B) on X as follows. The morphism $B\colon X \to X$ is given by $E\colon X \to X$ itself. The multiplication $\mu_B\colon B \circ B \to B$ is obtained by composing the canonical map $q\colon E \circ E \to E \circ_A E$ with $\mu\colon E \circ_A E \to E$, and the unit $\eta_B\colon 1_X \to B$ is defined to be the composite of the unit $\eta_A\colon 1_X \to A$ of the monad A with the unit $\eta\colon A \to E$ of the monad E. It is easy to verify that the monad axioms are satisfied. We can also define a monad map $\pi\colon A \to B$ by letting $\pi =_{\mathrm{def}} \eta$. It is now immediate that $B_A \cong E$, as required. ☐

Let $X \in \mathcal{E}$. Let $A = (A, \mu_A, \eta_A)$, $B = (B, \mu_B, \eta_B)$ be monads on X. If $\pi\colon A \to B$ is a map of monads, then the morphism $B\colon X \to X$ has the structure of both an (A, B)-bimodule and a (B, A)-bimodule. These bimodules will be denoted by $U\colon X/B \to X/A$ and $F\colon X/A \to X/B$, respectively. We wish to show that these morphisms are adjoint. In order to do so, let us define the unit of the adjunction $\eta\colon \mathrm{Id}_{X/A} \to U \circ_B F$ as the composite of $\pi\colon A \to B$ with the isomorphism $B \cong B \circ_B B$. We then define the counit of the adjunction $\varepsilon\colon F \circ_A U \to \mathrm{Id}_{X/B}$ as the multiplication $\mu^B\colon B \circ_A B \to B$ of the monad $B_A\colon X/A \to X/A$ defined above. It remains to verify the triangular laws, which in this case amounts to verifying the commutativity of the following diagrams:



For the diagram on the left-hand side, observe that $\mu^B \circ_B B = \mu^B$, since $\mu^B\colon B \circ_A B \to B$ is a map of right B-modules. Thus,

$$(\mu^B \circ_B B) \cdot (B \circ_A \pi) = \mu^B \cdot (B \circ_A \pi) = 1_B\,,$$

since π is the unit of the monad $B_A\colon X/A \to X/A$. For the diagram on the right-hand side, dually, we have $B \circ_B \mu^B = \mu^B$, since $\mu^B\colon B \circ_A B \to B$ is a map of left B-modules. Thus,

$$(U \circ_B \mu^B) \cdot (\pi \circ_A U) = \mu^B \cdot (\pi \circ_A B) = 1_B = 1_U\,,$$

since π is the unit of the monad B_A. We have therefore proved that $(F, U, \eta, \varepsilon)\colon X/A \to Y/B$ is an adjunction.

Proposition 5.3.6. *The adjunction* $(F, U, \eta, \varepsilon)\colon X/A \to X/B$ *defined above is monadic and the monad* $U \circ_B F\colon X/A \to X/A$ *is isomorphic to the monad* $B_A\colon X/A \to X/A$. *Hence, the bimodule* $U\colon X/B \to X/A$ *is an Eilenberg-Moore object for the monad* $B_A\colon X/A \to X/A$.

Proof. Let us show that the adjunction is monadic. By Proposition 5.3.1, the bicategory $\mathrm{Bim}(\mathcal{E})$ is tame and therefore, by Theorem 5.2.10, it suffices to show that the adjunction is effective. For this we have to show that the fork

$$F \circ_A U \circ_B F \circ_A U \underset{F\circ_A U \circ_B \mu^B}{\overset{\mu^B \circ_B F \circ_A U}{\rightrightarrows}} F \circ_A U \xrightarrow{\quad \mu^B \quad} B \tag{5.3.3}$$

is a coequalizer diagram in the category of (B, B)-bimodules. But the fork in (5.3.3) is isomorphic to the fork

$$B \circ_A B \circ_A B \underset{B\circ_A \mu^B}{\overset{\mu^B \circ_A B}{\rightrightarrows}} B \circ_A B \xrightarrow{\quad \mu^B \quad} B\,, \tag{5.3.4}$$

since $F = U = B$ and $B \circ_B B = B$. But the image of the fork in (5.3.4) under the forgetful functor $U\colon \mathcal{E}[X, X]^B_B \to \mathcal{E}[X, X]$ splits in the category $\mathcal{E}[X, X]$,

$$\begin{tikzcd} B \circ_A B \circ_A B \arrow[r, shift left, "\mu^{B/A}\circ_A B"] \arrow[r, shift right, "B\circ_A \mu^{B/A}"'] & B \circ_A B \arrow[r, "\mu^{B/A}"] \arrow[l, bend right=60, "\pi\circ_A B \circ_A B"'] & B \,. \arrow[l, bend right=60, "\pi\circ_A B"'] \end{tikzcd}$$

Since the functor U is monadic, as observed in Proposition 4.1.19, this shows that the fork in (5.3.4) is a coequalizer diagram. We have proved that the adjunction (F, U, π, μ^B) is monadic.

Finally, let us show that the monad $U \circ_B F\colon X/A \to X/A$ is isomorphic to $B_A\colon X/A \to X/A$. First of all, we have

$$U \circ_B F = B \circ_B B = B\,.$$

Secondly, the multiplication of $U \circ_B F$ is given by $U \circ_B \varepsilon \circ_B F\colon U \circ_B F \circ_A U \circ_B F \to U \circ_B F$, which is $B \circ_B \mu^B \circ_B B\colon B \circ_B B \circ_A B \circ_B B \to B$. But $B \circ_B \mu^B \circ_B = \mu^B$, as required. Finally, the units of $U \circ_B F$ and B_A coincide, since both are given by $\pi\colon A \to B$. □

Definition 5.3.7. We say that a bicategory $\mathcal{E}$ is *Eilenberg-Moore complete* (resp. *Kleisli complete*) if every monad in $\mathcal{E}$ admits an Eilenberg-Moore object (resp. Kleisli object).

For example, the 2-category Cat is both Eilenberg-Moore complete and Kleisli complete. A bicategory $\mathcal{E}$ is Eilenberg-Moore complete if and only if its opposite $\mathcal{E}^{\mathrm{op}}$ is Kleisli complete. Since Eilenberg-Moore and Kleisli objects coincide in a tame bicategory by Corollary 5.2.12, a tame bicategory is Eilenberg-Moore complete if and only if it is Kleisli complete.

Theorem 5.3.8. *For any tame category* $\mathcal{E}$, *the bicategory* $\mathrm{Bim}(\mathcal{E})$ *is Eilenberg-Moore complete.*

Proof. Let us show that every monad $E = (E, \mu, \eta)$ over an object $X/A \in \mathrm{Bim}(\mathcal{E})$ admits an Eilenberg-Moore object. By Lemma 5.3.5, we have $E \cong B_A$ for a monad (B, μ_B, η_B) on X and a map of monads $\pi\colon A \to B$ in $\mathrm{Mon}(X)$. It then follows from Proposition 5.3.6 that the bimodule $U\colon X/B \to X/A$ defined above is an Eilenberg-Moore object for the monad $B_A\colon X \to X$. □

PROPOSITION 5.3.9. *A tame bicategory $\mathcal{E}$ is Eilenberg-Moore complete if and only if the homomorphism* $\mathrm{J}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$ *is an equivalence.*

PROOF. If $\mathrm{J}_{\mathcal{E}}$ is an equivalence, then $\mathcal{E}$ is Eilenberg-Moore complete because $\mathrm{Bim}(\mathcal{E})$ is Eilenberg-Moore complete, as proved in Theorem 5.3.8. Conversely, let us assume that $\mathcal{E}$ is Eilenberg-Moore complete and show that $\mathrm{J}_{\mathcal{E}}$ is an equivalence. Recall that $\mathrm{J}_{\mathcal{E}}$ is full and faithful by Proposition 5.3.1. Hence it suffices to show that $\mathrm{J}_{\mathcal{E}}$ is essentially surjective. For this we have to show that every object $X/A \in \mathrm{Bim}(\mathcal{E})$ is equivalent to an object of $\mathcal{E}$. The monad (X, A) admits an Eilenberg-Moore object $U \colon X^A \to X$ in $\mathcal{E}$, since $\mathcal{E}$ is Eilenberg-Moore complete by hypothesis. This Eilenberg-Moore object is also an Eilenberg-Moore object in $\mathrm{Bim}(\mathcal{E})$ by Proposition 5.3.2. Hence we have an equivalence $X^A \simeq X/A$ in $\mathrm{Bim}(\mathcal{E})$, since any two Eilenberg-Moore objects for a monad are equivalent. □

5.4. Bicategories of bimodules as Eilenberg-Moore completions

Recall from Section 4.2 that, for tame bicategories $\mathcal{E}$ and $\mathcal{F}$, we write $\mathrm{REG}[\mathcal{E}, \mathcal{F}]$ for the full sub-bicategory of $\mathrm{HOM}[\mathcal{E}, \mathcal{F}]$ whose objects are tame homomorphisms. Clearly, the composite of two tame homomorphisms is proper.

DEFINITION 5.4.1. Given a tame bicategory $\mathcal{E}$ and a tame and Eilenberg-Moore complete bicategory $\mathcal{E}'$, we say that a tame homomorphism $J \colon \mathcal{E} \to \mathcal{E}'$ exhibits $\mathcal{E}'$ as the *Eilenberg-Moore completion of* $\mathcal{E}$ *as a tame bicategory* if the homomorphism

$$(-) \circ J \colon \mathrm{REG}[\mathcal{E}', \mathcal{F}] \to \mathrm{REG}[\mathcal{E}, \mathcal{F}]$$

is a biequivalence for any tame and Eilenberg-Moore complete bicategory $\mathcal{F}$.

It follows from this definition that such an Eilenberg-Moore completion of a tame bicategory, if it exists, is unique up to biequivalence. If $\mathcal{E}$ a tame bicategory, then the bicategory $\mathrm{Bim}(\mathcal{E})$ is tame by Theorem 5.3.1 and Eilenberg-Moore complete by Theorem 5.3.8 and the homomorphism $\mathrm{J}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$ is tame by Proposition 5.3.1. The next theorem is a special case of a result obtained independently in [**33**].

THEOREM 5.4.2. *For a tame bicategory* $\mathcal{E}$*, the homomorphism* $\mathrm{J}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$ *exhibits* $\mathrm{Bim}(\mathcal{E})$ *as the Eilenberg-Moore completion of* $\mathcal{E}$ *as a tame bicategory.*

PROOF. See Appendix B. □

REMARK 5.4.3. As shown in [**50**], for a 2-category $\mathcal{E}$, not necessarily tame, it is possible to define its Eilenberg-Moore completion $\mathrm{EM}(\mathcal{E})$, which comes equipped with a 2-functor $\mathrm{I}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{EM}(\mathcal{E})$ satisfying a suitable universal property. The definitions of $\mathrm{EM}(\mathcal{E})$ and $\mathrm{I}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{EM}(\mathcal{E})$ make sense also when $\mathcal{E}$ is a bicategory, in which case $\mathrm{EM}(\mathcal{E})$ is also a bicategory and $\mathrm{I}_{\mathcal{E}}$ is a homomorphism. We can then relate $\mathrm{EM}(\mathcal{E})$ and $\mathrm{Bim}(\mathcal{E})$ via a homomorphims $\Gamma \colon \mathrm{EM}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{E})$, defined below, which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\mathrm{I}_{\mathcal{E}}} & \mathrm{EM}(\mathcal{E}) \\ & \searrow^{\mathrm{J}_{\mathcal{E}}} & \downarrow{\scriptstyle \Gamma} \\ & & \mathrm{Bim}(\mathcal{E})\,. \end{array}$$

For a bicategory $\mathcal{E}$, the objects and the morphisms of $\mathrm{EM}(\mathcal{E})$ are the same as those of the bicategory $\mathrm{Mnd}(\mathcal{E})$ recalled in Section 4.3. Given morphisms $(M, \phi), (G, \psi) \colon (X, A) \to (Y, B)$, a 2-cell

$f\colon (M,\phi) \to (M',\phi')$ in $\mathrm{EM}(\mathcal{E})$, instead, is a 2-cell $f\colon M \to M' \circ A$ making the following diagram commute:

$$\begin{array}{ccccc} B \circ M & \xrightarrow{\phi} & M \circ A & \xrightarrow{f \circ A} & M' \circ A \circ A \\ {\scriptstyle B\circ f}\downarrow & & & & \downarrow{\scriptstyle M'\circ\mu} \\ B \circ M' \circ A & \xrightarrow[\phi' \circ A]{} & M' \circ A \circ A & \xrightarrow[M'\circ\mu]{} & M' \circ A\,. \end{array}$$

The homomorphism $\Gamma\colon \mathrm{EM}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{E})$ is defined exactly as the homomorphism $R\colon \mathrm{Mnd}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{E})$ of Section 4.3 on objects and morphisms. For a 2-cell $f\colon (M,\phi) \to (M',\phi')$ in $\mathrm{EM}(\mathcal{E})$, we define $\Gamma(f)\colon M \circ A \to M' \circ A$ as the composite

$$M \circ A \xrightarrow{f \circ A} M' \circ A \circ A \xrightarrow{M' \circ \mu_A} M' \circ A$$

The commutativity of the required diagrams follows easily.

Proposition 5.4.4. *Let $\mathcal{E} \subseteq \mathcal{F}$ be an inclusion of tame bicategories. If every object of $\mathcal{F}$ is an Eilenberg-Moore object (or, equivalently, a Kleisli object) for a monad in $\mathcal{E}$, then the induced inclusion* $\mathrm{Bim}(\mathcal{E}) \subseteq \mathrm{Bim}(\mathcal{F})$ *is an equivalence.*

Proof. We have the following diagram:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F} \\ {\scriptstyle \mathrm{J}_{\mathcal{E}}}\downarrow & & \downarrow{\scriptstyle \mathrm{J}_{\mathcal{F}}} \\ \mathrm{Bim}(\mathcal{E}) & \longrightarrow & \mathrm{Bim}(\mathcal{F})\,. \end{array}$$

We show that $\mathrm{Bim}(\mathcal{E}) \subseteq \mathrm{Bim}(\mathcal{F})$ is essentially surjective. Let $(Y,B) \in \mathrm{Bim}(\mathcal{F})$. By the hypothesis, $Y \in \mathcal{F}$ is an Eilenberg-Moore object for a monad $A\colon X \to X$ in $\mathcal{E}$. By Proposition 5.3.1, the homomorphism $\mathrm{J}_{\mathcal{F}}\colon \mathcal{F} \to \mathrm{Bim}(\mathcal{F})$ is tame and so, by Proposition 5.2.13, it preserves Eilenberg-Moore objects. Hence, $(Y,1_Y) \in \mathrm{Bim}(\mathcal{F})$ is an Eilenberg-Moore object for $A\colon (X,1_X) \to (X,1_X)$ in $\mathrm{Bim}(\mathcal{F})$. But also $(X,A) \in \mathrm{Bim}(\mathcal{E})$ is an Eilenberg-Moore object for the same monad by Lemma 5.3.3 and so it is also an Eilenberg-Moore object for it in $\mathrm{Bim}(\mathcal{F})$. Therefore, there is an equivalence $Y \simeq (X,A)$ in $\mathrm{Bim}(\mathcal{F})$ and so there is also an equivalence $(Y,B) \simeq ((X,A),E)$ in $\mathrm{Bim}(\mathcal{F})$ for some monad $E\colon (X,A) \to (X,A)$ in $\mathrm{Bim}(\mathcal{E})$. By Lemma 5.3.5, E must have the form $B'_A\colon (X,A) \to (X,A)$ for some monad $B'\colon X \to X$ and monad map $\pi\colon A \to B'$. By Proposition 5.3.6, an Eilenberg-Moore object for E is given by (X,B'), which is therefore equivalent to (Y,B). Since $(X,B') \in \mathrm{Bim}(\mathcal{E})$, we have the required essential surjectivity. □

Proposition 5.4.4 implies the known fact that the inclusion $\mathrm{Bim}(\mathcal{E}) \subseteq \mathrm{Bim}(\mathrm{Bim}(\mathcal{E}))$ is an equivalence [**18**]. In particular, we have that $\mathrm{Dist}_{\mathcal{V}} \subseteq \mathrm{Bim}(\mathrm{Dist}_{\mathcal{V}})$ is an equivalence. We now apply Proposition 5.4.4 to the inclusion $\mathrm{OpdBim}_{\mathcal{V}} \subseteq \mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})$ of (4.4.3).

Theorem 5.4.5. *The inclusion* $\mathrm{OpdBim}_{\mathcal{V}} \subseteq \mathrm{Bim}(\mathrm{CatSym}_{\mathcal{V}})$ *is an equivalence.*

Proof. By duality, it is sufficient to prove that the inclusion $\mathrm{Bim}(S\text{-}\mathrm{Mat}_{\mathcal{V}}) \subseteq \mathrm{Bim}(S\text{-}\mathrm{Dist}_{\mathcal{V}})$ is an equivalence. In order to do so, we apply Proposition 5.4.4 and show that every object of S-$\mathrm{Dist}_{\mathcal{V}}$ is a Kleisli object for a monad in S-$\mathrm{Mat}_{\mathcal{V}}$. Let $\mathbb{X}$ be a small $\mathcal{V}$-category. Since monads in S-$\mathrm{Mat}_{\mathcal{V}}$ are operads, we can regard $\mathbb{X}$ also as a monad in S-$\mathrm{Mat}_{\mathcal{V}}$. So, we show that $\mathbb{X}$, viewed as an object of S-Dist$_{\mathcal{V}}$, is a Kleisli object for $\mathbb{X}$, viewed as a monad in S-Mat$_{\mathcal{V}}$. In order to do so, we begin by defining a right $\mathbb{X}$-module with domain Obj($\mathbb{X}$),

$$F\colon \mathrm{Obj}(\mathbb{X}) \to \mathbb{X}\,,$$

in S-Dist$_{\mathcal{V}}$. The $\mathcal{V}$-functor $F\colon S(\mathbb{X})^{\mathrm{op}} \otimes \mathrm{Obj}(\mathbb{X}) \to \mathcal{V}$ is defined by letting

$$F[\bar{x}; x'] =_{\mathrm{def}} \begin{cases} \mathbb{X}[x, x'] & \text{if } \bar{x} = (x) \text{ for some } x \in \mathbb{X}\,, \\ 0 & \text{otherwise.} \end{cases}$$

The right $\mathbb{X}$-action is then defined by the composition operation of $\mathbb{X}$ in the evident way. In order to show that $F\colon \mathrm{Obj}(\mathbb{X}) \to \mathbb{X}$ is the required Kleisli object, we need to show that, for every small $\mathcal{V}$-category $\mathbb{K}$, the functor

$$S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathbb{X}, \mathbb{K}] \to S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathrm{Obj}(\mathbb{X}), \mathbb{K}]_{\mathbb{X}}\,,$$

defined by composition with F, is an equivalence of categories, where $S\text{-}\mathrm{Dist}_{\mathcal{V}}[\mathrm{Obj}(\mathbb{X}), \mathbb{K}]_{\mathbb{X}}$ denotes the category of right $\mathbb{X}$-modules with codomain $\mathbb{K}$. In order to see this, observe that to give a right $\mathbb{X}$-action on an S-distributor $M\colon \mathrm{Obj}(\mathbb{X}) \to \mathbb{K}$, i.e. a $\mathcal{V}$-functor $M\colon S(\mathbb{K})^{\mathrm{op}} \otimes \mathrm{Obj}(\mathbb{X}) \to \mathcal{V}$, is the same thing as extending M to a $\mathcal{V}$-functor $M'\colon S(\mathbb{K})^{\mathrm{op}} \otimes \mathbb{X} \to \mathcal{V}$. □

We can now prove the main result of the paper.

Theorem 5.4.6. *The bicategory* OpdBim$_{\mathcal{V}}$ *is cartesian closed.*

Proof. Recall that the bicategory CatSym$_{\mathcal{V}}$ is cartesian closed by Theorem 3.4.2 and so the associated bicategory of bimodules Bim(CatSym$_{\mathcal{V}}$) is cartesian closed by Theorem 5.1.2. The result follows since, as stated in Theorem 5.4.5, OpdBim$_{\mathcal{V}}$ is equivalent to Bim(CatSym$_{\mathcal{V}}$). □

We conclude by illustrating the cartesian closed structure of OpdBim$_{\mathcal{V}}$. For simplicily, let us now denote an operad by the name of its underlying symmetric sequence and omit the mention of its underlying set of sorts. Thus, we write simply A rather than (X, A). If we denote products, exponentials and the terminal object in OpdBim$_{\mathcal{V}}$ by

$$A \sqcap B\,, \quad B^A\,, \quad \top$$

respectively, the cartesian closed structure gives us equivalences

$$\begin{aligned} \mathrm{OpdBim}_{\mathcal{V}}[A, B_1 \sqcap B_2] &\simeq \mathrm{OpdBim}_{\mathcal{V}}[A, B_1] \times \mathrm{OpdBim}_{\mathcal{V}}[A, B_2]\,, \\ \mathrm{OpdBim}_{\mathcal{V}}[A \sqcap B, C] &\simeq \mathrm{OpdBim}_{\mathcal{V}}[A, C^B]\,, \\ \mathrm{OpdBim}_{\mathcal{V}}[A, \top] &\simeq 1\,. \end{aligned}$$

Furthermore, since the operad $\top$ has an empty set of sorts, we have that

$$\mathrm{OpdBim}_{\mathcal{V}}[\top, A] \simeq \mathrm{Alg}_{\mathcal{V}}(A)\,.$$

Therefore, as a special case of the natural isomorphisms characterizing products and exponentials, we have equivalences

$$\begin{aligned} \mathrm{Alg}_{\mathcal{V}}(A \sqcap B) &\simeq \mathrm{Alg}_{\mathcal{V}}(A) \sqcap \mathrm{Alg}_{\mathcal{V}}(B)\,, \\ \mathrm{Alg}_{\mathcal{V}}(B^A) &\simeq \mathrm{OpdBim}_{\mathcal{V}}(A, B)\,. \end{aligned}$$

This shows that the algebras for $A_1 \sqcap A_2$ are pairs consisting of an A-algebra and a B-algebra, while the algebras for B^A are the (B, A)-bimodules. These equivalences can actually be generalized by replacing $\mathcal{V}$ with an arbitrary $\mathcal{V}$-rig $\mathcal{R} = (\mathcal{R}, \diamond, e)$, thus obtaining the equivalences mentioned in the introduction, since the canonical $\mathcal{V}$-rig homomorphism from $\mathcal{V}$ to $\mathcal{R}$ (defined by mapping $X \in \mathcal{V}$ to $X \otimes e \in \mathcal{R}$) preserves all the relevant structure.

APPENDIX A

A compendium of bicategorical definitions

Definition. A *bicategory* $\mathcal{E}$ consists of the data in (i)-(vi) subject to the axioms in (vii)-(viii), below.

(i) A class $\mathrm{Obj}(\mathcal{E})$ of *objects*. We will write simply $X \in \mathcal{E}$ instead of $X \in \mathrm{Obj}(\mathcal{E})$.
(ii) For every $X, Y \in \mathcal{E}$, a category $\mathcal{E}[X, Y]$. An object of $\mathcal{E}[X, Y]$ is called a 1-*cell* or a *morphism* and written $A \colon X \to Y$, while a morphism $f \colon A \to B$ of $\mathcal{E}[X, Y]$ is called a 2-*cell* .
(iii) For every $X, Y, Z \in \mathcal{E}$, a functor

$$(-) \circ (-) : \mathcal{E}(Y, Z) \times \mathcal{E}(X, Y) \to \mathcal{E}(X, Z)$$

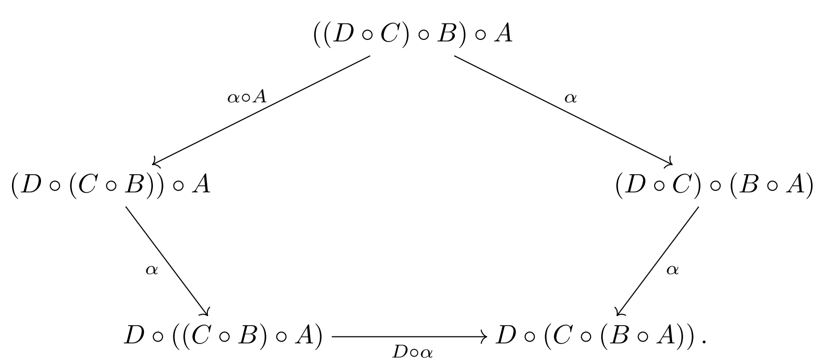
which associates a morphism $B \circ A \colon X \to Z$ to a pair of morphisms $A \colon X \to Y$, $B \colon Y \to Z$ and a 2-cell $f \circ g : B \circ A \to B' \circ A'$ to a pair of 2-cells $f \colon A \to A' \colon X \to Y$ and $g \colon B \to B' \colon Y \to Z$.
(iv) For every $X \in \mathcal{E}$, a morphism $1_X \colon X \to X$.
(v) A natural isomorphism with components

$$\alpha_{A,B,C} \colon (C \circ B) \circ A \to C \circ (B \circ A)$$

for $A \colon X \to Y$, $B \colon Y \to Z$ and $C \colon Z \to U$.
(vi) Two natural isomorphisms with components

$$\lambda_A \colon 1_Y \circ A \to A \, , \quad \rho_A \colon A \circ 1_X \to A$$

for $A \colon X \to Y$.
(vii) For all $A \colon X \to Y$, $B \colon Y \to Z$, $C \colon Z \to U$ and $D \colon U \to V$, the following diagram commutes:



(viii) For all $A\colon X \to Y$ and $B\colon Y \to Z$, the following diagram commutes:

$$\begin{array}{ccc} (B \circ 1_Y) \circ A & \xrightarrow{\ \alpha\ } & B \circ (1_Y \circ A) \\ & {\scriptstyle \rho \circ A} \searrow \quad \swarrow {\scriptstyle B \circ \lambda} & \\ & B \circ A\,. & \end{array}$$

Definition. A *homomorphism* $\Phi : \mathcal{E} \to \mathcal{F}$ of bicategories consists of the data in (i)-(iv) subject to axioms (v)-(vi) below.

(i) A function $\Phi : \mathrm{Obj}(\mathcal{E}) \to \mathrm{Obj}(\mathcal{F})$.

(ii) For every $X, Y \in \mathcal{E}$, a functor

$$\Phi_{X,Y} : \mathcal{E}(X, Y) \to \mathcal{F}(\Phi X, \Phi Y)\,.$$

Below, we write Φ instead of $\Phi_{X,Y}$.

(iii) A natural isomorphism with components

$$\phi_{B,A}\colon \Phi(B \circ A) \to \Phi(B) \circ \Phi(A)\,,$$

for $A\colon X \to Y$ and $B\colon Y \to Z$.

(iv) For each $X \in \mathcal{E}$, an isomorphism $\iota_X : \Phi(1_X) \to 1_{\Phi X}$.

(v) For every $A\colon X \to Y$, $B\colon Y \to Z$ and $C\colon Z \to U$, the following diagram commutes:

$$\begin{array}{ccc} & \Phi((C \circ B) \circ A) & \\ {\scriptstyle \phi} \swarrow & & \searrow {\scriptstyle \Phi(\alpha)} \\ \Phi(C \circ B) \circ \Phi(A) & & \Phi(C \circ (B \circ A)) \\ \downarrow {\scriptstyle \phi \circ \Phi(A)} & & \downarrow {\scriptstyle \phi} \\ (\Phi(C) \circ \Phi(B)) \circ \Phi(A) & & \Phi(C) \circ \Phi(B \circ A) \\ {\scriptstyle \alpha} \searrow & & \swarrow {\scriptstyle \Phi(C) \circ \phi} \\ & \Phi(C) \circ (\Phi(B) \circ \Phi(A))\,. & \end{array}$$

(vi) For every $A\colon X \to Y$, the following diagrams commute:

$$\begin{array}{ccc} \Phi(1_Y \circ A) & \xrightarrow{\ \phi\ } & \Phi(1_Y) \circ \Phi(A) \\ & & \downarrow {\scriptstyle \iota \circ \Phi(A)} \\ & & 1_{\Phi(Y)} \circ \Phi(A) \\ & {\scriptstyle \Phi(\lambda)} \searrow & \downarrow {\scriptstyle \lambda} \\ & & \Phi(A)\,. \end{array} \qquad \begin{array}{ccc} \Phi(A) \circ \Phi(1_X) & \xleftarrow{\ \phi\ } & \Phi(A \circ 1_X) \\ {\scriptstyle \Phi(A) \circ \iota} \downarrow & & \\ \Phi(A) \circ 1_{\Phi(X)} & & \\ {\scriptstyle \rho} \downarrow & \swarrow {\scriptstyle \Phi(\rho)} & \\ \Phi(A)\,. & & \end{array}$$

DEFINITION. A *pseudo-natural transformation* $P\colon \Phi \to \Psi$ between two homomorphism $\mathcal{E} \to \mathcal{F}$ consists of the data in (i)-(ii) subject to the axioms (iii)-(iv) below.

(i) For each $X \in \mathcal{E}$, a morphism $P(X)\colon \Phi X \to \Psi X$

(ii) For each morphism $A\colon X \to Y$, an invertible 2-cell

$$\begin{array}{ccc} \Phi X & \xrightarrow{P(X)} & \Psi X \\ {\scriptstyle \Phi(A)}\downarrow & \Downarrow p_A & \downarrow{\scriptstyle \Psi(A)} \\ \Phi Y & \xrightarrow[P(Y)]{} & \Psi Y\,. \end{array}$$

(iii) For every $A\colon X \to Y$ and $B\colon Y \to Z$, we have

$$\begin{array}{ccc} \Phi X & \xrightarrow{P(X)} & \Psi X \\ {\scriptstyle \Phi A}\downarrow & \Downarrow p_A & \downarrow{\scriptstyle \Psi A} \\ \Phi Y & \xrightarrow{P(Y)} & \Psi Y \\ {\scriptstyle \Phi B}\downarrow & \Downarrow p_B & \downarrow{\scriptstyle \Psi B} \\ \Phi Z & \xrightarrow[P(Z)]{} & \Psi Z \end{array} \quad = \quad \begin{array}{ccc} \Phi X & \xrightarrow{P(X)} & \Psi X \\ {\scriptstyle \Phi(B\circ A)}\downarrow & \Downarrow p_{B\circ A} & \downarrow{\scriptstyle \Psi(B\circ A)} \\ \Phi Z & \xrightarrow[P(Z)]{} & \Psi Z\,. \end{array}$$

(iv) For every $X \in \mathcal{E}$,

$$\begin{array}{ccc} \Phi X & \xrightarrow{P(X)} & \Psi X \\ {\scriptstyle \Phi(1_X)}\downarrow & \Downarrow p_{1_X} & \downarrow{\scriptstyle \Psi(1_X)} \\ \Phi X & \xrightarrow[P(X)]{} & \Psi X \end{array} \quad = \quad \begin{array}{ccc} \Phi X & \xrightarrow{P(X)} & \Psi X \\ {\scriptstyle 1_{\Phi X}}\downarrow & & \downarrow{\scriptstyle 1_{\Psi X}} \\ \Phi X & \xrightarrow[P(X)]{} & \Psi X\,. \end{array}$$

DEFINITION. Let $P, Q : \Phi \to \Psi$ be pseudo-natural transformations. A *modification* $\sigma : P \to Q$ consists of a family of 2-cells $\sigma_X : P(X) \to Q(X)$ such that

$$\begin{array}{ccc} \Phi X & \overset{P(X)}{\underset{Q(X)}{\overset{\Downarrow \sigma_X}{\rightrightarrows}}} & \Psi X \\ {\scriptstyle \Phi(A)}\downarrow & \Downarrow q_Y & \downarrow{\scriptstyle \Psi(A)} \\ \Phi Y & \xrightarrow[Q(Y)]{} & \Psi Y \end{array} \quad = \quad \begin{array}{ccc} \Phi X & \xrightarrow{P(X)} & \Psi X \\ {\scriptstyle \Phi A}\downarrow & \Downarrow p_A & \downarrow{\scriptstyle \Psi A} \\ \Psi X & \overset{P(Y)}{\underset{Q(Y)}{\overset{\Downarrow \sigma_Y}{\rightrightarrows}}} & \Psi Y\,. \end{array}$$

APPENDIX B

A technical proof

B.1. Preliminaries

In order to prove Theorem 5.4.2 we need to recall some facts about the the formal theory of monads. Let us consider a fixed bicategory $\mathcal{E}$, which for the moment we do not need to suppose being tame. The inclusion homomorphism

$$\mathrm{Inc}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Mnd}(\mathcal{E})$$

takes an object X to the pair $(X, 1_X)$, where 1_X is the identity monad on X.

As proved in [**69**] in the context of 2-categories, the bicategory $\mathcal{E}$ admits Eilenberg-Moore objects if and only if the inclusion homomorphism $\mathrm{Inc} \colon \mathcal{E} \to \mathrm{Mnd}(\mathcal{E})$ has a right adjoint

$$\mathrm{EM} \colon \mathrm{Mnd}(\mathcal{E}) \to \mathcal{E}\,.$$

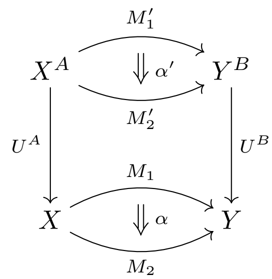
It will be useful to give an explicit description of it. The homomorphism $\mathrm{EM} \colon \mathrm{Mnd}(\mathcal{E}) \to \mathcal{E}$ takes a monad (X, A) to the Eilenberg-Moore object X^A and the counit of the adjunction is the monad morphism $(U^A, \lambda^A) \colon (X^A, 1_{X^A}) \to (X, A)$, where $\lambda^A \colon A \circ U^A \to U^A$ is the left action of A on $U^A \colon X^A \to X$ that is part of the structure of an Eilenberg-Moore object. The image by EM of a lax monad morphism $(M, \phi) \colon (X, A) \to (Y, B)$ is constructed as follows: the 2-cell

$$B \circ M \circ U^A \xrightarrow{\phi \circ U^A} M \circ A \circ U^A \xrightarrow{M \circ \lambda^A} M \circ U^A$$

is a left action of the monad B on $M \circ U^A$. There is then a morphism $M' \colon X^A \to Y^B$ together with an isomorphism of left B-modules

$$\begin{array}{ccc} X^A & \xrightarrow{U^A} & X \\ {\scriptstyle M'}\downarrow & \Downarrow \sigma_M & \downarrow{\scriptstyle M} \\ Y^B & \xrightarrow[U^B]{} & Y\,. \end{array}$$

By definition, $\mathrm{EM}(M, \phi) =_{\mathrm{def}} M'$. Finally, let us describe the image by EM of a monad 2-cell $\alpha \colon (M_1, \phi_1) \to (M_2, \phi_2)$. By construction, $\mathrm{EM}(\alpha)$ is the unique 2-cell $\alpha' \colon M_1' \to M_2'$ such that the following cylinder commutes:



B.2. The proof

Let us now recall the statement of Theorem 5.4.2 and give its proof.

Theorem B.2.1. *If $\mathcal{E}$ is a tame bicategory, then the homomorphism*

$$\mathrm{J}_{\mathcal{E}} \colon \mathcal{E} \to \mathrm{Bim}(\mathcal{E})$$

exhibits $\mathrm{Bim}(\mathcal{E})$ *as an Eilenberg-Moore completion of $\mathcal{E}$ as a tame bicategory.*

Proof. We will prove that the homomorphism

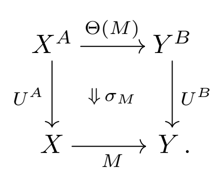
$$(-) \circ \mathrm{J}_{\mathcal{E}} \colon \mathrm{HOM}^p\big(\mathrm{Bim}(\mathcal{E}), \mathcal{F})\big) \to \mathrm{HOM}^p\big(\mathcal{E}, \mathcal{F}\big) \tag{B.2.1}$$

is a surjective equivalence for any Eilenberg-Moore complete tame bicategory $\mathcal{F}$.

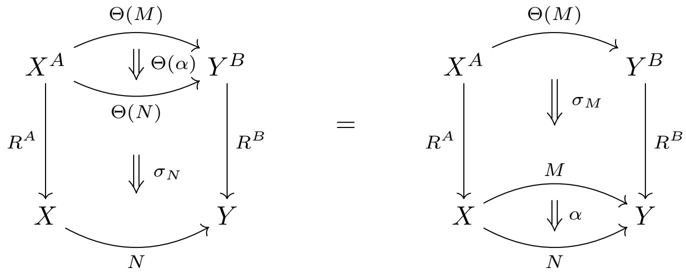
Let us first show that the homomorphism is surjective on objects. The homomorphism $\mathrm{J}_{\mathcal{F}} \colon \mathcal{F} \to \mathrm{Bim}(\mathcal{F})$ is an equivalence by Proposition 5.3.9, since the bicategory $\mathcal{F}$ is Eilenberg-Moore complete. Hence there is a homomorphism $\Theta \colon \mathrm{Bim}(\mathcal{F}) \to \mathcal{F}$ together with a pseudo-natural transformation

$$E \colon \mathrm{Id}_{\mathcal{F}} \to \Theta \circ \mathrm{J}_{\mathcal{F}}\,,$$

whose components are equivalences in $\mathcal{F}$. Let us show that the pair (Θ, E) can be chosen so that the pseudo-natural transformation E is the identity. The homomorphism Θ is constructed by first choosing an Eilenberg-Moore object $U^A \colon X^A \to X$ in $\mathcal{F}$ for each object $X/A \in \mathrm{Bim}(\mathcal{F})$ and letting $\Theta(X/A) = X^A$, and then by choosing for each morphism $M \colon X/A \to Y/B$ in $\mathrm{Bim}(\mathcal{F})$, a morphism $\Theta(M) \colon X^A \to Y^B$ in $\mathcal{F}$ together with an isomorphism of left B-modules $\sigma_M \colon R^B \circ \Theta(M) \to M \circ_A R^A$

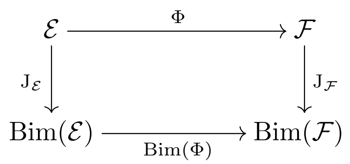


The value of Θ on 2-cells is determined by these choices afterward. More precisely, if $\alpha \colon M \to N$ is a 2-cell, then $\Theta(\alpha) \colon \Theta(M) \to \Theta(N)$ is the unique 2-cell such that the following equality of pasting diagrams holds:



If $A = 1_X$, we can choose $X^A = X$ and $U^A = 1_X$. And if $M\colon X/1_X \to Y/1_Y$ we can choose $\Theta(M) = M$ and σ_M to be the identity 2-cell. It follows from these choices that we have $\Theta(\alpha) = \alpha$ for every $\alpha\colon M \to N$. Thus, $\Theta \circ \mathrm{J}_{\mathcal{F}} = \mathrm{Id}_{\mathcal{F}}$.

Let us now show that every tame homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ can be extended as a tame homomorphism $\Phi'\colon \mathrm{Bim}(\mathcal{E}) \to \mathcal{F}$. As observed in Remark 4.2.7, the homomorphism $\Phi\colon \mathcal{E} \to \mathcal{F}$ has a natural extension $\mathrm{Bim}(\Phi)\colon \mathrm{Bim}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{F})$. Then, the following square commutes by construction



It is easy to verify that the homomorphism $\mathrm{Bim}(\Phi)$ is proper. Let us put $\Phi' = \Theta \circ \mathrm{Bim}(\Phi)$. Then we have

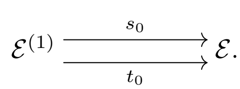
$$\Phi' \circ \mathrm{J}_{\mathcal{E}} = \Theta \circ \mathrm{Bim}(\Phi) \circ \mathrm{J}_{\mathcal{E}} = \Theta \circ \mathrm{J}_{\mathcal{E}} \circ \Phi = \mathrm{Id}_{\mathcal{F}} \circ \Phi = \Phi\,.$$

We have proved that the homomorphism in (B.2.1) is surjective on objects. For any tame homomorphism $\Phi\colon \mathrm{Bim}(\mathcal{E}) \to \mathcal{F}$, let us define $\Phi_{|\mathcal{E}} =_{\mathrm{def}} \Phi \circ \mathrm{J}_{\mathcal{E}}$. For a pair of tame homomorphisms $\Phi, \Psi\colon \mathrm{Bim}(\mathcal{E}) \to \mathcal{F}$, we can then define the restriction functor

$$\mathrm{Res}(\Phi, \Psi)\colon [\Phi, \Psi] \to [\Phi_{|\mathcal{E}}, \Psi_{|\mathcal{E}}]$$

in the evident way. It remains to show that the functor $\mathrm{Res}(\Phi, \Psi)$ is an equivalence of categories surjective on objects. We will prove that the functor $\mathrm{Res}(\Phi, \Psi)$ is surjective on objects in Lemma B.2.4, and that it is full and faithful in Lemma B.2.7. □

We need to prove a few intermediate results. We first recall the bicategory of 1-cells $\mathcal{E}^{(1)}$ of a bicategory $\mathcal{E}$. The bicategory $\mathcal{E}^{(1)}$ is equipped with a universal pseudo-natural transformation $U\colon s_0 \to t_0$ between two homomorphisms



The universality of U means that for any bicategory $\mathcal{F}$ and any pseudo-natural transformation $M\colon \Phi \to \Psi$ between a pair of homomorphisms $\Phi, \Psi\colon \mathcal{F} \to \mathcal{E}$, there exists a *unique* homomorphism

$\Theta\colon \mathcal{F} \to \mathcal{E}^{(1)}$ such that $s_0 \circ \Theta = \Phi$, $t_0 \circ \Theta = \Psi$ and $U\Theta = M$. By construction, an object of $\mathcal{E}^{(1)}$ is a 1-cell $M\colon X_0 \to X_1$ in the bicategory $\mathcal{E}$. A 1-cell $M \to N$ of $\mathcal{E}^{(1)}$ is a pseudo-commutative square

$$\begin{array}{ccc} X_0 & \xrightarrow{E_0} & Y_0 \\ {\scriptstyle M}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle N} \\ X_1 & \xrightarrow[E_1]{} & Y_1 \end{array}$$

defined by a triple (E_0, E_1, α), where $\alpha\colon E_1 \circ M \to N \circ E_0$ is an isomorphism. Composition of 1-cells is defined by pasting squares. If $(F_0, F_1, \beta)\colon M \to N$ is another pseudo-commutative square then a 2-cell $(E_0, E_1, \alpha) \to (F_0, F_1, \beta)$ in $\mathcal{E}^{(1)}$ is a cylinder

$$\begin{array}{ccc} X_0 & \overset{E_0}{\underset{F_0}{\overset{\Downarrow\gamma_0}{\rightrightarrows}}} & Y_0 \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle N} \\ X_1 & \overset{E_1}{\underset{F_1}{\overset{\Downarrow\gamma_1}{\rightrightarrows}}} & Y_1. \end{array}$$

defined by a pair of 2-cells $\gamma_0\colon E_0 \to F_0$, $\gamma_1\colon E_1 \to F_1$ such that the following square commutes,

$$\begin{array}{ccc} E_1 \circ M & \xrightarrow{\gamma_1 \circ M} & F_1 \circ M \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ N \circ E_0 & \xrightarrow[N \circ \gamma_0]{} & N \circ F_0 \end{array}$$

Vertical composition of 2-cells in $\mathcal{E}^{(1)}$ is defined component-wise. Observe that the source and target homomorphisms

$$\mathcal{E}^{(1)} \underset{t_0}{\overset{s_0}{\rightrightarrows}} \mathcal{E}$$

are preserving composition strictly. The pseudo-natural stansformation $U\colon s_0 \to t_0$ takes the object $M\colon X_0 \to X_1$ of $\mathcal{E}^{(1)}$ to the 1-cell $M\colon s_0(M) \to t_0(M)$ of $\mathcal{E}$, and it takes the 1-cell $(E_0, E_1, \alpha)\colon M \to N$ to the isomorphism $\alpha\colon E_1 \circ M \to N \circ E_0$.

Lemma B.2.2. *If $\mathcal{E}$ is tame, then so is its bicategory of 1-cells $\mathcal{E}^{(1)}$.*

Proof. If $M\colon X_0 \to X_1$ and $N\colon Y_0 \to Y_1$, then the bicategory $\mathcal{E}^{(1)}[M, N]$ is defined by a pseudo-pullback square

$$\begin{array}{ccc} \mathcal{E}^{(1)}[M, N] & \longrightarrow & \mathcal{E}[X_0, Y_0] \\ \downarrow & & \downarrow{\scriptstyle N \circ (-)} \\ \mathcal{E}[X_1, Y_1] & \xrightarrow[(-) \circ M]{} & \mathcal{E}[X_0, Y_1] \end{array}$$

It follows that the bicategory $\mathcal{E}^{(1)}[M,N]$ admits reflexive coequalizers, since the functors $N\circ(-)$ and $(-)\circ M$ are preserving reflexive coequalizers. Moreover, the functor

$$(s,t)\colon \mathcal{E}^{(1)}[M,N]\to \mathcal{E}[X_0,Y_0]\times\mathcal{E}[X_1,Y_1]$$

preserves and reflects reflexive coequalizers. It follows that the composition functor

$$(-)\circ(-)\colon \mathcal{E}^{(1)}[N,P]\times\mathcal{E}^{(1)}[M,N]\to\mathcal{E}^{(1)}[M,P]$$

preserves reflexive coequalizers in each variable. Hence the bicategory $\mathcal{E}^{(1)}$ is tame. □

A monad on an object $M\colon X_0\to X_1$ of the bicategory $\mathcal{E}^{(1)}$ is a 6-tuple

$$A=(A_0,\mu_0,\eta_0,A_1,\mu_1,\eta_1,\alpha)\,,$$

where $A_0=(A_0,\mu_0,\eta_0)$ is a monad on X_0, $A_1=(A_1,\mu_1,\eta_1)$ is a monad on X_1 and $\alpha\colon A_1\circ M\to M\circ A_0$ is an isomorphism satisfying the coherence conditions expressed by the diagrams:

$$\begin{array}{ccccc} A_1\circ A_1\circ M & \xrightarrow{A_1\circ\alpha} & A_1\circ M\circ A_0 & \xrightarrow{\alpha\circ A_0} & M\circ A_0\circ A_0 \\ {\scriptstyle \mu_1\circ M}\downarrow & & & & \downarrow{\scriptstyle M\circ\mu_0} \\ A_1\circ M & & \xrightarrow[\alpha]{} & & M\circ A_0 \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\eta_1\circ M} & A_1\circ M\,. \\ & {\scriptstyle M\circ\eta_0}\searrow & \downarrow{\scriptstyle\alpha} \\ & & M\circ A_0 \end{array}$$

It follows that we have a morphism $(M,\alpha)\colon(X_0,A_0)\to(X_1,A_1)$ in the bicategory $\mathrm{Mnd}(\mathcal{E})$, defined in Section 4.3.

LEMMA B.2.3. *If $\mathcal{E}$ is Eilenberg-Moore complete, then so is its bicategory of* 1*-cells $\mathcal{E}^{(1)}$.*

PROOF. Let us show that every monad in $\mathcal{E}^{(1)}$ admits an Eilenberg-Moore object. We will use the homomorphism $\mathrm{EM}\colon\mathrm{Mnd}(\mathcal{E})\to\mathcal{E}$. If $A=(A_0,\mu_0,\eta_0,A_1,\mu_1,\eta_1,\alpha)$ is a monad on an object $M\colon X_0\to X_1$ of the bicategory $\mathcal{E}^{(1)}$, then $(M,\alpha)\colon(X_0,A_0)\to(X_1,A_1)$ is a morphism in $\mathrm{Mnd}(\mathcal{E})$. Let $U_0\colon X_0^{A_0}\to X_0$ be an Eilenberg-Moore object for A_0 and $U_1\colon X_1^{A_1}\to X_1$ be an Eilenberg-Moore object for A_1. There is then a morphism $M^A=_{\mathrm{def}}\mathrm{EM}(M,\alpha)\colon X_0^{A_0}\to X_1^{A_1}$ in the bicategory $\mathcal{E}$ together with an isomorphism of left A_1-modules $\sigma\colon U_1\circ M^A\to M\circ U_0$,

$$\begin{array}{ccc} X_0^{A_0} & \xrightarrow{M^A} & X_1^{A_1} \\ {\scriptstyle U_0}\downarrow & \Downarrow\sigma & \downarrow{\scriptstyle U_1} \\ X_0 & \xrightarrow[M]{} & X_1. \end{array}$$

It is easy to verify that the morphism $(U_0,U_1,\sigma)\colon M^A\to M$ in $\mathcal{E}^{(1)}$ is an Eilenberg-Moore object for the monad A. □

LEMMA B.2.4. *The restriction functor* $\mathrm{Res}(\Phi,\Psi)\colon[\Phi,\Psi]\to[\Phi\mid\mathcal{E},\Psi\mid\mathcal{E}]$ *is surjective on objects for any pair of tame homomorphisms $\Phi,\Psi\colon\mathrm{Bim}(\mathcal{E})\to\mathcal{F}$.*

PROOF. Let us show that any pseudo-natural transformation $M\colon\Phi_{|\mathcal{E}}\to\Psi_{|\mathcal{E}}$ admits an extension $M'\colon\Phi\to\Psi$. The pseudo-natural transformation M is defining a homomorphism $M\colon\mathcal{E}\to\mathcal{F}^{(1)}$. The homomorphism M is proper, since the homomorphisms $s_0(M)=\Phi_{|\mathcal{E}}$ and $t_0(M)=\Psi_{|\mathcal{E}}$ are proper. It follows by the first part of the proof of Theorem B.2.1 that the homomorphism M can be extended as a tame homomorphism $M' \colon \mathrm{Bim}(\mathcal{E}) \to \mathcal{F}^{(1)}$. It remains to show that M' can be chosen so that $s_0(M') = \Phi$ and $t_0(M') = \Psi$.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{M} & \mathcal{F}^{(1)} \\ {\scriptstyle \mathrm{J}_{\mathcal{E}}}\downarrow & \overset{M'}{\nearrow} & \downarrow{\scriptstyle (s_0,t_0)} \\ \mathrm{Bim}(\mathcal{E}) & \xrightarrow[(\Phi,\Psi)]{} & \mathcal{F} \times \mathcal{F} \end{array}$$

The homomorphism $M \colon \mathcal{E} \to \mathcal{F}^{(1)}$ sends $X \in \mathcal{E}$ to a morphism $M(X) \colon \Phi(X) \to \Psi(X)$ in $\mathcal{F}$ and a monad A on X in $\mathcal{E}$, to a monad $M(A)$ on $(M(X)$ in $\mathcal{F}^{(1)}$. Thus,

$$M(A) = (\Phi(A), \Phi(\mu), \Phi(\eta), \Psi(A), \Psi(\mu), \Psi(\eta), \alpha)\,,$$

where $\Phi(A) = (\Phi(A), \Phi(\mu), \Phi(\eta))$ is a monad on $\Phi(X)$, $\Psi(A) = (\Psi(A), \Psi(\mu), \Psi(\eta))$ is a monad on $\Psi(X)$ and α is an invertible 2-cell

$$\alpha \colon \Psi(A) \circ M(X) \to M(X) \circ \Phi(A)$$

defining a monad morphism $(M(X), \alpha) \colon (\Phi(X), \Phi(A)) \to (\Psi(X), \Psi(A))$,

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\Phi(A)} & \Phi(X) \\ {\scriptstyle M(X)}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle M(X)} \\ \Psi(X) & \xrightarrow[\Psi(A)]{} & \Psi(X)\,. \end{array}$$

The morphism $A \colon X/A \to X/1_X$ in $\mathrm{Bim}(\mathcal{E})$ is an Eilenberg-Moore object for the monad $A \colon X \to X$ by Lemma 5.3.3. Hence, the morphism $\Phi(A) \colon \Phi(X/A) \to \Phi(X)$ is an Eilenberg-Moore object for the monad $\Phi(A)$ on the object $\Phi(X)$, since the homomorphism Φ is proper. Similarly, the morphism $\Psi(A) \colon \Psi(X/A) \to \Psi(X)$ is an Eilenberg-Moore object for the monad $\Psi(A)$, since the homomorphism Ψ is proper. The homomorphism $M' \colon \mathrm{Bim}(\mathcal{E}) \to \mathcal{F}^{(1)}$ can be constructed by choosing for each $X/A \in \mathrm{Bim}(\mathcal{E})$ a morphism $M'(X/A) \colon \Phi(X/A) \to \Psi(X/A)$ together with an isomorphism of left $\Psi(A)$-modules $\sigma_A \colon \Psi(A) \circ M(X/A) \to M(X) \circ \Phi(A)$,

$$\begin{array}{ccc} \Phi(X/A) & \xrightarrow{M'(X/A)} & \Psi(X/A) \\ {\scriptstyle \Phi(A)}\downarrow & \Downarrow \sigma_A & \downarrow{\scriptstyle \Psi(A)} \\ \Phi(X) & \xrightarrow[M(X)]{} & \Psi(X) \end{array}$$

We then have $s_0(M') = \Phi$ and $t_0(M') = \Psi$. If $A = 1_X$, we can choose $M(X/A) = M(X)$ and $\sigma_A = \mathrm{Id}$. In which case we have $M'_{|\mathcal{E}} = M$, as required. □

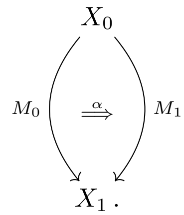
We now recall the definition of the bicategory of 2-cells $\mathcal{E}^{(2)}$ of the bicategory $\mathcal{E}$. The bicategory $\mathcal{E}^{(2)}$ is equipped with a universal modification $\alpha \colon s_1 \to t_1$ between a pair of pseudo-natural transformations

$$\mathcal{E}^{(2)} \underset{t_1}{\overset{s_1}{\rightrightarrows}} \mathcal{E}^{(1)}$$

such that

$$s_0\, s_1 = s_0\, t_1\,, \quad t_0\, s_1 = t_0\, t_1\,.$$

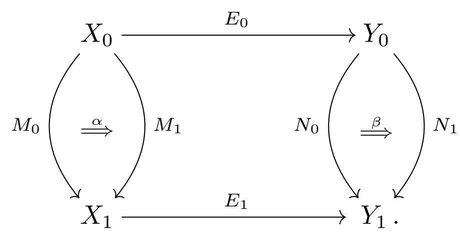
By construction, an object of $\mathcal{E}^{(2)}$ is a 2-cell $\alpha\colon M_0 \to M_1$ of the bicategory $\mathcal{E}$. We refer to α as a *disk* and represent it as follows:



If $\beta\colon N_0 \to N_1\colon Y_0 \to Y_1$ is another disk, then a 1-cell $(E_0, E_1, \gamma_0, \gamma_1)\colon \alpha \to \beta$ in $\mathcal{E}^{(2)}$ is a 4-tuple where $\gamma_0\colon E_1 \circ M_0 \to N_0 \circ E_0$ and $\gamma_1\colon E_1 \circ M_1 \to N_1 \circ E_0$ are invertible 2-cells such that the following square commutes,

$$\begin{array}{ccc} E_1 \circ M_0 & \xrightarrow{\gamma_0} & N_0 \circ E_0 \\ {\scriptstyle E_1 \circ \alpha}\downarrow & & \downarrow{\scriptstyle \beta \circ E_0} \\ E_1 \circ M_1 & \xrightarrow[\gamma_1]{} & N_1 \circ E_0 \,. \end{array}$$

We represent such a 2-cell by a cylinder



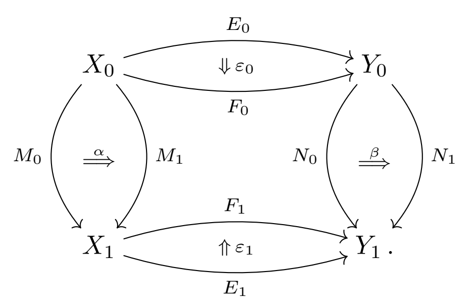
Composition of 1-cells in $\mathcal{E}^{(2)}$ is defined by pasting cylinders. A 2-cell

$$(\sigma_0, \sigma_1)\colon (E_0, E_1, \gamma_0, \gamma_1) \to (F_0, F_1, \delta_0, \delta_1)$$

in $\mathcal{E}^{(2)}$ is a pair of 2-cells $\varepsilon_0\colon E_0 \to F_0$ and $\varepsilon_1\colon E_1 \to F_1$ such that the following two squares commute:

$$\begin{array}{ccc} E_1 \circ M_0 & \xrightarrow{\gamma_0} & N_0 \circ E_0 \\ {\scriptstyle \varepsilon_1 \circ M_0}\downarrow & & \downarrow{\scriptstyle N_0 \circ \varepsilon_0} \\ F_1 \circ M_0 & \xrightarrow[\delta_0]{} & N_0 \circ F_0 \,, \end{array} \qquad \begin{array}{ccc} E_1 \circ M_1 & \xrightarrow{\gamma_1} & N_1 \circ E_0 \\ {\scriptstyle \varepsilon_1 \circ M_1}\downarrow & & \downarrow{\scriptstyle N_1 \circ \varepsilon_0} \\ F_1 \circ M_1 & \xrightarrow[\delta_1]{} & N_1 \circ F_0 \,. \end{array}$$

We represent such a 2-cell as a "deformation" of cylinders



Composition of 2-cells in $\mathcal{E}^{(1)}$ is defined component-wise. The source and target homomorphisms

$$\mathcal{E}^{(2)} \underset{t_1}{\overset{s_1}{\rightrightarrows}} \mathcal{E}^{(1)}$$

are preserving composition strictly.

Lemma B.2.5. *If $\mathcal{E}$ is tame, then so is its bicategory of 2-cells $\mathcal{E}^{(2)}$*

Proof. Similar to the proof of Lemma B.2.2. □

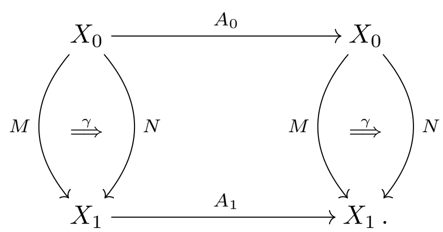
A monad on an object $\gamma\colon M \to N\colon X_0 \to X_1$ of the bicategory $\mathcal{E}^{(2)}$ is an 8-tuple

$$A = (A_0, \mu_0, \eta_0, A_1, \mu_1, \eta_1, \alpha, \beta)\,,$$

where (A_0, μ_0, η_0) is a monad on X_0, (A_1, μ_1, η_1) is a monad on X_1, $\alpha\colon A_1 \circ M \to M \circ A_0$ and $\beta\colon A_1 \circ N \to N \circ A_0$ are invertible 2-cells such that (M, α) and (N, β) are monad morphisms and the following diagram commutes,

$$\begin{array}{ccc} A_1 \circ M & \xrightarrow{\alpha} & M \circ A_0 \\ {\scriptstyle A_1 \circ \gamma}\downarrow & & \downarrow{\scriptstyle \gamma \circ A_0} \\ A_1 \circ N & \xrightarrow[\beta]{} & N \circ A_0 . \end{array}$$

It follows that we have a monad 2-cell $\gamma\colon (M, \alpha) \to (N, \beta)$ in $\mathrm{Mnd}(\mathcal{E})$,



Lemma B.2.6. *If $\mathcal{E}$ is Eilenberg-Moore complete, then so is its bicategory of 2-cells $\mathcal{E}^{(2)}$.*

Proof. Let us show that every monad in $\mathcal{E}^{(2)}$ admits an Eilenberg-Moore object. We will use the homomorphism $\mathrm{EM}\colon \mathrm{Mnd}(\mathcal{E}) \to \mathcal{E}$. Let $A = (A_0, \mu_0, \eta_0, A_1, \mu_1, \eta_1, \alpha, \beta)$ be a monad on an object $\gamma\colon M \to N$ of $\mathcal{E}^{(2)}$, where $M\, N\colon X_0 \to X_1$. Then, $\alpha\colon A_1 \circ M \to M \circ A_0$ determines a monad morphism $(M, \alpha)\colon A_0 \to A_1$ in $\mathrm{Mnd}(\mathcal{E})$, $\beta\colon A_1 \circ N \to N \circ A_0$ determines a monad morphism $(N, \beta)\colon A_0 \to A_1$, and $\gamma\colon M \to N$ determines a monad 2-cell $(M, \alpha) \to (N, \beta)$.

Let $U_0\colon X_0^{A_0} \to X_0$ be an Eilenberg-Moore object for A_0 and $U_1\colon X_1^{A_1} \to X_1$ be an Eilenberg-Moore object for A_1. There then is a morphism $M^A\colon X_0^{A_0} \to X_1^{A_1}$ in the bicategory $\mathcal{E}$, defined by letting $M^A =_{\mathrm{def}} \mathrm{EM}(M, \alpha)$, together with an isomorphism of left A_1-modules

$$\begin{array}{ccc} X_0^{A_0} & \xrightarrow{M^A} & X_1^{A_1} \\ {\scriptstyle U_0}\downarrow & \Downarrow \sigma_M & \downarrow{\scriptstyle U_1} \\ X_0 & \xrightarrow[M]{} & X_1 \,. \end{array}$$

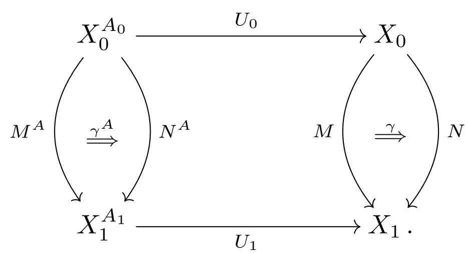
Similarly, there is a morphism $N^A\colon X_0^{A_0} \to X_1^{A_1}$ in $\mathcal{E}$ together with an isomorphism of left A_1-modules

$$\begin{array}{ccc} X_0^{A_0} & \xrightarrow{N^A} & X_1^{A_1} \\ {\scriptstyle R_0}\downarrow & \Downarrow \sigma_N & \downarrow{\scriptstyle R_1} \\ X_0 & \xrightarrow[N]{} & X_1\,. \end{array}$$

If we define $\gamma^A\colon M^A \to N^A$ by letting $\gamma^A =_{\mathrm{def}} \mathrm{EM}(\gamma)$, then the following square commutes

$$\begin{array}{ccc} U_1 \circ M^A & \xrightarrow{\sigma_M} & M \circ R_0 \\ {\scriptstyle R_1\circ\gamma^A}\downarrow & & \downarrow{\scriptstyle \gamma\circ R_0} \\ U_1 \circ N & \xrightarrow{\sigma_N} & N \circ R_0\,. \end{array}$$

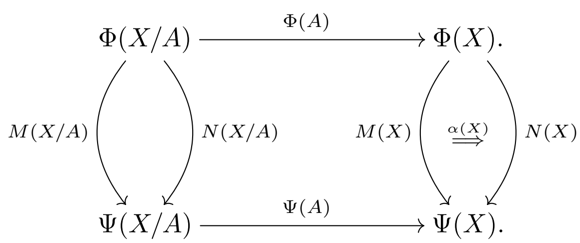
This means that the following cylinder in $\mathcal{E}$ commutes:



It is easy to verify that the morphism $(U_0, U_1, \sigma_M, \sigma_N)\colon \gamma^A \to \gamma$ in $\mathcal{E}^{(2)}$ is an Eilenberg-Moore object for the monad A. $\square$

LEMMA B.2.7. *The restriction functor* $\mathrm{Res}(\Phi, \Psi)\colon [\Phi, \Psi] \to [\Phi_{|\mathcal{E}}, \Psi_{|\mathcal{E}}]$ *is full and faithful for any pair of tame homomorphisms* $\Phi, \Psi\colon \mathrm{Bim}(\mathcal{E}) \to \mathcal{F}$.

PROOF. If $M, N\colon \Phi \to \Psi$ are pseudo-natural transformations, let us show that every modification $\alpha\colon M_{|\mathcal{E}} \to N_{|\mathcal{E}}$ admits a unique extension $\alpha'\colon M \to N$. For every $X/A \in \mathcal{E}$, we have an open cylinder,



with back and front faces given by isomorphisms

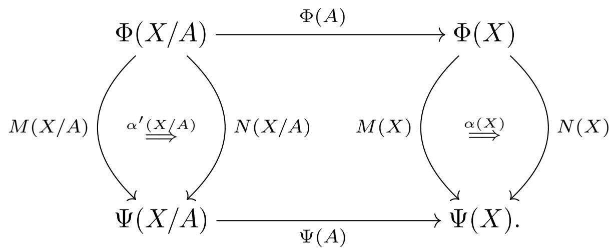
$$\sigma_M\colon \Psi(A) \circ M(X/A) \simeq M(X) \circ \Phi(A)\,, \quad \sigma_N\colon \Psi(A) \circ N(X/A) \simeq N(X) \circ \Phi(A)\,.$$

The morphism $\Phi(A)\colon \Phi(X/A) \to \Phi(X)$ is an Eilenberg-Moore object for $\Phi(A)$ and the morphism $\Psi(A)\colon \Psi(X/A) \to \Psi(X)$ is an Eilenberg-Moore object for $\Psi(A)$. The 2-cell $\alpha(X)\colon M(X) \to N(X)$

is a monad 2-cell. It follows that there is a unique 2-cell $\alpha'(X/A)\colon M(X/A) \to N(X/A)$ such that the following square commutes

$$\begin{array}{ccc}
\Psi(A) \circ M(X/A) & \xrightarrow{\sigma_M} & M(X) \circ \Phi(A) \\
{\scriptstyle \Psi(A)\circ\alpha'(X/A)}\downarrow & & \downarrow{\scriptstyle \alpha(X)\circ\Phi(A)} \\
\Psi(A) \circ N(X/A) & \xrightarrow[\sigma_N]{} & N(X) \circ \Phi(A)\,.
\end{array}$$

We obtain a full cylinder



This defines the extension $\alpha'\colon M \to N$ of the modification $\alpha\colon M_{|\mathcal{E}} \to N_{|\mathcal{E}}$. The uniqueness of α' is clear from the construction. $\square$

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