Published paper
Bavula, V.V. (2009) *The group of automorphisms of the first weyl algebra in prime characteristic and the restriction map.* Glasgow Mathematical Journal, 51 (2). pp. 263-274.
http://dx.doi.org/10.1017/S0017089508004680

The group of automorphisms of the first Weyl algebra in prime characteristic and the restriction map

V. V. Bavula

Abstract

Let K be a *perfect* field of characteristic $p>0$, $A_1 := K\langle x, \partial \,|\, \partial x - x\partial = 1\rangle$ be the first Weyl algebra and $Z := K[X := x^p, Y := \partial^p]$ be its centre. It is proved that (i) the restriction map $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \mathrm{Aut}_K(Z)$, $\sigma \mapsto \sigma|_Z$, is a monomorphism with $\mathrm{im(res)} = \Gamma := \{\tau \in \mathrm{Aut}_K(Z) \,|\, \mathcal{J}(\tau) = 1\}$ where $\mathcal{J}(\tau)$ is the Jacobian of τ (note that $\mathrm{Aut}_K(Z) = K^* \ltimes \Gamma$ and if K is *not* perfect then $\mathrm{im(res)} \neq \Gamma$); (ii) the bijection $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \Gamma$ is a monomorphism of infinite dimensional algebraic groups which is *not* an isomorphism (even if K is algebraically closed); (iii) an explicit formula for res^{-1} is found via differential operators $\mathcal{D}(Z)$ on Z and negative powers of the Fronenius map F. Proofs are based on the following (non-obvious) equality proved in the paper:

$$(\frac{d}{dx} + f)^p = (\frac{d}{dx})^p + \frac{d^{p-1}f}{dx^{p-1}} + f^p, \quad f \in K[x].$$

Key Words: the Weyl algebra, the group of automorphisms, the restriction map, the Jacobian.

Mathematics subject classification 2000: 14J50, 16W20, 14L17, 14R10, 14R15, 14M20.

Contents

1 Introduction

Let $p>0$ be a prime number and $\mathbb{F}_p := \mathbb{Z}/\mathbb{Z}p$. Let K be a commutative $\mathbb{F}_p$-algebra, $A_1 := K\langle x, \partial \,|\, \partial x - x\partial = 1\rangle$ be the first Weyl algebra over K. In order to avoid awkward expressions we use y instead of ∂ sometime, i.e. $y = \partial$. The centre Z of the algebra A_1 is the polynomial algebra $K[X, Y]$ in two variables $X := x^p$ and $Y := \partial^p$. Let $\mathrm{Aut}_K(A_1)$ and

$\mathrm{Aut}_K(Z)$ be the groups of K-automorphisms of the algebras A_1 and Z respectively. They contain the subgroups of affine automorphisms $\mathrm{Aff}(A_1) \simeq \mathrm{SL}_2(K)^{op} \ltimes K^2$ and $\mathrm{Aff}(Z) \simeq \mathrm{GL}_2(K)^{op} \ltimes K^2$ respectively. If K is a field of arbitrary characteristic then the group $\mathrm{Aut}_K(K[X,Y])$ of automorphisms of the polynomial algebra $K[X,Y]$ is generated by two of its subgroups, namely, $\mathrm{Aff}(K[X,Y])$ and $U(K[X,Y]) := \{\phi_f : X \mapsto X, Y \mapsto Y + f \,|\, f \in K[X]\}$. This was proved by H. W. E. Jung [5] in characteristic zero and by W. Van der Kulk [7] in general.

If K is a field of characteristic zero J. Dixmier [4] proved that the group $\mathrm{Aut}_K(A_1)$ is generated by its subgroups $\mathrm{Aff}(A_1)$ and $U(A_1) := \{\phi_f : x \mapsto x, \partial \mapsto \partial + f \,|\, f \in K[x]\}$. If K is a field of characteristic $p > 0$ L. Makar-Limanov [8] proved that the groups $\mathrm{Aut}_K(A_1)$ and $\Gamma := \{\tau \in \mathrm{Aut}_K(K[X,Y]) \,|\, \mathcal{J}(\tau) = 1\}$ are isomorphic as *abstract* groups where $\mathcal{J}(\tau)$ is the *Jacobian* of τ. In his paper he used the restriction map

$$\mathrm{res} : \mathrm{Aut}_K(A_1) \to \mathrm{Aut}_K(Z), \quad \sigma \mapsto \sigma|_Z. \tag{1}$$

In this paper, we study this map in detail. Recently, the restriction map (for the nth Weyl algebra) appeared in papers of Y. Tsuchimoto [12], A. Belov-Kanel and M. Kontsevich [2], K. Adjamagbo and A. van den Essen [1]. Let us describe some of the results proved in the paper.

Theorem 1.1 *Let K be a perfect field of characteristic $p > 0$. Then the restriction map* res *is a group monomorphism with* $\mathrm{im}(\mathrm{res}) = \Gamma$.

Note that $\mathrm{Aut}_K(Z) = K^* \ltimes \Gamma$ where $K^* \simeq \{\tau_\lambda : X \mapsto \lambda X, Y \mapsto Y \,|\, \lambda \in K^*\}$.

If K is not perfect then Theorem 1.1 is *not* true as one can easily show that the automorphism $\Gamma \ni s_\mu : X \mapsto X + \mu$, $Y \mapsto Y$, does not belong to the image of res provided $\mu \in K \backslash F(K)$ where $F : a \mapsto a^p$ is the Frobenius map. So, in the case of a perfect field we have another proof of the result of L. Makar-Limanov (in both proofs the results of Jung-Van der Kulk are essential).

The groups $\mathrm{Aut}_K(A_1)$, $\mathrm{Aut}_K(Z)$ and Γ are infinite dimensional algebraic groups over K in the sense of I. Shafarevich [10], [11] (see also T. Kambayashi [9]).

Corollary 1.2 *Let K be a perfect field of characteristic $p > 0$. Then the bijection* res : $\mathrm{Aut}_K(A_1) \to \Gamma$, $\sigma \mapsto \sigma|_Z$, *is a monomorphism of algebraic groups over K which is not an isomorphism of algebraic groups.*

The proofs of Theorem 1.1 and Corollary 1.2 are based on the following (non-obvious) formula which allows us to find the inverse map $\mathrm{res}^{-1} : \Gamma \to \mathrm{Aut}_K(A_1)$ (using differential operators $\mathcal{D}(Z)$ on Z), see (14) and Proposition 2.2.

Theorem 1.3 *Let K be a reduced commutative $\mathbb{F}_p$-algebra and $A_1(K)$ be the first Weyl algebra over K. Then*

$$(\partial + f)^p = \partial^p + \frac{d^{p-1}f}{dx^{p-1}} + f^p$$

for all $f \in K[x]$. In more detail, $(\partial + f)^p = \partial^p - \lambda_{p-1} + f^p$ where $f = \sum_{i=0}^{p-1} \lambda_i x^i \in K[x] = \oplus_{i=0}^{p-1} K[x^p]x^i$, $\lambda_i \in K[x^p]$.

Remark. We used the fact that $\frac{d^{p-1}f}{dx^{p-1}} = (p-1)!\lambda_{p-1}$ and $(p-1)! \equiv -1 \mod p$. Theorem 1.3 generalizes the following equality obtained by A. Belov-Kanel and M. Kontsevich in [3]: *if K is a field of characteristic $p > 0$ and $f = \frac{dg}{dx}$ for some polynomial $g \in K[x]$, then* $(\partial + f)^p = \partial^p + f^p$.

The group Γ is generated by its two subgroups $U(Z)$ and

$$\Gamma \cap \mathrm{Aff}(Z) = \{\sigma_{A,a} : \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \mapsto A\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) + a \,|\, A \in \mathrm{SL}_2(K), a \in K^2\} \simeq \mathrm{SL}_2(K)^{op} \ltimes K^2.$$

Recall that the group $\mathrm{Aut}_K(A_1)$ is generated by its two subgroups $U(A_1)$ and

$$\mathrm{Aff}(A_1) = \{\sigma_{A,a} : \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + a \,|\, A \in \mathrm{SL}_2(K), a \in K^2\} \simeq \mathrm{SL}_2(K)^{op} \ltimes K^2.$$

If K is a perfect field of characteristic $p > 0$ then Theorem 1.3 shows that

$$\mathrm{res}(\mathrm{Aff}(A_1)) = \Gamma \cap \mathrm{Aff}(Z), \quad \text{and} \quad \mathrm{res}(U(A_1)) = U(Z).$$

In more detail,

$$\mathrm{res} : \mathrm{Aff}(A_1) \to \Gamma \cap \mathrm{Aff}(Z), \quad \sigma_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} e \\ f \end{smallmatrix}\right)} \mapsto \begin{cases} \sigma_{\left(\begin{smallmatrix} a^p & b^p \\ c^p & d^p \end{smallmatrix}\right), \left(\begin{smallmatrix} e^p \\ f^p \end{smallmatrix}\right)}, & \text{if } p > 2, \\ \sigma_{\left(\begin{smallmatrix} a^2 & b^2 \\ c^2 & d^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} e^2+ab \\ f^2+cd \end{smallmatrix}\right)}, & \text{if } p = 2, \end{cases}$$

see Lemma 3.1 and (11); and

$$\mathrm{res} : U(A_1) \to U(Z), \quad \phi_f \mapsto \phi_{\theta(f)},$$

where the map $\theta := F + \frac{d^{p-1}}{dx^{p-1}} : K[x] \to K[x^p]$ is a bijection. An explicit formula for the inverse map θ^{-1} is found (Proposition 2.2) via differential operators $\mathcal{D}(Z)$ on Z and negative powers of the Frobenius map F. As a consequence, a formula for the inverse map $\mathrm{res}^{-1} : \Gamma \to \mathrm{Aut}_K(A_1)$ is given, see (14).

2 Proof of Theorem 1.3 and the inverse map θ^{-1}

In this section, a proof of Theorem 1.3 is given and an inversion formula for a map θ is found which is a key ingredient in the inversion formula for the restriction map.

Proof of Theorem 1.3. The Weyl algebra $A_1(K) \simeq K \otimes_{\mathbb{F}_p} A_1(\mathbb{F}_p)$, the Frobenius $F : a \mapsto a^p$ and $\frac{d^{p-1}}{dx^{p-1}}$ are well behaved under ring extensions, localizations and algebraic closure of the coefficient field. So, without loss of generality we may assume that K is an algebraically closed field of characteristic $p > 0$: the commutative $\mathbb{F}_p$-algebra K is reduced, $\cap_{\mathfrak{p} \in \mathrm{Spec}(K)} \mathfrak{p} = 0$, and $A_1(K)/A_1(K)\mathfrak{p} \simeq A_1(K/\mathfrak{p})$, and so we may assume that K is a domain; then $A_1(K) \subseteq A_1(\mathrm{Frac}(K)) \subseteq A_1(\overline{\mathrm{Frac}(K)})$ where $\mathrm{Frac}(K)$ is the field of fractions of K and $\overline{\mathrm{Frac}(K)}$ is its algebraic closure.

First, let us show that the map $L : K[x] \to K[x^p]$, $f \mapsto L(f)$, defined by the rule

$$(\partial + f)^p = \partial^p + L(f) + f^p,$$

is well defined and additive, i.e. $L(f+g) = L(f) + L(g)$. The map

$$K[x] \to \mathrm{Aut}_K(A_1), f \mapsto \sigma_f : x \mapsto x, \partial \mapsto \partial + f,$$

is a group homomorphism, i.e. $\sigma_{f+g} = \sigma_f \sigma_g$. Since $\partial^p \in Z(A_1) = K[x^p, \partial^p]$ and $(\partial + f)^p = \sigma(\partial)^p = \sigma(\partial^p) \in Z(A_1)$, the map L is well defined, i.e. $L(f) \in K[x^p]$. Comparing both ends of the series of equalities proves the additivity of the map L:

$$\begin{aligned} \partial^p + L(f+g) + f^p + g^p &= \sigma_{f+g}(\partial)^p = \sigma_{f+g}(\partial^p) = \sigma_f \sigma_g(\partial^p) = \sigma_f(\partial^p + L(g) + g^p) \\ &= \partial^p + L(f) + f^p + L(g) + g^p. \end{aligned}$$

In a view of the decomposition $K[x] = \oplus_{i=0}^{p-1} K[x^p]x^i$ and the additivity of the map L, it suffices to prove the theorem for $f = \lambda x^m$ where $m = 0, 1, \dots, p-1$ and $\lambda \in K[x^p]$. In addition, we may assume that $\lambda \in K$. This follows directly from the natural $\mathbb{F}_p$-algebra epimorphism

$$A_1(K[t]) \to A_1(K), \ \ t \mapsto \lambda, \ \ x \mapsto x, \ \ \partial \mapsto \partial,$$

and the fact that the polynomial algebra $K[t]$ is a domain (hence, reduced). Therefore, it suffices to prove the theorem for $f = \lambda x^m$ where $m = 0, 1, \dots, p-1$ and $\lambda \in K^*$.

The result is obvious for $m = 0$. So, we fix the natural number m such that $1 \leq m \leq p-1$. Then

$$l_m(\lambda) := L(\lambda x^m) = \sum_{k=0}^{m-1} l_{mk}(\lambda) x^{kp}$$

is a sum of *additive* polynomials $l_{mk}(\lambda)$ in λ of degree $\leq p-1$ (by the very definition of $L(\lambda x^m)$ and its additivity). Recall that a polynomial $l(t) \in K[t]$ is additive if $l(\lambda + \mu) = l(\lambda) + l(\mu)$ for all $\lambda, \mu \in K$. By Lemma 20.3.A [6], each additive polynomial $l(t)$ is a *p-polynomial*, i.e. a linear combination of the monomials t^{p^r}, $r \geq 0$. Hence, $l_m(\lambda) = a_m \lambda$ for some polynomial $a_m = \sum_{k=0}^{m-1} a_{mk} x^{kp}$ where $a_{mk} \in K$, i.e.

$$(\partial + \lambda x^m)^p = \partial^p + \lambda \sum_{k=0}^{m-1} a_{mk} x^{kp} + (\lambda x^m)^p.$$

Applying the K-automorphism $\gamma : x \mapsto \mu x$, $\partial \mapsto \mu^{-1}\partial$, $\mu \in K^*$, of the Weyl algebra A_1 to the equality above, we have

$$\begin{aligned} \mathrm{LHS} &= (\mu^{-1}\partial + \lambda \mu^m x^m)^p = \mu^{-p}(\partial + \lambda \mu^{m+1} x^m)^p \\ &= \mu^{-p}(\partial^p + \lambda \mu^{m+1} \sum_{k=0}^{m-1} a_{mk} x^{kp} + (\lambda \mu^{m+1} x^m)^p) \\ \mathrm{RHS} &= \mu^{-p}\partial^p + \lambda \sum_{k=0}^{m-1} a_{mk} \mu^{kp} x^{kp} + (\lambda \mu^m x^m)^p. \end{aligned}$$

Equating the coefficients of x^{kp} gives $\lambda a_{mk}\mu^{m+1-p} = \lambda a_{mk}\mu^{kp}$. If $a_{mk} \neq 0$ then $\mu^{m+1-p} = \mu^{kp}$ for all $\mu \in K^*$, i.e. $m+1-p = kp$. The maximum of $m+1-p$ is 0 at $m = p-1$, the minimum of kp is 0 at $k = 0$. Therefore, $a_{mk} = 0$ for all $(m,k) \neq (p-1,0)$.

For $(m,k) = (p-1,0)$, let $a := a_{p-1,0}$. Then

$$(\partial + \lambda x^{p-1})^p = \partial^p + \lambda a + (\lambda x^{p-1})^p.$$

In order to find the coefficient $a \in K$, consider the left A_1-module

$$V := A_1/(A_1x^p + A_1\partial) \simeq K[x]/K[x^p] = \oplus_{i=0}^{p-1} K\overline{x}^i$$

where $\overline{x}^i := x^i + A_1x^p + A_1\partial$. An easy induction on i gives the equalities:

$$(\partial + \lambda x^{p-1})^i \overline{x}^{p-1} = (p-1)(p-2)\cdots(p-i)\overline{x}^{p-1-i}, \;\; i = 1,2,\dots,p-1.$$

Now,

$$(\partial+\lambda x^{p-1})^p\overline{x}^{p-1} = (\partial+\lambda x^{p-1})(\partial+\lambda x^{p-1})^{p-1}\overline{x}^{p-1} = (\partial+\lambda x^{p-1})(p-1)!\overline{1} = (p-1)!\lambda\overline{x}^{p-1}.$$

On the other hand,

$$(\partial^p + \lambda a + (\lambda x^{p-1})^p)\overline{x}^{p-1} = \lambda a\overline{x}^{p-1},$$

and so $a = (p-1)! \equiv -1 \mod p$. This finishes the proof of Theorem 1.3. $\square$

The map θ and its inverse. Let K be a commutative $\mathbb{F}_p$-algebra. The polynomial algebra $K[x] = \oplus_{i\geq 0} Kx^i$ is a positively graded algebra and a positively filtered algebra $K[x] = \cup_{i\geq 0} K[x]_{\leq i}$ where $K[x]_{\leq i} := \oplus_{j=0}^i Kx^j = \{f \in K[x] \mid \deg(f) \leq i\}$. Similarly, the polynomial algebra $K[x^p]$ in the variable x^p is a positively graded algebra $K[x^p] = \oplus_{i\geq 0} Kx^{pi}$ and a positively filtered algebra $K[x^p] = \cup_{i\geq 0} K[x^p]_{\leq i}$ where $K[x^p]_{\leq i} := \oplus_{j=0}^i Kx^{pj} = \{f \in K[x^p] \mid \deg_{x^p}(f) \leq i\}$. The associated graded algebras $\mathrm{gr}\, K[x]$ and $\mathrm{gr}\, K[x^p]$ are canonically isomorphic to $K[x]$ and $K[x^p]$ respectively. For a polynomial $f = \sum_{i=0}^d \lambda_i x^i \in K[x]$ (resp. $g = \sum_{i=0}^d \mu_i x^{pi} \in K[x^p]$) of degree d, $\lambda_d x^d$ (resp. $\mu_d x^{pd}$) is called the leading term of f (resp. g) denoted $l(f)$ (resp. $l(g)$). Consider the $\mathbb{F}_p$-linear map (see Theorem 1.3)

$$\theta : F + \frac{d^{p-1}}{dx^{p-1}} : K[x] \to K[x^p], f \mapsto f^p + \frac{d^{p-1}f}{dx^{p-1}}, \tag{2}$$

where $F : f \mapsto f^p$ is the Frobenius ($\mathbb{F}_p$-algebra monomorphism). In more detail,

$$\theta : K[x] = \oplus_{i=0}^{p-1} K[x^p]x^i \to K[x^p] = \oplus_{i=0}^{p-1} K[x^{p^2}]x^{pi}, \;\; \sum_{i=0}^{p-1} a_i x^i \mapsto \sum_{i=0}^{p-1} a_i^p x^{pi} - a_{p-1},$$

where $a_i \in K[x^p]$. This means that the map θ respects the filtrations of the algebras $K[x]$ and $K[x^p]$, $\theta(K[x]_{\leq j}) \subseteq K[x^p]_{\leq j}$ for all $j \geq 0$, and so the associated graded map $\mathrm{gr}(\theta) : K[x] \to K[x^p]$ coincides with the Frobenius F,

$$\mathrm{gr}(\theta) = F. \tag{3}$$

Lemma 2.1 *Let K be a perfect field of characteristic $p > 0$. Then*

1. $\mathrm{gr}(\theta) = F : K[x] \to K[x^p]$ *is an isomorphism of* $\mathbb{F}_p$*-algebras.*

2. $\theta : K[x] \to K[x^p]$ *is an isomorphism of vector spaces over* $\mathbb{F}_p$ *such that* $\theta(K[x]_{\leq i}) = K[x^p]_{\leq i}$, $i \geq 0$.

3. *For each* $f \in K[x]$, $l(\theta(f)) = l(f)^p$.

Proof. Statement 1 is obvious since K is a perfect field of characteristic $p > 0$ ($F(K) = K$). Statements 2 and 3 follow from statement 1. $\square$

Remark. The problem of finding the inverse map res^{-1} of the group isomorphism $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \Gamma$, $\sigma \mapsto \sigma|_Z$, is essentially equivalent to the problem of finding θ^{-1}, see (14).

The inversion formula for θ^{-1} (Proposition 2.2) is given via certain differential operators. We recall some facts on differential operators that are needed in the proof of Proposition 2.2.

Let K be a field of characteristic $p > 0$ and $\mathcal{D}(K[x]) = \oplus_{i\geq 0} K[x]\partial^{[i]}$ be the ring of differential operators on the polynomial algebra $K[x]$ where $\partial^{[i]} := \frac{\partial^i}{i!}$. The algebra $K[x]$ is a left $\mathcal{D}(K[x])$-module (in the usual sense):

$$\partial^{[i]}(x^j) = \binom{j}{i} x^{j-i} \ \text{ for all } \ i, j \geq 0.$$

In particular,

$$\partial^{[pi]}(x^{pj}) = \binom{pj}{pi} x^{p(j-i)} = \binom{j}{i} x^{p(j-i)} \ \text{ for all } \ i, j \geq 0.$$

The subalgebra $K[x^p] = \oplus_{i=0}^{p-1} K[x^{p^2}]x^{pi}$ of $K[x]$ is $x^p\partial^{[p]}$-invariant and, for each $i = 0, 1, \ldots, p-1$, $K[x^{p^2}]x^{pi}$ is the eigenspace of the element $x^p\partial^{[p]}$ that corresponds to the eigenvalue i. Let $J(i) := \{0, 1, \ldots, p-1\}\backslash\{i\}$. Then

$$\pi_i := \partial^{[pi]} \frac{\prod_{j\in J(i)}(x^p\partial^{[p]} - j)}{\prod_{j\in J(i)}(i-j)} : K[x^p] \to K[x^{p^2}], \ \sum_{i=0}^{p-1} a_i x^{pi} \mapsto a_i, \tag{4}$$

where all $a_i \in K[x^{p^2}]$ (since the map $\frac{\prod_{j\in J(i)}(x^p\partial^{[p]}-j)}{\prod_{j\in J(i)}(i-j)} : K[x^p] \to K[x^p]$ is the projection onto the summand $K[x^{p^2}]x^{pi}$ in the decomposition $K[x] = \oplus_{i=0}^{p-1} K[x^{p^2}]x^{pi}$ and $\partial^{[pi]}(a_i x^{pi}) = a_i$).

Let K be a perfect field of characteristic $p > 0$. Consider the $\mathbb{F}_p$-linear map

$$\partial^{[(p-1)p]} F^{-1} : K[x^{p^2}] \to K[x^{p^2}], \ \sum_{i\geq 0} a_i x^{p^2 i} \mapsto \sum_{i\geq 0} a_{p-1+pi}^{\frac{1}{p}} x^{p^2 i}, \tag{5}$$

where $a_i \in K$. By induction on a natural number n, we have

$$(\partial^{[(p-1)p]} F^{-1})^n (\sum_{i\geq 0} a_i x^{p^2 i}) = \sum_{i\geq 0} a^{p^{-n}}_{(p-1)(1+p+\cdots+p^{n-1})+p^n i} x^{p^2 i}, \ \ n \geq 1. \tag{6}$$

This shows that the map $\partial^{[(p-1)p]}F^{-1}$ is a *locally nilpotent map*. This means that $K[x^{p^2}] = \cup_{n\geq 1}\ker(\partial^{[(p-1)p]}F^{-1})^n$, i.e. for each element $a \in K[x^{p^2}]$, $(\partial^{[(p-1)p]}F^{-1})^n(a) = 0$ for all $n \gg 0$. Hence, the map $1-\partial^{[(p-1)p]}F^{-1}$ is invertible and its inverse is given by the rule

$$(1-\partial^{[(p-1)p]}F^{-1})^{-1} = \sum_{j\geq 0}(\partial^{[(p-1)p]}F^{-1})^j. \tag{7}$$

The following proposition gives an explicit formula for θ^{-1}.

Proposition 2.2 *Let K be a perfect field of characteristic $p>0$. Then the inverse map $\theta^{-1} : K[x^p] = \oplus_{i=0}^{p-1}K[x^{p^2}]x^{pi} \to K[x] = \oplus_{i=0}^{p-1}K[x^p]x^i$, $\sum_{i=0}^{p-1}\mu_i x^{pi} \mapsto \sum_{i=0}^{p-1}\lambda_i x^i$, $\mu_i \in K[x^{p^2}]$, $\lambda_i \in K[x^p]$, is given by the rule*

1. *for $i = 0,1,\ldots,p-2$, $\lambda_i = \mu_i^{\frac{1}{p}} + F^{-1}\pi_i F^{-1}\sum_{j\geq 0}(\partial^{[(p-1)p]}F^{-1})^j(\mu_{p-1})$,*
2. *$\lambda_{p-1} = (\sum_{i=0}^{p-2}x^{pi}\pi_i F^{-1}\sum_{j\geq 0}(\partial^{[(p-1)p]}F^{-1})^j + x^{p(p-1)}\sum_{j\geq 1}(\partial^{[(p-1)p]}F^{-1})^j)(\mu_{p-1})$ where π_i is defined in (4).*

Proof. Let $g = \sum_{i=0}^{p-1}\mu_i x^{pi} \in K[x^p]$, $\mu_i \in K[x^{p^2}]$; $f = \sum_{i=0}^{p-1}\lambda_i x^i \in K[x]$, $\lambda_i \in K[x^p]$; and $\lambda_{p-1} = \sum_{i=0}^{p-1}a_i x^{pi}$, $a_i \in K[x^{p^2}]$. Then $\theta^{-1}(g) = f$ iff $g = \theta(f)$ iff $F^{-1}(g) = F^{-1}\theta(f)$ iff

$$\sum_{i=0}^{p-1}F^{-1}(\mu_i)x^i = F^{-1}(F(f)-\lambda_{p-1}) = f - F^{-1}(\lambda_{p-1}) = \sum_{i=0}^{p-1}(\lambda_i - F^{-1}(a_i))x^i$$

iff

$$\lambda_i = F^{-1}(\mu_i + a_i), \;\; i = 0,1,\ldots,p-1. \tag{8}$$

For $i = p-1$, the equality (8) can be rewritten as follows

$$\sum_{i=0}^{p-2}a_i x^{pi} + a_{p-1}x^{p(p-1)} = F^{-1}(\mu_{p-1} + a_{p-1}). \tag{9}$$

For each $i = 0,1,\ldots,p-2$, applying the map π_i (see (4)) to (9) gives the equality $a_i = \pi_i F^{-1}(\mu_{p-1}+a_{p-1})$, and so the equalities (8) can be rewritten as follows

$$\lambda_i = F^{-1}(\mu_i + \pi_i F^{-1}(\mu_{p-1}+a_{p-1})), \;\; i = 0,1,\ldots,p-2. \tag{10}$$

Applying $\partial^{[(p-1)p]}$ to (9) yields $a_{p-1} = \partial^{[(p-1)p]}F^{-1}(\mu_{p-1}+a_{p-1})$, and so $(1-\Delta)a_{p-1} = \Delta(\mu_{p-1})$ where $\Delta := \partial^{[(p-1)p]}F^{-1}$. By (7), $a_{p-1} = \sum_{j\geq 1}\Delta^j(\mu_{p-1})$. Putting this expression in (10) yields,

$$\lambda_i = F^{-1}(\mu_i) + F^{-1}\pi_i F^{-1}\sum_{j\geq 0}\Delta^j(\mu_{p-1}), \;\; i = 0,1,\ldots,p-2.$$

This proves statement 1. Finally,

$$\begin{aligned}
\lambda_{p-1} &= \sum_{i=0}^{p-1} a_i x^{pi} = \sum_{i=0}^{p-2} a_i x^{pi} + a_{p-1}x^{p(p-1)} \\
&= \sum_{i=0}^{p-2} x^{pi}\pi_i F^{-1}(\mu_{p-1}+a_{p-1}) + x^{p(p-1)}\sum_{j\geq 1}\Delta^j(\mu_{p-1}) \\
&= \sum_{i=0}^{p-2} x^{pi}\pi_i F^{-1}\sum_{j\geq 0}\Delta^j(\mu_{p-1}) + x^{p(p-1)}\sum_{j\geq 1}\Delta^j(\mu_{p-1}) \\
&= (\sum_{i=0}^{p-2} x^{pi}\pi_i F^{-1}\sum_{j\geq 0}(\partial^{[(p-1)p]}F^{-1})^j + x^{p(p-1)}\sum_{j\geq 1}(\partial^{[(p-1)p]}F^{-1})^j)(\mu_{p-1}). \quad \square
\end{aligned}$$

3 The restriction map and its inverse

In this section, Theorem 1.1, 3.4 and Corollary 1.2 are proved. An inversion formula for the restriction map $\mathrm{res} : \mathrm{Aut}_K(A_1) \to \Gamma$ is found, see (14).

The group of affine automorphisms. Let K be a perfect field of characteristic $p > 0$. Each element a of the Weyl algebra $A_1 = \oplus_{i,j\in\mathbb{N}} Kx^iy^i$ is a unique sum $a = \sum \lambda_{ij}x^iy^j$ where all but finitely many scalars $\lambda_{ij} \in K$ are equal to zero. The number $\deg(a) := \max\{i + j \,|\, \lambda_{ij} \neq 0\}$ is called the degree of a, $\deg(0) := -\infty$. Note that $\deg(ab) = \deg(a) + \deg(b)$, $\deg(a + b) \leq \max\{\deg(a), \deg(b)\}$, and $\deg(\lambda a) = \deg(a)$ for all $\lambda \in K^*$. For each $\sigma \in \mathrm{Aut}_K(A_1)$,

$$\deg(\sigma) := \max\{\deg(\sigma(x)), \deg(\sigma(y))\}$$

is called the *degree* of σ. The set (which is obviously a subgroup of $\mathrm{Aut}_K(A_1)$) $\mathrm{Aff}(A_1) = \{\sigma \in \mathrm{Aut}_K(A_1) \,|\, \deg(\sigma) = 1\}$ is called the group of affine automorphisms of the Weyl algebra A_1. Clearly,

$$\mathrm{Aff}(A_1) = \{\sigma_{A,a} : (\begin{smallmatrix} x \\ y \end{smallmatrix}) \mapsto A(\begin{smallmatrix} x \\ y \end{smallmatrix}) + a \,|\, A \in \mathrm{SL}_2(K), a \in K^2\}, \;\; \sigma_{A,a}\sigma_{B,b} = \sigma_{BA,Ba+b}.$$

For each group G, let G^{op} be its *opposite* group ($G^{op} = G$ as sets but the product ab in G^{op} is equal to ba in G). The map $G \to G^{op}$, $g \mapsto g^{-1}$, is a group automorphism. The group $\mathrm{Aff}(A_1)$ is the semi-direct product $\mathrm{SL}_2(K)^{op} \ltimes K^2$ of its subgroups $\mathrm{SL}_2(K)^{op} = \{\sigma_{A,0} \,|\, A \in \mathrm{SL}_2(K)\}$ and $K^2 \simeq \{\sigma_{1,a} \,|\, a \in K^2\}$ where K^2 is the normal subgroup of $\mathrm{Aff}(A_1)$ since $\sigma_{A,0}\sigma_{1,a}\sigma_{A,0}^{-1} = \sigma_{1,A^{-1}a}$. It is obvious that the group $\mathrm{Aff}(A_1)$ is generated by the automorphisms:

$$s : x \mapsto y, \; y \mapsto -x; \;\; t_\mu : x \mapsto \mu x, \; y \mapsto \mu^{-1}y; \;\; \phi_{\lambda x^i} : x \mapsto x, \; y \mapsto y + \lambda x^i,$$

where $\lambda \in K$, $\mu \in K^*$ and $i = 0, 1$.

Recall that the centre Z of the Weyl algebra A_1 is the polynomial algebra $K[X,Y]$ in $X := x^p$ and $Y := y^p$ variables. Let $\deg(z)$ be the total degree in X and Y of a polynomial $z \in Z$. For each automorphism $\sigma \in \mathrm{Aut}_K(Z)$,

$$\deg(\sigma) := \max\{\deg(\sigma(X)), \deg(\sigma(Y))\}$$

is called the *degree* of σ.

$$\mathrm{Aff}(Z) := \{\sigma \in \mathrm{Aut}_K(Z) \,|\, \deg(\sigma) = 1\} = \{\sigma_{A,a} : \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \mapsto A\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) + a \,|\, A \in \mathrm{GL}_2(K), a \in K^2\}$$

is the group of affine automorphisms of Z, $\sigma_{A,a}\sigma_{B,b} = \sigma_{BA,Ba+b}$. The group $\mathrm{Aff}(A_1)$ is the semi-direct product $\mathrm{GL}_2(K)^{op} \ltimes K^2$ of its subgroups $\mathrm{GL}_2(K)^{op} = \{\sigma_{A,0} \,|\, A \in \mathrm{GL}_2(K)\}$ and $K^2 \simeq \{\sigma_{1,a} \,|\, a \in K^2\}$ where K^2 is a normal subgroup of $\mathrm{Aff}(Z)$ since $\sigma_{A,0}\sigma_{1,a}\sigma_{A,0}^{-1} = \sigma_{1,A^{-1}a}$.

A group G is called an *exact* product of its subgroups G_1 and G_2 denoted $G = G_1 \times_{ex} G_2$ if each element $g \in G$ is a unique product $g = g_1g_2$ for some elements $g_1 \in G_1$ and $g_2 \in G_2$. Then $\mathrm{GL}_2(K)^{op} = K^* \times_{ex} \mathrm{SL}_2(K)^{op}$ where $K^* \simeq \{\gamma_\mu : X \mapsto \mu X,\ Y \mapsto Y \,|\, \mu \in K^*\}$, $\gamma_\mu\gamma_\nu = \gamma_{\mu\nu}$. Clearly, $\mathrm{Aff}(Z) = (K^* \times_{ex} \mathrm{SL}_2(K)^{op}) \ltimes K^2$, and so the group $\mathrm{Aff}(Z)$ is generated by the following automorphisms (where $\lambda \in K$, $\mu \in K^*$ and $i = 0, 1$):

$$s : X \mapsto Y,\ Y \mapsto -X;\quad t_\mu : X \mapsto \mu X,\ Y \mapsto \mu^{-1}Y; \phi_{\lambda X^i} : X \mapsto X,\ Y \mapsto Y + \lambda X^i;\ \text{ and } \gamma_\mu.$$

The automorphisms t_μ and γ_ν commute.

Lemma 3.1 *Let K be a perfect field of characteristic $p > 0$. Then the restriction map $\mathrm{res}_{aff} : \mathrm{Aff}(A_1) \to \mathrm{Aff}(Z)$, $\sigma \mapsto \sigma|_Z$, is a group monomorphism with* $\mathrm{im}(\mathrm{res}_{aff}) = \mathrm{SL}_2(K)^{op} \ltimes K^2$.

Proof. Since $\mathrm{res}_{aff}(s) = s$; $\mathrm{res}_{aff}(t_\mu) = t_{\mu^p}$; for $i = 0, 1$, $\mathrm{res}_{aff}(\phi_{\lambda x^i}) = \phi_{\lambda^p X^i}$ if $p > 2$ and $\mathrm{res}_{aff}(\phi_{\lambda x^i}) = \phi_{\lambda^2 X^i + \delta_{i,1}\lambda}$ if $p = 2$ where $\delta_{i,1}$ is the Kronecker delta (Theorem 1.3), i.e.

$$\mathrm{res}_{aff}(\sigma_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} e \\ f \end{smallmatrix}\right)}) = \begin{cases} \sigma_{\left(\begin{smallmatrix} a^p & b^p \\ c^p & d^p \end{smallmatrix}\right), \left(\begin{smallmatrix} e^p \\ f^p \end{smallmatrix}\right)}, & \text{if } p > 2, \\ \sigma_{\left(\begin{smallmatrix} a^2 & b^2 \\ c^2 & d^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} e^2+ab \\ f^2+cd \end{smallmatrix}\right)}, & \text{if } p = 2, \end{cases} \tag{11}$$

the result is obvious. $\square$

Lemma 3.2 *The automorphisms of the algebra Z: s, t_μ, $\phi_{\lambda X^i}$ and γ_μ satisfy the following relations:*

1. $st_\mu = t_{\mu^{-1}}s$ *and* $s\gamma_\mu = \gamma_\mu t_{\mu^{-1}}s$.
2. $\phi_{\lambda X^i}t_\mu = t_\mu\phi_{\lambda\mu^{-i-1}X^i}$ *and* $\phi_{\lambda X^i}\gamma_\mu = \gamma_\mu\phi_{\lambda\mu^{-i}X^i}$.
3. $s^2 = t_{-1}$, $s^{-1} = t_{-1}s : X \mapsto -Y$, $Y \mapsto X$.

Proof. Straightforward. $\square$

The map

$$K[X] \to \mathrm{Aut}(Z), \;\; f \mapsto \phi_f : X \mapsto X, \;\; Y \mapsto Y + f,$$

is a group monomorphism ($\phi_{f+g} = \phi_f \phi_g$). For $\sigma \in \mathrm{Aut}(Z)$, $\mathcal{J}(\sigma) := \det\begin{pmatrix} \frac{\partial\sigma(X)}{\partial X} & \frac{\partial\sigma(X)}{\partial Y} \\ \frac{\partial\sigma(Y)}{\partial X} & \frac{\partial\sigma(Y)}{\partial Y} \end{pmatrix}$ is the Jacobian of σ. It follows from the equality (which is a direct consequence of the chain rule) $\mathcal{J}(\sigma\tau) = \mathcal{J}(\sigma)\sigma(\mathcal{J}(\tau))$ that $\mathcal{J}(\sigma) \in K^*$ (since $1 = \mathcal{J}(\sigma\sigma^{-1}) = \mathcal{J}(\sigma)\sigma(\mathcal{J}(\sigma^{-1}))$ in $K[X, Y]$), and so the kernel $\Gamma := \{\sigma \in \mathrm{Aut}_K(Z) \,|\, \mathcal{J}(\sigma) = 1\}$ of the group epimorphism $\mathcal{J} : \mathrm{Aut}(Z) \to K^*$, $\sigma \mapsto \mathcal{J}(\sigma)$, is a normal subgroup of $\mathrm{Aut}_K(Z)$. Hence,

$$\mathrm{Aut}_K(Z) = K^* \ltimes \Gamma \tag{12}$$

is the semi-direct product of its subgroups Γ and $K^* \simeq \{\gamma_\mu \,|\, \mu \in K^*\}$.

Corollary 3.3 *Let K be a field of characteristic $p > 0$. Then*

1. *Each automorphism $\sigma \in \mathrm{Aut}_K(Z)$ is a product $\sigma = \gamma_\mu t_\nu \phi_{f_1} s \phi_{f_2} s \ldots \phi_{f_{n-1}} s \phi_{f_n}$ for some $\mu, \nu \in K^*$ and $f_i \in K[x]$.*

2. *Each automorphism $\sigma \in \Gamma$ is a product $\sigma = t_\nu \phi_{f_1} s \phi_{f_2} s \ldots \phi_{f_{n-1}} s \phi_{f_n}$ for some $\nu \in K^*$ and $f_i \in K[x]$.*

Proof. 1. Statement 1 follows at one from Lemma 3.2 and the fact that the group $\mathrm{Aut}_K(Z)$ is generated by $\mathrm{Aff}(Z)$ and $\phi_{\lambda X^i}$, $\lambda \in K$, $i \in \mathbb{N}$.

2. Statement 2 follows from statement 1: $\sigma = \gamma_\mu t_\nu \phi_{f_1} s \phi_{f_2} s \ldots \phi_{f_{n-1}} s \phi_{f_n} \in \Gamma$ iff

$$1 = \mathcal{J}(\sigma) = \mathcal{J}(\gamma_\mu t_\nu \phi_{f_1} s \phi_{f_2} s \ldots \phi_{f_{n-1}} s \phi_{f_n}) = \mathcal{J}(\gamma_\mu)\gamma_\mu(1) = \mu$$

iff $\sigma = t_\nu \phi_{f_1} s \phi_{f_2} s \ldots \phi_{f_{n-1}} s \phi_{f_n}$. $\square$

Proof of Theorem 1.1. *Step 1:* res *is a monomorphism.* It is obvious that

$$\deg \mathrm{res}(\sigma) = \deg \sigma, \;\; \sigma \in \mathrm{Aut}_K(A_1). \tag{13}$$

The map res is a group homomorphism, so we have to show that $\mathrm{res}(\sigma) = \mathrm{id}_Z$ implies $\sigma = \mathrm{id}_{A_1}$ where id_Z and id_{A_1} are the identity maps on Z and A_1 respectively. By (13), $\mathrm{res}(\sigma) = \mathrm{id}_Z$ implies $\deg(\sigma) = 1$. Then, by (11), $\sigma = \mathrm{id}_{A_1}$.

Step 2: $\Gamma \subseteq \mathrm{im(res)}$. By Corollary 3.3.(2), each automorphism $\sigma \in \Gamma$ is a product $\sigma = t_\nu \phi_{f_1} s \ldots \phi_{f_{n-1}} s \phi_{f_n}$. Since $\mathrm{res}(t_{\nu^{\frac{1}{p}}}) = t_\nu$, $\mathrm{res}(\phi_{\theta^{-1}(f_i)}) = \phi_{f_i}$ and $\mathrm{res}(s) = s$, we have $\sigma = \mathrm{res}(t_{\nu^{\frac{1}{p}}} \phi_{\theta^{-1}(f_1)} s \ldots \phi_{\theta^{-1}(f_{n-1})} s \phi_{\theta^{-1}(f_n)})$, and so $\Gamma \subseteq \mathrm{im(res)}$.

Step 3: $\Gamma = \mathrm{im(res)}$. Let $\sigma \in \mathrm{im(res)}$. By Corollary 3.3.(1),

$$\mathrm{res}(\sigma) = \gamma_\mu t_\nu \phi_{f_1} s \ldots \phi_{f_{n-1}} s \phi_{f_n} = \gamma_\mu \mathrm{res}(\tau)$$

for some $\tau \in \mathrm{Aut}_K(A_1)$ such that $\mathrm{res}(\tau) \in \Gamma$, by Step 2. Then $\mathrm{res}(\sigma\tau^{-1}) = \gamma_\mu$. By (13), $\deg(\sigma\tau^{-1}) = \deg \mathrm{res}(\sigma\tau^{-1}) = \deg \gamma_\mu = 1$, and so $\sigma\tau^{-1} \in \mathrm{Aff}(A_1)$. By Lemma 3.1, $\gamma_\mu = 1$, and so $\sigma = \tau$, hence $\mathrm{res}(\sigma) = \mathrm{res}(\tau) \in \Gamma$. This means that $\Gamma = \mathrm{im(res)}$. $\square$

If K is a *perfect* field of characteristic $p > 0$ we obtain the result of L. Makar-Limanov.

Theorem 3.4 *Let K be a perfect field of characteristic $p > 0$. Then the group $\mathrm{Aut}_K(A_1)$ is generated by $\mathrm{Aff}(A_1) \simeq \mathrm{SL}_2(K)^{op} \ltimes K^2$ and the automorphisms $\phi_{\lambda x^i}$, $\lambda \in K^*$, $i = 2, 3, \ldots$.*

Proof. By Theorem 1.1, the map res : $\mathrm{Aut}_K(A_1) \to \Gamma$ is the isomorphism of groups. By Corollary 3.3.(2), each element $\gamma \in \Gamma$ is a product

$$\gamma = t_\nu \phi_{f_1} s \ldots \phi_{f_{n-1}} s \phi_{f_n} = \mathrm{res}(t_{\nu^{\frac{1}{p}}} \phi_{\theta^{-1}(f_1)} s \ldots \phi_{\theta^{-1}(f_{n-1})} s \phi_{\theta^{-1}(f_n)}).$$

Now, it is obvious that the group $\mathrm{Aut}_K(A_1)$ is generated by $\mathrm{Aff}(A_1)$ and the automorphisms $\phi_{\lambda x^i}$, $\lambda \in K^*$, $i = 2, 3, \ldots$. $\Box$

The inverse map $\mathrm{res}^{-1} : \Gamma \to \mathrm{Aut}_K(A_1)$. By Corollary 3.3.(2), each element $\gamma \in \Gamma$ is a product $\gamma = t_\nu \phi_{f_1} s \ldots \phi_{f_{n-1}} s \phi_{f_n}$. By Proposition 2.2, the inverse map for res is given by the rule

$$\mathrm{res}^{-1} : \Gamma \to \mathrm{Aut}_K(A_1), \ \gamma = t_\nu \phi_{f_1} s \ldots \phi_{f_{n-1}} s \phi_{f_n} \mapsto t_{\nu^{\frac{1}{p}}} \phi_{\theta^{-1}(f_1)} s \ldots \phi_{\theta^{-1}(f_{n-1})} s \phi_{\theta^{-1}(f_n)}. \quad (14)$$

Proof of Corollary 1.2. The group $\mathrm{Aut}_K(A_1)$ (resp. $\mathrm{Aut}_K(Z)$) are infinite-dimensional algebraic groups over K (and over $\mathbb{F}_p$) where the coefficients of the polynomials $\sigma(x)$ and $\sigma(y)$ where $\sigma \in \mathrm{Aut}_K(A_1)$ (resp. of $\tau(X)$ and $\tau(Y)$ where $\tau \in \mathrm{Aut}_K(Z)$) are coordinate functions (see [10] and [11]). The group Γ is a closed subgroup of $\mathrm{Aut}_K(Z)$. By the very definition, the map res : $\mathrm{Aut}_K(A_1) \to \Gamma$ is a polynomial map (i.e. a morphism of algebraic varieties). By (14) and Proposition 2.2, res^{-1} is not a polynomial map over K (and over $\mathbb{F}_p$ as well). $\Box$

4 The image of the restriction map res_n

Let K be a field of characteristic $p > 0$ and $A_n = K\langle x_1, \ldots, x_{2n}\rangle$ be the nth Weyl algebra over K: for $i, j = 1, \ldots, n$,

$$[x_i, x_j] = 0, \ \ [x_{n+i}, x_{n+j}] = 0, \ \ [x_{n+i}, x_j] = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. The centre Z_n of the algebra A_n is the polynomial algebra $K[X_1, \ldots, X_{2n}]$ in $2n$ variables where $X_i := x_i^p$. The groups of K-automorphisms $\mathrm{Aut}_K(A_n)$ and $\mathrm{Aut}_K(Z_n)$ contain the affine subgroups $\mathrm{Aff}(A_n) = \mathrm{Sp}_{2n}(K)^{op} \ltimes K^n$ and $\mathrm{Aff}(Z_n) = \mathrm{GL}_n(K)^{op} \ltimes K^n$ respectively. Clearly, $\mathrm{Aff}(A_n) = \{\sigma \in \mathrm{Aut}_K(A_n) \,|\, \deg(\sigma) = 1\}$ and $\mathrm{Aff}(Z_n) = \{\tau \in \mathrm{Aut}_K(Z_n) \,|\, \deg(\tau) = 1\}$ where $\deg(\sigma)$ (resp. $\deg(\tau)$) is the (total) degree of σ (resp. τ) defined in the obvious way. The kernel Γ_n of the group epimorphism $\mathcal{J} : \mathrm{Aut}_K(Z_n) \to K^*$, $\tau \mapsto \mathcal{J}(\tau) := \det(\frac{\partial \tau(X_i)}{\partial X_j})$ is the normal subgroup $\Gamma_n := \{\tau \in \mathrm{Aut}_K(Z_n) \,|\, \mathcal{J}(\tau) = 1\}$, and $\mathrm{Aut}_K(Z_n) = K^* \ltimes \Gamma_n$ is the semi-direct product of $K^* \simeq \{\gamma_\mu \,|\, \gamma_\mu(X_1) = \mu X_1, \gamma_\mu(X_j) = X_j, j = 2, \ldots, 2n; \mu \in K^*\}$ and Γ_n.

By considering leading terms of the polynomials $\sigma(X_i)$, it follows as in the case $n = 1$ that the restriction map

$$\mathrm{res}_n : \mathrm{Aut}_K(A_n) \to \mathrm{Aut}_K(Z_n), \ \ \sigma \mapsto \sigma|_{Z_n},$$

is a group monomorphism. If K is a perfect field then

$$\mathrm{res}_n(\mathrm{Aff}(A_n)) = \mathrm{Sp}_{2n}(K)^{op} \ltimes K^{2n} \subset \mathrm{Aff}(Z_n) = \mathrm{GL}_{2n}(K)^{op} \ltimes K^{2n}.$$

This follows from the fact that for any element of $\mathrm{Aff}(A_n)$, $\sigma_{A,a} : x \mapsto Ax + a$ where $A = (a_{ij}) \in \mathrm{Sp}_{2n}(K)$ and $a = (a_i) \in K^{2n}$,

$$\mathrm{res}_n(\sigma_{A,a}) = \begin{cases} \sigma_{(a_{ij}^p),(a_i^p)} & \text{if } p > 2, \\ \sigma_{(a_{ij}^2),(a_i^2+\sum_{j=1}^n a_{ij}a_{i,n+j})} & \text{if } p = 2, \end{cases} \tag{15}$$

which can be proved in the same fashion as (11). Since $\mathrm{Sp}_{2n}(K) \subseteq \mathrm{SL}_{2n}(K)$ (any symplectic matrix $S \in \mathrm{Sp}_{2n}(K)$ has the from $S = TJT^{-1}$ for some matrix $T \in \mathrm{GL}_{2n}(K)$ where $J = \mathrm{diag}((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), \dots, (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))$, n times, hence $\det(S) = 1$),

$$\mathrm{res}_n(\mathrm{Aff}(A_n)) \subseteq \mathrm{SL}_{2n}(K)^{op} \ltimes K^{2n} \subset \Gamma_n.$$

Question 1. For an algebraically closed field K of characteristic $p > 0$, is $\mathrm{im}(\mathrm{res}_n) \subseteq \Gamma_n$*?*

Question 2. For an algebraically closed field K of characteristic $p > 0$, is the injection

$$\mathrm{Aff}(Z_n)/\mathrm{res}_n(\mathrm{Aff}(A_n)) \simeq \mathrm{GL}_{2n}(K)^{op}/\mathrm{Sp}_{2n}(K)^{op} \to \mathrm{Aut}_K(Z_n)/\mathrm{im}(\mathrm{res}_n)$$

a bijection?

The next corollary follows from Theorem 1.3.

Corollary 4.1 *Let K be a reduced commutative $\mathbb{F}_p$-algebra, $A_n(K)$ be the Weyl algebra, and $\partial_i := x_{n+i}$. Then*

$$(\partial_i + f)^p = \partial_i^p + \frac{\partial^{p-1} f}{\partial x_i^{p-1}} + f^p$$

for all polynomials $f \in K[x_1, \dots, x_n]$.

Proof. Without loss of generality we may assume that $i = 1$. Since $K[x_2, \dots, x_n]$ is a reduced commutative $\mathbb{F}_p$-algebra and $\partial_1 + f \in A_1(K[x_2, \dots, x_n])$, the result follows from Theorem 1.3. $\square$

References

[1] K. Adjamagbo and A. van den Essen, On equivalence of the Jacobian, Dixmier and Poisson Conjectures in any characteristic, arXiv:math. AG/0608009.

[2] A. Belov-Kanel and M. Kontsevich, The Jacobian conjecture is stably equivalent to the Dixmier Conjecture, *Mosc. Math. J.*, **7** (2007), no. 2, 209–218, arXiv:math. RA/0512171.

[3] A. Belov-Kanel and M. Kontsevich, Automorphisms of the Weyl algebra, *Lett. Math. Phys.*, **74** (2005), no. 2, 181–199, arXiv:math. RA/0512169.

[4] J. Dixmier, Sur les algèbres de Weyl, *Bull. Soc. Math. France*, **96** (1968), 209–242.

[5] H. W. E. Jung, Uber ganze birationale Transformationen der Eben, *J. Reiner Angew. Math.*, **184** (1942), 161–174.

[6] J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, 1975.

[7] W. Van der Kulk, On polynomial ring in two variables, *Niew. Arch. Wisk.*, **1** (1953), 33–41.

[8] L. Makar-Limanov, On automorphisms of Weyl algebra, *Bull. Soc. Math. France*, **112** (1984), 359–363.

[9] T. Kambayashi, Some results on pro-affine algebras and ind-affine schemes, *Osaka J. Math.*, **40** (2003), 621–638.

[10] I. R. Shafarevich, On some infinite-dimensional groups, *Rend. Mat. Appl.*, **25** (1966), 208-212.

[11] I. R. Shafarevich, On some infinite-dimensional groups-II, *Math. USSR-Izvestija*, **18** (1982), 185–194.

[12] Y. Tsuchimoto, Endomorphisms of Weyl algebra and p-curvatures. *Osaka J. Math.* **42** (2005), no. 2, 435–452.

Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk