COMPUTATION AND CHARACTERIZATION OF THE ZEROS OF LINEAR MULTIVARIABLE SYSTEMS

by

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Abstract

A natural extension of the results of Kouvaritakis and MacFarlane on the calculation of multivariable zeros is described.

Kouvaritakis and MacFarlane[1,2] have suggested a conceptually simple numerical technique for the calculation of the invariant zeros of the ℓ-input/m-output linear, time-invariant left-invertible system S(A,B,C),

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t) \in R^n$$
$$y(t) = Cx(t) \quad , \quad y(t) \in R^m \quad , \quad u(t) \in R^\ell \tag{1}$$

when, in particular, $m \geq \ell$, rank $B = \ell$, rank $C = m$. The purpose of this note is to describe a natural extension of their results that can make possible a reduction in dimension of the defining relationships.

Let k^* be the uniquely defined integer ≥ 1 such that

$$C A^{i-1} B = 0 \quad , \quad 1 \leq i \leq k^*-1 \quad , \quad C A^{k^*-1} B \neq 0 \tag{2}$$

Lemma 1

The subspaces

$$V_o \triangleq \{0\} \quad , \quad V_k \triangleq R(B) \oplus AR(B) \oplus .. \oplus A^{k-1}R(B) \quad (k \geq 1) \tag{3}$$

have dimension $k\ell$ for k in the range $0 \leq k \leq k^*$. By suitable choice of subspace W, the state space has the direct sum decomposition

$$R^n = V_{k^*} \oplus \nu^* \oplus W \tag{4}$$

where ν^* is the maximal {A,B}-invariant subspace in the kernel of C. Sufficient conditions for W = {0} are that $m = \ell$ and $|CA^{k^*-1}B| \neq 0$.

Proof

Follows directly from theorem 5 in ref. 3.

This fundamental lemma leads to the following construction

Lemma 2

Let $1 \le k \le k^*$ and, for each complex scalar λ, define

$$S_k(\lambda) \triangleq \{x \in \bigcap_{i=1}^{k} \ker CA^{i-1} : (\lambda I_n - A)x \in V_k\} \tag{5}$$

Then

$$S_k(\lambda) = \omega_1(\lambda) \oplus \{ V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\}\} \tag{6}$$

where $\quad \omega_1(\lambda) = \{(\lambda I_n - A)^{-1} R(B)\} \cap \ker C.$

Proof

It is known[3] that $\omega_1(\lambda) \subset \nu^*$ and, using equation (2), $\nu^* \subset \bigcap_{i=1}^{k} \ker CA^{i-1}$. By definition $(\lambda I_n - A)\omega_1(\lambda) \subset V_1 \subset V_k$ from which we obtain $\omega_1(\lambda) \subset S_k(\lambda)$. It is also easily verified that $(\lambda I_n - A) V_{k-1} \subset V_k$ and hence that

$$\omega_1(\lambda) \oplus \{V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\}\} \subset S_k(\lambda) \tag{7}$$

(the sum being direct by lemma one). Now take $x \in S_k(\lambda)$ and write

$$(\lambda I_n - A)x = B\alpha_1 + AB\alpha_2 + \ldots + A^{k-1}B\alpha_k \tag{8}$$

This expression can be written in the form

$$(\lambda I_n - A)(x + A^{k-2}B\alpha_k) = B\alpha_1 + AB\alpha_2 + \ldots + A^{k-2}B(\alpha_{k-1} + \lambda\alpha_k)$$

and hence (by induction) in the form

$$(\lambda I_n - A)(x-b) \in R(B)$$

for some $b \in V_{k-1}$ ie $x-b \in \omega_1(\lambda)$. In particular, it is seen that $b \in S_k(\lambda)+\omega_1(\lambda) = S_k(\lambda)$ from which

$$b \in V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\}$$

and hence

$$x \in \omega_1(\lambda) \oplus \{ V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\}\}$$

which reverses the inclusion in equation (7). This proves the lemma.

We obtain immediately the following geometric characterization of the invariant zeros of $S(A,B,C)$.

Theorem 1

Let $1 \le k \le k^*$. Then the complex number λ is an invariant zero of $S(A,B,C)$ if, and only if,

$$\dim S_k(\lambda) > \dim V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\} \qquad (9)$$

Proof

Follows directly from lemma 2 noting[3] that λ is a zero if, and only if, $\omega_1(\lambda) \neq \{0\}$.

This geometric formulation can be converted into algebraic relations paralleling those of Kouvaritakis and MacFarlane by the following constructions. Define the matrices

$$B_k \triangleq [B, AB, .., A^{k-1}B] \quad , \quad C_k \triangleq \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} \tag{10}$$

when, taking $1 \le k \le k^*$, lemma one indicates that we can define full rank matrices N_k, M_k of dimension $(n-k\ell)$xn and nx$(n-\text{rank } C_k)$ respectively satisfying the relations

$$N_k B_k = 0 \quad , \quad C_k M_k = 0 \tag{11}$$

Lemma 3

rank $B_k = k\ell \le$ rank C_k , $1 \le k \le k^*$, equality holding if $m = \ell$.

Proof

rank $B_k = k\ell$, $1 \le k \le k^*$, follows directly from lemma one. If $m = \ell$ then a dual argument on the system $S(A^T, C^T, B^T)$ yields rank $C_k = km$ as required. Finally, taking $m > \ell$ and choosing an ℓxm matrix K such that the square system $S(A,B,KC)$ is invertible, we see that

$$k\ell = \text{rank} \begin{pmatrix} KC \\ KCA \\ \vdots \\ KCA^{k-1} \end{pmatrix} \le \text{rank } C_k \quad , \quad 1 \le k \le k^*$$

as required.

Defining, for notational convenience, $N_o = I_n$, we now prove the following main result of this paper.

Theorem 2

The complex number λ is an invariant zero of $S(A,B,C)$ if, and only if,

$$\text{rank } N_k(\lambda I_n - A)M_k < \text{rank } N_{k-1}M_k \tag{12}$$

for any $1 \leq k \leq k^*$.

Proof

The proof is obtained by evaluation of the expressions occurring in equation (9). Firstly, we see from the definitions and lemma three that

$$\dim S_k(\lambda) = \dim \{x \in \bigcap_{i=1}^{k} \ker CA^{i-1} : N_k(\lambda I - A)x = 0\}$$

$$= \dim \ker N_k(\lambda I_n - A)M_k$$

$$= \text{rank } M_k - \text{rank } N_k(\lambda I_n - A)M_k \tag{13}$$

Secondly, we can, without loss of generality, assume that $M_k = [X_k, Y_k]$ where the columns of X_k are a basis for $V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\}$ and $\ker Y_k = \{0\}$. It follows directly that $\text{rank } N_{k-1}M_k = \text{rank } N_{k-1}Y_k$. Consider now the relation $N_{k-1}Y_k\beta = 0$. We see immediately that $Y_k\beta \in V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\}$ ie $Y_k\beta = 0$ (from the definition of X_k and Y_k) and hence $\beta = 0$. It follows that

$$\text{rank } N_{k-1}M_k = \text{rank } M_k - \dim V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\} \tag{14}$$

The theorem follows by combining equations (9), (13) and (14).

The results of Kouvaritakis and MacFarlane[1-3] are obtained by setting $k = 1$ when equation (12) reduces to

$$\text{rank } N_1(\lambda I_n - A)M_1 < \text{rank } M_1 = n - m \tag{15}$$

or, in the important case of $m = \ell$ when (Lemma 3) the matrices are square, we recover the well-known relation[1]

$$|\lambda N_1 M_1 - N_1 A M_1| = 0 \quad (16)$$

For those systems with $k^* > 1$, the choice of $k > 1$ represents a reduction in dimension of the relationships defining the zeros.

The rank conditions in equation (12) can be simplified when $m = \ell$ and $|CA^{k^*-1}B| \neq 0$. In this case, applying lemma three rank M_k = rank $N_k = n-km$, $1 \leq k \leq k^*$. Noting that (using equation (2) and lemma one)

$$V_{k-1} \cap \{\bigcap_{i=1}^{k} \ker CA^{i-1}\} = V_{\min\{(k-1),(k^*-k)\}} \quad (17)$$

and that $\dim V_k = km$, $1 \leq k \leq k^*$, then a combination of equations (9) and (13) indicates that λ is a zero of $S(A,B,C)$ if, and only if,

$$\text{rank } N_k(\lambda I_n - A)M_k < n-km-\min\{(k-1),(k^*-k)\}m \quad (18)$$

In particular, if $k = k^*$ this reduces to the $(n-k^*m)\times(n-k^*m)$ determinental relationship

$$z_o(\lambda) \triangleq |\lambda N_{k^*} M_{k^*} - N_{k^*} A M_{k^*}| = 0 \quad (19)$$

As $|CA^{k^*-1}B| \neq 0$, then it is easily verified that $N_{k^*}M_{k^*}$ is non-singular and hence that, if N_{k^*} is replaced by $\bar{N}_{k^*} = (N_{k^*}M_{k^*})^{-1}N_{k^*}$ in equation (19), then the invariant zeros are the <u>eigenvalues</u> of $\bar{N}_{k^*}AM_{k^*}$. This result is a direct generalization of the result[1] that the invariant zeros are the eigenvalues of $N_1 A M_1$ if $|CB| \neq 0$ and we choose N_1 such that $N_1 M_1 = I_{n-m}$.

Finally, note that equation (19) will also give the correct algebraic multiplicity of each zero. To prove this, note that the above discussion indicates that $z_o(\lambda)$ has degree equal to $n-k^*m$. The decomposition of lemma one makes possible the consideration of arbitrarily small perturbations δ to A such that $\delta\nu^* \subset \nu^*$ and $\delta[V_{k^*}] = \{0\}$. It is easily verified that the number of zeros including multiplicities (ie[3] $\dim \nu^* = n-k^*m$) is unaffected by any such perturbation, that the invariant zeros vary continuously with δ and that both N_{k^*} and M_{k^*} are unchanged. The invariant zeros are hence defined by the relation

$$z_\delta(\lambda) \triangleq |\lambda N_{k^*}M_{k^*} - N_{k^*}(A+\delta)M_{k^*}|$$

$$\equiv |N_{k^*}M_{k^*}|\cdot|\lambda I_{n-k^*m} - \bar{N}_{k^*}(A+\delta)M_{k^*}|$$

$$= 0 \qquad (20)$$

Note that the degree of the polynomial $n-k^*m$ is independent of δ and that the $n-k^*m$ zeros are distinct for 'almost all' choices of δ. A simple continuity argument now proves that $z_o(s)$ provides the correct algebraic multiplicity.

To illustrate the results, consider the system defined by

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 3 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 1 & 2 & 1 & -1 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & -4 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (21)$$

The system has one zero at the point $\lambda = -4$ and $k^* = 3$.

Consider now the application of equation (12) to the calculation of the zero. Take initially $k = 2$, then $C_1 = C$,

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B_2^T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad (22)$$

yielding

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$N_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_2^T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (23)$$

Substituting into equation (12) indicates that λ is a zero if, and only if,

$$\text{rank}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \lambda+4 \end{bmatrix} < \text{rank}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (24)$$

ie $\lambda = -4$ is the only zero. Considering now the case of $k = k^* = 3$ we obtain

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_3^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (25)$$

Substituting into equation (12) indicates that λ is a zero if, and only if,

$$\text{rank}\begin{bmatrix} 0 & 0 \\ 0 & \lambda+4 \end{bmatrix} < \text{rank}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (26)$$

again verifying that $\lambda = -4$ is the only zero of the system.

References

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(3) D.H. Owens, 'Invariant zeros of linear multivariable systems: a geometric analysis', Int. J. Contr., Vol.26, No.4, pp.537-548, 1977.