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On the Manipulation of Optimal System

Asymptotic Root-loci

by

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Abstract

This note outlines a procedure for manipulating the asymptotic directions of the optimal closed-loop poles of a time-invariant linear regulator as the weight of the input in the performance criterion approaches zero.

It is known[1,2] that the stabilizable and detectable ℓ-input/m-output time-invariant linear system S(A,B,C) with state feedback controller minimizing

$$J = \int_0^\infty \{y^T(t)Qy(t) + p^{-1}u^T(t)R\, u(t)\}dt \qquad \ldots(1)$$

(where both Q and R are symmetric positive definite and $p>0$) has closed-loop poles equal to the left-half plane solutions of the equation

$$|I_\ell + p\, G^T(-s)G(s)| = 0 \qquad \ldots(2)$$

where, if $Q^{\frac{1}{2}}$ and $R^{\frac{1}{2}}$ the symmetric, positive-definite square-roots of Q and R respectively

$$G(s) = Q^{\frac{1}{2}}C(sI_n - A)^{-1}BR^{-\frac{1}{2}} \qquad \ldots(3)$$

Attention has been focussed[1-5] on the unbounded solutions of equation (3) as $p\to+\infty$, which take the form[2]

$$s_{j\ell r}(p) = p^{1/2k_j}\eta_{j\ell r} + \alpha_{jr} + \varepsilon_{j\ell r}(p)$$

$$\lim_{p\to+\infty} \varepsilon_{j\ell r}(p) = 0 \;,\; 1 \le \ell \le k_j \;,\; 1 \le r \le d_j \;,\; 1 \le j \le q \qquad \ldots(4)$$

for suitable choice of integers q, k_j and d_j, $1\le j\le q$. Each α_{jr} is pure imaginary and the $\eta_{j\ell r}$, $1\le\ell\le k_j$, take the form $\lambda_{jr}\mu_{j\ell}$ where λ_{jr}

is real and strictly positive and the $\mu_{j\ell}$, $1\leq\ell\leq k_j$, are the distinct left-half-plane $2k_j$th roots of $(-1)^{k_j+1}$.

It is the purpose of this note to point out a systematic method for the systematic modification of the Q and R matrices to provide the required asymptotic properties. More precisely, for the case of $m\geq\ell$ and $S(A,B,C)$ left-invertible, we consider the systematic modification of the R matrix to change the asymptotic parameters λ_{jr}, $1\leq r\leq d_j$, $1\leq j\leq q$, into 'desired' parameters $\tilde{\lambda}_{jr}$, $1\leq r\leq d_j$, $1\leq j\leq q$. The results represent a generalization of recent work[6] from the case of $m = \ell$ and $|CB| \neq 0$ to the case defined above.

The following lemma is fundamental:

Lemma 1: Equation (2) remains valid if $G(s)$ is replaced by

$$\tilde{G}(s) \quad = \quad Q^{\frac{1}{2}}\, C(sI_n-A)^{-1}BV \qquad \ldots(5)$$

where V is any matrix such that $VV^T = R^{-1}$.

Proof: It is easily verified that $VV^T = R^{-1}$ if, and only if, $V = R^{-\frac{1}{2}}U$ for some orthogonal matrix U. The result then follows from the identity $|I_\ell+pG^T(-s)G(s)| \equiv |I_\ell+p\tilde{G}^T(-s)\tilde{G}(s)|$.

We also need the following construction:

Lemma 2:[2] There exists a real orthogonal transformation T_1 and a unimodular polynomial matrix of the form

$$M(s) \quad = \quad \begin{pmatrix} I_{d_1} & 0(s^{-1}) & \cdots & \cdots & 0(s^{-1}) \\ 0 & I_{d_2} & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & 0(s^{-1}) \\ 0 & \cdots & \cdots & 0 & I_{d_q} \end{pmatrix} \qquad \ldots(6)$$

such that

$$M^T(-s)T_1^T G^T(-s)G(s)T_1 M(s) = \text{block diag}\{Q_j(s)\}_{1\le j\le q} + O(s^{-(2k_q+2)}) \qquad \ldots(7)$$

where the $d_j x d_j$ transfer function matrices $Q_j(s)$ have uniform rank[7,8] $2k_j$ and take the form $N_j^T(-s)N_j(s)$ for some mxd_j left-invertible transfer function matrices $N_j(s)$, $1\le j\le q$.

In fact, applying known techniques[7], the characterization of equation (4) follows quite simply[2]. In particular, the following result is easily proved:

<u>Lemma 3</u>: The real, strictly positive numbers $\lambda_{jr}^{2k_j}$, $1\le r\le d_j$, are the eigenvalues of the real, symmetric positive-definite matrix

$$Q_j^{(2k_j)} \triangleq \lim_{|s|\to\infty} s^{2k_j} Q_j(s)(-1)^{k_j} \qquad \ldots(8)$$

Consider now the real constant nonsingular matrix

$$L = \text{block diag}\ \{L_j\}_{1\le j\le q} \qquad \ldots(9)$$

where the nonsingular matrices L_j have dimensions $d_j x d_j$, $1\le j\le q$. Multiplying equation (7) from the left and right by L^T and L respectively yields

$$\tilde{M}^T(-s)T_1^T(T_1 L^T T_1^T G^T(-s)G(s)T_1 L T_1^T)T_1\tilde{M}(s)$$

$$= \text{block diag}\ \{L_j^T Q_j(s)L_j\}_{1\le j\le q} + O(s^{-(2k_q+2)}) \qquad \ldots(10)$$

where $\tilde{M}(s)$ (defined by $M(s)L \equiv L\tilde{M}(s)$) has the same structure as $M(s)$.

In fact, we obtain the following main result of this paper:

Theorem: If $\tilde{G}(s) \triangleq G(s)T_1LT_1^T$, then the solutions of the relation

$$|I_\ell + p\tilde{G}^T(-s)\tilde{G}(s)| = 0 \qquad \ldots(11)$$

characterize the stability of S(A,B,C) with state feedback controller minimizing the performance criterion of equation (1) with R replaced by R_o where

$$R_o^{-1} \triangleq R^{-\frac{1}{2}}T_1LL^TT_1^TR^{-\frac{1}{2}} \qquad \ldots(12)$$

Moreover, the unbounded solutions of (11) have the form of (4) but where, in particular, the parameters λ_{jr}, $1\le r\le d_j$, $1\le j\le q$, are replaced by the real, strictly positive parameters $\tilde{\lambda}_{jr}$, $1\le r\le d_j$, $1\le j\le q$. The real, strictly positive numbers $\tilde{\lambda}_{jr}^{2k_j}$, $1\le r\le d_j$, are the eigenvalues of

$$\tilde{Q}_j^{(2k_j)} \triangleq L_j^T Q_j^{(2k_j)} L_j \quad , \qquad \ldots(13)$$

$1\le j\le q$.

Proof: The first part of the result follows from the definition of G and $\tilde{G}$, bearing in mind lemma 1. Equation (10) then implies that $\tilde{G}$ satisfies lemma 2 with M and Q_j, $1\le j\le q$, replaced by $\tilde{M}$ and $\tilde{Q}_j = L_j^TQ_jL$, $1\le j\le q$. Standard results[2] then indicate that the general characterization of equation (4) remains valid with (lemma 3) λ_{jr}, $1\le r\le d_j$, replaced by the eigenvalues of $\lim_{s\to\infty} s^{2k_j}L_j^TQ_j(s)L_j = L_j^TQ_j^{(2k_j)}L_j$.

The theorem provides an explicit method for manipulation of the asymptotic directions of the optimal root-locus. For example, suppose that a given choice of Q and R yield infinite zeros with, in particular, parameters λ_{jr}, $1 \leq r \leq d_j$, $1 \leq j \leq q$. These can be obtained by application of known numerical algorithms[2,7] to compute (amongst other things) the matrix T_1 and the Markov parameters $Q_j^{(2k_j)}$, $1 \leq j \leq q$, and subsequent application of lemma 3. Suppose that it is desired that the parameters $\{\lambda_{jr}\}$ be replaced by $\tilde{\lambda}_{jr}$, $1 \leq r \leq d_j$, $1 \leq j \leq q$. Write,

$$Q_j^{(2k_j)} = U_j \operatorname{diag} \{\lambda_{jr}^{2k_j}\}_{1 \leq r \leq d_j} U_j^T , \quad 1 \leq j \leq q \quad \ldots(14)$$

where U_j is the orthogonal eigenvector matrix of $Q_j^{(2k_j)}$ and set

$$L_j = U_j \operatorname{diag} \{\tilde{\lambda}_{jr}^{k_j} / \lambda_{jr}^{k_j}\}_{1 \leq r \leq d_j} W_j^T , \quad 1 \leq j \leq q \quad \ldots(15)$$

where W_j is a real orthogonal matrix, $1 \leq j \leq q$. It is trivially verified that

$$\tilde{Q}_j^{(2k_j)} = W_j \operatorname{diag} \{\tilde{\lambda}_{jr}^{2k_j}\}_{1 \leq r \leq d_j} W_j^T , \quad 1 \leq j \leq q \ldots(16)$$

and hence, by the theorem, that the desired objective has been achieved.

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