Monograph:

Owens, D.H. (1979) On the Inversion of Uniform Rank Multi-Variable Systems. Research Report. ACSE Research Report 104 . Department of Automatic Control and Systems Engineering

ON THE INVERSION OF UNIFORM RANK MULTIVARIABLE SYSTEMS

by

D. H. Owens B.Sc., A.R.C.S., Ph.D., A.F.I.M.A., C.Eng., M.I.E.E.

Department of Control Engineering,
University of Sheffield,
Mappin Street,
Sheffield S1 3JD.

Research Report No. 104

November 1979

Abstract

A simple numerical procedure is presented for the inversion of a system S(A,B,C) satisfying the constraints $CA^{i-1}B = 0$, $1 \leqslant i \leqslant k-1$, and $|CA^{k-1}B| \neq 0$ for some $k \geqslant 1$. The results generalize the work of Kouvaritakis for the case of $k = 1$.

The inversion of the m-input-m-output strictly proper, linear, time-invariant system $S(A,B,C)$ of state dimension n is of great theoretical and practical interest[1,2]. Although general techniques are available[3,4], great simplification in computational procedures are possible in special cases[5] of practical interest. The purpose of this note is to extend these results to the case of a mxm square system of uniform rank k (see, for example, ref. 2). That is a system satisfying the relations,

$$\begin{aligned} CA^{i-1}B &= 0 \qquad 1 \leqslant i \leqslant k-1 \\ |CA^{k-1}B| &\neq 0 \end{aligned} \tag{1}$$

for some integer $k \geqslant 1$. In this case it is trivially verified that the inverse of the system transfer function matrix $G(s) = C(sI_n - A)^{-1}B$ takes the form

$$G^{-1}(s) = s^k A_o + s^{k-1}A_1 + \ldots + s\,A_{k-1} + A_k + A_oH(s) \tag{2}$$

where $H(s)$ is strictly proper and $|A_o| \neq 0$. The calculation of the inverse system hence reduces to the calculation of $A_o, A_1, \ldots, A_k$ and $H(s)$.

Without loss of generality[7] suppose that $S(A,B,C)$ has the special form

$$A = \begin{pmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & & \vdots & \vdots \\ \vdots & & & & 0 & 0 \\ \vdots & & & 0 & I_m & 0 \\ -A_o^{-1}A_k & \cdots & \cdots & -A_o^{-1}A_2 & -A_o^{-1}A_1 & -C_2 \\ B_2 & 0 & \cdots & 0 & 0 & A_2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0_{(k-1)m\times m} \\ A_o^{-1} \\ 0_{(n-km)\times m} \end{pmatrix} \qquad C = \begin{bmatrix} I_m & & 0 \end{bmatrix} \tag{3}$$

by choice of state of the form $x^T(t) = [y^T(t), \dot{y}^T(t), \ldots \overset{(k-1)}{y}{}^T(t), z^T(t)]$.

It is easily verified that, in this basis,

$$H(s) = C_2 (sI_{n-km} - A_2)^{-1} B_2 \qquad (4)$$

$$C A^{i-1} = [0_{mx(i-1)m} \; I_m \quad 0] \quad , \quad 1 \leq i \leq k \qquad (5)$$

$$C A^k = - [A_o^{-1} A_k, \ldots, A_o^{-1} A_1, C_2] \qquad (6)$$

$$C A^{k-1} B = A_o^{-1} \qquad (7)$$

Following recent results[6], it is convenient to define the <u>full rank</u>[7] matrices

$$C_k = \begin{pmatrix} C \\ C A \\ \vdots \\ C A^{k-1} \end{pmatrix} \quad , \quad B_k = [B, AB, \ldots, A^{k-1} B] \qquad (8)$$

In particular, in the defined basis, these matrices take the form

$$C_k = [I_{km} \quad 0] \quad , \quad B_k = \begin{pmatrix} C_k B_k \\ 0 \end{pmatrix} \qquad (9)$$

and hence, by combination with (6) and (7), the relation

$$[A_k \; A_{k-1} \ldots A_2 \; A_1] = - (CA^{k-1}B)^{-1} C A^k B_k (C_k B_k)^{-1} \qquad (10)$$

together with (7) defines the matrices $A_o, A_1, \ldots, A_k$ uniquely. In fact, the right-hand side of (10) and $CA^{k-1} B$ are independent of basis and hence can be used directly without the need to transform to the special form defined by (3).

Consider now the construction[6] of full rank nx(n-km) and (n-km)xn annihilators M_k and N_k respectively, satisfying the relations

$$C_k M_k = 0 \quad , \quad N_k B_k = 0 \quad , \quad N_k M_k = I_{n-km} \qquad (11)$$

In the defined basis, we can always choose (eqn. (5)) $M_k^T = [0 \quad I_{n-km}] = N_k$ when a simple calculation yields the relations

$$C_2 = - C A^k M_k \quad , \quad B_2 = N_k A^k B (C A^{k-1} B)^{-1}$$
$$A_2 = N_k A M_k \qquad (12)$$

and hence, using (4),

$$H(s) = - C A^k M_k (s I_{n-km} - N_k A M_k)^{-1} N_k A^k B (C A^{k-1} B)^{-1} \qquad (13)$$

Again this relation is basis independent and independent of the precise choice of N_k and M_k satisfying equation (11) as can be verified by noting that transformations of the form $A \to T^{-1} A T$, $B \to T^{-1} B$, $C \to C T$ 'induce' the 'transformations' $N_k \to L N_k T$, $M_k \to T^{-1} M_k L^{-1}$ where the matrix L is nonsingular.

In summary, we have proved the following basis independent result

---

Theorem

Let S(A,B,C) have uniform rank k and define full rank matrices C_k and B_k by equation (8). The system inverse transfer function matrix then takes the form given in equation (2) with

$$A_o = (C A^{k-1} B)^{-1} \qquad (14)$$

$$[A_k, A_{k-1}, \ldots, A_1] = -(C A^{k-1} B)^{-1} C A^k B_k (C_k B_k)^{-1} \qquad (15)$$

and

$$H(s) = - C A^k M_k (sI_{n-km} - N_k A M_k)^{-1} N_k A^k B (C A^{k-1} B)^{-1} \qquad (16)$$

where the annihilators N_k, M_k are any solutions of the relations

$$C_k M_k = 0 \quad , \quad N_k B_k = 0 \quad , \quad N_k M_k = I_{km} \qquad (17)$$

This result reduces to previous work[5] in the case of k=1. It also has the same overall structure of previous work with the simple operations involved being easily implemented on a digital computer.

References

(1) H.H. Rosenbrock: 'Computer-aided control systems design', Academic Press, 1974.

(2) D.H. Owens: 'Feedback and multivariable systems', Peter Peregrinus, 1978.

(3) L.M. Silverman: 'Inversion of multivariable linear systems' I.E.E.E. Trans. AC - 14, 3, 1969, pp. 270-276.

(4) G. Bengtsson: 'Minimal system inverses for linear multivariable systems', J. Math. Anal., Appl., 1974, 46, pp. 261-274.

(5) B. Kouvaritakins: 'A geometric approach to the inversion of multivariable systems', Int. J. Control, 1976, 24, 5, pp. 609-626.

(6) D.H. Owens: 'On the computation and characterization of the zeros of linear multivariable systems', Proc. I.E.E., to appear.

(7) D.H. Owens: 'Invariant zeros of linear multivariable systems: a geometric analysis', Int. J. Control, 1977, 26, 4, pp. 537-548.