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Optimality in the frequency domain with cross-product weighting

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Summary

The frequency domain representation of the steady-state matrix Riccati equation which defines the solution of the linear optimal control problem is illustrated with a cross-product weighting term included in the quadratic performance index. Conditions which have been applied to the magnitude of the return-difference matrix in the complex plane[2] are shown to apply similarly with such weighting.

Introduction The linear optimal control problem with an infinite-time quadratic performance index is associated with a steady-state matrix Riccati equation, and can also be defined in terms of an equivalent equation in the frequency domain[1-3]. Conditions imposed by the steady-state matrix Riccati equation have been given in terms of the magnitude of the determinant of the return-difference function in the complex frequency plane, for both the single-[1] and multiple-input[2] cases with quadratic weighting of the states and inputs. A similar condition is now investigated with the performance index containing cross-product weighting of the states and inputs, which can be associated with the minimisation of a power function for the physical system[4]. Similar weighting also appears in the model following problem[5]. It can then be shown that the previous condition on the magnitude of the return-difference matrix in the frequency plane applies similarly with cross-product weighting.

Linear optimal control with cross-product weighting. Consider the problem of determining the control vector u(t) in the n-state, r-input, m-output linear system

$$\mathring{x}(t) = Ax + Bu , \qquad y = Cx \tag{1}$$

which minimises the quadratic performance functional

$$J = \int_{t_o}^{T_f} f_o(y(t),u(t))dt , \quad f_o = (y^tQy + u^tRu + 2y^tWu)/2 \tag{2}$$

where Q is an mxm positive semi-definite matrix and R is an rxr positive definite matrix. A scalar product of the state and control variables is included with the mxr matrix W. In the maximum principle a Hamiltonian is defined

$$H(p,x,u) = p_o f_o + p^t f \tag{3}$$

with $\quad \mathring{x} = \partial H/\partial p , \qquad \mathring{p} = -\partial H/\partial x$

For the linear system with quadratic performance and appropriate boundary conditions

$$H = -f_o + p^t(Ax + Bu) , \quad f_o = (x^tC^tQCx + u^tRu + 2x^tC^tWu)/2 \tag{4}$$

$$\mathring{p} = C^tQCx + C^tWu - A^tp \tag{5}$$

Then differentiation of eqn 4 with respect to u gives

$$u(t) = R^{-1}(B^tp - W^tCx) \tag{6}$$

The optimal Hamiltonian system may be represented in terms of the state and adjoint variables by the 2n-dimensional differential equations

$$\begin{bmatrix} \mathring{x}(t) \\ \mathring{p}(t) \end{bmatrix} = \begin{bmatrix} (A - BR^{-1}W^tC) , & BR^{-1}B^t \\ C^t(Q-WR^{-1}W^t)C , & -(A^t-C^tWR^{-1}B^t) \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \quad \text{or } \mathring{h}(t) = Mh(t) \tag{7}$$

The optimally regulated trajectory is then given by the solution of eqn 7 with the two-point boundary conditions, $x(t_o) = 0$, $p(T_f) = 0$. Including a state-adjoint variable relation $p(t) = -Px(t)$ in eqn 7 gives the nonlinear matrix Riccati differential equation

$$\mathring{P} = (PB + C^tW)R^{-1}(B^tP + W^tC) - PA - A^tP - C^tQC \tag{8}$$

where the nxn symmetrical matrix P is a unique positive definite solution for all positive definite matrices $\bar{Q} = C^t(Q - WR^{-1}W^t)C$.

A frequency-domain representation of the infinite-time linear optimal control problem associated with the performance functional of eqn 2 may now be developed using the steady-state matrix Riccati equation from eqn 8, which may be stated in the form

$$-P\bar{A} - \bar{A}^t P + PBR^{-1}B^t P = \bar{Q} \qquad (9)$$

where $\bar{A} = A - BR^{-1}W^t C$, $\bar{Q} = C^t(Q - WR^{-1}W^t)C$

Then by the usual procedure,[1]

$$P(sI - \bar{A}) + (-sI - \bar{A}^t)P = \bar{Q} - PBR^{-1}B^t P \qquad (10)$$

and multiplying from the left by $B^t(-sI - \bar{A}^t)^{-1}$ and from the right by $(sI - \bar{A})^{-1}B$ gives

$$B^t(-sI-\bar{A}^t)^{-1}PB + B^t P(sI-\bar{A})^{-1}B = B^t(-sI-\bar{A}^t)^{-1}C^t(Q-WR^{-1}W^t)C(sI-\bar{A})^{-1}B$$
$$- B^t(-sI-\bar{A}^t)^{-1}PBR^{-1}B^t P(sI-\bar{A})^{-1}B \qquad (11)$$

Now define the 'plant' and dual-system transfer matrices

$$\bar{G}(s) = C(sI - \bar{A})^{-1}B , \quad \bar{G}^t(-s) = B^t(-sI - \bar{A}^t)^{-1}C^t \qquad (12)$$

and including the control-law matrix from eqn 6

$$K = R^{-1}(B^t P + W^t C) \qquad (13)$$

for eliminating P gives

$$B^t(-sI - \bar{A}^t)^{-1}(K^t R - C^t W) + (RK - W^t C)(sI - \bar{A})^{-1}B \qquad (14)$$
$$= \bar{G}^t(-s)(Q-WR^{-1}W^t)\bar{G}(s) - B^t(-sI-\bar{A}^t)^{-1}(K^t R-C^t W)R^{-1}(RK-W^t C)(sI-\bar{A})^{-1}B$$

Then

$$\bar{G}^t(-s)(Q-WR^{-1}W^t)\bar{G}(s)+R = [I + B^t(-sI-\bar{A}^t)^{-1}(K^t-C^t WR^{-1})]R[I+(K-R^{-1}W^t C)(sI-\bar{A})^{-1}B] \qquad (15)$$

or $$\bar{G}^t(-s)(Q - WR^{-1}W^t)\bar{G}(s) + R = \bar{F}^t(-s)R\bar{F}(s) \qquad (16)$$

where $\bar{F}(s)$ defines a return-difference-type matrix for the optimally controlled system. Eqn 16 represents the condition for optimality in the frequency domain and includes nxr nonlinear algebraic equations which can, in principle, be used for determining the elements of the control matrix K. However, the solution will usually be difficult and not unique.

The transfer matrices $\bar{G}(s)$ and $\bar{G}^t(-s)$ may be related to the conventional plant transfer matrices $G(s) = C(sI-A)^{-1}B$ and $G^t(-s) = B^t(-sI-A^t)^{-1}C^t$ by expanding the forms of eqn 12 using the matrix inversion identity, to give

$$\bar{G}(s) = C[(sI-A) + (B)R^{-1}(W^t C)]^{-1}B = G(s) - G(s)[R+W^t G(s)]^{-1}W^t G(s) \qquad (17)$$
$$\bar{G}^t(-s) = G^t(-s) - G^t(-s)W[R + G^t(-s)W]^{-1}G^t(-s) \qquad (18)$$

Now the combined optimally regulated system matrix M of eqn 7 has a characteristic equation given by

$$|sI - M| = \begin{vmatrix} sI-\bar{A} , & -BR^{-1}B^t \\ -\bar{Q} , & sI+\bar{A}^t \end{vmatrix} \qquad (19)$$

Then using the result for partitioned determinants[6]

$$\begin{vmatrix} P & Q \\ R & S \end{vmatrix} = |P - QS^{-1}R||S| \qquad (20)$$

we obtain

$$|sI-M| = |(sI-\bar{A}) - BR^{-1}B^t(sI+\bar{A}^t)^{-1}\bar{Q}||sI+\bar{A}^t| \qquad (21)$$

$$= |I - BR^{-1}B^t(sI+\bar{A}^t)^{-1}C^t(Q-WR^{-1}W^t)C(sI-\bar{A})^{-1}||sI-\bar{A}||sI+\bar{A}^t| \qquad (22)$$

$$= |I + R^{-1}G^t(-s)(Q-WR^{-1}W^t)G(s)||sI-\bar{A}||sI+\bar{A}^t| \qquad (23)$$

Then from eqn 16

$$\frac{|sI-M|}{|sI-\bar{A}||sI+\bar{A}^t|} = |\bar{F}^t(-s)||\bar{F}(s)| = L(s^2) \qquad (24)$$

where $L(s^2)$ is a scalar function of the squared complex frequency s^2. The eigenvalues of the matrix M are symmetrically disposed about the imaginary axis and, for $s = j\omega$, the function L is real for all ω. Also, the magnitude $|\det \bar{F}(j\omega)|$ is real, thus illustrating the existence of a zero quadrature component of phase shift in the Hamiltonian system, with the dual system acting as a phase compensator, as previously discussed.[2] The form of the optimally controlled system with cross-product weighting is shown in FIG 1 which illustrates, particularly, the corresponding roles of the matrices W^t and W in the system and dual system respectively.

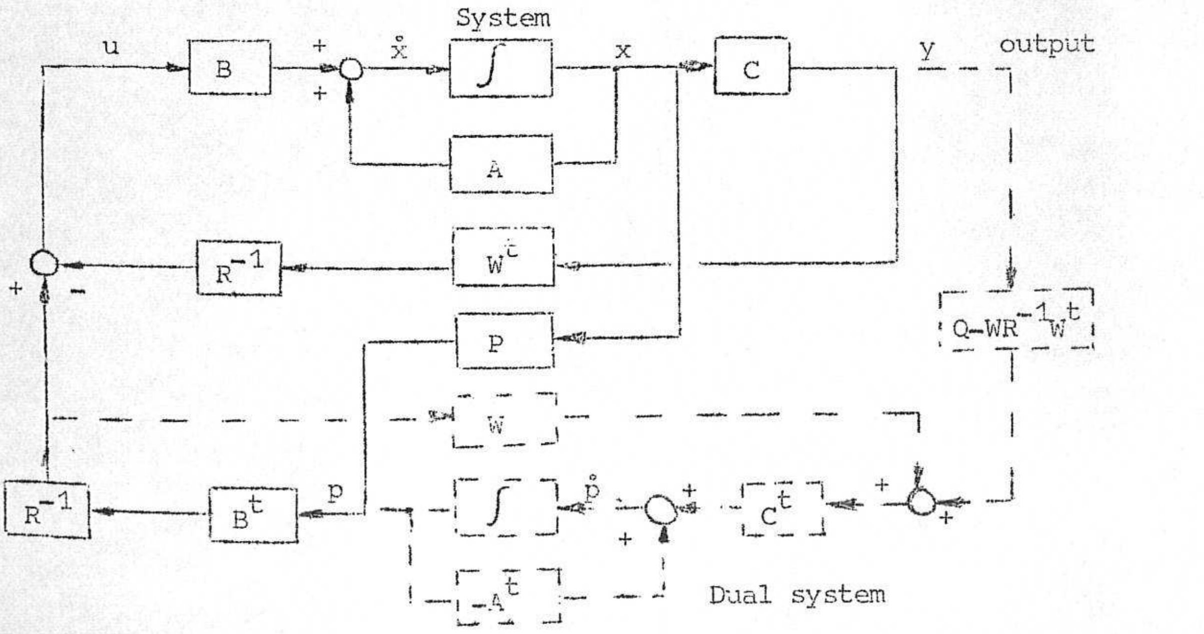

FIG 1. Optimally controlled linear system including cross-product weighting

Now since det $\bar{F}(s)$, and also det $\bar{G}(s)$, are complex quantities

$$\left|\det \bar{F}(s)\right|^2 = \left|\det \bar{F}(s) \det \bar{F}^t(-s)\right| \qquad (25)$$

For the system without cross-product weighting ($W = 0$), eqn 25 thus leads to the condition that $\left|\det F(j\omega)\right|^2$ ($= \det[F(j\omega)F^t(-j\omega)]$) is always a positive number for all ω, and, as in the single variable case[1], $\left|\det F(j\omega)\right| > 1$ for all ω. For the system with $W \neq 0$, this condition will also exist with a positive definite matrix $\bar{Q}$. For example, consider the system

$$\mathring{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

with $V = \frac{1}{2}\int_0^\infty (x^t x + u^t u + 2x^t Wu)dt$, $Q = I$, $R = I$, $C = I$, $W = \mathrm{diag}[w_i]$

Then

$$\bar{A} = \begin{bmatrix} -w_1 & 1 \\ 0 & -w_2 \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} 1-w_1^2 & 0 \\ 0 & 1-w_2^2 \end{bmatrix}$$

$$\bar{G}(s) = (sI-\bar{A})^{-1} = \begin{bmatrix} \frac{1}{s+w_1}, & \frac{1}{(s+w_1)(s+w_2)} \\ 0 , & \frac{1}{s+w_2} \end{bmatrix}, \quad \bar{G}^t(-s) = \begin{bmatrix} -\frac{1}{s-w_1} & , & 0 \\ \frac{1}{(s-w_1)(s-w_2)}, & -\frac{1}{s-w_2} \end{bmatrix}$$

$$\bar{F}(s) = \begin{bmatrix} \frac{s+k_{11}}{s+w_1} , & \frac{k_{11}-w_1+k_{12}(s+w_1)}{(s+w_1)(s+w_2)} \\ \frac{k_{21}}{s+w_1} , & \frac{s^2+s(w_1+k_{22})+k_{21}+k_{22}w_1}{(s+w_1)(s+w_2)} \end{bmatrix}$$

$$\bar{F}^t(-s) = \begin{bmatrix} \frac{s-k_{11}}{s-w_1} , & \frac{-k_{21}}{s-w_1} \\ \frac{k_{11}-w_1-k_{12}(s-w_1)}{(s-w_1)(s-w_2)} , & \frac{s^2-s(w_1+k_{22})+k_{21}+k_{22}w_1}{(s-w_1)(s-w_2)} \end{bmatrix}$$

$$\det \bar{F}(s) = [s^2 + s(k_{11} + k_{22}) + k_{11}k_{22} + k_{21} - k_{21}k_{12}]/[(s + w_1)(s + w_2)]$$
$$\det \bar{F}^t(-s) = [s^2 - s(k_{11} + k_{22}) + k_{11}k_{22} + k_{21} - k_{21}k_{12}]/[(s - w_1)(s - w_2)]$$

Then for the optimally controlled combined system

$$M = \begin{bmatrix} -w_1 & 1 & 1 & 0 \\ 0 & -w_2 & 0 & 1 \\ 1-w_1^2 & 0 & w_1 & 0 \\ 0 & 1-w_2^2 & -1 & w_2 \end{bmatrix}$$

giving the characteristic equation

$$|sI-M| = \begin{vmatrix} s+w_1 & -1 & -1 & 0 \\ 0 & s+w_2 & 0 & -1 \\ w_1^2-1 & 0 & s-w_1 & 0 \\ 0 & w_2^2-1 & 1 & s-w_2 \end{vmatrix} = s^4 - 2s^2 + 2 - w_1^2$$

Thus the eigenvalues for the particular combined system are not affected by the cross-product component w_2. Also

$$|sI-\bar{A}| = (s+w_1)(s+w_2), \qquad |sI+\bar{A}^t| = (s-w_1)(s-w_2)$$

and $\det[R + \bar{G}^t(-s)\bar{Q}\bar{G}(s)] = \det[\bar{F}^t(-s)R\bar{F}(s)] = (s^4 - 2s^2 + 2-w_1^2)/[(s^2-w_1^2)(s^2-w_2^2)]$

which checks with eqn 24. Then

$$\det[\bar{F}^t(-j\omega)\bar{F}(j\omega)] = 1 + [\omega^2(2-w_1^2-w_2^2)+(2-w_1^2-w_1^2w_2^2)]/[(\omega^2+w_1^2)(\omega^2+w_2^2)]$$

Thus the magnitude condition $|\det F(j\omega)|^2 > 1$ will exist for all ω with the elements of W (in present ex. $0 \leq w_i^2 < 1$) ensuring that $\bar{Q}$ is positive definite, and, for optimality with cross-product weighting, the plot for $\det F(j\omega)$ does not enter the interior of the unit circle.

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