Monograph:
Nicholson, H. (1974) System Concepts in Kron's Polyhedron Model and the Scattering Problem. Research Report. ACSE Research Report 26 . Department of Automatic Control and Systems Engineering

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System concepts in Kron's polyhedron model and the scattering problem

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Research Report No. 26

May 1974

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Summary

Kron's polyhedron model based on a sequence of transformations related to the orthogonal electrical network problem was proposed for a wide range of system problems. The scattering formulation for a flow process introduces a similar sequence of 'obstacles' and can form an analytical basis for many of the concepts discussed by Kron. It can be associated with the system problems incorporating an optimality condition and the updating of a priori information, and can also be identified in the sweep method of solution of the two-point boundary value problems for optimal control and estimation, which would appear to be related to the concept of wave propagation in Kron's model. The sequential networks in the polyhedron model and the scattering problem thus provide a basic analytical structure which has important applications in general system theory.

1. Introduction

Kron developed the concept of the complete orthogonal network incorporating nonsingular transformations in his pioneering work on the generalised theory of rotating electrical machines and stationary electrical networks, in which the overall system is related to a single primitive form. Although the electrical network problem had been studied extensively, Kron appeared to be the first to develop a complete formulation including the symmetry of the orthogonal network solution. This then provided a basic framework for the development of the method of tearing and the polyhedron model. The significance of the orthogonal network defined in terms of closed- and open-path variables has not been widely accepted and the broad concepts associated with the polyhedron model representing a multidimensional space filter have been largely unexplored, possibly because of the lack of an adequate analytical basis which does not appear in the published work. However, Kron's work represents an important contribution which now appears to be particularly relevant in general systems theory.

Kron proposed a wide field of application for the polyhedron model which consists essentially of a set of orthogonal network structures connected sequentially according to the geometrical properties of points, lines, surfaces and volume elements. Electrical, electromagnetic, thermodynamic, fluid flow, chemical and other types of physical wave phenomena were associated with the model, together with a procedure for obtaining an improved least squares estimate, similar to the process of updating a priori information within a sequential recursive algorithm.

The general scattering problem, concerned with propagation through media containing obstacles or distributed constants and characterised by the effects of incident and reflected waves, has been shown to possess similar properties[1,2]. It also provides an algebraic framework related to a sequential network structure which supports the validity of many of the concepts associated with the polyhedron model. The interconnection of obstacles corresponds to the procedure for updating a priori information in the least squares estimation problem, and can be associated with a process of smoothing which also exists in the polyhedron model. The properties of a star product define the connection of adjacent obstacles in the scattering problem and in the continuum the system is represented by a set of quadratic and linear matrix differential equations. A similar star product and differential equation set are also associated with the linear two-point boundary value problems concerned with optimal control and estimation. The sweep method of solution combining a forward and reverse integration or filtering procedure as in the optimal smoothing problem, would also appear to be relevant to the concept of multidimensional wave propagation in the polyhedron model. The scattering problem and Kron's model thus possess similar basic features which are of fundamental importance, particularly in the study of system structure.

The present work illustrates the significance of Kron's polyhedron model and the scattering problem, and the resulting sequential forms are also shown to exist in the linear quadratic control and estimation problems. It is believed that the concepts associated with the interconnection of mutually interacting subsystems or sequential networks will stimulate the development of generalised solutions and computational techniques required in the analysis of large scale system problems.

2. Kron's polyhedron model[3-9]

Kron's polyhedron network model or 'automaton' was proposed for a wide range of system problems, concerned for example with multidimensional curve fitting, the phenomena in molecules and crystals, brain modelling and crystal computers. The model was endowed with self-adaptive properties and assumed to accommodate many physical concepts concerned with multidimensional 'generalised' electrical machines, thermodynamics, fluid flow and statistical phenomena.

The model consists essentially of a sequence of multidimensional networks, each represented by the solution of an electrical network defined with closed- and open-paths, which Kron called an orthogonal network. The adjacent networks are interconnected using boundary operators and incidence matrices related to points, lines, surfaces and volume elements forming a multidimensional space. The resulting topological structure can accommodate superimposed electrical and electromagnetic variables, and the geometrical properties of the surfaces passing through the given data points provide additional information based on physical laws, which may be used particularly for obtaining improved curve fitting.

2.1 The orthogonal electrical network problem[1] The general electrical network problem includes the connection of b-primitive branches specified by

$$E + e = Z(I + i) \quad \text{or} \quad V = ZJ \tag{1}$$

where Z is a symmetrical impedance matrix for the primitive network and e,I represent branch voltage and current source vectors respectively. In the orthogonal formulation, the branch connections are defined in terms of square, nonsingular connection matrices

$$C = \begin{bmatrix} C_c & C_o \end{bmatrix} \quad , \quad A = \begin{bmatrix} A^c & A^o \end{bmatrix} = (C^T)^{-1}$$

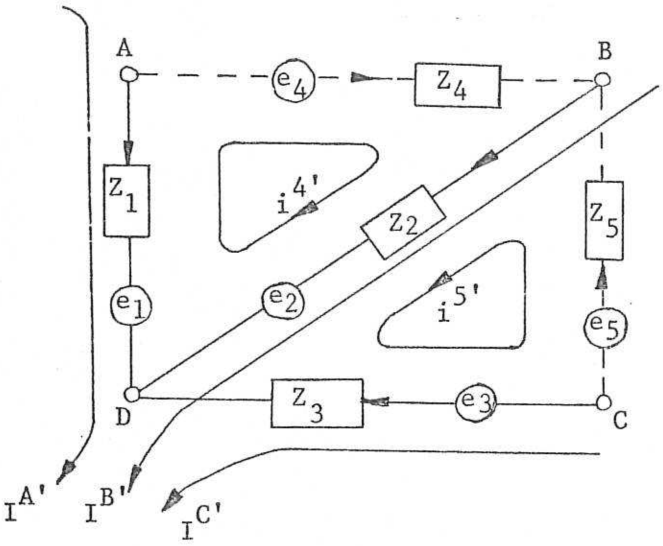
related to specified closed- and open-paths containing variables $i^{c'}$, $e_c{}'$ and $I^{o'}$, $E_o{}'$ respectively. In the connected network

$$J = b\overset{\quad m \quad\; p}{\begin{bmatrix} C_c & C_o \end{bmatrix}} \begin{bmatrix} i^{c'} \\ I^{o'} \end{bmatrix} \qquad V = b\overset{\quad m \quad\; p}{\begin{bmatrix} A^c & A^o \end{bmatrix}} \begin{bmatrix} e_c{}' \\ E_o{}' \end{bmatrix} \tag{2}$$

and with specified trees and links

$$C = \begin{array}{c} \\ T \\ L \end{array} \begin{array}{c} \begin{array}{cc} c & o \end{array} \\ \begin{bmatrix} C_T & B_T \\ C_L & 0 \end{bmatrix} \end{array} \qquad A = \begin{array}{c} \\ T \\ L \end{array} \begin{array}{c} \begin{array}{cc} c & o \end{array} \\ \begin{bmatrix} 0 & A_T \\ \delta_m & A_L \end{bmatrix} \end{array}$$

A typical set of paths and connection matrices for the unit-tree, unit-link case are illustrated in FIG 1.



$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 4' & 5' & A' & B' & C' \end{array} \\ \left[\begin{array}{cc|ccc} -1 & . & 1 & . & . \\ 1 & 1 & . & 1 & . \\ . & -1 & . & . & 1 \\ \hline 1 & . & . & . & . \\ . & 1 & . & . & . \end{array}\right] \end{array}$$

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 4' & 5' & A' & B' & C' \end{array} \\ \left[\begin{array}{cc|ccc} . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \\ \hline 1 & . & 1 & -1 & . \\ . & 1 & . & -1 & 1 \end{array}\right] \end{array}$$

FIG 1 Network specified with two closed- and three open-paths

The branch variables E,i and the tree-branch voltages $E_o{}'$ and mesh currents $i^{c'}$ are related by

$$E = \begin{bmatrix} E_T \\ E_L \end{bmatrix} = A^o E_o{}' \qquad i = \begin{bmatrix} i^T \\ i^L \end{bmatrix} = C_c i^{c'}$$

The branch variables are also constrained by the Kirchoff laws

$$C_c^T E = 0 \qquad (A^o)^T i = 0 \qquad E^T i = 0$$

New equivalent sources e_L, I^T are then referred to the links and open-paths or tree branches respectively, with

$$\tilde{e} = \begin{bmatrix} e_T \\ e_L \end{bmatrix} = A^c e_c' = \begin{bmatrix} 0 \\ C_c^T e \end{bmatrix} , \quad \tilde{I} = \begin{bmatrix} I^T \\ I^L \end{bmatrix} = C_o I^{o'} = \begin{bmatrix} B_T (A^o)^T I \\ 0 \end{bmatrix}$$

where $\quad e_c' = C_c^T e \qquad I^{o'} = (A^o)^T I$

represent equivalent induced mesh-voltage and nodal-current sources respectively, with arbitrary sources e,I. Combining eqns 1 and 2 and rearranging them gives the solutions

$$\begin{bmatrix} i^{c'} \\ E_o' \end{bmatrix} = \begin{bmatrix} -Z_1^{-1} Z_2 & Z_1^{-1} \\ Z_4 - Z_3 Z_1^{-1} Z_2 & Z_3 Z_1^{-1} \end{bmatrix} \begin{bmatrix} I^{o'} \\ e_c' \end{bmatrix} = \begin{bmatrix} Y_2 Y_4^{-1} & Y_1 - Y_2 Y_4^{-1} Y_3 \\ Y_4^{-1} & -Y_4^{-1} Y_3 \end{bmatrix} \begin{bmatrix} I^{o'} \\ e_c' \end{bmatrix} = N \begin{bmatrix} I^{o'} \\ e_c' \end{bmatrix} \tag{3}$$

where $\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = C^T Z C = \bar{Z} \quad , \quad \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} = A^T Y A = \bar{Y}$

and with $Z = \mathrm{diag}(Z_T, Z_L)$, $Y = \mathrm{diag}(Y_T, Y_L)$

$$Z_1^{-1} = (Z_L + A_L Z_T A_L^T)^{-1} \qquad Y_4^{-1} = (Y_T + A_L^T Y_L A_L)^{-1}$$

The usual mesh- and node-solutions are included in eqn 3 with

$$i = L(e - Z\tilde{I}) \qquad E = M(I - Y\tilde{e})$$

where $\quad L = C_c (C_c^T Z C_c)^{-1} C_c^T , \quad M = A^o [(A^o)^T Y A^o]^{-1} (A^o)^T = Z - ZLZ$

represent the branch-admittance and branch-impedance matrices respectively. The matrices M,L are significant in many linear system problems, and in the particular form possess properties similar to those of the generalised inverse matrix[10].

Equation 3 represents the orthogonal formulation of the electrical network problem, and a similar form is used to characterise each 'isolated' higher-dimensional network in the polyhedron model, as illustrated in FIG 2. Incidence matrices M_{i+1}^{i} interconnect the spatial elements in adjacent networks, such as branches with nodes or planes, and are related to the boundary

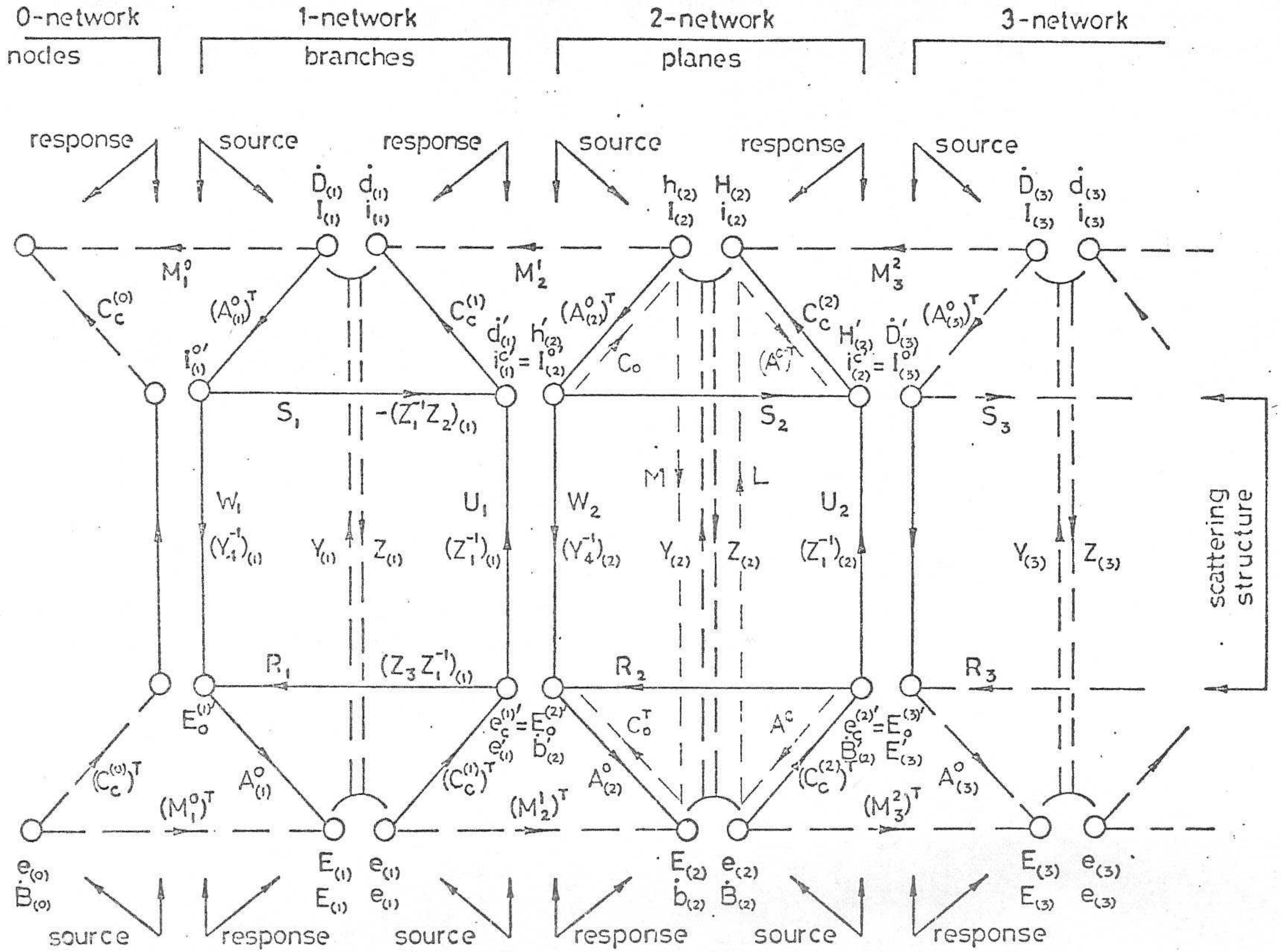


FIG 2 Transformation diagram for sequence of orthogonal networks

operators or partial connection matrices C_c, A^o for the i- and i+1-networks, with

$$C_c^{(i)}(A^o_{(i+1)})^T = M^i_{i+1} \;,\; (C_c^{(i)})^T A^o_{(i)} = 0 \;,\; M^{i-1}_i M^i_{i+1} = 0 \;,\; L_i(M^{i-1}_i)^T = 0$$

The transformation diagram includes residual-type operators directed horizontally between similar closed- and open-path variables, with impedance/admittance-type projection operators directed vertically between the closed- and open-path dual variables.

2.2 Wave propagation in the polyhedron model Kron introduced a concept of wave propagation into the polyhedron model, with a transverse electromagnetic wave associated with closed-path dielectric and magnetic variables $(\dot{d}_{(i)}, e_{(i)})$ and $(h_{(i)}, \dot{b}_{(i)})$ respectively, and a longitudinal wave associated with open-path dielectric and magnetic variables $(\dot{D}_{(i)}, E_{(i)})$ and $(H_{(i)}, \dot{B}_{(i)})$ respectively. Propagation proceeds from the 0-dimensional points to the higher-dimensional network elements, and induces electrical and electromagnetic variables, as shown in FIG 2, consistent with the form of Maxwell's field equations. A cycle of 'open-circuit' wave propagation repeats after each two dimensions, and is represented by the general steps

$$\dot{b}_{(i+1)} + \dot{B}_{(i+1)} = Z_{(i+1)}(M^i_{i+1})^{\dagger} Y_{(i)}(M^{i-1}_i)^T \left[\dot{b}_{(i-1)} + \dot{B}_{(i-1)}\right] \;,\; \dot{b}_{(o)} = 0, \quad i = 1,3...$$

where $(M^i_{i+1})^{\dagger}$ represents an equivalent inverse defined by

$$(M^i_{i+1})^{\dagger} = \left[C_c^{(i)}(A^o_{(i+1)})^T\right]^{\dagger} \equiv C_o^{(i+1)}(A^c_{(i)})^T$$

which can be identified with the operation curl^{-1}.

Wave propagation induces physical variables with properties which can be associated with higher-order divided differences[5,6,8,9], representing generalisations of similar quantities used in the calculus of finite differences. These are proposed for obtaining improved curve fitting with nonlinear functions, although the precise relevance of these variables is not discussed in detail and requires further investigation. Similar properties may be derived using the scattering representation of a flow process, and this also provides an analytical basis for many of the physical concepts discussed by Kron. The scattering formulation includes inherently the effects of interaction between coupled obstacles or 'networks', compared to the 'open-circuit' propagation across the polyhedron with a 'pathway' defined by the incidence matrices M^i_{i+1}, which apparently avoids the necessity for considering such interaction.

3. The scattering problem[1,2]

The interconnection of physical components will in general introduce an effect of mutual interaction or feedback, with the subsystems being influenced by and reacting with the adjacent subsystems. The scattering problem, concerned with the interconnection of 'obstacles' in a flow process defined in terms of incident and reflected variables, introduces similar effects of reaction within a combined scattering matrix relating the input and output variables. The representation has application to energy transfer and wave propagation in inhomogeneous media, and is particularly relevant in network and transmission line theory, neutron diffusion and radiative transfer.

The problem is represented by the connection of two obstacles, as in FIG 3.

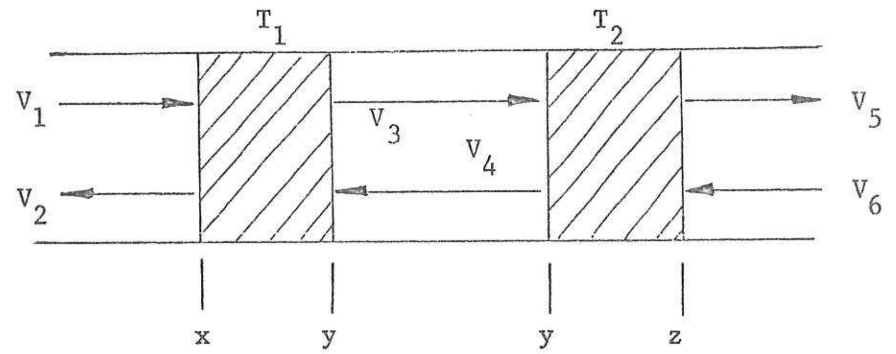


FIG 3 Propagation through adjacent obstacles

The n-component reflected and incident waves are related by

$$\begin{bmatrix} V_3 \\ V_2 \end{bmatrix} = T_1 \begin{bmatrix} V_1 \\ V_4 \end{bmatrix}, \quad \begin{bmatrix} V_5 \\ V_4 \end{bmatrix} = T_2 \begin{bmatrix} V_3 \\ V_6 \end{bmatrix}, \quad T_j = \begin{bmatrix} S_j & U_j \\ W_j & R_j \end{bmatrix}, \; j = 1,2 \qquad (4)$$

where T_j defines the scattering matrix for the jth obstacle, and $S_j(x,y)$, $R_j(x,y)$ and $U_j(x,y)$, $W_j(x,y)$ are nxn transmission and reflection matrix functions, with spatial coordinates (x,y). The system variables are also illustrated in the transformation diagram of FIG 4.

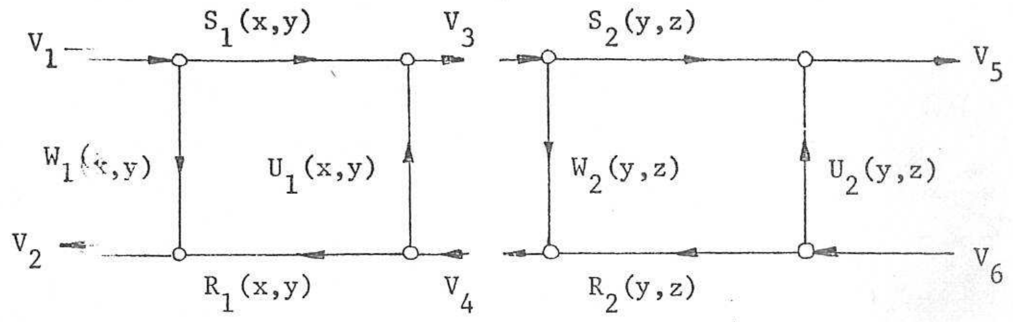


FIG 4 Transformation diagram representing the scattering process

Eliminating the connecting variables in eqn 4 then gives the combined system response

$$\begin{bmatrix} V_5 \\ V_2 \end{bmatrix} = \begin{bmatrix} S_2(\delta - U_1W_2)^{-1}S_1, & U_2+S_2U_1(\delta - W_2U_1)^{-1}R_2 \\ W_1+R_1W_2(\delta - U_1W_2)^{-1}S_1, & R_1(\delta - W_2U_1)^{-1}R_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_6 \end{bmatrix} = T_1{*}T_2 \begin{bmatrix} V_1 \\ V_6 \end{bmatrix} \quad (5)$$

where $T_1{*}T_2$ defines the star product or combined scattering matrix for two adjacent obstacles. For a series of obstacles, the continued star product is given by $T = T_1{*}T_2{*}..{*}T_n$. The input and output variables may also be related by

$$\begin{bmatrix} V_2 \\ V_1 \end{bmatrix} = \begin{bmatrix} R_1 - W_1S_1^{-1}U_1, & W_1S_1^{-1} \\ -S_1^{-1}U_1 & , S_1^{-1} \end{bmatrix} \begin{bmatrix} V_4 \\ V_3 \end{bmatrix} = L_1 \begin{bmatrix} V_4 \\ V_3 \end{bmatrix}$$

with a combined transformation given by the continued matrix product $L = L_1L_2..L_n$.

The matrix L can be identified with the orthogonal network matrix N of eqn 3 with the scattering matrix T corresponding to $\bar{Z}$ and $\bar{Y}$. The matrix N also decomposes into a star product with

$$N = \begin{bmatrix} Z_2 & 0 \\ Z_4 & Z_3 \end{bmatrix} * \begin{bmatrix} -Z_1^{-1} & Z_1^{-1} \\ -Z_1^{-1} & Z_1^{-1} \end{bmatrix} = \begin{bmatrix} C_T^TZ_T & 0 \\ Z_T & Z_TC_T \end{bmatrix} * \begin{bmatrix} -Z_1^{-1} & Z_1^{-1} \\ -Z_1^{-1} & Z_1^{-1} \end{bmatrix} \quad (6)$$

or

$$N = \begin{bmatrix} Y_4^{-1} & -Y_4^{-1} \\ Y_4^{-1} & -Y_4^{-1} \end{bmatrix} * \begin{bmatrix} Y_2 & Y_1 \\ 0 & Y_3 \end{bmatrix} = \begin{bmatrix} Y_4^{-1} & -Y_4^{-1} \\ Y_4^{-1} & -Y_4^{-1} \end{bmatrix} * \begin{bmatrix} Y_LA_L & Y_L \\ 0 & A_L^TY_L \end{bmatrix}$$

The star product can thus represent the form of an orthogonal network solution and eqn 6 illustrates a basis for interconnecting network tree and link elements.

The components of the combined star product and corresponding elements of the orthogonal network sequence of FIG 2 are illustrated in FIG 5. The contributions to the response variables are indicated, together with the unit-block and residue-type transformations $(\delta - U_i W_{i+1})^{-1}$ representing the connecting zone, which also introduces the properties of a return-difference operator. Kron's orthogonal network sequence can thus be represented as a scattering problem, although it may appear that the solution does not introduce inherently the properties of a return difference operator, with U_i or $W_{i+1} = 0$. This would also be supported by Kron's reference to the polyhedron model as an open-circuited structure of networks. However, the orthogonal network solution can be considered to contain a return difference effect within the off-diagonal elements of N, and thus includes the ability to incorporate a priori information (as Y_L or Z_T). In the scattering problem, this effect and the return difference operator are introduced into each successive stage of the multi-stage process by the effects of the connecting zone between adjacent obstacles, in contrast to the connection of adjacent orthogonal networks in the polyhedron model based on the incidence matrices M^i_{i+1}.

The solution of the scattering problem for adjacent obstacles extends in the continuum to form a set of functional equations which generalise to a matrix differential system given by

$$T_y(x,y) = \begin{bmatrix} S_y & U_y \\ W_y & R_y \end{bmatrix} = \begin{bmatrix} (D+UC)S, & A+DU+UB+UCU \\ RCS & R(B+CU) \end{bmatrix} \qquad (7)$$

where $T_y \equiv \partial T/\partial y$ and A,B,C,D are complex nxn matrix functions of the real variable y, given by $(T_y)_o = \begin{bmatrix} D & A \\ C & B \end{bmatrix}$ and $T(x,x) = \delta_{2n}$. The projection matrices U,W are obtained as solutions of a Riccati-type equation and a quadrature and possess properties of covariance or admittance/impedance-type operators. Matrices S,R are obtained as solutions of linear differential equations with coefficients depending on U. A linear matrix differential equation is also defined in the scattering problem, with

$$H_y = \begin{bmatrix} -B & -C \\ A & D \end{bmatrix} H(x,y) \; , \quad H = \begin{bmatrix} R^{-1}, & -R^{-1}W \\ UR^{-1}, & S-UR^{-1}W \end{bmatrix} \; , \quad H(x,x) = \delta_{2n}$$

Also $H^{-1} = L$, and the components of H can be identified with the orthogonal network solution. A similar structure appears in a variational problem defined by an integral in the dependent variable and its derivative given by

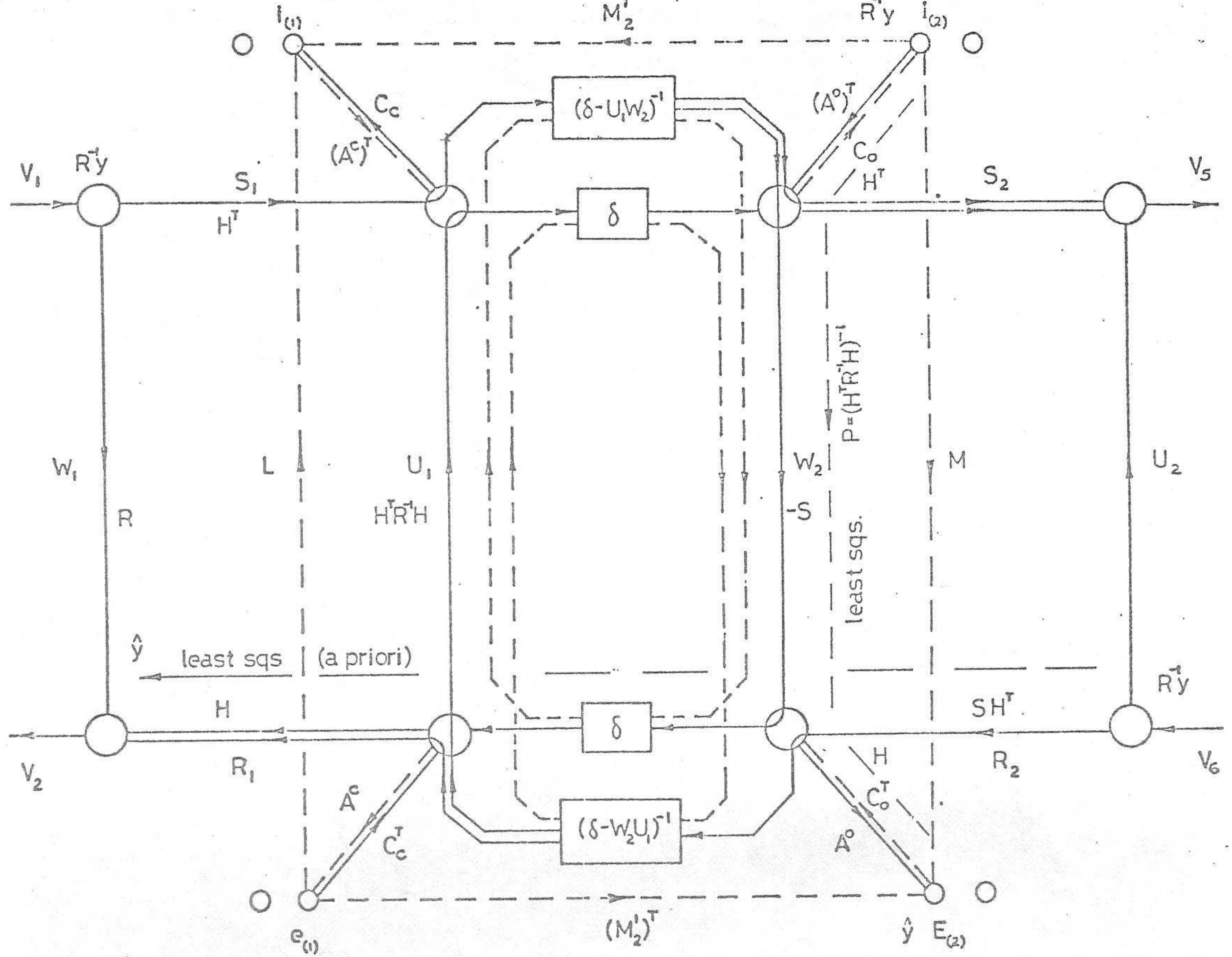


FIG 5 Internal structure of combined scattering matrix

$$\int_{y_1}^{y_2} J dy \ , \quad J = \begin{bmatrix} \alpha' \\ \alpha \end{bmatrix}^T \begin{bmatrix} F & E \\ E^T & G \end{bmatrix} \begin{bmatrix} \alpha' \\ \alpha \end{bmatrix} = \begin{bmatrix} \alpha' \\ \alpha \end{bmatrix}^T \begin{bmatrix} \beta \\ \beta' \end{bmatrix}$$

The definition of variables β,β' represents an L-type operator which transforms to a scattering T-form to give the Euler differential equation in the canonical variables α,β

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} -F^{-1}E & F^{-1} \\ G-E^TF^{-1}E & E^TF^{-1} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

A similar formulation appears in the linear two-point boundary value problems for optimal control and estimation, which also contain the matrix differential system of eqn 7. These problems introduce a constraint of minimum quadratic performance which exists inherently in the electrical network problem as a minimum power condition, and such a condition also appears in the formulation of the scattering problem.

3.1 <u>The scattering process as a lattice-type structure</u> The component transformations in the scattering representation may also be formed into a lattice-type structure as illustrated in FIG 6. The lattice cross connections define the input to output through-transformations, and with the interconnection of adjacent lattice-type 'obstacles' as in the scattering problem, the overall structure appears as an interwoven double helix consisting of the transformations $R_n..R_2,R_1$ and $S_1,S_2..S_n$ with interconnections $U_1,W_2,U_2,W_3\cdots$. The overall lattice illustratesparticularly the significance of the interconnecting zones, with alternate clockwise and anticlockwise transformations which introduce the return-difference-type operators $(\delta - U_iW_{i+1})$. They are also associated inherently with the process of updating a priori information into the adjacent stages and with the method of tearing and interconnection.

It is of interest to note that the lattice sequence appears to have a similar form to the double helix model for the DNA molecule[33], with U_i,W_{i+1} forming a core of transformations between the coiled vertical transformations $R_n..R_1$ and $S_1..S_n$ in reverse directions. A coupled lattice-type model with reverse cross connections and reciprocal feedback repressor paths representing a sequence of undamped nonlinear oscillators, has also been used to represent the dynamic properties of a control mechanism for macromolecular synthesis in cells[32]. The modelling of continuous biochemical control systems in cells and the pattern of molecular interactions requires a highly organised and integrated spatial-time structure and a formalism of

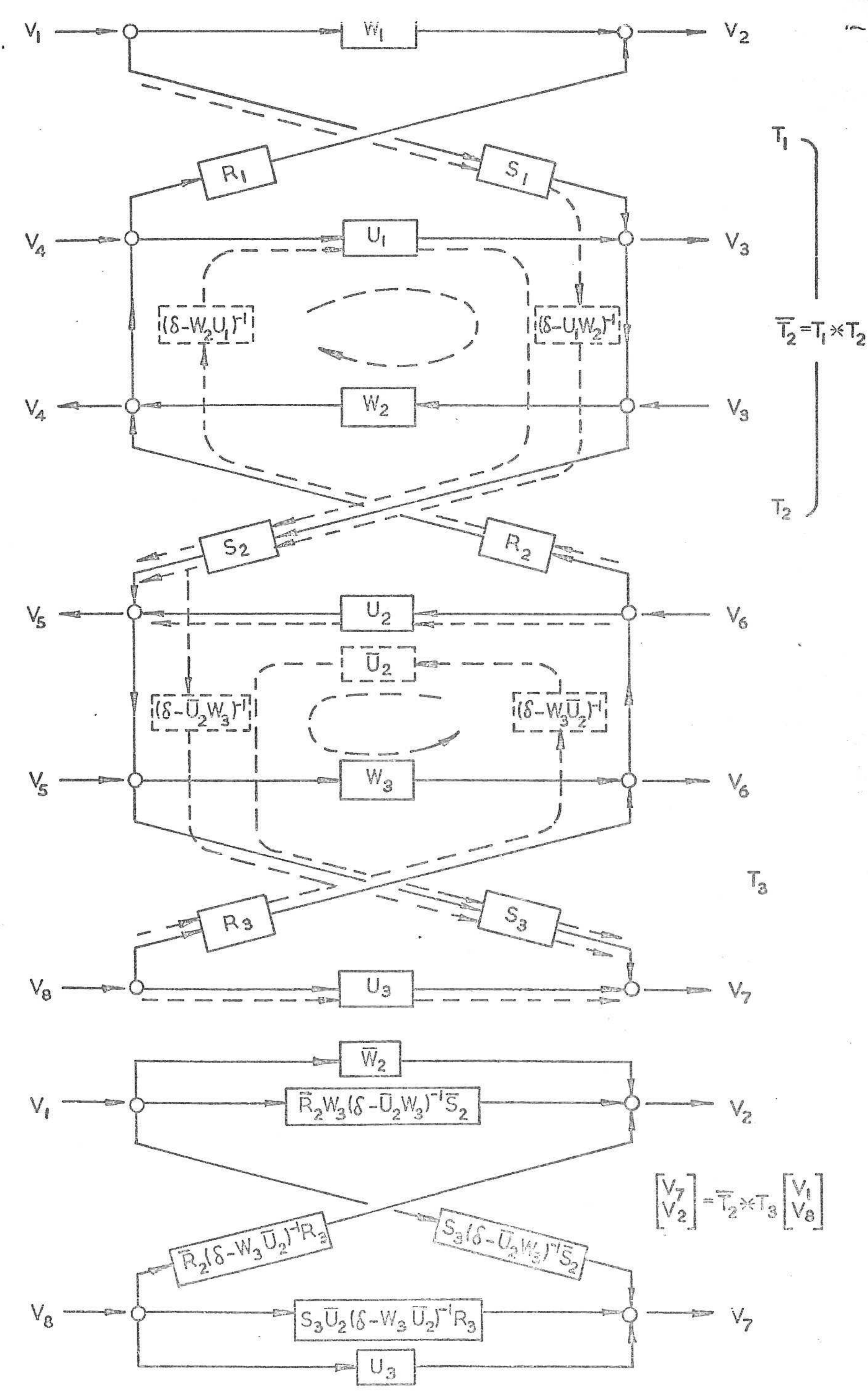


FIG.6 INTERCONNECTED LATTICE - TYPE SEQUENCE

functional analysis and automata theory with continuous signals. The form of the interconnected lattice and the scattering problem would now appear to provide a basic framework with important applications in the study of complex cellular structure and organisation.

4. Optimality conditions in the polyhedron model and scattering problem

The transformations forming the structure of Kron's polyhedron model and the variables in each network can be associated with an inherent minimum power condition. The scattering problem similarly contains an algebraic framework with properties of a return difference matrix or feedback effect which can be associated with a minimum quadratic condition. Both representations thus include features of a least squares solution, and the interacting sequential networks introduce dimensionless and impedance/admittance or variance-type operators in a multi-stage solution procedure which evolves by updating a priori information. A similar structure is shown to exist in the linear two-point boundary value problems for optimal control and estimation, and similarly in the classical least squares problem incorporating a priori information concerning the unknown parameter vector.

4.1 Least-squares estimation A least squares estimate for the parameter vector x in the measurement equation

$$y = Hx + v \tag{8}$$

with known mean and covariance properties

$$E[v] = 0 , \quad E[vv^T] = R , \quad E[xx^T] = S , \quad E[vx^T] = 0$$

is obtained by minimising the function

$$J(x) = \|y-Hx\|^2_{R^{-1}} + \|x\|^2_{S^{-1}}$$

where $\|v\|^2_{R^{-1}} = v^T R^{-1} v$. Then

$$\hat{x} = PH^T R^{-1} y , \quad P = (S^{-1}+H^T R^{-1} H)^{-1} = S-SH^T LHS$$

and $$\hat{y} = H\hat{x} = MR^{-1}y , \quad M = H(S^{-1}+H^T R^{-1} H)^{-1} H^T$$

where $$L = R^{-1} - R^{-1} M R^{-1} = R^{-1} F^{-1} = (R+HSH^T)^{-1} , \quad M = R-RLR$$

and $$F = (\delta - MR^{-1})^{-1} = \delta + HSH^T R^{-1}$$

represents a return-difference-type matrix.

We can then identify the form of the orthogonal network matrix N of eqn 3, with

$$\begin{bmatrix} 0 \\ \hat{y}-y \end{bmatrix} = \begin{bmatrix} -H^T & P^{-1} \\ -R & H \end{bmatrix} \begin{bmatrix} R^{-1}y \\ \hat{x} \end{bmatrix} = N \begin{bmatrix} R^{-1}y \\ \hat{x} \end{bmatrix} , \quad \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \equiv \begin{bmatrix} S^{-1} & H^T R^{-1} \\ R^{-1}H & -R^{-1} \end{bmatrix} \tag{9}$$

The particular analogy requires the condition $i^{c'} = 0$, and a return difference-type matrix also appears in the electrical network problem with

$$F = [\delta + Y_4^{-1} Y_3 (Y_1 - Y_2 Y_4^{-1} Y_3)^{-1} Y_2]^{-1} = \delta - Y_4^{-1} Y_3 Y_1^{-1} Y_2$$

The properties of a scattering matrix also appear with

$$\begin{bmatrix} R^{-1}(\hat{y}-y) \\ \hat{y}-y \end{bmatrix} = N_1 \begin{bmatrix} R^{-1}y \\ \hat{y} \end{bmatrix} , \quad \begin{bmatrix} 0 \\ \hat{y} \end{bmatrix} = N_2 \begin{bmatrix} R^{-1}(\hat{y}-y) \\ \hat{x} \end{bmatrix} \tag{10}$$

where $\quad N = N_1 * N_2 = \begin{bmatrix} -\delta & R^{-1} \\ -R & \delta \end{bmatrix} * \begin{bmatrix} H^T & S^{-1} \\ 0 & H \end{bmatrix}$

The star product transformations are illustrated in FIG 7.

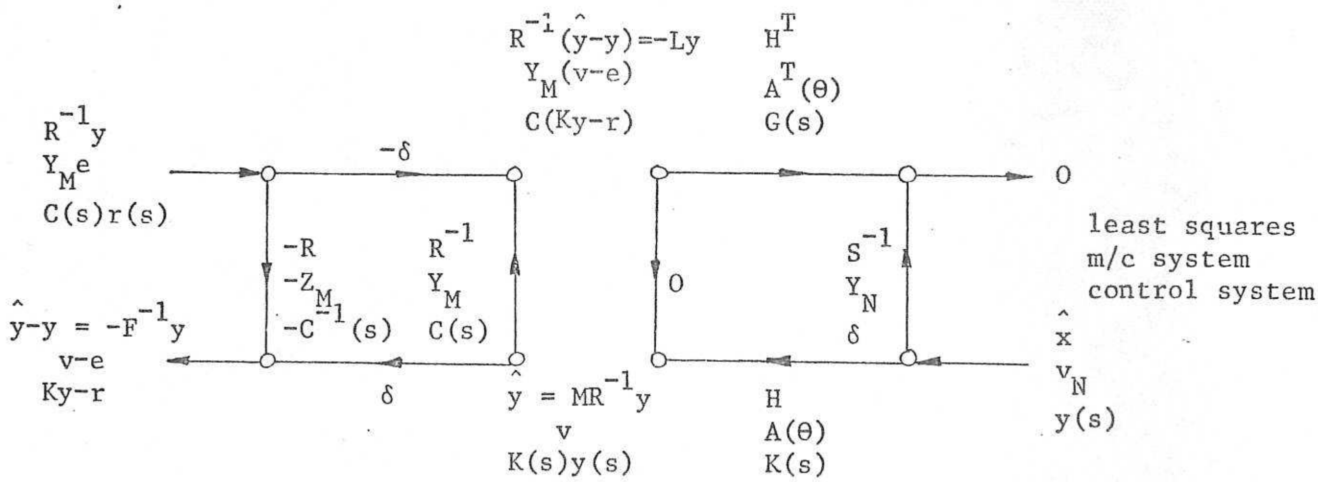


FIG 7 Transformation diagram for least squares, multimachine and control system problems

The transformation operators M,L associated with the least squares solution without a priori information and the conventional network problem appear within the component blocks of FIG 2, thus indicating that Kron's vertical propagation within each polyhedron component introduces effectively a least squares operation, with the measurement and covariance matrices defined by the properties of the higher-dimensional elements. The improved estimate obtained by incorporating a priori information (S) may then be identified with the orthogonal electrical network, although it does not exist within the solution of the conventional electrical network problem, as Kron appreciated by physical reasoning. The interconnection of obstacles in the scattering problem similarly introduces the effects of a priori or previous-stage covariance-type information based on a return difference or feedback-type operator associated with the adjacent stages. The polyhedron model and the scattering problem thus include a structure for updating a least squares solution[12], which also forms the basis of Kron's method of tearing which is highlighted by the analogy between the components Y_1, Y_4 and the covariance operators S^{-1}, R^{-1} in eqn 9.

The form of eqn 9 and the star product components of eqn 10 and the associated properties also exist in the multimachine system and multivariable feedback control problems.

4.2 Multimachine system The connection of generators with a system network is defined by the relationships[1,11,12]

$$i_N = Y_N v_N \ , \quad v = e - Z_M i \ , \quad v = A(\theta) v_N \ , \quad i_N = A^T(\theta) i$$

where θ represents machine load angle and $A(\theta)$ the generator-network node connection matrix. The generator variables are then given by

$$v = M(\theta) Y_M e = (A Z_N A^T) i \ , \qquad i = Le$$

where $$M = A(Y_N + A^T Y_M A)^{-1} A^T \ , \ L = Y_M - Y_M M Y_M = Y_M F^{-1} = (Z_M + A Z_N A^T)^{-1}$$

The solution includes transformation matrices M,L with impedance and admittance properties respectively and a dimensionless return difference matrix F as in the least squares problem, with implicit feedback introduced by the machine-network interconnection and machine equations. The machine-network relationships may then be specified in the form of eqns 9 and 10, with the variables defined in FIG 7. The machine system can thus be identified with the orthogonal network and scattering problems, with

$$N = \begin{bmatrix} -A^T(\theta) & Y_N + A^T(\theta) Y_M A(\theta) \\ -Z_M & A(\theta) \end{bmatrix} , \ N_1 = \begin{bmatrix} -\delta & Y_M \\ -Z_M & \delta \end{bmatrix} , \ N_2 = \begin{bmatrix} A^T(\theta) & Y_N \\ 0 & A(\theta) \end{bmatrix}$$

The star product then effects an interconnection of the machine and network parameters by means of the connection matrix or transmission operator $A(\theta)$ and by analogy the network admittance matrix Y_N introduces a priori information concerning the 'estimate' for network voltages. The orthogonal electrical network problem also forms a basis for the analogy between the machine system problem and the concept of wave propagation in the polyhedron model[1].

4.3 <u>Multivariable control problem</u> The closed-loop response of the linear multivariable system represented by a plant transfer function matrix G(s), forward controller C(s) and feedback controller K(s), with reference input r(s) is given by

$$y(s) \quad = \quad (\delta + GCK)^{-1}GCr(s)$$

Transformation operators may then be defined, as previously, with

$$M(s) \quad = \quad K(\delta + GCK)^{-1}G \;, \qquad L(s) \quad = \quad C-CMC \quad = \quad CF^{-1}$$

where F(s) represents a return-difference matrix. The solution can then be represented in terms of an orthogonal network matrix N and the components of a star product as in eqns 9 and 10, with the variables defined in FIG 7, provided C^{-1} exists. Then

$$N \quad = \quad \begin{bmatrix} -G & \delta+GCK \\ -C^{-1} & K \end{bmatrix} \quad = \quad \begin{bmatrix} -\delta & C \\ -C^{-1} & \delta \end{bmatrix} * \begin{bmatrix} G & \delta \\ 0 & K \end{bmatrix}$$

It is significant to note that the controller matrix C(s) may be associated with the reciprocal covariance properties of the error signal, by comparison with the estimation problem. The solutions of the least squares, machine and control system problems are also defined by the respective quadratic functions

$$\|v\|^2_{R^{-1}} + \|\hat{x}\|^2_{S^{-1}} \equiv \|i\|^2_{Z_M} + \|v_N\|^2_{Y_N} \equiv \|e(s)\|^2_{C(s)} + \|y(s)\|^2$$

The transformation diagram of FIG 7 incorporates a basic topological and algebraic structure defined by the interrelationships between similar and conjugate sets of variables, and defines the available forms of solution for a wide range of linear system problems.

The general linear feedback system may also be represented as an equivalent signal-flow graph in a lattice or return-difference operator form as illustrated in FIG 8, where u(s),y(s),d(s) represent transformed input, output and internal dynamical variables respectively, and disturbance n(s) is included as an output variable[31].

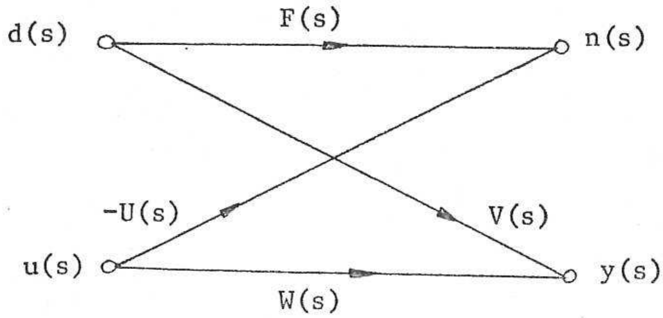


FIG 8 System representation in return-difference operator form

The feedback system is characterised by the operator matrices U(s),V(s) and W(s), and $F(s) = \delta - T(s)$ represents the return-difference operator matrix where T(s) is the return-ratio operator matrix. The output is then given by

$$y(s) = \left[V(s)F^{-1}(s)U(s)+W(s)\right]u(s) + V(s)F^{-1}(s)n(s) = G(s)u(s) + V(s)F^{-1}(s)n(s)$$

The transformations can be illustrated in the scattering forms of FIG 9.

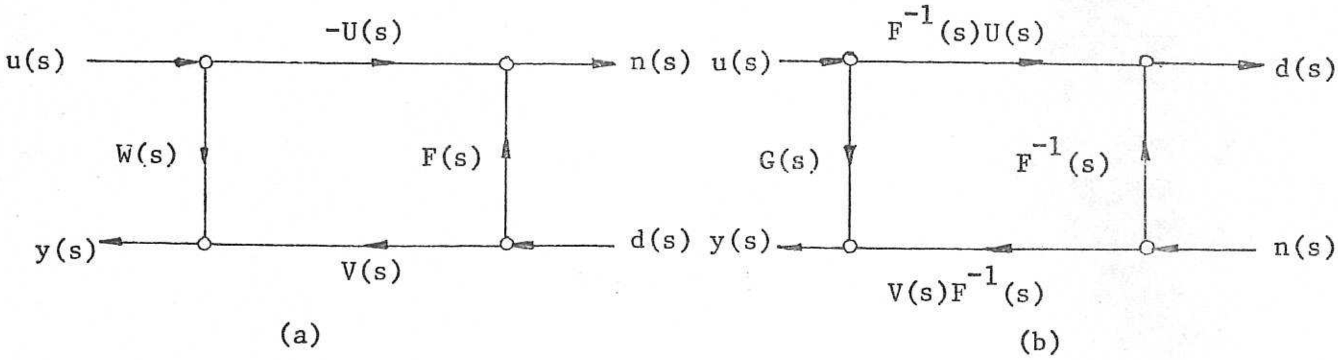


FIG 9 System representations in scattering form

The scattering or system matrix associated with FIG 9(a) is then given by

$$P(s) = \begin{bmatrix} -U(s) & F(s) \\ W(s) & V(s) \end{bmatrix}$$

The scattering matrix for FIG 9(b) also includes a star product given by

$$N(s) = \begin{bmatrix} F^{-1}(s)U(s) & F^{-1}(s) \\ W(s)+V(s)F^{-1}(s)U(s) & V(s)F^{-1}(s) \end{bmatrix} = \begin{bmatrix} -U(s) & 0 \\ W(s) & V(s) \end{bmatrix} * \begin{bmatrix} -F^{-1}(s) & F^{-1}(s) \\ -F^{-1}(s) & F^{-1}(s) \end{bmatrix}$$

The matrix N(s) can then be identified with the orthogonal network matrix N, with impedances and source and response variables defined by

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \equiv \begin{bmatrix} F(s) & -U(s) \\ V(s) & W(s) \end{bmatrix} , \begin{bmatrix} I^{o'} \\ e_c' \end{bmatrix} \equiv \begin{bmatrix} u(s) \\ n(s) \end{bmatrix} , \begin{bmatrix} i^{c'} \\ E_o' \end{bmatrix} \equiv \begin{bmatrix} d(s) \\ y(s) \end{bmatrix}$$

4.4 Multistage optimal control problem The solution of the control problem concerned with minimising the N-stage index

$$J = \tfrac{1}{2}x^T(N)Q(N)x(N) + \tfrac{1}{2} \sum_{i=o}^{N-1} \left[x^T(i)Q(i)x(i) + u^T(i)R(i)u(i)\right]$$

for the discrete linear system

$$x(i+1) = \Phi(i)x(i) + \Delta(i)u(i) , \qquad x(o) \text{ given}$$

can be formulated by defining the H^i sequence

$$H^i = \tfrac{1}{2} x^T(i)Q(i)x(i) + \tfrac{1}{2} u^T(i)R(i)u(i) + \lambda^T(i+1)\left[\Phi(i)x(i) + \Delta(i)u(i)\right]$$

The condition $\partial H^i/\partial u(i) = 0$ then gives the minimising sequence

$$u(i) = -R^{-1}(i)\Delta^T(i)\lambda(i+1)$$

The optimal trajectory and undetermined multipliers are defined by a two-point boundary value problem represented by the coupled linear difference equations

$$\begin{bmatrix} x(i+1) \\ \lambda(i) \end{bmatrix} = \begin{bmatrix} \Phi(i) & -\Delta(i)R^{-1}(i)\Delta^T(i) \\ Q(i) & \Phi^T(i) \end{bmatrix} \begin{bmatrix} x(i) \\ \lambda(i+1) \end{bmatrix} , \begin{bmatrix} x(o) \\ \lambda(N) \end{bmatrix} = \begin{bmatrix} x(o) \\ Q(N)x(N) \end{bmatrix} , \quad i = 0..N-1 \tag{11}$$

The sweep method of solution is then based on the transformation

$$\lambda(i) = P(i)x(i) \tag{12}$$

Combining eqns 11 and 12 gives the backward recursive relationship

$$P(i) = \Phi^T(i)\left[P^{-1}(i+1)+\Delta(i)R^{-1}(i)\Delta^T(i)\right]^{-1}\Phi(i)+Q(i) \ , \ i = N-1,...0, \ P(N) = Q(N)$$

Then
$$x(i+1) = \left[\delta + \Delta(i)R^{-1}(i)\Delta^T(i)P(i+1)\right]^{-1}\Phi(i)x(i) = M(i)x(i)$$
$$u(i) = -\left[R(i) + \Delta^T(i)P(i+1)\Delta(i)\right]^{-1}\Delta^T(i)P(i+1)\Phi(i)x(i)$$

An orthogonal network matrix and the components of a star product are identified, as in eqns 9 and 10, with

$$\begin{bmatrix} x(i+1) \\ 0 \end{bmatrix} = \begin{bmatrix} -\Delta(i) & P^{-1}(i+1)+\Delta(i)R^{-1}(i)\Delta^T(i) \\ -R(i) & \Delta^T(i) \end{bmatrix} \begin{bmatrix} -u(i) \\ \lambda(i+1) \end{bmatrix} = N \begin{bmatrix} -u(i) \\ \lambda(i+1) \end{bmatrix}$$

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \equiv \begin{bmatrix} P^{-1}(i+1) & \Delta(i)R^{-1}(i) \\ R^{-1}(i)\Delta^T(i) & -R^{-1}(i) \end{bmatrix}$$

$$N = N_1 * N_2 = \begin{bmatrix} -\delta & R^{-1}(i) \\ -R(i) & \delta \end{bmatrix} * \begin{bmatrix} \Delta(i) & P^{-1}(i+1) \\ 0 & \Delta^T(i) \end{bmatrix}$$

The augmented system matrix of eqn 11 can also be specified as a scattering-type matrix which combines with a previous-stage matrix in a star product to form the state transition-type matrix $M(i)$ and current-stage covariance matrix $P(i)$, with

$$\begin{bmatrix} x(i+1) \\ \lambda(i) \end{bmatrix} = \begin{bmatrix} M(i) & 0 \\ P(i) & 0 \end{bmatrix} \begin{bmatrix} x(i) \\ \lambda(i+1) \end{bmatrix} = T(i) \begin{bmatrix} x(i) \\ \lambda(i+1) \end{bmatrix}$$

where $$T(i) = \begin{bmatrix} \Phi(i) & -\Delta(i)R^{-1}(i)\Delta^T(i) \\ Q(i) & \Phi^T(i) \end{bmatrix} * \begin{bmatrix} \delta & 0 \\ P(i+1) & 0 \end{bmatrix}$$

The component transformations associated with the sweep method of solution are illustrated in FIG 10.

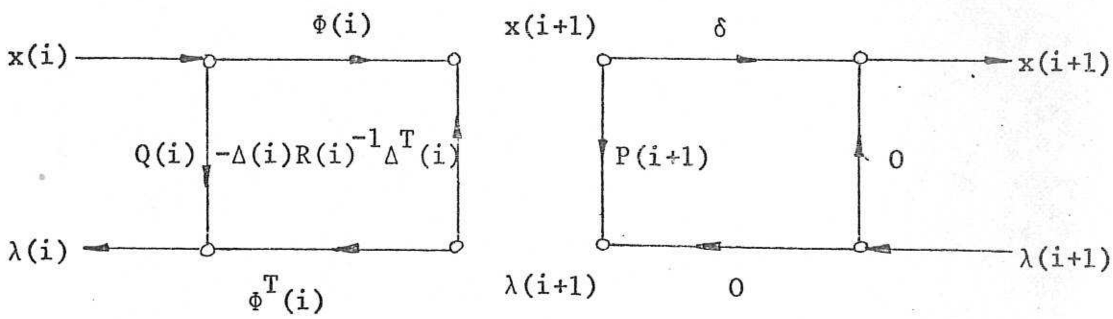


FIG 10 Sweep method of solution in scattering form

The multi-stage minimisation procedure and associated dynamic programming algorithms thus operate according to the properties of the star product in the scattering process, which essentially incorporates the effects of a priori information by updating the previous-stage covariance operators. A similar multi-stage procedure also exists inherently in the polyhedron model.

4.5 Linear optimal control with terminal constraint[11] Optimal control of the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \quad y(t) = Cx(t) \ , \quad x(t_o) = x_o$$

with minimum performance index

$$J(u) = \tfrac{1}{2} \int_{t_o}^{t_1} (y^T Qy + u^T Ru)dt + \tfrac{1}{2} y^T(t_1)Fy(t_1)$$

subject to the terminal vector constraint $z = Zy(t_1)$

is defined by the solution of the two-point boundary value problem

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^T QC & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} , \quad \begin{bmatrix} x(t_o) \\ p(t_1) \end{bmatrix} = \begin{bmatrix} x_o \\ C^T FCx(t_1) + C^T Z^T \lambda \end{bmatrix} \qquad (13)$$

where λ is a vector multiplier. Adjoint-state variable and terminal constraint relationships are then introduced with

$$p(t) = P(t)x(t) + G(t)\lambda \qquad z = G^T(t)x(t) + N(t)\lambda \ , \quad G^T(t_1) = ZC \ , \ N(t_1) = 0$$

giving

$$\begin{bmatrix} \lambda \\ p(t) \end{bmatrix} = \begin{bmatrix} -N^{-1}G^T & N^{-1} \\ P-GN^{-1}G^T & GN^{-1} \end{bmatrix} \begin{bmatrix} x(t) \\ z \end{bmatrix} = T_1 \begin{bmatrix} x(t) \\ z \end{bmatrix} \qquad (14)$$

$$\begin{bmatrix} p(t_1) \\ z \end{bmatrix} = \begin{bmatrix} C^T Z^T & C^T FC \\ 0 & ZC \end{bmatrix} \begin{bmatrix} \lambda \\ x(t_1) \end{bmatrix} = T_2 \begin{bmatrix} \lambda \\ x(t_1) \end{bmatrix} \qquad (15)$$

Equations 14 and 15 will now combine as a star product to form a composite object with external 'reflected' and 'incident' variables in the form

$$\begin{bmatrix} p(t_1) \\ p(t) \end{bmatrix} = T_1 {*} T_2 \begin{bmatrix} x(t) \\ x(t_1) \end{bmatrix} \quad \text{where } T_1 {*} T_2 = \begin{bmatrix} -C^T Z^T N^{-1} G^T & C^T (F + Z^T N^{-1} Z)C \\ P-GN^{-1}G^T & GN^{-1}ZC \end{bmatrix}$$

The corresponding transformation diagram is illustrated in FIG 11.

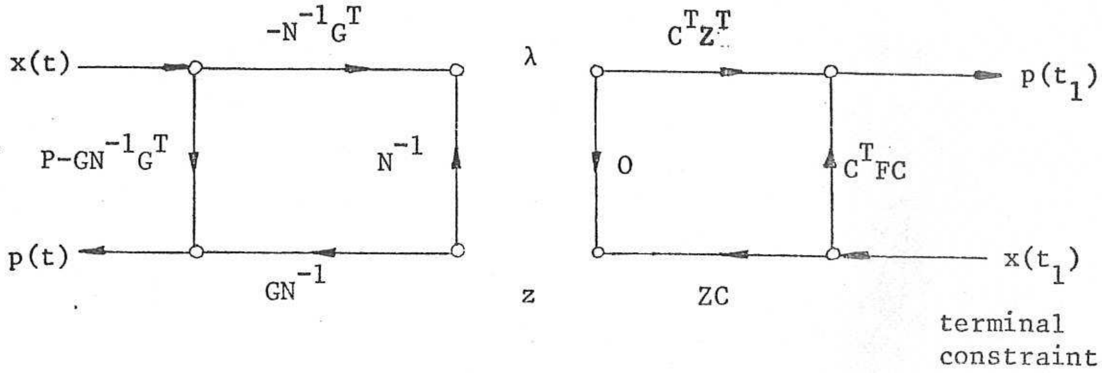


FIG 11 Scattering representation of optimal control problem

The state variables act as 'incident' variables in the forward direction to the left of the 'obstacle' and the adjoint variables act as left reflected variables in the reverse direction. The terminal constraint is also defined with external state and adjoint variables, and variables λ, z appear in the interconnecting zone. The combined scattering matrix is also represented by

$$T_1 * T_2 = \begin{bmatrix}\delta & O\\ P & \delta\end{bmatrix} * \begin{bmatrix}G^T & O\\ O & G\end{bmatrix} * \begin{bmatrix}-N^{-1} & N^{-1}\\ -N^{-1} & N^{-1}\end{bmatrix} * \begin{bmatrix}Z^T & O\\ O & Z\end{bmatrix} * \begin{bmatrix}C^T & C^TFC\\ O & C\end{bmatrix}$$

The form of an orthogonal network can also be identified in eqns 14,15 with

$$\bar{Z}_1 = \begin{bmatrix}Z_1 Z_2\\ Z_3 Z_4\end{bmatrix} \equiv \begin{bmatrix}N & G^T\\ G & P\end{bmatrix}, \quad \bar{Z}_2 \equiv \begin{bmatrix}Z_1 Z_2\\ Z_3 Z_4\end{bmatrix} \equiv \begin{bmatrix}(C^TFC)^{-1} & -(C^TFC)^{-1}C^TZ^T\\ ZC(C^TFC)^{-1} & -ZC(C^TFC)^{-1}C^TZ^T\end{bmatrix}$$

The variables $\lambda, p(t)$ associated with the 'obstacle' T_1 may then be identified with the closed- and open-path response variables $i_c{}', E_o{}'$ and $x(t), z$ with the open- and closed-path source variables $I_o{}', e_c{}'$ respectively.

The relationships of eqn 13 also lead to a set of matrix differential equations which define the optimal control problem in the form

$$\begin{bmatrix}\dot{G} & \dot{P}\\ \dot{N} & \dot{G}^T\end{bmatrix} = \begin{bmatrix}(PBR^{-1}B^T-A^T)G, & -PA-A^TP+PBR^{-1}B^TP-C^TQC\\ G^TBR^{-1}B^TG \;, & G^T(BR^{-1}B^TP-A)\end{bmatrix}, \quad \begin{bmatrix}G(t_1)P(t_1)\\ N(t_1)G^T(t_1)\end{bmatrix} = \begin{bmatrix}C^TZ^T & C^TFC\\ 0 & ZC\end{bmatrix} \qquad (16)$$

and $\quad u(t) = -R^{-1}B^TG(t)N^{-1}(t)z - R^{-1}B^T\left[P(t)-G(t)N^{-1}(t)G^T(t)\right]x(t)$

Equations 16 can be integrated backwards to yield $P(t), G(t)$ and $N(t)$, then eqn 13 is integrated forward as an initial value problem[13]. The differential equation set appears similarly in the optimal tracking problem and in the terminal control problem requiring $x_i(t_1) = 0$.[13] It also appears inherently in the scattering problem in eqn 7, with the space argument (x,y) translated to a time interval. The linear optimal control problem thus incorporates concepts of scattering with interconnected obstacles, and the associated algebraic relationships also appear within the structure of the orthogonal network and thus similarly in Kron's polyhedron model.

4.6 Kalman-Bucy filter Consider the random process x(t) and measurement z(t) generated by the system

$$\dot{x}(t) = Fx(t) + Gu(t) \qquad z(t) = Hx(t) + v(t) \tag{17}$$

where u(t),v(t) represent independent zero-mean white Gaussian noise vectors with covariance matrices $Q\delta(t-\tau)$, $R\delta(t-\tau)$ respectively. The minimum-variance estimate $\hat{x}(t)$ is generated by the Kalman-Bucy filter equations

$$\dot{\hat{x}}(t) = F\hat{x}(t) + K(t)[z(t) - H\hat{x}(t)] \quad , \quad K(t) = P(t)H^TR^{-1}$$

where the state error covariance matrix P(t) is given by the solution of a matrix Riccati differential equation. For the stationary filter, the Laplace transformed solution is given by

$$\hat{x}(s) = [\delta + \Phi(s)PH^TR^{-1}H]^{-1}\Phi(s)PH^TR^{-1}z(s) \; , \quad \Phi(s) = (s\delta - F)^{-1}$$

$$\hat{y}(s) = H\hat{x}(s) = MR^{-1}z(s)$$

where $\quad M = H[P^{-1}\Phi^{-1}(s) + H^TR^{-1}H]^{-1}H^T \; , \; L = R^{-1} - R^{-1}MR^{-1} = [R + H\Phi(s)PH^T]^{-1}$

A return-difference matrix F(s) also exists, with the tracking error given by

$$e(s) = z(s) - H\hat{x}(s) = RLz(s) = F^{-1}(s)z(s) \; , \; F(s) = (\delta - MR^{-1})^{-1} = \delta + H\Phi(s)K$$

The solution can now be defined directly by the form of the least squares estimation eqns 9 and 10, with a priori covariance $S \equiv \Phi(s)P$. The stationary optimal filter problem thus also includes a structure which is associated with the orthogonal network and the scattering problem. The star product effects a combination of the measurement noise covariance with the 'previous-stage' transformed error covariance operator $\Phi(s)P$ representing a priori information. A direct analogy also exists with the machine system problem with $R \equiv Z_M$, $H \equiv A(\theta)$, $\Phi(s)P \equiv Z_N$. The condition of optimality for the filter problem in the frequency domain, represented by the spectral density for the observation[14], can also be defined by means of a star product in the form

$$\begin{bmatrix} -G(s) & R+G(s)QG^T(-s) \\ -Q^{-1} & G^T(-s) \end{bmatrix} = \begin{bmatrix} -\delta & Q \\ -Q^{-1} & \delta \end{bmatrix} * \begin{bmatrix} G(s) & R \\ 0 & G^T(-s) \end{bmatrix} = N .$$

The scattering matrix N is similarly identified as an orthogonal network matrix with

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \equiv \begin{bmatrix} R & G(s)Q \\ QG^T(-s) & -Q \end{bmatrix}$$

4.7 The optimal smoothing problem[13,15-25] Optimal smoothing provides a state estimate at intermediate time points t, based on all measurements within the range $t_o \leqslant t \leqslant T$. It gives an improved estimate, with reduced variance, compared to the filter estimate at time t based on data obtained before the time t. A solution is formulated in terms of a two-point boundary value problem which can be solved using a forward and reverse sweep or integration technique incorporating properties of the filtering solution.

4.7.1 Continuous-time fixed-interval smoothing The problem is stated- given the measurement vector $\{z(t),\ t_o \leqslant t \leqslant T\}$, determine the minimum variance estimate of the random state vector $\{x(t),\ t_o \leqslant t \leqslant T\}$ generated by the system defined by eqn 17 with similar noise properties and known initial state mean and error covariance values. A solution can be formulated in terms of the Hamiltonian function

$$H = L + p^T(t)\left[Fx(t) + Gu(t)\right] , \quad L = \tfrac{1}{2}\{ \|u\|^2_{Q^{-1}} + \|z-Hx\|^2_{R^{-1}} \}$$

The condition $\partial H/\partial u = 0$ and the canonical equations $\dot{x} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial x$ lead to the two-point boundary value problem represented by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} F & -GQG^T \\ -H^TR^{-1}H & -F^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} 0 \\ H^TR^{-1}z(t) \end{bmatrix} , \quad \begin{bmatrix} x(t_o) \\ p(T) \end{bmatrix} = \begin{bmatrix} x_o - P_o p(t_o) \\ 0 \end{bmatrix} \qquad (18)$$

An off-line sweep method of solution for the smoothed estimate is then formulated with the transformation[18]

$$p(t) = -P^{-1}(t)\{x(t) - g(t)\} \qquad (19)$$

Combining eqns 18 and 19 then gives P(t),g(t) as solutions of an initial value problem, over the forward interval $(t_o - T)$, represented by

$$\dot{P} = FP + PF^T - PH^TR^{-1}HP + GQG^T \;, \quad P(t_o) = P_o \tag{20}$$

$$\dot{g} = (F - PH^TR^{-1}H)g + PH^TR^{-1}z \;, \quad g(t_o) = \bar{x}_o \tag{21}$$

where $P(t) \equiv P(t/t)$ represents a filtered error covariance. The smoothed trajectory $\hat{x}(t/T)$ is then obtained by backward integration of eqn 17, from T to t_o, in the form

$$\dot{\hat{x}}(t/T) = (F + GQG^TP^{-1})\hat{x} - GQG^TP^{-1}g \tag{22}$$

with a smoothed terminal state given by the optimal filtered estimate $\hat{x}(T/T) = g(T) = \hat{x}(T)$, and from eqn 21, for all T,

$$\frac{d\hat{x}(T)}{dT} = F\hat{x}(T) + K(T)\left[z(T) - H\hat{x}(T)\right] \;, \quad K(T) = P(T)H^TR^{-1} \;, \quad \hat{x}(t_o) = \bar{x}_o \tag{23}$$

The solution thus includes a forward sweep to the end conditions defined by the filtered estimate $\hat{x}(T)$, followed by a reverse sweep to an intermediate stage giving a correction of the corresponding filtered estimate. The smoothing solution thus only requires the solution for the filtered estimate $g(t) = \hat{x}(t)$ and $P(t)$ forward in time, then eqn 22 is solved backward in time to determine the smoothed estimate[25].

The optimum linear smoother may also be considered as an optimal combination of two independent filter estimates, based on a forward estimate at time t using data in the range t_o-t and a backward estimate at time t using data in the range t-T[17,22,26]. The forward filter of eqns 20,23 with a priori knowledge $\hat{x}_1(t_o), P_1(t_o)$ will give $\hat{x}_1(t)$ and P_1. The backward estimate can be obtained by replacing x_1 by $\hat{x}_2$, K by -K and P_1 by $-P_2$ in the forward filter equations, with $P_2^{-1}(T) = 0$, giving an independent estimate $\hat{x}_2(t)$ by tracing backward for the period T-t. The estimates with zero-mean Gaussian statistics are then combined to give an unbiassed smoothed minimum-variance estimate of the form[27]

$$\hat{x}(t/T) = P(t/T)\left[P_1^{-1}(t)\hat{x}_1(t) + P_2^{-1}(t)\hat{x}_2(t)\right] \;, \quad P(t/T) = \left[P_1^{-1}(t) + P_2^{-1}(t)\right]^{-1}$$

or

$$\hat{x}(t/T) = \hat{x}_1(t) - P_1(t)\left[P_1(t) + P_2(t)\right]^{-1}\left[\hat{x}_1(t) - \hat{x}_2(t)\right]$$

$$= \hat{x}_2(t) + P_2(t)\left[P_1(t) + P_2(t)\right]^{-1}\left[\hat{x}_1(t) - \hat{x}_2(t)\right]$$

and

$$P(t/T) = P_1(t) - P_1(t)\left[P_1(t)+P_2(t)\right]^{-1}P_1(t) = P_2(t)\left[P_1(t)+P_2(t)\right]^{-1}P_1(t)$$

A scattering formulation can then be introduced with

$$\begin{bmatrix} \hat{x}(t/T) \\ \hat{x}(t/T)-\hat{x}_1(t) \end{bmatrix} = \begin{bmatrix} (\delta+P_1P_2^{-1})^{-1} & P_1(\delta+P_2^{-1}P_1)^{-1}P_2^{-1} \\ -P_1P_2^{-1}(\delta+P_1P_2^{-1})^{-1} & P_1(\delta+P_2^{-1}P_1)^{-1}P_2^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = T \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}(t/T) \\ \hat{x}(t/T)-\hat{x}_1(t) \end{bmatrix} = T_1 \begin{bmatrix} \hat{x}_1(t) \\ P_1^{-1}(t)\{\hat{x}(t/T)-\hat{x}_1(t)\} \end{bmatrix}, \quad \begin{bmatrix} \hat{x}(t/T) \\ P_1^{-1}(t)\{\hat{x}(t/T)-\hat{x}_1(t)\} \end{bmatrix} = T_2 \begin{bmatrix} \hat{x}(t/T) \\ \hat{x}_2(t) \end{bmatrix}$$

where $$T = T_1 * T_2 = \begin{bmatrix} \delta & P_1(t) \\ 0 & P_1(t) \end{bmatrix} * \begin{bmatrix} \delta & 0 \\ -P_2^{-1}(t) & P_2^{-1}(t) \end{bmatrix}$$

The combination of estimates $\hat{x}_1(t)$ and $\hat{x}_2(t)$ based on the star product of the basic components T_1, T_2 is illustrated in FIG 12.

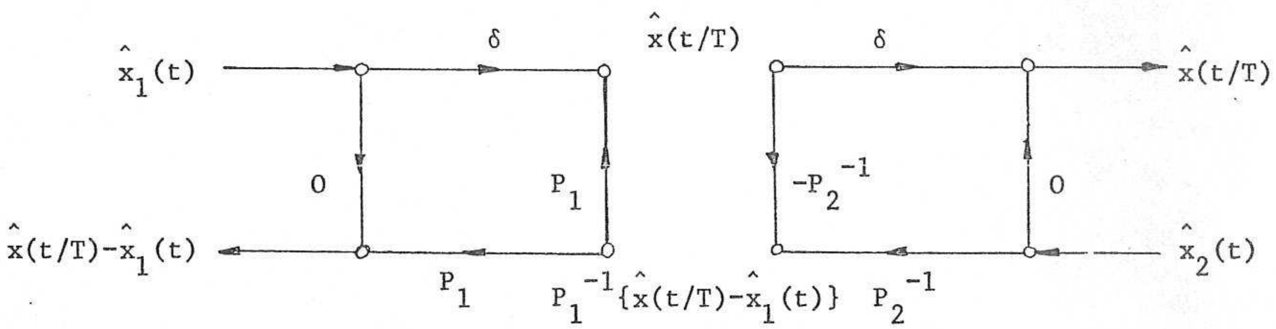


FIG 12 Optimal combination of estimates in scattering form

The two-filter solution also produces a differential equation set[17] as defined by the scattering problem.

4.7.2 <u>Discrete-time fixed-interval smoothing</u>[15,17,20,23,24] The discrete-time problem includes the vector difference equations

$$x(i+1) = \Phi(i)x(i) + \Delta(i)u(i) \;, \quad z(i) = H(i)x(i) + v(i)$$

where $u(i), v(i)$ represent independent zero-mean white Gaussian noise sequences with covariance matrices $Q(i)\delta_{ji}$, $R(i)\delta_{ji}$ respectively. A solution for the minimum-variance estimate of the state sequence $\{x(i),\ i = 0..N\}$ based on the observed sequence $\{z(i),\ i = 0..N\}$ is formulated by minimising the loss function[15]

$$J = \tfrac{1}{2}\,\|x(o)-\bar{x}_o\|^2_{P_o^{-1}} + \sum_{i=o}^{N} \tfrac{1}{2}\,\|z(i)-H(i)x(i)\|^2_{R^{-1}(i)} + \sum_{i=o}^{N-1} \{\tfrac{1}{2}\,\|u(i)\|^2_{Q^{-1}(i)} + \lambda^T(i)\left[x(i+1) - \Phi(i)x(i) - \Delta(i)u(i)\right]\}$$

Setting dJ = 0 then gives the two-point boundary value problem

$$\begin{bmatrix} \hat{x}(i+1/N) \\ \lambda(i-1) \end{bmatrix} = \begin{bmatrix} \Phi(i) & \Delta(i)Q(i)\Delta^T(i) \\ -H^T(i)R^{-1}(i)H(i) & \Phi^T(i) \end{bmatrix} \begin{bmatrix} \hat{x}(i/N) \\ \lambda(i) \end{bmatrix} + \begin{bmatrix} 0 \\ H^T(i)R^{-1}(i)z(i) \end{bmatrix} \quad (24)$$

where $\hat{x}(i/N)$ represents the fixed-interval smoothed estimate of $x(i)$. The boundary condition is represented by

$$\lambda(o) = -P_o^{-1}\left[\bar{x}_o - \hat{x}(o/N)\right] \quad , \quad \lambda(N) = 0$$

Then by induction, the smoothing solution reduces to the form

$$\hat{x}(i/N) = \hat{x}(i/i)+\left[\delta+P(i)H^T(i)R^{-1}(i)H(i)\right]^{-1}P(i)\Phi^T(i)\lambda(i) = \Phi(i-1)\hat{x}(i-1/i-1)+P(i)\lambda(i-1) \quad (25)$$

$$P(i+1) = \Phi(i)\left[\delta+P(i)H^T(i)R^{-1}(i)H(i)\right]^{-1}P(i)\Phi^T(i)+\Delta(i)Q(i)\Delta^T(i) \quad (26)$$

$$\hat{x}(i/i) = \Phi(i-1)\hat{x}(i-1/i-1)+\left[\delta+P(i)H^T(i)R^{-1}(i)H(i)\right]^{-1}P(i)H^T(i)R^{-1}(i)\left[z(i)-H(i)\Phi(i-1)\hat{x}(i-1/i-1)\right] \quad (27)$$

Equations 26,27 represent a forward recursive solution of the linear filtering problem and the terminal stage estimate $\hat{x}(N)$ gives the smoothed estimate $\hat{x}(N/N)$, and with $\lambda(N) = 0$ for fixed N the solution is transformed to an initial value problem. Backsweeping using eqn 25 and λ in eqn 24 then gives the fixed interval smoothing solution.

The augmented system matrix of eqn 24 can be considered as a scattering matrix, and combines with the current-stage covariance matrix to give the relationship of eqn 26 in the star product

$$\begin{bmatrix} - & P(i+1) \\ - & - \end{bmatrix} = \begin{bmatrix} - & P(i) \\ - & - \end{bmatrix} * \begin{bmatrix} \Phi(i) & \Delta(i)\,Q(i)\Delta^T(i) \\ -H^T(i)R^{-1}(i)H(i) & \Phi^T(i) \end{bmatrix} \quad (28)$$

The forward filtered estimate of eqn 27 incorporating the measured data and previous-stage filtered estimate, and the backward smoothed estimate of eqn 25 incorporating the filtered estimate and vector $\lambda(i)$, may also be represented in the scattering form

$$\begin{bmatrix} \hat{x}(i/i) \\ P^{-1}(i)\{\hat{x}(i/i)-\Phi(i-1)\hat{x}(i-1/i-1)\} \end{bmatrix} = T\begin{bmatrix} R^{-1}(i)z(i) \\ \hat{x}(i-1/i-1) \end{bmatrix} \quad , \quad T = \begin{bmatrix} H^T(i) & -H^T(i)R^{-1}(i)H(i) \\ H^T(i) & -H^T(i)R^{-1}(i)H(i) \end{bmatrix} * \begin{bmatrix} P(i) & \Phi(i-1) \\ P(i) & \Phi(i-1) \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}(i/N) \\ P^{-1}(i)\{\hat{x}(i/N)-\hat{x}(i/i)\} \end{bmatrix} = T\begin{bmatrix} \lambda(i) \\ \hat{x}(i/i) \end{bmatrix} \quad , \quad T = \begin{bmatrix} \Phi^T(i) & -H^T(i)R^{-1}(i)H(i) \\ \Phi^T(i) & -H^T(i)R^{-1}(i)H(i) \end{bmatrix} * \begin{bmatrix} P(i) & \delta \\ P(i) & 0 \end{bmatrix}$$

The system matrix of eqn 24 can be associated with the impedance matrix $\bar{Z}$ of eqn 3. An orthogonal network-type matrix N_o then exists in the linear two-point boundary value problem with

$$\begin{bmatrix} \hat{x}(i/N) \\ \lambda(i-1) \end{bmatrix} = N_o\begin{bmatrix} \lambda(i) \\ \hat{x}(i+1/N) \end{bmatrix} + \begin{bmatrix} 0 \\ H^T(i)R^{-1}(i)z(i) \end{bmatrix} = N_o\begin{bmatrix} \lambda(i)+Z_4^{-1}\,H^T(i)R^{-1}(i)z(i) \\ \hat{x}(i+1/N)+Z_2Z_4^{-1}H^T(i)R^{-1}(i)z(i) \end{bmatrix}$$

$$\text{where}\quad N_o = \begin{bmatrix} -Z_1^{-1}Z_2 & Z_1^{-1} \\ Z_4-Z_3Z_1^{-1}Z_2 & Z_3Z_1^{-1} \end{bmatrix} \equiv$$

$$\begin{bmatrix} -\Phi^{-1}(i)\Delta(i)Q(i)\Delta^T(i) & \Phi^{-1}(i) \\ \Phi^T(i)+H^T(i)R^{-1}(i)H(i)\Phi^{-1}(i)\Delta(i)Q(i)\Delta^T(i) & -H^T(i)R^{-1}(i)H(i)\Phi^{-1}(i) \end{bmatrix}$$

Closed- and open-path response and source variables may then be related to the smoothing problem variables. A similar representation has been considered as the basis of a parallel filter-smoother algorithm[28], with the elements of a matrix corresponding to $\bar{Z}$ forming sweep quantities which can be obtained from a parallel step solution of a differential equation set similar to eqn 7. The smoothed estimate may also be obtained as an optimal combination of two filtering estimates, as in Section 4.7.1. The corresponding covariance matrices can then be obtained from

$$P_1(i) = -A^{-1}(i)B(i) \qquad P_2^{-1}(i) = J^{-1}(i)K(i)$$

$$\text{where}\quad \begin{bmatrix} B(i) & A(i) \\ K(i) & J(i) \end{bmatrix} = \begin{bmatrix} A(i-1) & B(i-1) \\ J(i-1) & K(i-1) \end{bmatrix} N_o(i) \;,\; \begin{bmatrix} B(o) & A(o) \\ K(f) & J(f) \end{bmatrix} = \begin{bmatrix} P_o^{-1} & \delta \\ 0 & \delta \end{bmatrix} \qquad (29)$$

with Z_2, Z_3 symmetrical. Expansion of eqn 29 then gives

$$P_1(i) = Z_2 + Z_1P_1(i-1)\left[\delta - Z_3P_1(i-1)\right]^{-1}Z_4$$
$$P_2(i-1) = -Z_3 + Z_4P_2(i)\left[\delta + Z_2P_2(i)\right]^{-1}Z_1$$

The covariance matrices are also defined by the star product of eqn 28 and by

$$\begin{bmatrix} Z_1(i+1) & -Z_2(i+1) \\ -Z_3(i+1) & Z_4(i+1) \end{bmatrix} * \begin{bmatrix} - & - \\ P_2(i+1) & - \end{bmatrix} = \begin{bmatrix} - & - \\ P_2(i) & - \end{bmatrix}$$

The corresponding transformation diagrams are illustrated in FIG 13.

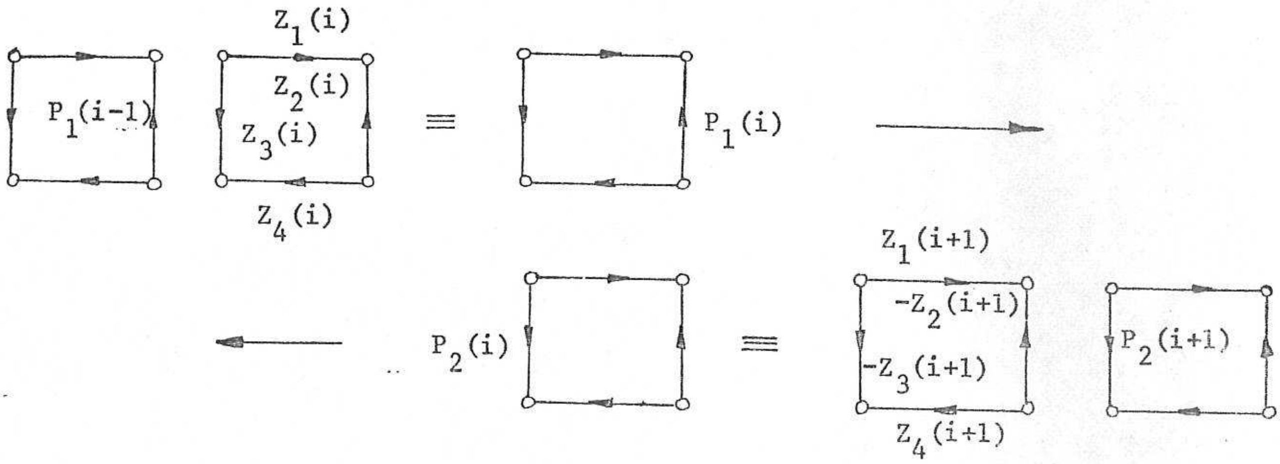


FIG 13 Foward and reverse sweeps in scattering form

The scattering formulation and the solution of the orthogonal electrical network problem which forms the basis of Kron's polyhedron model exist within the two-point boundary value problem, and the sweep method of solution incorporating a combination of forward and reverse-time optimal filters appears to have relevance to the concept of wave propagation in Kron's model. The formalism is also similar to the procedure of invariant imbedding which has been applied in a number of different fields, including wave propagation and neutron transport processes, and can be used for reducing coupled differential equations subject to two-point boundary conditions to two sets of initial value problems using a sweep method[29,30].

5. Conclusions

Kron made a significant contribution by formulating the electrical network problem in terms of closed- and open-path variables and in proposing a wide application for its basic structure. The resulting orthogonal network was developed into a polyhedron model representing a geometrical structure which could accommodate the dielectric and electromagnetic phenomena in electrical networks and wave propagation in higher dimensional structures. The general application of this work was discussed at an early stage, before the development of other aspects of system theory, essentially on the basis of physical intuition and a deep understanding of the behaviour of electrical networks and electromagnetic wave propagation. However, the relevance of Kron's work has not been fully appreciated and accepted possibly because of the emphasis on physical reasoning and the apparent lack of an adequate mathematical description.

The polyhedron model includes a cascade connection of multidimensional networks and can be associated with the scattering representation of a flow process incorporating a functional equation related to the behaviour of the subunits, which also appears as an underlying concept in Kron's work. The basic structure has also been shown to exist in the least squares problem, and exists inherently in the linear two-point boundary value problem concerned with optimal control and estimation. The forward and reverse integration procedures forming the sweep method of solution then appear similar to the process of wave propagation in Kron's model. The orthogonal network and scattering problems thus possess similar basic characteristics within an algebraic framework which incorporates a condition of optimality. They are also consistent with the concept of dynamic programming and invariant imbedding in which a complex system is represented as a sequence of lower dimensional processes, and could have an important application in the structuring of system problems for parallel computation.

The solution of the discrete optimal control and estimation problems includes the updating of a priori information in a stage-wise procedure, and a similar recurrence-type process is incorporated inherently within the general scattering problem. The subunits interact according to the properties of reflection operators which introduce an implicit feedback or return-difference effect and in the continuum the solution reduces to a differential law. Kron's network sequence is connected using the properties of incidence matrices associated with a physical topological structure, and the a priori-type information appears to be contained within the orthogonal network solutions. The polyhedron model thus provides an improved least squares estimate which Kron considered to be an essential property of the model, particularly in its application to nonlinear curve fitting using divided differences related to physical variables.

Kron introduced many concepts into the polyhedron model based on a wide knowledge of physical phenomena, and although certain of these concepts concerned for example with oscillatory behaviour and multidimensional generalised machines lack an analytical basis, the relevance of the model to the general scattering problem would support many of Kron's proposals. The polyhedron now appears to have wide application, particularly in non-linear distributed parameter systems, although further understanding is required. The proposed application in nonlinear estimation, based on the concept of introducing the structural properties of a physical system into an abstract data fitting problem, is also of considerable significance.

Recent work indicates that the scattering formulation can provide a unified reference frame for combining the interacting space- and time-coordinate vectors in dynamical systems. It introduces a recurrence form in which the discrete-time response of mutually interacting subsystems evolve as a stage-wise scattering process. Kron also used the orthogonal network problem and spatially orthogonal sequences in the polyhedron model, combined with the technique of tearing, as a space-time configuration for the analysis of distributed parameter systems. The scattering formulation based on a single-dimensional, mutually-interacting sequence, can also be extended into a multidimensional cellular space and could have many interesting and important applications in the analysis of cellular automata representing large numbers of interacting elements.

Kron's work includes many original and inspiring ideas and concepts relevant to the structure of complex systems. However, the potential of the polyhedron model and also the extended scattering problem have still to be realised, particularly in the study of large scale systems. This may require the symbolic language of a tensor notation as used by Kron, in order to group the vast array of space-time variables defined within a cellular space. The polyhedron model and the general theory of scattering can then provide a basic framework for the development of a theory of complex systems which is required in many fields of study.

6. Acknowledgments

Stimulating discussions with Dr. J. W. Lynn of the Department of Electrical Engineering and Electronics, University of Liverpool, provided an understanding and appreciation of Kron's work which initiated the present study, and the contribution is gratefully acknowledged.

7. References

1. NICHOLSON, H.: 'Structure of Kron's polyhedron model and the scattering problem', Proc.IEE, 1973, 120, 12, pp.1545-1557.

2. NICHOLSON, H.: 'Concepts of scattering in the linear optimal-control problem', Proc.IEE, 1971, 118, 12, pp.1823-1829.

3. KRON, G.: 'Multidimensional space filters', The Matrix and Tensor Quarterly, 1958, 9, 2, pp.40-46.

4. KRON, G.: 'Basic concepts of multidimensional space filters', Trans.AIEE, Part 1, Commns. and Electronics, 1959, 78, pp.554-561.

5. KRON, G.: 'Power-system type self-organizing automata', RAAG Memoirs, Vol.3, 1962, pp.392-417.

6. KRON, G.: 'Multidimensional curve-fitting with self-organizing automata', Jl. of Mathl. Analysis and Applicns., 1962, 5, pp.46-69.

7. KRON, G.: 'Camouflaging electrical l-networks as graphs', Quarterly of Applied Maths., 1962, 20, pp.161-174.

8. KRON, G.: 'A generalization of the calculus of finite differences to nonuniformly spaced variables', Trans.AIEE, Part 1, Commns. and Electronics, 1958, No.38, pp.539-544.

9. KRON, G.: 'Self-organizing, dynamo-type automata', The Matrix and Tensor Quarterly, 1960, 11, 2, pp.42-55.

10. NICHOLSON, H.: 'Properties of the generalised inverse matrix in the electrical network problem', Electronics Letters, 1972, 8, (10), pp.267-268.

11. NICHOLSON, H.: 'Transformations of linear-graph theory in multivariable-system problems', Electronics Letters, 1972, 8, (10), pp.268-270.

12. NICHOLSON, H.: 'Sequential least-squares estimation and the electrical-network problem', Proc.IEE, 1971, 118, (11), pp.1635-1641.

13. BRYSON, A.E., and HO, Y.C.: 'Applied optimal control', Ginn & Co., 1969.

14. MACFARLANE, A.G.J.: 'Return-difference matrix properties for optimal stationary Kalman-Bucy filter', Proc.IEE, 1971, 118, (2), pp.373-376.

15. COX, H.: 'On the estimation of state variables and parameters for noisy dynamic systems', Trans.IEEE AC-9, 1964, 1, pp.5-12.

16. COX, H.: 'Estimation of state variables via dynamic programming', JACC, 1964, Stanford Univ., pp.376-381.

17. MAYNE, D.Q.: 'A solution of the smoothing problem for linear dynamic systems', Automatica, 1966, 4, pp.73-92.

18. PEARSON, J.D.: 'On the duality between estimation and control', Jl. SIAM Control, 1966, 4, 4, pp.594-600.

19. SAATY, T.L., and BRAM, J.: 'Nonlinear Mathematics', McGraw-Hill, 1964, p.357.

20. SAGE, A.P.: 'Maximum a posteriori filtering and smoothing algorithms', Intl. Jl. Control, 1970, 11, 4, pp.641-658.

21. MEDITCH, J.S.: 'A survey of data smoothing for linear and nonlinear dynamic systems', Automatica, 1973, 9, pp.151-162.

22. KAILATH, T., and FROST, P.: 'An innovations approach to least-squares estimation Part II: Linear smoothing in additive white noise', Trans.IEEE AC-13, 1968, 6, pp.655-660.

23. MEHRA, R.K.: 'On a limiting procedure in linear recursive estimation theory', Jl. of Mathl. Analysis & Applicns., 1969, 26, pp.1-8.

24. RAUCH, H.E.: 'Solutions to the linear smoothing problem', Trans.IEEE AC-8, 1963, 4, pp.371-372.

25. SAGE, A.P.: 'Optimum systems control', Prentice-Hall, 1968, p.236.

26. FRASER, D.C., and POTTER, J.E.: 'The optimum linear smoother as a combination of two optimum linear filters', Trans.IEEE AC-14, 1969, 4 , pp.387-390.

27. DEUTSCH, R.: 'Estimation theory', Prentice-Hall, 1966, p.108.

28. McREYNOLDS, S.R.: 'Parallel data processing algorithms', Proc.IEEE Conf. on Decision and Control and 11th Sympm. on Adaptive Processes, New Orleans, 1972, pp.657-660.

29. ANGEL, E., and KALABA, R.: 'A one-sweep numerical method for vector-matrix difference equations with two-point boundary conditions', Jl. Optmn. Theory & Applns., 1970, 6, 5, pp.345-355.

30. ANGEL, E., and BELLMAN, R.: 'Dynamic programming and partial differential equations', Academic Press, 1972.

31. MACFARLANE, A.G.J.: 'A survey of some recent results in linear multi-variable feedback theory', Automatica, 1972, 8, pp.455-492.

32. GOODWIN, B.C.: 'Temporal organization in cells', Academic Press, 1963, pp.23-54.

33. WATSON, J.D.: 'The double helix', Weidenfeld & Nicolson, 1968, p.202.