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Stability of Systems with Multiple Time Delays

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Stability of Systems with Multiple Time Delays

Abstract

Systems whose dynamical representations involve multiple transportation lags are generally difficult to analyse. Such systems give rise to infinite strings of poles in the complex plane and subsequently, transfer functions of a transcendental nature. A Nyquist stability analysis of a system with two delay terms reveals a central region of stability even though the system has an infinite number of poles with positive real parts. An input-output stability criterion due to Zadeh is used to confirm the Nyquist analysis and show that systems with an infinite number of poles are not necessarily subject to the same pole location restrictions as systems with a finite number of poles.

Many industrial processes involve transportation lags and when these lags occur in both the states and inputs to the system, quite difficult nonrational transfer functions occur. Various authors have considered systems with constant time lags as inputs only and in particular Ansoff [1], [2], develops stability criterion for linear oscillating systems with a time delay in the feedback term. Many standard methods of stability analysis have been applied to time delayed systems with Laplace transforms [3], Nyquist analysis [4], Root Locus [5] and Lyapunov's method [6], [7] all yielding useful stability criterions.

Edwards [8], [9] has examined multipass systems in which the material or workpiece involved is processed by a sequence of passes of the processing tool. In particular, we consider an underground coal cutter which is a single input single output system described by the difference equation

$$y(x) = y(x-T_1) + u(x-T_2) \qquad (1)$$

where x is a distance variable, u is an input to the system and T_1, T_2 are arbitrary delay distances chosen such that $T_1 >> T_2$. The input u acts as a control element and can be written as

$$u(x-T_2) = K_1 [y_{ref} - y(x-T_2)] \qquad (2)$$

where y_{ref} is the initial value of y and K_1 is a constant gain factor.

Combining (1) and (2) and applying the Laplace transform with respect to x yields the system transfer function

$$F(s) = \frac{K_1}{1-e^{-T_1 s} + K_1 e^{-T_2 s}} \qquad (3)$$

Nyquist Analysis

Edwards[9] has derived the open loop transfer function of the system in (1) to be

$$G(s) = \frac{K_1 e^{-T_2 s}}{1-e^{-T_1 s}} \tag{4}$$

and applied an inverse Nyquist plot of (4) to obtain Figure 1. The process is clearly unstable for all practical values of T_2 since the plot shows the critical point to be encircled for all gains K_1.

Series Expansion

Since the useful stability criterion developed by Routh and Hurwitz can only be applied to polynomials and hence to systems with rational transfer functions, we may obtain approximate results for nonrational transfer functions by using series expansions of the exponential terms in the characteristic equation

$$F_1(s) = 1 - e^{-T_1 s} + K_1 e^{-T_2 s} \tag{5}$$

Thus we can write

$$F_1(s) = 1 - \left[1-T_1 s + \frac{T_1^2 s^2}{2!} - \frac{T_1^3 s^3}{3!} + \ldots\right]$$

$$+ K_1\left[1-T_2 s + \frac{T_2^2 s^2}{2!} - \frac{T_2^3 s^3}{3!} + \ldots\right] \tag{6}$$

and grouping in powers of s yields

$$F_1(s) = K_1+(T_1-K_1T_2)s + \frac{(-T_1^2+K_1T_2^2)s^2}{2!} + \frac{(T_1^3-K_1T_2^3)s^3}{3!}$$

$$+ \ldots \tag{7}$$

It is immediately clear that the system in (1) has an infinite number of poles. For these poles to all be in the same half of the complex plane, it can easily be shown from the Routh criterion that all the coefficients in (7) must be of the same sign. Since K_1 is positive, each coefficient must be positive, ie

$$\frac{T_1^k - K_1 T_2^k}{k!} > 0 \qquad (8)$$

for any odd integer k, and

$$\frac{K_1 T_2^{k+1} - T_1^{k+1}}{(k+1)!} > 0 \qquad (9)$$

must be satisfied. Combining (8) and (9) yields

$$\frac{T_1^{k+1}}{T_2^{k+1}} < K_1 < \frac{T_1^k}{T_2^k} \qquad (10)$$

which clearly cannot be satisfied for any choice of K_1 since $T_1 >> T_2$. Thus, the system in (1) has poles in both halves of the complex plane.

Pole Location

A great deal of literature is available on the location of poles of systems with delays [3], [10]. We define an exponential polynomial to be of the form

$$F_1(s) = \sum_{i=o}^{n} a_i e^{c_i s} \qquad (11)$$

where a_i and c_i are real constants. If there is some real α such that

$$c_i = \alpha p_i \qquad i = 1,2,...,n \qquad (12)$$

for integral values p_i, then

$$F_1(s) = \sum_{i=o}^{n} a_i \, (e^{\alpha s})^{p_i} \quad , \quad p_o = 0 \tag{13}$$

If we can find the p_n roots of the polynomial in $e^{\alpha s}$ defined as $\xi_1, \xi_2, \ldots, \xi_{p_n}$; then it can be shown [3] that the zeros of $F_1(s)$ are given by

$$s_i = \frac{1}{\alpha} \left[2m\pi j + \log|\xi_i|\right] \qquad i = 1,2,\ldots,p_n \quad m = 0,\pm1,\pm2,\ldots \tag{14}$$

Since T_1, T_2 and K_1 are arbitrary, we can put (5) in the exponential polynomial form

$$F_1(s) = (e^{\alpha s})^{p_n} - K_1(e^{\alpha s}) - 1 = 0 \tag{15}$$

where $\alpha = -T_2$.

Thus the zeros of $F_1(s)$ will lie along p_n lines, normal to the real axis of the complex plane defined by

$$Re(s_i) = \frac{-\log|\xi_i|}{T_2} \quad , \quad i = 1,2,\ldots,p_n \tag{16}$$

If we take $p_n = 2$, the roots of (15) are

$$\xi_1 = \frac{K_1 + \sqrt{K_1^2 + 4}}{2} \quad , \quad \xi_2 = \frac{K_1 - \sqrt{K_1^2 + 4}}{2}$$

Now since $\log|\xi_1|$ and $\log|\xi_2|>0$, the two lines of zeros specified by $p_n = 2$ will both lie in the left hand plane for $K_1<1$. If $K_1>1$, then $\log|\xi_2|<0$ and one line of zeros will lie in the right half plane. The Nyquist analysis shows that the system will be unstable for any choice of K_1. Thus, for a system with an infinite number of closed loop poles, it is not a sufficient condition of stability that all the poles have negative real parts.

A question little considered by other authors is whether the location of closed loop poles is a necessary condition for stability of systems with an infinite number of closed loop poles. We will consider a system which has an infinite number of poles in both halves of the complex plane, yet has a region of stability in which the system can be said to be input-output stable.

The system in question arises in a metal rolling multipass process and has been studied by Edwards [9]. The system open loop transfer function is of the form

$$G_2(s) = \frac{K_1 e^{-T_2 s}}{1-K_2 e^{-T_1 s}} \qquad (17)$$

and is the same as that of system in (1) except for the nonunity gain factor K_2 in the interpass feedback loop. T_1 and T_2 are delay distances as before and K_1 a constant gain factor. Edwards [9] has used an inverse Nyquist plot of this system, shown in Figure 2, to reveal a central region of stability for a suitable choice of $K_1, K_2 < 1$. It can easily be shown that the closed loop transfer function of this system with the gain factor K_2 in the interpass feedback loop is of the form

$$H(s) = \frac{K_1}{1-K_2 e^{-T_1 s} + K_1 e^{-T_2 s}} \qquad (18)$$

We can utilize the series expansion of the characteristic equation of H(s) denoted by $H_1(s)$ to examine the poles of H(s). Therefore

$$H_1(s) = (1-K_2+K_1)+(K_2T_1-K_1T_2)s + \frac{(-K_2T_1{}^2+K_1T_2{}^2)s^2}{2!} + \ldots \qquad (19)$$

which yields the inequality

$$\frac{T_1^{k+1}}{T_2^{k+1}} < \frac{K_1}{K_2} < \frac{T_1^{k}}{T_2^{k}} \tag{20}$$

where k is any odd integer. The inequality is (20) cannot be satisfied for any choice of $K_1, K_2 < 1$. Thus the system with characteristic equation $H_1(s)$ appears to have poles in both halves of the complex plane.

We can locate these poles as before by writing $H_1(s)$ as an exponential polynomial of the form

$$H_1(s) = (e^{\alpha s})^{p_n} - \frac{K_1}{K_2}(e^{\alpha s}) - \frac{1}{K_2} = 0 \tag{21}$$

where $\alpha = -T_2$. As before, choosing $p_n = 2$ the roots of the polynomial in $e^{\alpha s}$ are

$$\xi_1 = \frac{\frac{K_1}{K_2} + \sqrt{K_1^2/K_2^2 + 4/K_2}}{2} \qquad \xi_2 = \frac{\frac{K_1}{K_2} - \sqrt{K_1^2/K_2^2 + 4/K_2}}{2}$$

and $\xi_2 < 0$ ensures that $H_1(s)$ will have at least one infinite line of zeros in the right half plane for any K_1, K_2.

Thus it appears that the system above has both a region of stability for suitable choice K_1 and K_2 as well as poles with positive real parts.

Impulse Response Criterion

Many studies of stability of variable systems are based upon boundedness and asymptotic behaviour of solutions. For the remainder of this paper we shall be concerned only with the so-called input-output stability of a system. In particular, we consider a forced system, the stability of which is defined as follows:

Definition

A forced system is stable with respect to a set of inputs $U = [u(x)]$, if and only if the system output is bounded for all u(x) in the set U for all $x>x_o$. That is, there exists a constant Q such that

$$\|y(x)\| \leq Q < \infty$$

for all u(x) in the set U for all $x>x_o$.

This definition, introduced by James and Weiss [14] leads naturally to the basic stability theorem for forced systems developed by Zadeh [11].

Theorem I

A variable forced system is stable with respect to a set of bounded inputs $U = [u(x)]$ if, and only if its' impulse response $\bar{H}(x,\tau)$ is integrable with respect to τ for all values of x; that is

$$\int_0^\infty |\bar{H}(x,\tau)|\,d\tau < \infty \qquad \text{for all } x.$$

Zadeh [11] has shown this to be a necessary and sufficient condition for input-output stability. The impulse response stability criterion has received much comment by various authors [12], [13]. Desoer and Wu [15] have found a form of a system's open loop impulse response which can be related directly to a Nyquist diagram of the open loop gain to give stability criterion for the closed loop system. This approach is not satisfactory for the system in question since we wish to examine closed loop characteristics and in particular the closed loop impulse response.

The criterion in Theorem I will be of great use to us under the following conditions

1) The system closedloop transfer function is known explicitly.
2) It is possible to analytically invert the system transfer function to yield an explicit expression of the system's closed loop impulse response.

The first condition above can be satisfied with the system transfer function

$$H(s) = \frac{K_1}{1-K_2e^{-T_1s}+K_1e^{-T_2s}} \tag{22}$$

To satisfy the second condition we can use the complex Laplace inversion formula as follows

$$h(x) = L^{-1}[H(s)] = \lim_{Y\to\infty} \frac{1}{2\pi j} \int_{\alpha-jY}^{\alpha+jY} \frac{e^{xs}K_1}{1-K_2e^{-T_1s}+K_1e^{-T_2s}}\, ds \tag{23}$$

We know that H(s) has an infinite number of simple poles $\pm s_i$ where $i = 1,2,\ldots$ If we consider a closed curve Γ inside of which H(s) is single-valued and analytic everywhere except at the singularities $s = \pm s_i (i = 1,2,\ldots,n)$, then the Cauchy residue theorem yields

$$\frac{1}{2\pi j} \int_\Gamma e^{xs}H(s)ds = \text{sum of residues of } e^{xs}H(s) \text{ at } s = \pm s_i \quad i = 1,2,\ldots,n \tag{24}$$

Even though H(s) has an infinite number of poles, only a finite number s_n need be considered to affect the stability of H(s). We calculate the residues in (24) by the following theorem.

Theorem II

If $H_1(s)$ and $H_2(s)$ are analytic in the neighbourhood of s_o and if $H_1(s_o) \neq 0$, but $H_2(s_o)$ has a simple zero at s_o, then the residue of $H_1(s)/H_2(s)$ at s_o is equal to $H_1(s_o)/H'_2(s_o)$.

Thus we can write

$$\frac{1}{2\pi j} \int_\Gamma e^{xs}H(s)ds = \sum_{i=1}^{n} \frac{K_1e^{xs_i}}{K_2T_1e^{-T_1s_i}-K_1T_2e^{-T_2s_i}} \tag{25}$$

Now let R tend to ∞ in Figure 3. Then by the inversion integral, AB tends to h(x). The behaviour of the integral along CE is found from the following theorem.

Theorem III

If for $r\to\infty$, $H(s) = H(re^{j\phi})$ tends to zero uniformly in ϕ in the left half plane $Re(s)<0$, ie $\pi/2<\phi<3\pi/2$, and if δ is a semicircle in the left hand plane of radius r around the origin then

$$\int_\delta e^{xs}H(s)ds \to 0 \qquad \text{for } r \to \infty$$

when $x>0$.

For s in the left hand plane

$$e^{-T_1 s} > 1 \qquad , \qquad e^{-T_2 s} > 1$$

and thus, in the lefthand plane, when $|s|\to\infty$, H(s) tends uniformly in ϕ to zero. The integral along CE tends to zero as $R\to\infty$ for $x>0$. e^{xs}, $e^{-T_1 s}$ and $e^{-T_2 s}$ are bounded along arcs BC and EA and these contributions to the integral also converge to zero since the length of the path of integration is bounded.

Thus

$$h(x) = \sum_{i=1}^{n} \frac{K_1 e^{xs_i}}{K_2T_1e^{-T_1s_i} - K_1T_2e^{-T_2s_i}} \qquad (26)$$

and we can write the system impulse response $\bar{H}$ as

$$\bar{H}(x,\tau) = \sum_{i=1}^{n} \frac{K_1 e^{(x-\tau)s_i}}{K_2T_1e^{-T_1s_i} - K_1T_2e^{-T_2s_i}} \qquad (27)$$

To establish stability, it is simply left to show that $\bar{H}(x,\tau)$ is integrable, which will be the case if h(x) is bounded. We can simply consider the expression for h(x) in (26) by considering only the s_m poles with positive real parts. If the system has k lines of poles in the right half complex plane, each with ℓ poles with positive real part a_k, then we can let

$$s_i = a_i + jy_i \tag{28}$$

where $\quad i = 1,2,\ldots,m$

$$a_1 = a_2 = \ldots = a_\ell$$

and $\quad k\ell = m$

If we use the well known identity

$$\exp(\pm jx) = \cos x \pm j \sin x \tag{29}$$

then

$$h(x) = \sum_{i=1}^{m} \frac{K_1 e^{a_i x}(\cos y_i x + j \sin y_i x)}{K_2 T_1 e^{-T_1 a_i}(\cos T_1 y_i - j\sin T_1 y_i) - K_1 T_2 e^{-T_2 a_i}(\cos T_2 y_i - j \sin T_2 y_i)} \tag{30}$$

Since from (14), the complex poles occur in conjugate pairs,

$$h(x) = 2 \times \mathrm{Re}\left[h(x)\right] \tag{31}$$

Thus, separating (30) into real and imaginary parts and applying (31) yields

$$h(x) = 2 \times \sum_{i=1}^{m/2} K_1 e^{a_i x}\left[R_i(x)\right] \tag{32}$$

where

$$R_i(x) = \frac{A_i \cos\left[y_i \cdot (T_1 + x)\right] - B_i \cos\left[y_i \cdot (T_2 + x)\right]}{A_i^2 + B_i^2 - 2A_i B_i \cos\left[y_i \cdot (T_1 + T_2)\right]} \tag{33}$$

and

$$A_i = K_2 T_1 e^{-T_1 a_i}$$

$$B_i = K_1 T_2 e^{-T_2 a_i}$$

For h(x) to be bounded, it is sufficient to show that there exists a constant M such that

$$|R_i(x)| < M < \infty, \quad i = 1,2,\ldots,m/2 \tag{34}$$

If T_1 and T_2 are chosen so that $|R_i(x)|$ is maximized, then

$$\begin{aligned} \cos\left[y_i \cdot (T_1+x)\right] &= +1 \\ \cos\left[y_i \cdot (T_2+x)\right] &= -1 \\ \cos\left[y_i \cdot (T_1+T_2)\right] &= +1 \end{aligned} \tag{35}$$

and

$$\max\left[R_i(x)\right] = \frac{A_i+B_i}{A_i^2+B_i^2-2A_iB_i} \tag{36}$$

Substituting in (36) for A_i and B_i yields

$$\max\left[R_i(x)\right] = \frac{K_2T_1e^{-T_1a_i}+K_1T_2e^{-T_2a_i}}{K_2^2T_1^2e^{-2T_1a_i}+K_1^2T_2^2e^{-2T_2a_i}-2K_2K_1T_1T_2e^{-a_i(T_1+T_2)}} \tag{37}$$

Thus, each line of poles in the right half complex plane will contribute one value of a, $\ell/2$ times and if there are k lines, then each line will have a constant value of R associated with it of the form

$$R = \frac{K_2T_1e^{-T_1a}+K_1T_2e^{-T_2a}}{K_2^2T_1^2e^{-2T_1a}+K_1^2T_2^2e^{-2T_2a}-2K_2K_1T_1T_2e^{-a(T_1+T_2)}} \tag{38}$$

since K_2, K_1, T_1, T_2 and a are all constants.

Now R can only be infinite if its denominator is zero. Thus we must show that the equality

$$K_2^2T_1^2e^{-2T_1a}+K_1^2T_2^2e^{-2T_2a} = 2K_2K_1T_1T_2e^{-a(T_1+T_2)} \qquad (39)$$

cannot be satisfied.

If we let $T_1 = T_2^t$ in (39) and divide each term by T_2^2, then the equality becomes

$$K_2^2T_2^te^{-2T_2^ta}+K_1^2e^{-2T_2a} = 2K_2K_1T_2^{(t-1)}e^{-a(T_2^t+T_2)} \qquad (40)$$

Now multiplying each term in (40) by $e^{2T_2^ta}$ yields

$$K_2^2T_2^t+K_1^2e^{2a(T_2^t-T_2)} = 2K_2K_1T_2^{(t-1)}e^{a(T_2^t-T_2)} \qquad (41)$$

If we let $W = a(T_2^t-T_2)$ then (41) becomes

$$K_2^2T_2^t+K_1^2e^{2W} = 2K_2K_1T_2^{(t-1)}e^W \qquad (42)$$

Writing E(W) for e^W and E(2W) for e^{2W} and rearranging (42) yields

$$\frac{K_2T_2}{2K_1} + \frac{K_1}{2K_2T_2^{(t-1)}}E(2W) = E(W) \qquad (43)$$

Since W will be some constant value dependent on T_2 and a, there exists a constant N such that

$$E(2W) = N \cdot E(W) \qquad (44)$$

where $E(W), E(2W), N > 1$.

Substituting (44) into (43) yields

$$\frac{K_2 T_2}{2K_1} = \left[1 - \frac{N \cdot K_1}{2K_2 T_2^{(t-1)}} \right] E(W) \qquad (45)$$

If we choose $K_1, K_2 < 1$, then for a fixed K_2 we can ensure that the equality in (45) does not hold by choosing

$$K_1 < 1 - K_2 \qquad (46)$$

for a suitable value of T_2. If the criterion in (46) is satisfied, then the right hand side of (45) will tend to E(W) and the left hand side to a factor of T_2. Since

$$E(W) > \left[\frac{K_2}{2K_1} \right] T_2 \qquad (47)$$

for any $K_1 < 1 - K_2$, the equality will not hold. There may be some small value of T_2 which satisfies the equality for ill-chosen K_1 and K_2, since E(W) will grow large faster than T_2 for large T_2. But in general, suitable T_2 can ensure that even the largest R will be bounded by some constant M and

$$\int_0^\infty \bar{H}(x,\tau)\, d\tau < \infty \qquad (48)$$

The criterion in (46) is the same as that developed by Edwards [9] and input-output stability is clearly dependent on suitable choice of K_1, K_2 and T_2. This stability demonstrates that for a system with an infinite number of closed loop poles, pole location to the left of the imaginary axis in the complex plane is not a necessary condition of stability.

It is expected that the same result would be achieved by examining the impulse response of the open loop system and using the result of Desoer and Wu [15], although this has not been attempted here.

Conclusions

A system whose open loop characteristics have yielded an inverse Nyquist plot with a central region of stability has been shown to have an infinite number of closed loop poles in both halves of the complex plane. It appears that systems with lines of infinite closed loop poles can exhibit stable properties when excited by bounded inputs, even when at least one line of poles lies to the right of the imaginary axis. The result may be due to the impulsive nature of the system transient response and further study of such responses is required.

Since it is possible to generate an explicit series representation of the impulse response of systems with infinite numbers of poles, the impulse response criterion used here may be generalized and standard criterion for systems of this type developed. This work could fill a large gap in the understanding of systems with multiple time delays.

Fig. 1 Inverse Nyquist Diagram for Coal Cutting Process

$12T_2 = T_1$

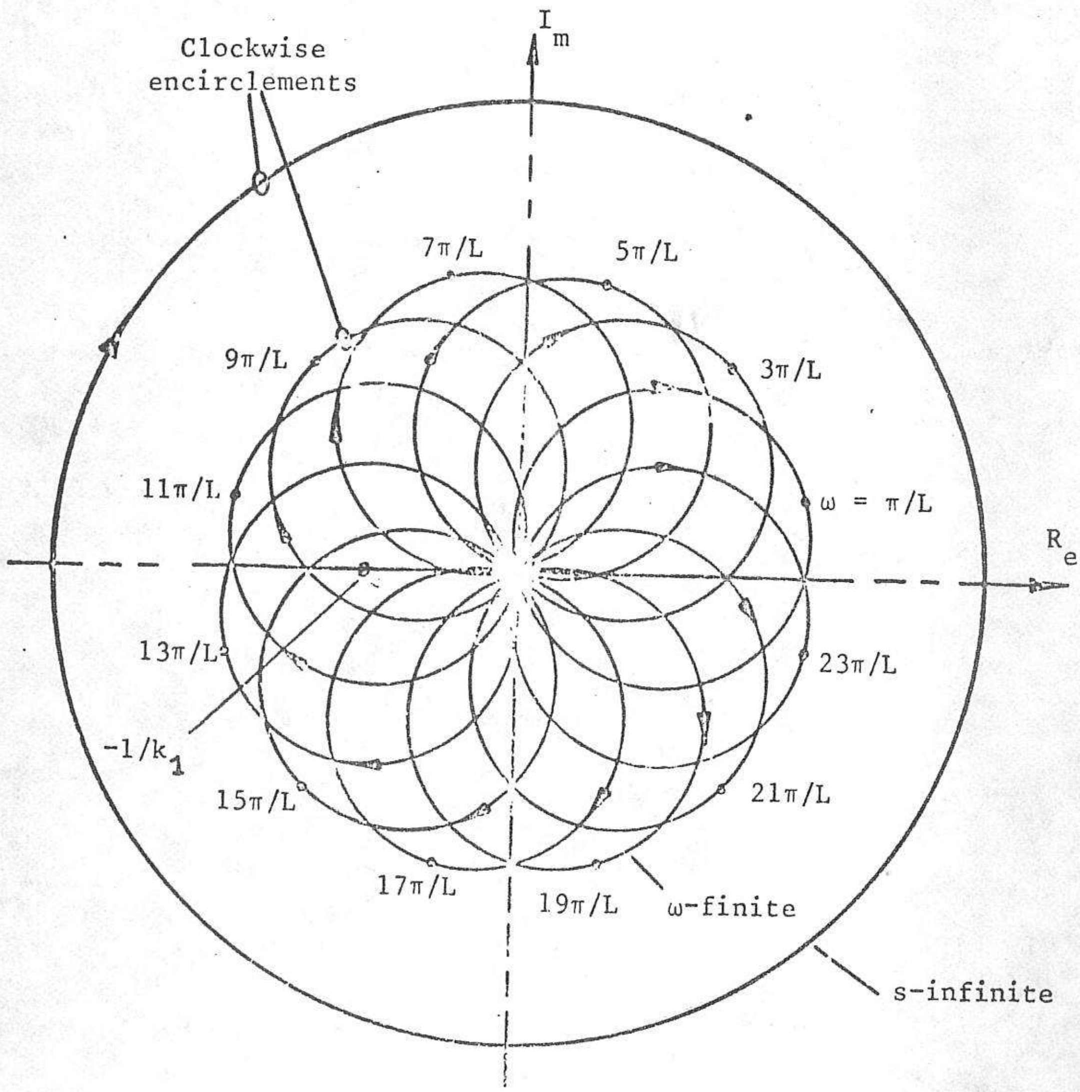


Fig. 2 Inverse Nyquist Diagram for Rolling Process

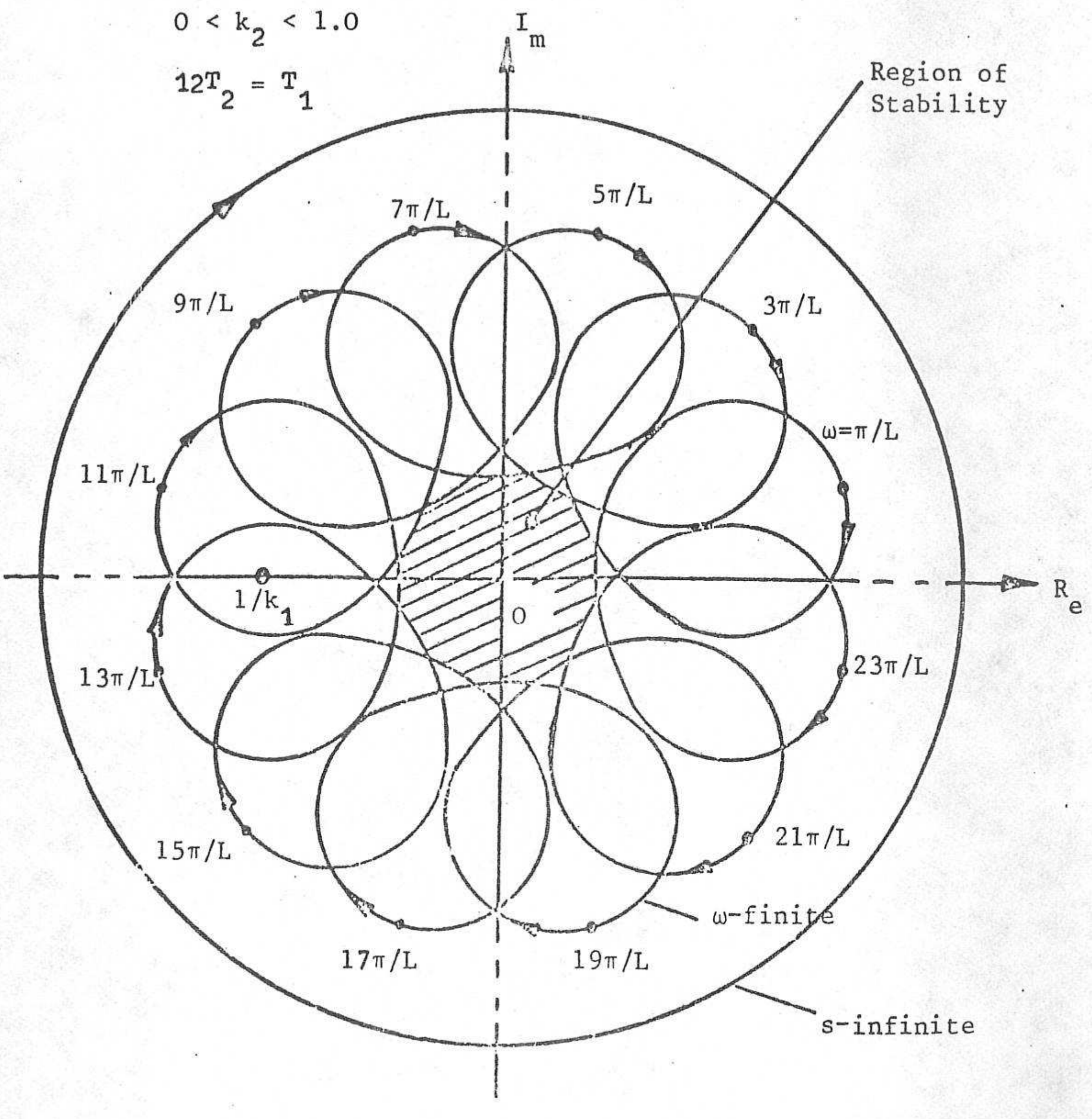
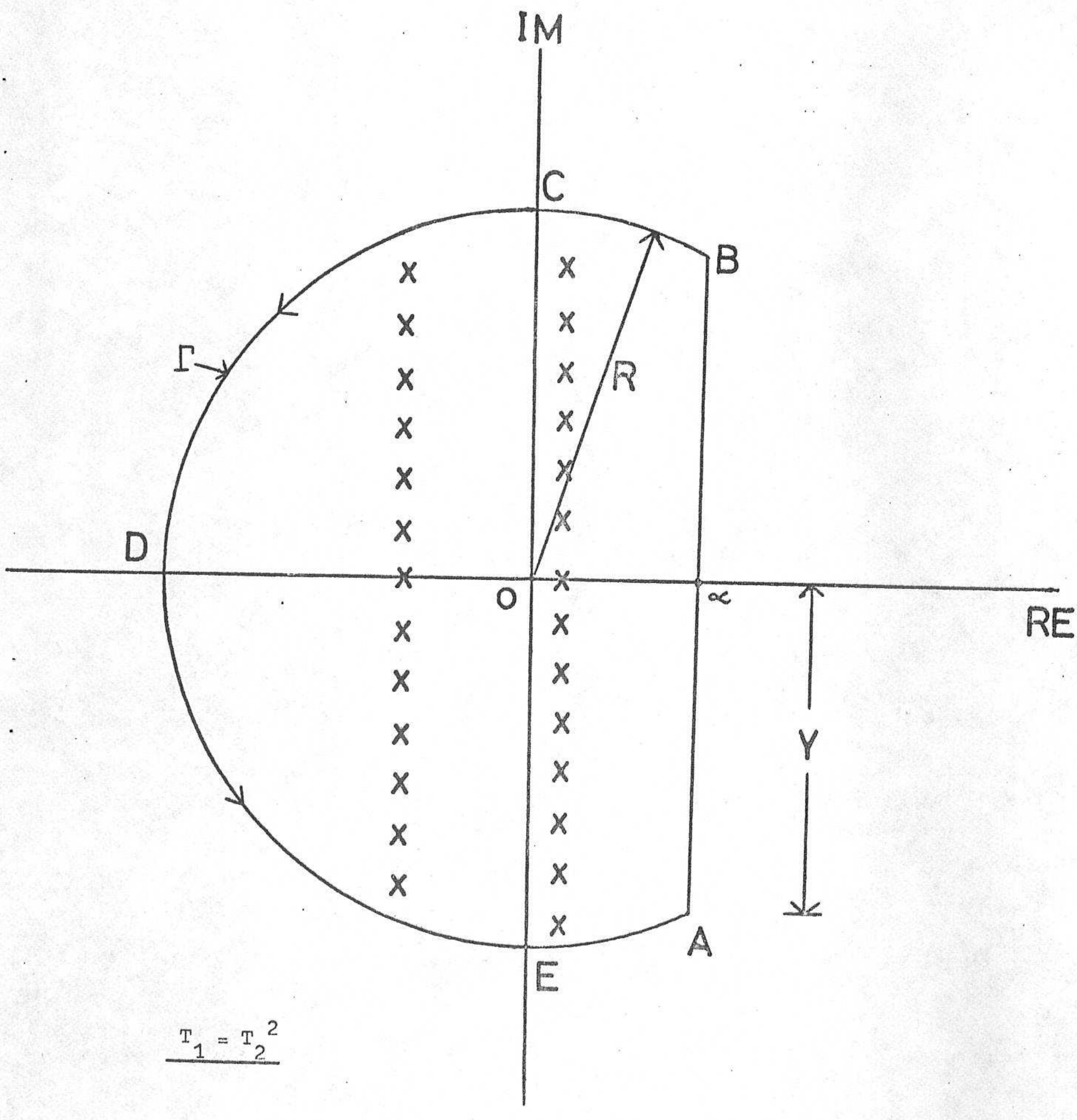


FIGURE 3



CONTOUR Γ : H(s) ANALYTIC INSIDE Γ EXCEPT AT POLES(denoted by X)

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