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ON QUANTIZATION AND OTHER BOUNDED NONLINEARITIES IN FIRST ORDER DISCRETE MULTIVARIABLE CONTROL

by

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Abstract

Recent work on the use of discrete first-order approximate models of plant dynamics in multivariable process control is extended to consider the effect on transient performance nonlinearities such as measurement quantization or deadzone at the implementation stage. Explicit upper bands on the transient error induced by the nonlinearity are obtained and their validity illustrated by numerical examples. In particular, at fast sampling rates, it is seen that, not only does the first order controller produce a closed-loop system with rapid, non-oscillating responses and small interaction effects, it also ensures that peak transient errors due to the nonlinearity are less than the error involved in approximating the nonlinearity by a unity gain matrix. For example, if the nonlinearity is measurement quantization, the peak transient effects of the quantization will be bounded by the quantization error itself at fast sampling rates, and will not be amplified by system dynamics.

1. Introduction

In a recent paper[1] the concept of a continuous first order lag[2,3,4] was extended to the sampled-data/discrete case by defining an m-input/m-output discrete first order lag to be a controllable and observable discrete system with mxm inverse transfer function matrix of the form

$$G^{-1}(z) = (z-1)\, B_o + B_1 \tag{1}$$

where B_o, B_1 are real mxm matrices and $|B_o| \neq 0$. A detailed analysis of the unity negative feedback system shown in Fig. 1 was provided. In particular, the proportional controller

$$K(z) = B_o \operatorname{diag}\{1-k_j\}_{1 \leq j \leq m} - B_1 \tag{2}$$

$$|k_j| < 1 \quad , \quad 1 \leqslant j \leqslant m$$

was shown to generate a stable closed-loop system and that both steady state errors and transient interaction effects in response to unit step demands are reduced and real-time closed-loop response speeds increased as the sampling rate is increased. Steady state errors can also be eliminated by the introduction of summation terms in the form

$$K(z) = B_o \operatorname{diag}\{1-k_j c_j + \frac{(1-k_j)(1-c_j)z}{(z-1)}\}_{1 \leq j \leq m} - B_1 \tag{3}$$

where $|c_j| < 1$, $1 \leqslant j \leqslant m$.

An important aspect of the results described[1] is that, by suitable use of approximation concepts, their applicability can be extended to provide a control synthesis procedure for an mxm, invertible, minimum phase discrete plant $S(\Phi, \Delta, C)$

$$x_{k+1} = \Phi\, x_k + \Delta\, u_k \quad , \quad x_k \in R^n$$
$$y_k = C\, x_k \quad , \quad k \geqslant o \tag{4}$$

with inverse z-transfer function matrix

$$G^{-1}(z) = (z-1)\, B_o + B_1 + B_o\, H(z) \tag{5}$$

with $|B_o| \neq 0$ and $H(z)$ proper, provided that $H(z)$ satisfies a contractionmapping condition. In particular, it was shown that the desired conditions are satisfied if $S(\Phi,\Delta,C)$ can be regarded as being derived from a continuous system $S(A,B,C)$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) \qquad (6)$$

with sampling period $h > o$, $x_k \triangleq x(kh)$ $(k \geqslant o)$ and piecewise constant input $u(t) = u_k$, $kh \leqslant t < (k+1)h$, $k \geqslant o$ and if the following conditions are satisfied,

(a) $|CB| \neq 0$

(b) $S(A,B,C)$ is minimum phase

(c) the sampling period h is 'small enough'

This paper describes the result of further theoretical investigations into the properties of such control systems and, in particular, the important problem of assessing the effects on theoretically predicted closed-loop <u>transient</u> performance of the introduction (at the implementation stage) of a memoryless nonlinearity

$$\hat{y}_k = f(y_k) \quad , \quad k \geqslant o \qquad (7)$$

into the feedback loop as illustrated in Fig. 2(a). It is assumed that $f(\cdot)$ satisfies the norm constraint

$$||y - f(y)||_m \leqslant q/2 \qquad (8)$$

for some constant $q \geqslant 0$ (independent of the choice of m-vector y) where $||\cdot||_m$ is the normal uniform norm on R^m defined by $||x||_m = \max_{1\leqslant j\leqslant m} |x_j|$. In effect, $f(\cdot)$ can be approximated by a unit diagonal gain matrix with a maximum error bound $\frac{1}{2}q$. Such a structure describes (and, indeed, was motivated by the need to consider) the effects of quantization on closed-loop dynamics although the results do have application to other nonlinearities such as measurement dead-zone. The validity of equation (8) in these two cases for the case of m=1 is illustrated schematically in Fig. 3.

The basic mathematical framework is set up in section 2 and the elementary operator theoretic bounds on the transient effects of the nonlinearity are described. In section 3, these bounds are evaluated for the cases when G(z) is a first order lag with controllers given by equations (2) and (3) and it is seen that these controllers are capable of reducing the transient errors induced by f(·) to a peak magnitude of ½q provided that the sampling rate is high enough. Finally, in section 4, it is demonstrated that all these results carry over to the case of the discrete plant of equations (4) and (5) if the underlying continuous system of equation (6) satisfies the conditions (a) - (c) described above.

A number of numerical examples illustrating the results are described.

2. Upper Bounds for the Transient Effects of the Nonlinearity

The general problem considered in this section is the numerical assessment of the transient errors involved in the prediction of the closed-loop response of the system of Fig. 2(a) using the linear configuration of Fig. 1, when the nonlinearity satisfies the bound of equation (8). This problem is best formulated in functional analysis terms as a problem in the Banach space E_∞^m of infinite sequences $s = \{s_o, s_1, s_2, \ldots\} = \{s_k\}_{k\geq o}$ of vectors $s_k \in R^m$ with the norm

$$||s||_\infty \triangleq \sup_{k\geq o} ||s_k||_m \qquad (9)$$

and the obvious definitions of addition and multiplication by scalars. With these definitions the (assumed stable) linear closed-loop configuration of Fig. 1 with zero initial conditions can be regarded as a bounded linear operator L_c mapping E_∞^m into itself by mapping bounded demand sequences $r = \{r_k\}_{k\geq o}$ into bounded output sequences $y = \{y_k\}_{k\geq o}$. The nonlinearity f(·) can also be regarded as a mapping $\tilde{f}$ of E_∞^m into itself, defined by $\tilde{y} = \tilde{f}(y)$ iff $\tilde{y}_k = f(y_k)(k\geq o)$ and it is easily seen that

$$||y - \tilde{f}(y)||_\infty \leq q/2 \ \forall \ y \in E_\infty^m \qquad (10)$$

It is trivially verified that the configurations of Fig. 2(a) and 2(b) are equivalent and hence that the closed-loop response is described by

$$y = L_c r + L_c \{y - \tilde{f}(y)\} + \eta$$
$$= y_L + L_c \{y - \tilde{f}(y)\} \quad (11)$$

where η is a term describing the effect of initial state conditions and $y_L = L_c r + \eta$ is the equivalent response of the linear configuration of Fig. 1. Writing this equation in the form

$$y - y_L = L_c \{y - \tilde{f}(y)\} \quad (12)$$

and noting that $||y-y_L||_\infty$ is simply the peak transient error involved in the prediction of the closed-loop response of Fig. 2(a) using the linear configuration of Fig. 1, we can provide an upper bound for this error in the normal way using equation (10)

$$\begin{aligned} ||y-y_L||_\infty &= ||L_c \{y-\tilde{f}(y)\}||_\infty \\ &\leq ||L_c||_\infty \, ||y-\tilde{f}(y)||_\infty \\ &\leq ||L_c||_\infty \frac{q}{2} \end{aligned} \quad (13)$$

where $||L_c||_\infty$ is the operator norm of L_c induced by the E_∞^m norm.

It follows immediately from equation (13) that

(a) the stability of the configuration is unaffected by the introduction of the nonlinearity[5,6]

(b) the peak transient error is bounded by the nonlinearity parameter q/2 amplified by the linear system parameter $||L_c||_\infty$, and

(c) the upper bound is independent of initial conditions and demand signals.

The investigation of the transient effects of the nonlinearity now reduce to the evaluation of $||L_c||_\infty$. This problem is undertaken in the following sections for a large class of systems of practical interest.

3. Nonlinear Effects on Discrete First Order Lags

Consider the case of the first order plant with inverse z-transfer function matrix given by equation (1). This system has a minimal realization of the form

$$x_{k+1} = (I - B_o^{-1}B_1)x_k + B_o^{-1}u_k \quad , \quad x_o = 0$$

$$y_k = x_k \quad , \quad k \geq o \tag{14}$$

Considering the use of the proportional controller of equation (2), it is easily verified that the closed-loop operator L_c can be realized as the map $r \to y$ defined by

$$x_{k+1} = \text{diag}\,\{k_j\}_{1\leq j\leq m}\, x_k + (\text{diag}\,\{1-k_j\}_{1\leq j\leq m} - B_o^{-1}B_1)r_k \quad , \quad x_o=0$$

$$y_k = x_k \quad , \quad k \geq o \tag{15}$$

or, equivalently,

$$y_k = \sum_{i=1}^{k} \text{diag}\,\{k_j^{k-i}\}_{1\leq j\leq m}(\text{diag}\{1-k_j\}_{1\leq j\leq m} - B_o^{-1}B_1)r_{i-1} \tag{16}$$

It follows directly that

Theorem 1

The configuration of Fig. 1 with plant and proportional controller given by equations (1) and (2) has norm

$$||L_c||_\infty = ||\text{diag}\{\frac{1}{1-|k_j|}\}_{1\leq j\leq m} \quad (\text{diag}\,\{1-k_j\}_{1\leq j\leq m} - B_o^{-1}B_1)||_m \tag{17}$$

(Note: the matrix norm $||\cdot||_m$ induced by the uniform norm on R^n is defined by $||M||_m \triangleq \max_{1\leq j\leq m} \sum_{i=1}^{m} |M_{ji}|$ for any real mxm matrix M)

The value of $||L_c||_\infty$ given in equation (17) is easily evaluated from the plant and controller data and, combined with the analysis of section 2, can yield information on the transient effect of the nonlinearity.

Suppose now that the discrete first order lag is derived from the continuous system S(A,B,C) of state dimension n=m and, without loss of generality, take $C = I_m$ when[1]

$$\Phi = e^{Ah} = I-B_o^{-1}B_1 \quad , \quad \Delta = B_o^{-1} = e^{Ah}\int_o^h e^{-Ah}\,Bdt \tag{18}$$

where $h > 0$ is the sampling interval. In particular, it is seen that

$$\lim_{h\to o+} B_o^{-1} B_1 = 0 \tag{19}$$

The following result now follows directly from theorem one and equation (19)

Theorem 2

With the assumptions of theorem 1 and the above construction

$$\lim_{h\to o+} ||L_c||_\infty = \max_{1\leq j\leq m} \frac{(1-k_j)}{(1-|k_j|)} \tag{20}$$

In particular

Corollary

If $0 \leq k_j < 1$ $(1 \leq j \leq m)$, then $\lim_{h\to o+} ||L_c||_\infty = 1$

It is known[1] that reasonably fast sampling is required if the (linear) closed-loop system is to have fast response speeds and small steady state errors and transient interaction effects in response to unit step demands. In such a situation equation (20) provides a simplified estimate for $||L_c||_\infty$. The theorem and its corollary also suggest the general rules.

(a) The use of $k_j < o$ (and hence oscillatory closed-loop responses) can lead to excessive amplification of the effects of the nonlinearity. For example, if $m = 2$, and $k_1 = 0.9 = -k_2$ then, at fast sampling $||L_c||_\infty \simeq 19.0$.

(b) The use of all $o \leq k_j < 1$ and fast sampling provides no amplification of the effects of the nonlinearity (by the corollary).

In particular, if we interpret f in terms of quantization with quantization level q, then the transient effects of this nonlinearity are bounded by the measurement quantization error q/2 under fast sampling conditions.

The inclusion of summation control action as represented by the controller of equation (3) can be analysed if a similar, but rather more involved, manner. In practice, however, summation action[1] would normally only be introduced in cases where the proportional design displays large steady state errors. Also reset times will tend to be considerably longer than rise times in the sense that[1]

$$|1 - c_j| << |1 - k_j| \quad , \quad 1 \leqslant j \leqslant m \tag{21}$$

In such cases the inclusion of summation action is a small perturbation to L_c and hence the estimates described by theorems 1 and 2 and the corollary will still be adequate for applications.

To illustrate the application of the results suppose that the continuous, invertible system[1] specified by the matrix triple

$$A = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} , \; B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} , \; C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{22}$$

is to be controlled by discrete proportional feedback with sampling period h. It follows[1] that

$$B_o^{-1} B_1 = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} (1 - e^{-3h}) \tag{23}$$

Following previous analyses, we suppose that similar response speeds are required from each channel and set $k_1 = k_2 = 0.5$ with a sampling period of $h = \frac{1}{30}$. It is known[1] that the resulting closed-loop system responds rapidly to unit step demands with transient interaction effects and steady state errors of peak magnitude 0.13. We can assess the tolerance to bounded feedback nonlinearities by using equation (17) to deduce that

$$||L_c||_\infty = 1 \tag{24}$$

That is, the maximum error in the prediction of closed-loop transient performance using the linear model must be less than the peak error in the approximation of the feedback nonlinearity by a unit gain!

4. Effects of Nonlinearity on Higher Order Plant under Fast Sampling Conditions

Suppose now that the plant is described by the mxm invertible, minimum-phase discrete plant model $S(\Phi,\Delta,C)$ given in equation (4) having inverse z-transfer function matrix $G^{-1}(z)$ given by equation (5) with $|B_o| \neq 0$ and $H(z)$ proper. Following the technique of reference (1), the plant is conceptually approximated by the mxm first order lag $G_A(z)$ defined by

$$G_A^{-1}(z) = (z - 1)B_o + B_1 \qquad (25)$$

and which is used as the basis for the choice of proportional or proportional plus summation controllers of the forms given in equations (2) and (3) respectively. The approximation G_A is a good approximation to the high frequency plant behaviour and, if we arrange[(1)] that $H(1) = 0$, it can also be a good approximation to plant steady state characteristics. The approximation contains no information on plant zero structure however.

It is known[(1)] that the procedure outlined above can provide a highly successful technique for guaranteeing the desired closed-loop performance if the underlying continuous plant $S(A,B,C)$ satisfies certain simple structural constraints and the sampling rate is high enough. A comparison of this work with theorem 2 and the following result indicates that the first order approximation can also be successfully used to estimate the transient effects of feedback nonlinearities at fast sampling rates:

Theorem 3 (see Appendix 8)

Suppose that the plant defined by equation (5) is to be controlled by the configuration of Fig. 1 with proportional controller $K(z)$ given by equation (2). Suppose also that this model is derived from the minimum-phase continuous model S(A,B,C) with a sampling interval $h > o$ and that $|CB| \neq 0$ and $\lim_{h \to o+} B_o^{-1} B_1 = 0$. Then

$$\lim_{h \to o+} ||L_c||_\infty = \max_{1 \leq j \leq m} \frac{(1-k_j)}{(1-|k_j|)} \qquad (26)$$

In effect the values of the norm at fast sampling is essentially the same for the real and approximate closed-loop systems, and hence the upper bounds (equation (13)) on the effects of the nonlinearity on both systems are essentially identical. The conditions on S(A,B,C) and sampling rate are precisely those used previously[1] to guarantee the stability of the closed-loop system and, as such, theorem 3 augments previous work. The conditions on $B_o^{-1} B_1$ can always be achieved[1] by ensuring for example that the approximate and real plants have identical steady state characteristics. It is particularly significant that the norm $||L_c||_\infty$ can be estimated knowing only the parameters $\{k_j\}_{1 \leq j \leq m}$ characterizing the closed-loop response times[1] and that the controller can be constructed from this information and the matrices B_o, B_1. These matrices can be estimated either from the plant model or from experimental transient tests[1]. In this sense the design technique[1] and the bounds described in this paper can be applied even if a detailed plant model is not available, provided that the underlying continuous plant satisfies the structural constraint of theorem 3 and suitably fast sampling is implemented.

The inclusion of summation control action in the form of the controller of equation (3) will, as discussed in the previous section, only generate a small perturbation to L_c if the condition expressed by equation (21) is implemented. In such cases, equation (26) will be a good approximation to the norm $||L_c||_\infty$ at high sampling rates.

To illustrate the validity and application of the results consider the open-loop unstable system S(A,B,C) described by the triple

$$A = \begin{pmatrix} 2 & 1 & 0.1 \\ 1 & -1 & 0.2 \\ 0.5 & 0.8 & -1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{27}$$

and note that $|CB| \neq 0$ and that the system is minimum phase with a zero at $s = -1 < 0$, as required by theorem 3. Discretizing the system with a sample interval of $h = \frac{1}{20}$, the approximating first order lag matching the high frequency and steady state plant characteristics is defined by the data,

$$B_o = \begin{pmatrix} 19.52 & -21.0 \\ -0.50 & 20.51 \end{pmatrix} \quad , \quad B_1 = \begin{pmatrix} -0.95 & -1.92 \\ -1.1 & 0.84 \end{pmatrix} \tag{28}$$

Assuming, for simplicity, that similar response speeds are required from both loops, we choose $k_1 = k_2 = 0.25$. The response of the closed-loop system with the controller of equation (2) is shown in Fig. 4. The response is seen to be rapid with steady state errors and transient interaction effects of peak magnitude 0.14. Overall this is a good design on paper.

Consider now the inclusion of feedback quantization errors. More precisely, consider the configuration of Fig. 2 where f is a diagonal nonlinearity representing the inclusion of identical quantization effects of the form shown in Fig. 3(a) in each loop. For illustrative purposes, we choose very coarse quantization with $q = 0.4$. Noting that the sampling interval is quite short, the peak transient effect of the nonlinearity can be estimated using theorem 3 and equation (13). That is, $||L_c||_\infty \simeq 1$, suggesting that the peak transient errors introduced by the nonlinearity are of the order of $||L_c||_\infty \frac{q}{2} \simeq 0.2$.

The responses of the linear and nonlinear feedback systems to periodic alternating unit step input in channel one of period 10 sampling intervals are shown in Fig. 5. They indicate that the upper bound 0.2 is a good estimate of the transient errors introduced by the quantization (the peak error is, in fact, 0.162).

The equivalent responses with $k_1 = k_2 = -0.25$ are shown in Fig. 6. The upper bound obtained from equation (13) and theorem 3 is $||L_c||_\infty \frac{q}{2} \simeq 0.333$, indicating a potential 60% magnification of the measurement quantization errors. The actual peak error is 0.253.

5. Conclusions

The paper has provided a theoretical analysis of the properties of first-order multivariable feedback systems[1] designed on a linear basis but which, at the implementation stage, include bounded nonlinearities such as measurement quantization or deadzone in the feedback loop. Such nonlinearities have no effect on the stability of the closed-loop system but can, a priori, have significant effects on transient performance, the magnitude of the effects being affected by the properties of the designed linear control system. The availability of estimates of the peak transient errors is hence important for assessing the quality of the designed controller on implementation.

If the plant is first order an explicit estimate of the peak transient errors is provided by equation (13) and theorem 1. This bound can be easily computed from plant and controller data and (theorem 2), under conditions of fast sampling and non-oscillation, reduces to unity i.e. the first order controller produces a closed-loop system[1] with rapid, non-oscillatory response, small interaction effects and peak transient errors due to the nonlinearity less that the error involved in approximating the nonlinearity by a unity gain matrix. For example, if the nonlinearity is measurement quantization, the peak transient effects of the quantizer will be bounded by the peak quantization error q/2!

If the plant is not first-order, satisfies[1] certain generic structural constraints (see theorem 3) and controller design is undertaken based on first order approximation[1], then explicit (computable) bounds on the transient effect of the nonlinearity cannot be obtained. However, under fast sampling conditions, it has been shown (theorem 3) that the transient errors are asymptotically ($h \to o+$) bounded by the bound obtained from the first order approximation. In other words, not only can the first order approximation be used (under fast sampling conditions) to design a rapid, non-oscillatory closed-loop system with small interaction effects for the original large-scale system, it can also be used to assess the transient effect of bounded feedback nonlinearities. This is particularly significant if a detailed high-order analytical model is not known but[1] a simple first order plant model has been estimated from experimental data.

Finally, the nature of the transient error has not been investigated. In particular, the closed-loop system may not be asymptotically stable (although it is always bounded-input-bounded-output stable) indicating the possibility of limit cycles. In this situation the results described in this paper represent bounds on the amplitude of any limit cycle. In many situations (such as the assessment of the effect of measurement quantization) the results will be used simply to assess whether or not the limit cycles are of sufficiently small magnitude to be ignored. More generally, the results will provide useful information to initiate a search for the limit cycle form.

6. Acknowledgement

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12. Appendix

Proof of Theorem 3:

The proof is based on the observation that (equation (5) and (25)) the linear configuration of Fig. 1 can be regarded as the minor loop system of Fig. 7(a) and hence, by interchange of the feedback loops, as the configuration of Fig. 7(b) with forward path system equal to the approximate unity negative feedback system used for the basis of the design study.

Suppose initially that $H(z)$ is strictly proper, then, applying known results[7]

$$B_o = (C\Delta)^{-1} \tag{29}$$

$$B_1 = -(C\Delta)^{-1} C\{\Phi - I\} \Delta (C\Delta)^{-1} \tag{30}$$

$$H(z) = -C\{\Phi - I\} M (zI_{n-m} - N\Phi M)^{-1} N \{\Phi - I\} \Delta (C\Delta)^{-1} \tag{31}$$

where N,M are $(n-m)$ xn and nx $(n-m)$ full rank matrices respectively satisfying the relations

$$CM = 0 \quad , \quad N\Delta = 0 \quad , \quad NM = I_{n-m} \tag{32}$$

Noting that[1]

$$\lim_{h\to o+} h^{-1} C\Delta = C B \tag{33}$$

is nonsingular by assumption and writing $\Phi = e^{Ah}$ it is easily verified that

$$\lim_{h\to o+} B_o^{-1} B_1 = 0 \tag{34}$$

The linear map $y \to \hat{y}$ induced by $K^{-1}(z) B_o H(z)$ is defined by the relations

$$z_{k+1} = N\Phi M z_k - N \{\Phi - I\} \Delta(C\Delta)^{-1} y_k \quad , \quad z_o = 0$$

$$\hat{y}_k = (B_o \operatorname{diag}\{1 - k_j\}_{1\leq j\leq m} - B_1)^{-1} B_o C\{\Phi - I\} M z_k \tag{35}$$

and will be denoted L_f. It is a bounded linear map of E_∞^m into itself (as $h\to o+$) as we can always[1] suppose that M is constant and that $\lim_{h\to o+} N = N_o$ where $N_oB = 0$ and $N_oM = I_{n-m}$, i.e.

$$N \Phi M = I_{n-m} + h N_o A M + O(h^2) \tag{36}$$

whose eigenvalues lie in the open unit circle on the complex plane for all fast enough sampling rates as the eigenvalues of $N_o AM$ (i.e. the zeros[7] of $S(A,B,C)$) all have strictly negative real parts.

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$$\lim_{h \to o+} ||L_f||_\infty = 0 \tag{37}$$

Proof

Using (33), it is easily verified that $N\{\Phi - I\} \Delta(C\Delta)^{-1} = O(h)$ and that $(B_o \text{ diag } \{1-k_j\}_{1 \leq j \leq m} - B_1)^{-1} B_o C(\Phi - I)M = O(h)$. Noting that the norm of the linear map $u \to v$ defined by

$$V_{k+1} = N \Phi M V_k + w_k \quad , \quad V_o = 0 \ , \ k > o \tag{38}$$

has norm $O(h^{-1})$ (without loss of generality, suppose that $N_o A M$ is in diagonal or Jordan canonical form) then the lemma follows as L_f is the composition of the three defined maps. Q.E.D.

The lemma is also true for any choice of B_1 such that $\lim_{h \to o+} B_o^{-1} B_1 = 0$. To prove this, let $\tilde{B}_1$ be one such choice so that $H(z)$ must be replaced by $\tilde{H}(z) = H(z) + B_o^{-1} \{B_1 - \tilde{B}_1\}$ and note that the correction term is $O(h)$.

To complete the proof, note that the configuration of Fig. 7(b) is characterized in E_∞^m by the bounded operator

$$L_c = (I + L_A L_f)^{-1} L_A \tag{39}$$

where L_A is the linear map in $E_\infty^{(m)}$ induced by the forward path system. This map is first order and hence, using theorem 2,

$$\lim_{h \to o+} ||L_A||_\infty = \max_{1 \leq j \leq m} \frac{(1-k_j)}{1-|k_j|} \tag{40}$$

In particular, it follows that $\lim_{h \to o+} ||L_A L_f|| = 0$. The result follows by deducing that

$$\lim_{h \to o+} ||L_C - L_A||_\infty = 0 \tag{41}$$

from the identity

$$L_C - L_A = -(I+L_AL_f)^{-1} L_AL_fL_A \qquad (42)$$

and using equation (40). Q.E.D.

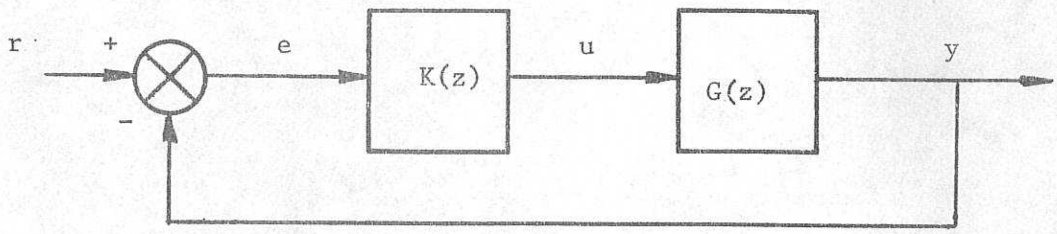
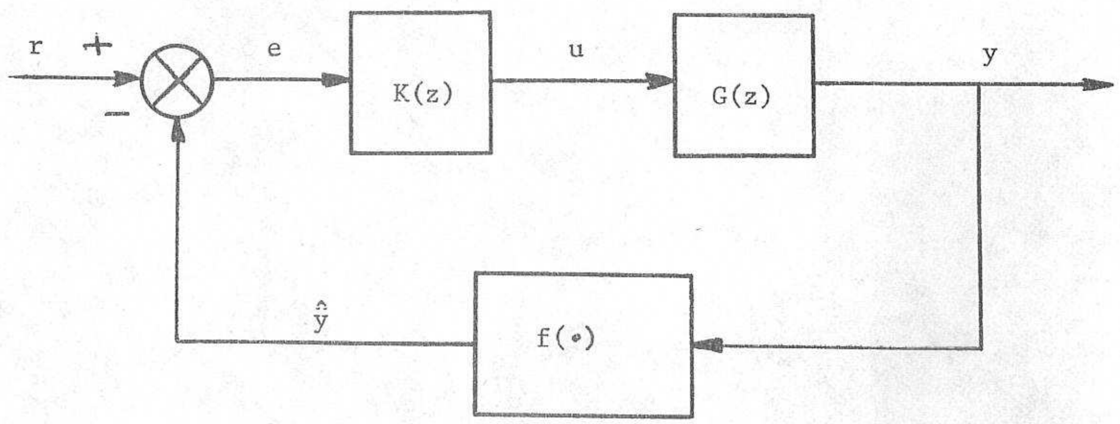
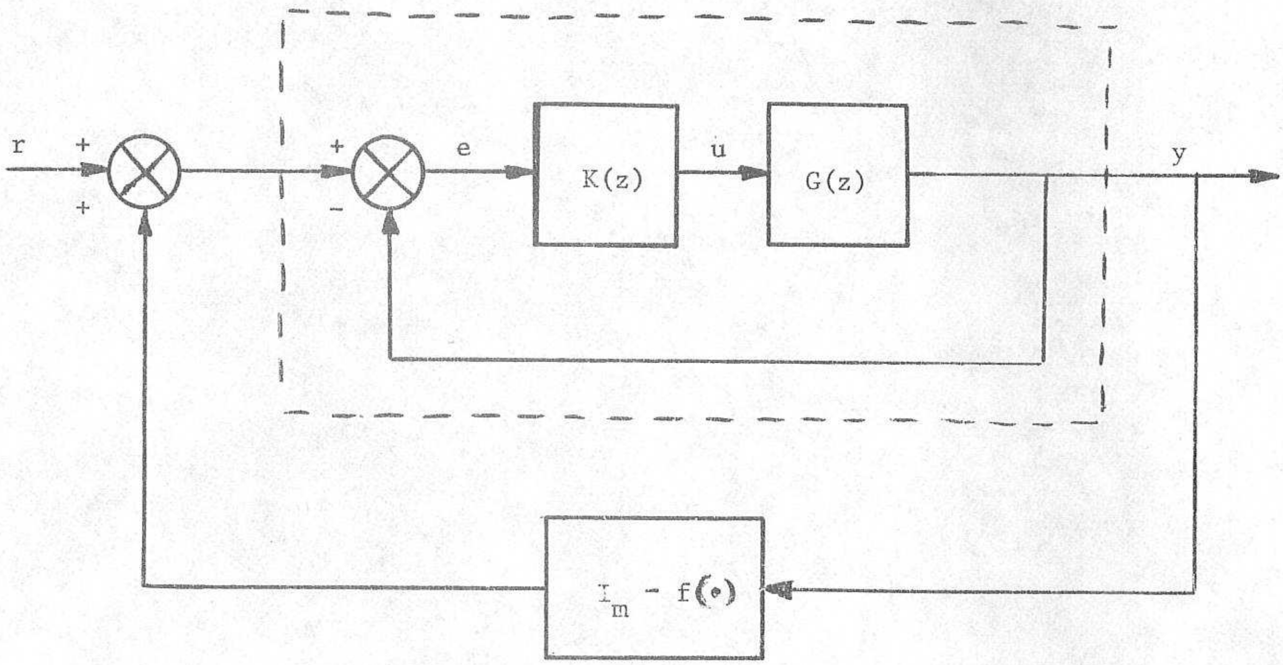


Fig. 1. Unity Feedback System



(a)



(b)

Fig. 2. Nonlinear Feedback Systems

(a) Closed-loop system with feedback linearity

(b) Equivalent transformed system

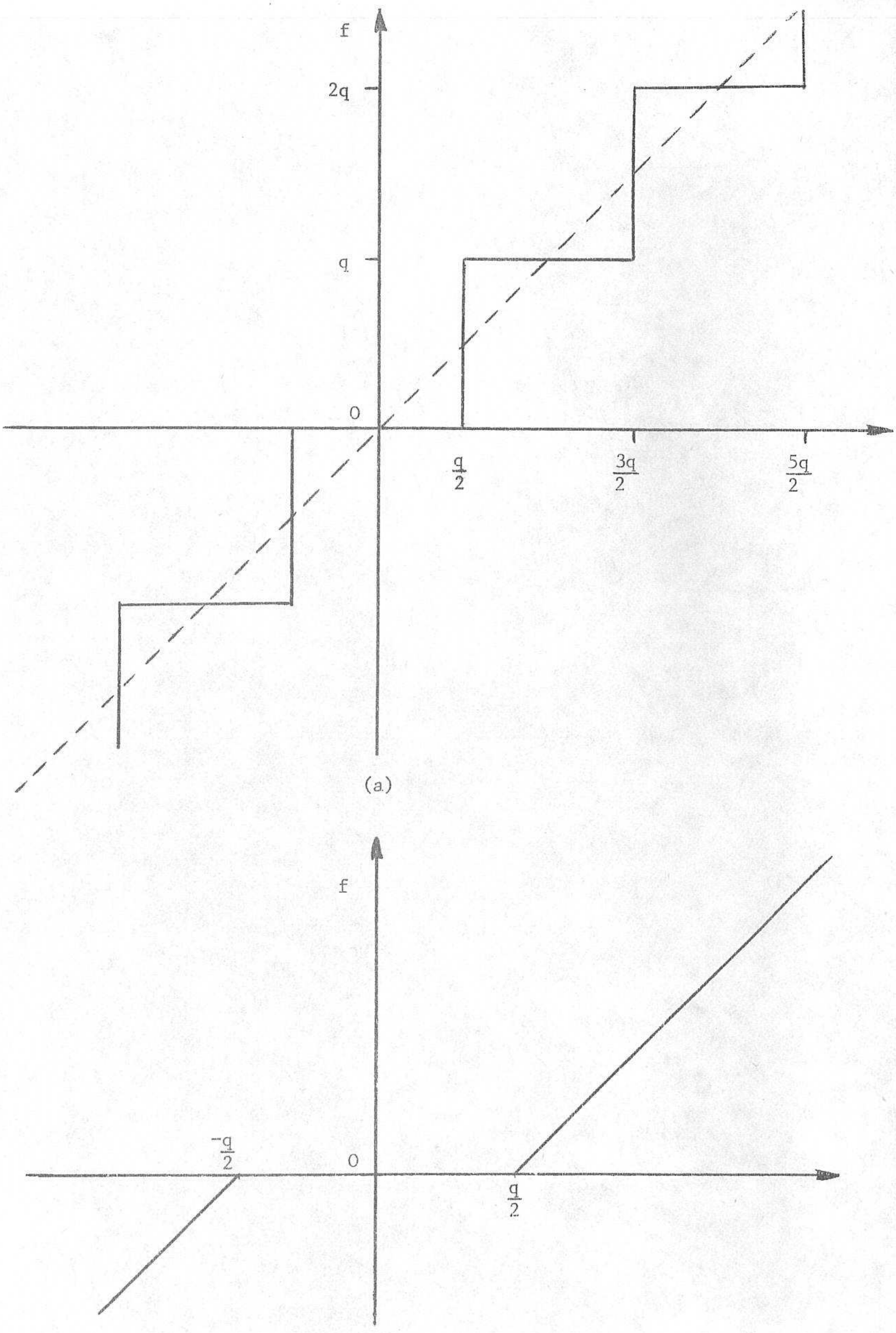


Fig. 3. Bounded Nonlinearities

(a) Quantization with quantization interval q

(b) Dead-zone of width q

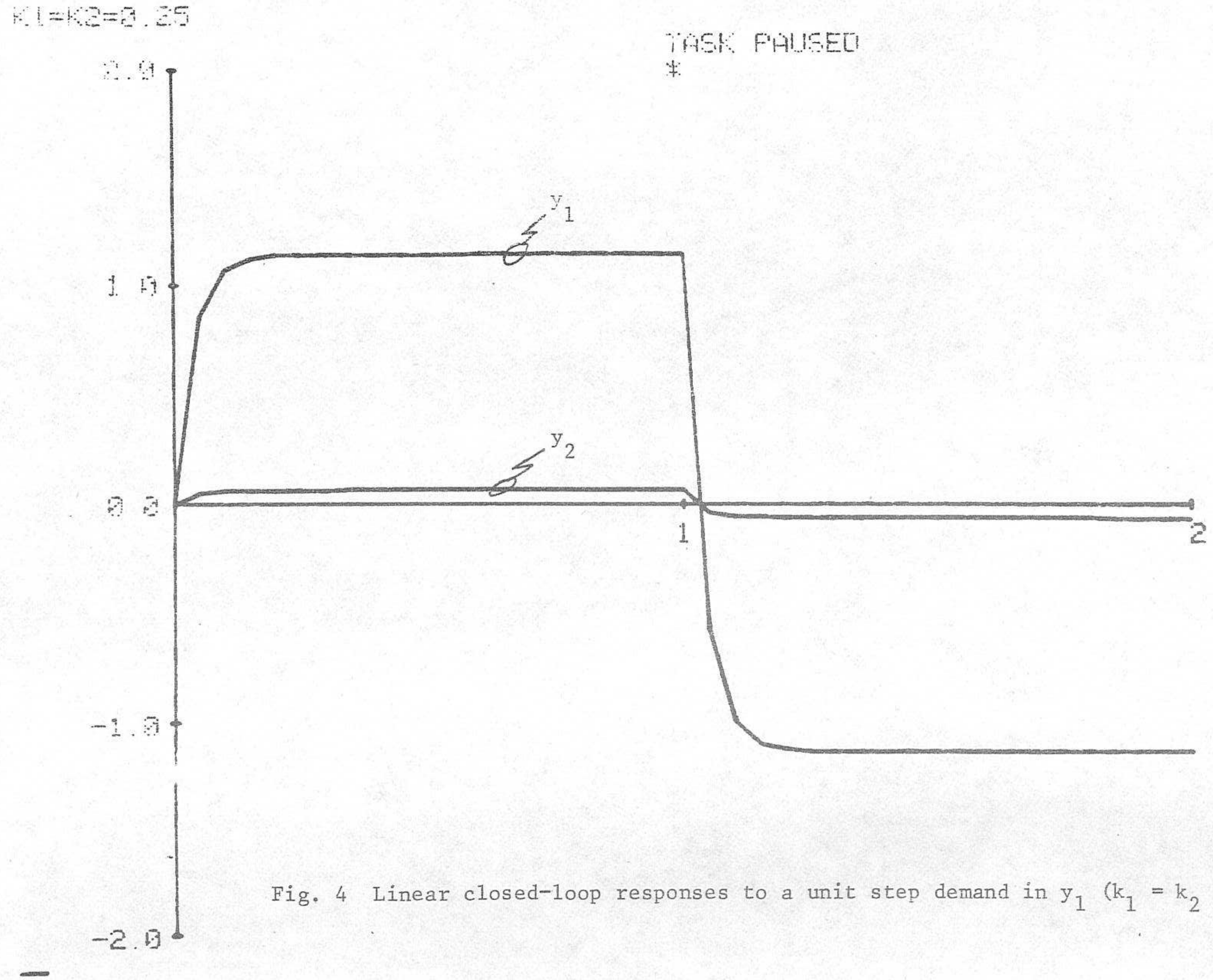


Fig. 4 Linear closed-loop responses to a unit step demand in y_1 ($k_1 = k_2 = 0.25$)

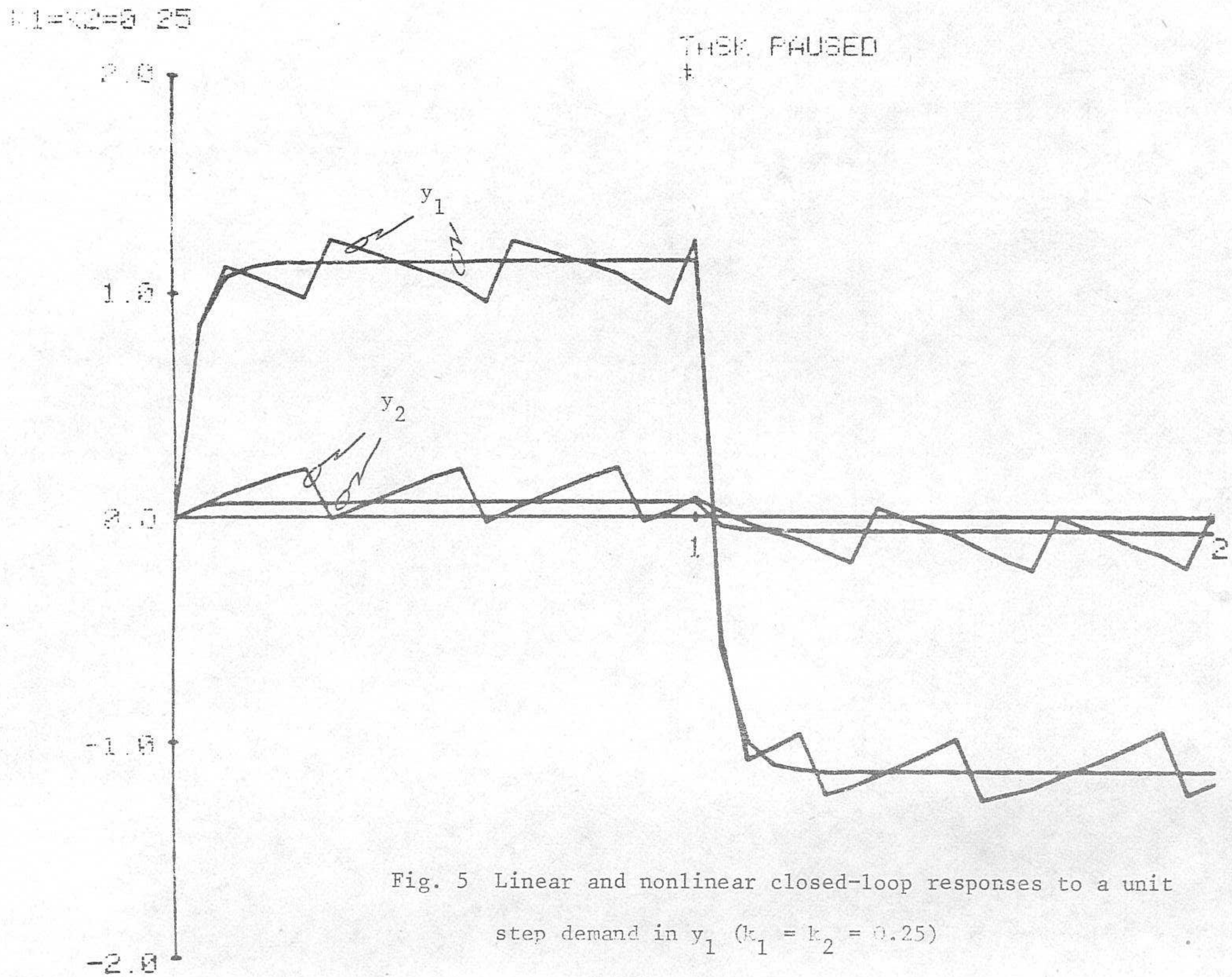


Fig. 5 Linear and nonlinear closed-loop responses to a unit step demand in y_1 ($k_1 = k_2 = 0.25$)

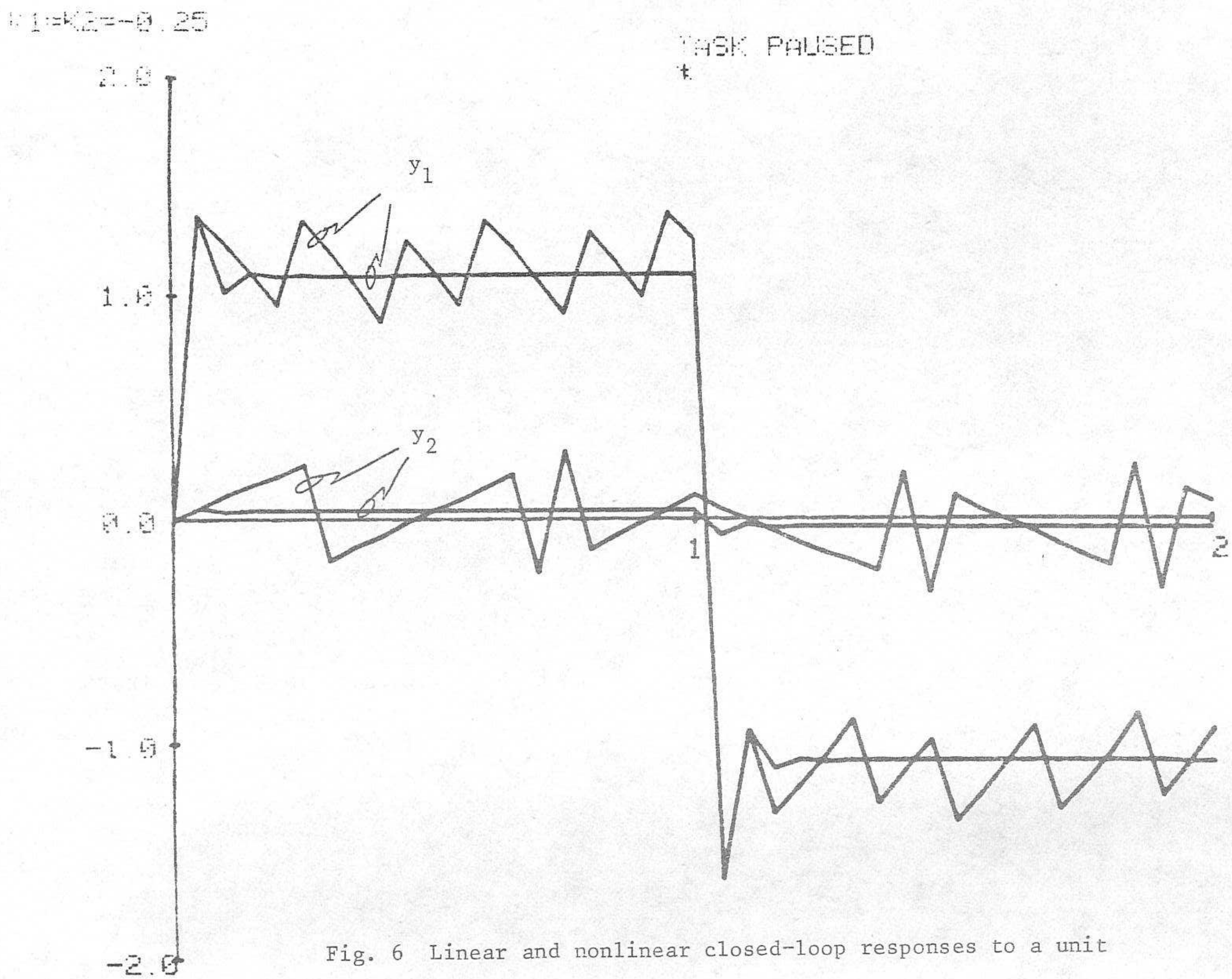
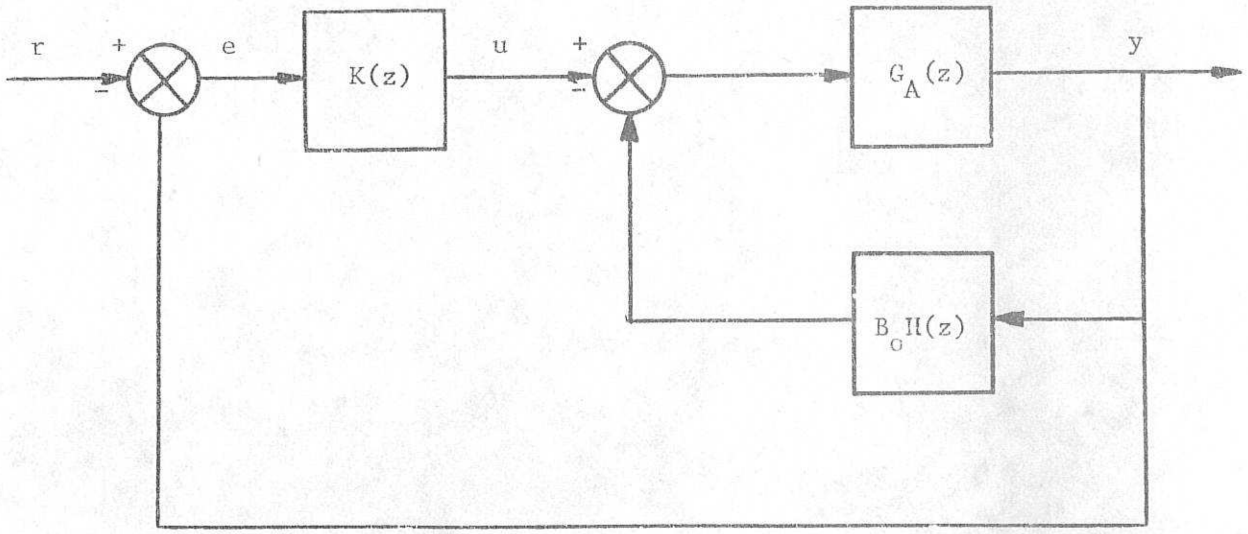
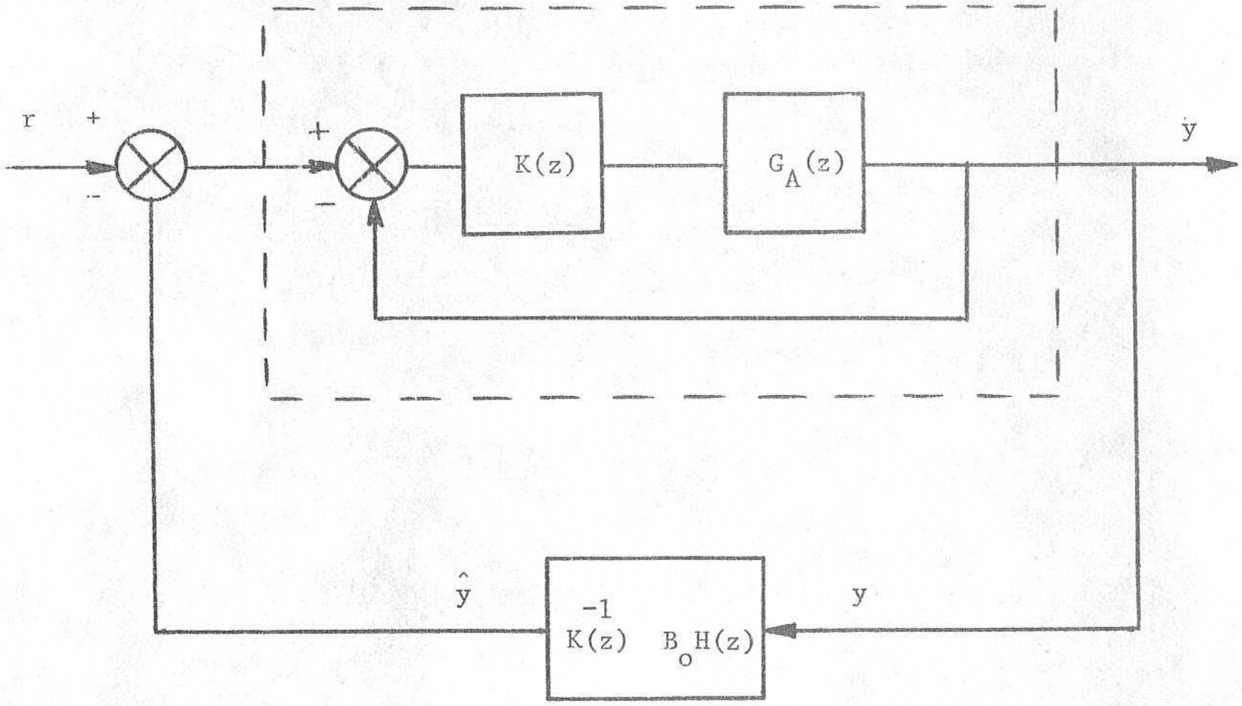


Fig. 6 Linear and nonlinear closed-loop responses to a unit step demand in y_1 ($k_1 = k_2 = -0.25$)



(a)



(b)

Fig. 7. Closed-loop System Representatives

(a) Representation as plant approximation plus minor loop

(b) Interchange of feedback loops