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Affine extensions of non-crystallographic Coxeter groups induced by projection

Pierre-Philippe Dechant[a)]

Ogden Centre for Fundamental Physics, Department of Physics,
University of Durham, South Rd, Durham DH1 3LE, UK and
Department of Mathematics, York Centre for Complex Systems Analysis,
University of York, Heslington, York, UK

Céline Bœhm[b)]

Ogden Centre for Fundamental Physics, Department of Physics, University of Durham,
South Rd, Durham DH1 3LE, UK

Reidun Twarock[c)]

Department of Mathematics, York Centre for Complex Systems Analysis,
University of York, Heslington, York, UK

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In this paper, we show that affine extensions of non-crystallographic Coxeter groups can be derived via Coxeter-Dynkin diagram foldings and projections of affine extended versions of the root systems E_8, D_6 and A_4. We show that the induced affine extensions of the non-crystallographic groups H_4, H_3 and H_2 correspond to a distinguished subset of the Kac-Moody-type extensions considered in[1]. This class of extensions was motivated by physical applications in icosahedral systems in biology (viruses), physics (quasicrystals) and chemistry (fullerenes). By connecting these here to extensions of E_8, D_6 and A_4, we place them into the broader context of crystallographic lattices such as E_8, suggesting their potential for applications in high energy physics, integrable systems and modular form theory. By inverting the projection, we make the case for admitting different number fields in the Cartan matrix, which could open up enticing possibilities in hyperbolic geometry and rational conformal field theory.

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[a)]Electronic mail: pierre-philippe.dechant@durham.ac.uk

[b)]Electronic mail: c.m.boehm@durham.ac.uk

[c)]Electronic mail: rt507@york.ac.uk

I. INTRODUCTION

The classification of finite-dimensional simple Lie algebras by Cartan and Killing is one of the mile stones of modern mathematics. The study of these algebras is essentially reduced to that of root systems and their Weyl groups, and all their geometric content is contained in Cartan matrices and visualised in Dynkin diagrams. The problem ultimately amounts to classifying all possible Cartan matrices[2].

Coxeter groups describe (generalised) reflections[3], and thus encompass the above Weyl groups, which are the reflective symmetry groups of the relevant root systems. In fact, the finite Coxeter groups are precisely the finite Euclidean reflection groups[4]. However, since the root systems arising in Lie Theory are related to lattices, the Weyl groups are automatically crystallographic in nature. Non-crystallographic Coxeter groups, i.e. those that do not stabilise any lattice (in the dimension equal to their rank), therefore cannot arise in the Lie Theory context, and as a consequence, they have not been studied as intensely. They include the groups H_2, H_3 and the largest non-crystallographic group H_4; the icosahedral group H_3 and its rotational subgroup I are of particular practical importance as H_3 is the largest discrete symmetry group of physical space. Thus, many 3-dimensional systems with 'maximal symmetry', like viruses in biology[5–9], fullerenes in chemistry[10–13] and quasicrystals in physics[14–17], can be modeled using Coxeter groups.

Affine Lie algebras have also been studied for a long time, and many of the salient features of the theory of simple Lie algebras carry over to the affine case. More recently, Kac-Moody Theory has provided another framework in which generalised Cartan matrices induce interesting algebraic structures that preserve many of the features encountered in the simple and affine cases[18]. However, such considerations again only give rise to extensions of crystallographic Coxeter groups. These infinite Coxeter groups are usually constructed directly from the finite Coxeter groups by introducing affine reflection planes (planes not containing the origin). While these infinite counterparts to the crystallographic Coxeter groups have been intensely studied[19], much less is known about their non-crystallographic counterparts[20]. Recently, we have derived novel affine extensions of the non-crystallographic Coxeter groups H_2, H_3 and H_4 in two, three and four dimensions, based on an extension of their Cartan matrices following the Kac-Moody formalism in Lie Theory[1].

In this paper, we develop a different approach and induce such affine extensions of the non-crystallographic groups H_2, H_3 and H_4 from affine extensions of the crystallographic groups A_4, D_6 and E_8, via projection from the higher-dimensional setting. Specifically, there exists a projection from the root system of E_8, the largest exceptional Lie algebra, to the root system of H_4, the largest non-crystallographic Coxeter group[19], and, due to the inclusions $A_4 \subset D_6 \subset E_8$ and $H_2 \subset H_3 \subset H_4$, also corresponding projections for the other non-crystallographic Coxeter groups.

We apply these projections here to the extended root systems of the groups A_4, D_6 and E_8. As expected, extending by a single node recovers only those affine extensions known in the literature. However, we also consider simply-laced extensions with two additional nodes in the Kac-Moody formalism, and consider their compatibility with the projection formalism. Specifically, we use the projection of the affine root as an affine root for the projected root system, and thereby find a distinguished subset of the solutions in the classification scheme presented in[1].

The E_8 root system, and the related structures: the E_8 lattice, the Coxeter group, the Lie algebra and the Lie group, are 'exceptional' structures, and are of critical importance in mathematics and in theoretical physics[2]. For instance, they occur in the context of Lie algebras, simple group theory and modular form theory, as well as lattice packing theory[21–23]. In theoretical physics, E_8 is central to String Theory, as it is the gauge group for the $E_8 \times E_8$ heterotic string[24]. More recently, via the Hořava-Witten picture[25–27] and other developments[28–32], E_8 and its affine extensions and overextensions (e.g. E_8^+ and E_8^{++}) have emerged as the most likely candidates for the underlying symmetry of M-Theory. It is also fundamental in the context of Grand Unified Theories[33–35], as it is the largest irreducible group that can accommodate the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$. Our new link between affine extensions of crystallographic Coxeter groups such as E_8^+ and their non-crystallographic counterparts could thus turn out to be important in High Energy Physics, e.g. in String Theory or in possible extensions of the Standard Model above the TeV scale after null findings at the LHC.

The structure of this paper is as follows. Section II reviews some standard results to provide the necessary background for our novel construction. Section II A discusses the basics of Coxeter groups. Section II B introduces the relationship between E_8 and H_4, and discusses how it manifests itself on the level of the root systems, the representation theory, and the Dynkin diagram foldings and projection formalism. Section II C introduces affine extensions of crystallographic Coxeter groups, and presents the standard affine extensions of the groups relevant in our context. In Section III A, we compute where the affine roots of the standard extensions of the crystallographic groups map under the projection formalism and examine the resulting induced affine extensions of the non-crystallographic groups. Section III B discusses Coxeter-Dynkin diagram automorphisms of the simple and affine groups, and shows that the induced affine extensions are invariant under these automorphisms. In Section III C, we consider affine extending the crystallographic groups by two nodes and show that these do not induce any further affine extensions. In Section IV A, we briefly review the novel Kac-Moody-type extensions of non-crystallographic Coxeter groups from a recent paper and compare the induced extensions with the classification scheme presented there (Section IV B). In Section V, we conclude that in a wide class of extensions (single extensions or simply-laced double extensions with trivial projection kernel), the ten induced cases considered here are the only ones that are compatible with the projection. We also discuss how lifting affine extensions of non-crystallographic groups to the crystallographic setting, as well as symmetrisability of the resulting matrices, motivate a study of Cartan matrices over extended number fields.

II. MATHEMATICAL PRELIMINARIES

In this section, we introduce the context of our construction, with the relevant concepts and the known links between them, as illustrated in Fig. 1. We introduce Coxeter groups and their root systems in Section II A, and discuss how certain crystallographic and non-crystallographic groups are related via projection (Section II B). Affine extensions of the crystallographic Coxeter groups are introduced in Section II C. Affine extensions of the non-crystallographic Coxeter groups in dimensions two, three and four have been discussed in our previous papers[1,20] (see Section IV A). Here, we present a different construction of such affine extensions, by inducing them from the known affine extensions of the crystallographic Coxeter groups via projection from the higher-dimensional setting. These induced extensions will be shown to be a subset of those derived in[1].

A. Finite Coxeter groups and root systems

Definition II.1 (Coxeter group)**.** *A* Coxeter group *is a group generated by some involutive generators $s_i, s_j \in S$ subject to relations of the form $(s_i s_j)^{m_{ij}} = 1$ with $m_{ij} = m_{ji} \geq 2$ for $i \neq j$. The matrix A with entries $A_{ij} = m_{ij}$ is called the* Coxeter matrix.

The finite Coxeter groups have a geometric representation where the involutions are realised as reflections at hyperplanes through the origin in a Euclidean vector space $\mathscr{E}$. In particular, let $(\cdot|\cdot)$ denote the inner product in $\mathscr{E}$, and λ, $\alpha \in \mathscr{E}$.

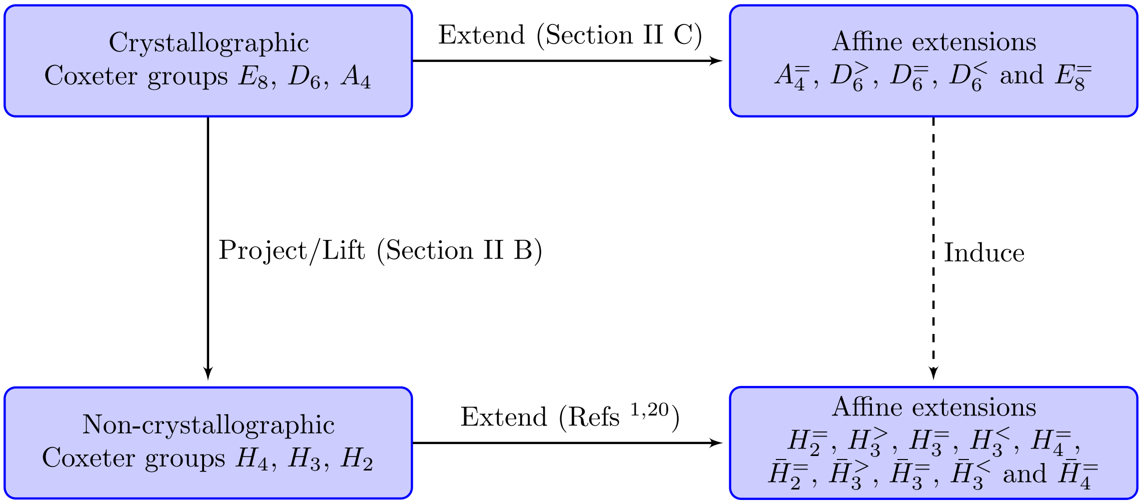


FIG. 1. Context of this paper: Section II A introduces Coxeter groups (left), and Section II B discusses how certain crystallographic and non-crystallographic groups are related via projection (left arrow). Section II C discusses the known affine extensions of the crystallographic Coxeter groups (upper arrow), and affine extensions of non-crystallographic Coxeter groups have been discussed in[1,20] (lower arrow). In this paper, we present a novel way of inducing affine extensions of the non-crystallographic groups via projection from the affine extensions of the crystallographic groups (dashed arrow on the right), yielding a distinguished subset of those derived in[1].

Definition II.2 (Reflection)**.** *The generator s_α corresponds to the* reflection

$$s_\alpha : \lambda \to s_\alpha(\lambda) = \lambda - 2\frac{(\lambda|\alpha)}{(\alpha|\alpha)}\alpha \tag{1}$$

at a hyperplane perpendicular to the root vector α.

The action of the Coxeter group is to permute these root vectors, and its structure is thus encoded in the collection $\Phi \in \mathscr{E}$ of all such root vectors, the root system:

Definition II.3 (Root system)**.** *A* root system Φ *is a finite set of non-zero vectors in $\mathscr{E}$ such that the following two conditions hold:*

1. *Φ only contains a root α and its negative, but no other scalar multiples: $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}\ \forall\ \alpha \in \Phi$.*

2. *Φ is invariant under all reflections corresponding to vectors in Φ: $s_\alpha\Phi = \Phi\ \forall\ \alpha \in \Phi$.*

For a crystallographic Coxeter group, a subset Δ of Φ, called *simple roots*, is sufficient to express every element of Φ via a $\mathbb{Z}$-linear combination with coefficients of the same sign. Φ is therefore completely characterised by this basis of simple roots, which in turn completely characterises the Coxeter group. In the case of the non-crystallographic Coxeter groups H_2, H_3 and H_4, the same holds for the extended integer ring $\mathbb{Z}[\tau]=\{a+\tau b|a,b\in\mathbb{Z}\}$, where τ is the golden ratio $\tau=\frac{1}{2}(1+\sqrt{5})$. Note that together with its Galois conjugate $\tau'\equiv\sigma=\frac{1}{2}(1-\sqrt{5})$, τ satisfies the quadratic equation $x^2=x+1$. In the following, we will call the exchange of τ and σ *Galois conjugation*, and denote it by $x\rightarrow\bar{x}=x(\tau\leftrightarrow\sigma)$.

The structure of the set of simple roots is encoded in the Cartan matrix, which contains the geometrically invariant information of the root system as follows:

Definition II.4 (Cartan matrix and Coxeter-Dynkin diagram)**.** *The* Cartan matrix *of a set of simple roots $\alpha_i\in\Delta$ is defined as the matrix*

$$A_{ij}=2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}. \tag{2}$$

A graphical representation of the geometric content is given by Coxeter-Dynkin diagrams*, in which nodes correspond to simple roots, orthogonal roots are not connected, roots at $\frac{\pi}{3}$ have a simple link, and other angles $\frac{\pi}{m}$ have a link with a label m.*

Note that Cartan matrix entries of τ and σ yield Coxeter diagram labels of 5 and $\frac{5}{2}$, respectively, since in the simply-laced setting $A_{ij}=-2\cos\frac{\pi}{m_{ij}}$, $\tau=2\cos\frac{\pi}{5}$ and $-\sigma=2\cos\frac{2\pi}{5}$. Such fractional values can also be understood as angles in hyperbolic space[36]. By the crystallographic restriction theorem, there are no lattices (i.e. periodic structures) with such non-crystallographic symmetry H_2, H_3 and H_4 in two, three and four dimensions, respectively. For these non-crystallographic Coxeter groups one therefore needs to move from a lattice to a quasilattice setting.

B. From E_8 to H_4: standard Dynkin diagram foldings and projections

The largest exceptional (crystallographic) Coxeter group E_8 and the largest non-crystallographic Coxeter group H_4 are closely related. This connection between E_8 and H_4 can be exhibited in various ways, including Coxeter-Dynkin diagram foldings in the Coxeter group picture[37], relating the root systems[16,38,39], and in terms of the representation theory[14,16,37–39]. For illustrative purposes, we focus on the folding picture first.

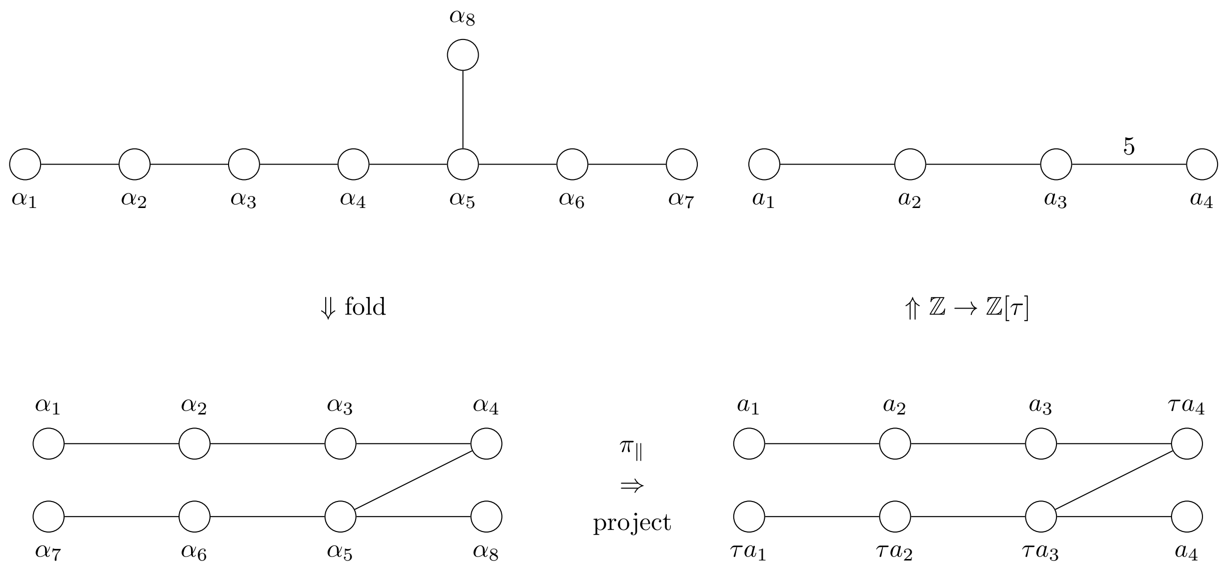


FIG. 2. Coxeter-Dynkin diagram folding and projection from E_8 to H_4. The nodes correspond to simple roots and links labeled m encode an angle of $\frac{\pi}{m}$ between the root vectors, with m omitted if the angle is $\frac{\pi}{3}$ and no link shown if $\frac{\pi}{2}$. Note that deleting nodes α_1 and α_7 yields corresponding results for $D_6 \to H_3$, and likewise for $A_4 \to H_2$ by further removing α_2 and α_6.

Following[37], we consider the Dynkin diagram of E_8 (top left of Fig. 2), where we have labeled the simple roots α_1 to α_8. We fold the diagram suggestively (bottom left of Fig. 2), and define the combinations $s_{\beta_1} = s_{\alpha_1}s_{\alpha_7}$, $s_{\beta_2} = s_{\alpha_2}s_{\alpha_6}$, $s_{\beta_3} = s_{\alpha_3}s_{\alpha_5}$ and $s_{\beta_4} = s_{\alpha_4}s_{\alpha_8}$. It can be shown that the subgroup with the generators β_i is in fact isomorphic to H_4 (top right)[37,40]. This amounts to demanding that the simple roots of E_8 project onto the simple roots of H_4 and their τ-multiples, as denoted on the bottom right of Fig. 2. One choice of simple roots for H_4 is $a_1 = \frac{1}{2}(-\sigma, -\tau, 0, -1)$, $a_2 = \frac{1}{2}(0, -\sigma, -\tau, 1)$, $a_3 = \frac{1}{2}(0, 1, -\sigma, -\tau)$ and $a_4 = \frac{1}{2}(0, -1, -\sigma, \tau)$, and in that case the highest root is $\alpha_H = (1, 0, 0, 0)$. In the bases of simple roots α_i and a_i, the projection is given by

$$\pi_{\|} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 & 0 & \tau & 0 & 0 \\ 0 & 0 & 1 & 0 & \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

There are similar diagrams for A_4 and D_6 that can be obtained from the E_8 diagram by deleting nodes. We display these in Fig. 3 in order to set out our notation, as the conventional way of numbering the roots in the Dynkin diagrams differs from the natural numbering in the folding

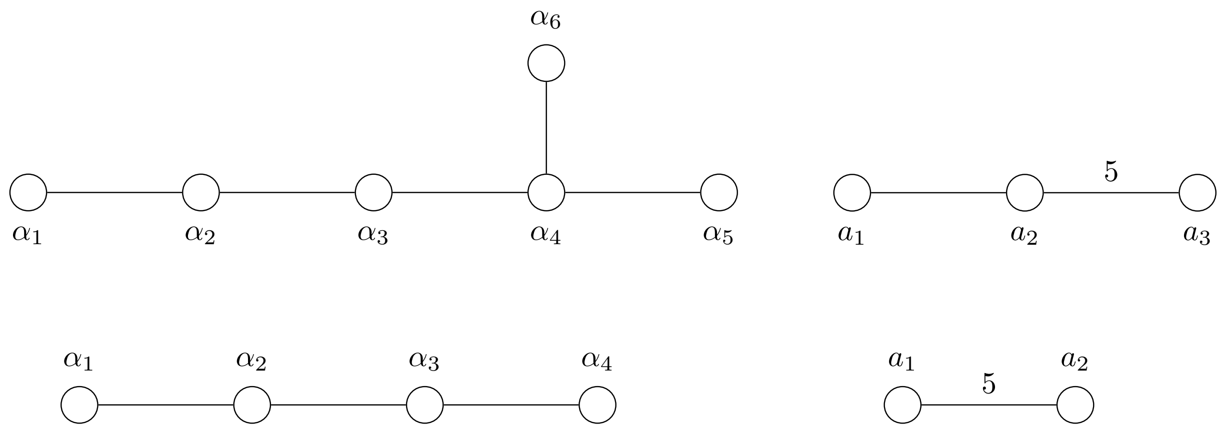


FIG. 3. We display the Dynkin diagrams for D_6 (top left), A_4 (bottom left), H_3 (top right) and H_2 (bottom right), in order to fix the labelling of the roots, such that in the following the Cartan matrices can be read more easily.

picture. The only non-trivial Coxeter relation is the one corresponding to the 5-fold rotation (e.g. the relation between β_3 and β_4 for the case of E_8). The additional β-generators in the higher-dimensional cases are trivial, as they can be straightforwardly shown to satisfy the relevant Coxeter relations (corresponding to 3-fold rotations) directly from the original Coxeter relations for the α_is of the larger groups.

It has been observed that the E_8 root vectors can be realised in terms of unit quaternions with coefficients in $\mathbb{Z}[\tau]$[16,38,39,41]. Specifically, the set of 120 icosians forms a discrete group under standard quaternionic multiplication, and is a realisation of the H_4 root system[16,42,43]. The 240 roots of E_8 have been shown to be in 1-1-correspondence with the 120 icosians and their 120 τ-multiples, so that, schematically, $E_8 \sim H_4 + \tau H_4$ holds. The projection considered above therefore exhibits this mapping of the simple roots of E_8 onto the simple roots of H_4 and their τ-multiples. Corresponding results hold for the other groups D_6 and H_3, as well as A_4 and H_2 by inclusion.

From a group theoretic point of view, E_8 has two conjugate H_4-invariant subspaces. Following the terminology in[37], we make the following definition.

Definition II.5 (Standard and non-standard representations)**.** *We denote by the* standard representation *of a Coxeter group the representation generated by mirrors forming angles* $\frac{\pi}{m_{ij}}$*, where* m_{ij} *are the entries of the Coxeter matrix, i.e.* 5, 3 *and* 2 *for the cases relevant here. One can also achieve a* non-standard representation *of the Coxeter groups* H_i *by instead taking mirrors at angles* $2\pi/5$*, i.e. by schematically going from a pentagon to a pentagram.*

Note that these therefore have Coxeter diagram labels of 5 and $\frac{5}{2}$, respectively. These are non-equivalent irreducible representations of the non-crystallographic Coxeter groups, i.e. there is no similarity transformation that takes one to the other. Their characters are exchanged under the Galois automorphism $\tau \leftrightarrow \sigma$, and the simple roots in the non-standard representation are the Galois conjugates of the simple roots of the standard representation, for instance $a_1 = \frac{1}{2}(-\sigma, -\tau, 0, -1) \leftrightarrow \bar{a}_1 = \frac{1}{2}(-\tau, -\sigma, 0, -1)$. This different set of simple roots $\bar{a}_i$ yields a Cartan matrix that is the Galois conjugate of the Cartan matrix of the simple roots a_i, and leads to the label of $\frac{5}{2}$ in the Coxeter diagram. However, in this finite-dimensional case the groups generated by the two sets of generators are isomorphic, as both sets of roots give rise to the same root system, but positioned differently in space. The non-trivial Coxeter relations are therefore in both cases $(s_i s_j)^{m_{ij}} = 1$ with $m_{ij} = 5$ for $A_{ij} = -\tau, \sigma$. Hence the Coxeter matrix is identical in both cases; in particular, it is still symmetric. Therefore, one usually restricts analysis to the standard-representation, where the simple roots form obtuse angles, and whose Cartan matrix therefore satisfies the usual negativity requirements. The projection $\pi_\perp$ into the second H_4-invariant subspace in terms of the bases α_i and $\bar{a}_i$ is the Galois conjugate of $\pi_\parallel$ in Eq. (3), which, however, is with respect to the bases α_i and a_i. The relationship between E_8 and H_4 is best understood as the standard representation of E_8 inducing both the standard and non-standard representations of the subgroup H_4. That is, E_8 decomposes under an H_4 subgroup as $\mathbf{8} = \mathbf{4} + \bar{\mathbf{4}}$ (c.f. also[38,39], in particular in the wider context of similar constructions unified in the Freudenthal-Tits magic square[44]). Equivalent statements $\mathbf{6} = \mathbf{3} + \bar{\mathbf{3}}$ and $\mathbf{4} = \mathbf{2} + \bar{\mathbf{2}}$ hold true for the lower-dimensional cases, and have found applications in the quasicrystal literature[14,15]. In the quasicrystal setting, one usually only considers the projection $\pi_\parallel$; in our setting one can consider projection into either invariant subspace, using $\pi_\parallel$ as well as $\pi_\perp$.

C. Affine extensions of crystallographic Coxeter groups

For a crystallographic Coxeter group, an affine Coxeter group can be constructed by defining affine hyperplanes $H_{\alpha_0,i}$ as solutions to the equations $(x|\alpha_0) = i$, where $x \in \mathscr{E}$, $\alpha_0 \in \Phi$ and $i \in \mathbb{Z}$[45]. The nontrivial isometry of $\mathscr{E}$ that fixes $H_{\alpha_0,i}$ pointwise is unique and called an affine reflection $s^{aff}_{\alpha_0,i}$.

Definition II.6 (Affine Coxeter group)**.** *An* affine Coxeter group *is the extension of a Coxeter* *group by an* affine reflection $s_{\alpha_0}^{aff}$ *whose geometric action is given by*

$$s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0|v)}{(\alpha_0|\alpha_0)}\alpha_0, \tag{4}$$

and is generated by the extended set of generators including the new affine reflection associated with the affine root α_0. *This operation is not distance-preserving, and hence the group is no longer compact. The* affine Cartan matrix *of the affine Coxeter group is the Cartan matrix associated with the extended set of roots. The non-distance preserving nature of the affine reflection entails that the affine Cartan matrix is degenerate (positive semi-definite), and thus fulfils* $\det A = 0$. *If the group contains both* $s_{\alpha_0}^{aff}$ *and* s_{α_0}, *it also includes the translation generator* $Tv = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v$*; otherwise,* $s_{\alpha_0}^{aff} s_{-\alpha_0}^{aff}$ *acts as a translation of twice the length.*

It is in fact possible to construct the affine Coxeter group directly from an extension of the Cartan matrix.

Definition II.7 (Kac-Moody-type affine extension)**.** *A* Kac-Moody-type affine extension A^{aff} of a Cartan matrix *is an extension of the Cartan matrix A of a Coxeter group by further rows and columns such that the following conditions hold:*

- *The diagonal entries are normalised as* $A_{ii}^{aff} = 2$ *according to the definition in Eq. (2).*
- *The additional matrix entries of* A^{aff} *take values in the same integer ring as the entries of A. This includes potentially integer rings of extended number fields as in the case of* H_3.
- *For off-diagonal entries we have* $A_{ij}^{aff} \leq 0$*; moreover,* $A_{ij}^{aff} = 0 \Leftrightarrow A_{ji}^{aff} = 0$.
- *The affine extended matrix fulfils the* determinant constraint $\det A^{aff} = 0$.

In our previous paper[1] we have laid out a rationale for Kac-Moody-type extensions of Cartan matrices, as well as consistency conditions that lead to a somewhat improved algorithm for numerically searching for such matrices. This was necessitated by our search for novel asymmetric affine extensions of H_2, H_3 and H_4. Here, our algorithm simply recovers the affine extensions of E_8, D_6 and A_4 that are well known in the literature for affine extensions by a single node. However, based on Definition II.7, we will also consider extending by two nodes in the context of the projection.

We begin with the case of E_8, which is the most interesting from a high energy physics point of view, and the largest exceptional Coxeter group. Various notations are used in the literature to denote its unique (standard) affine extension, but here we shall use $E_8^=$, where the equality sign is

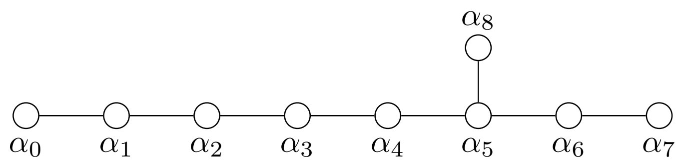
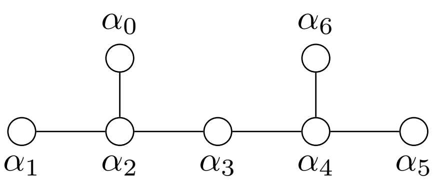


FIG. 4. Dynkin diagram for the standard affine extension of E_8, here denoted $E_8^{=}$.



$$A\left(D_6^{=}\right) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

FIG. 5. Dynkin diagram and Cartan matrix for the simply-laced standard affine extension of D_6 (here denoted $D_6^{=}$).

meant to signify that the extra root has the same length as the other roots, i.e. the affine extension is simply-laced (see Fig. 4 for our notation). The affine root α_0 that gives rise to this affine extension can be expressed in terms of the root vectors of E_8 as

$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8, \tag{5}$$

which will prove important in the projection context later.

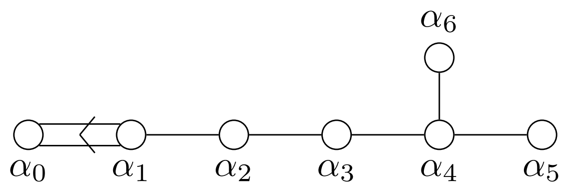
Likewise, D_6 has a simply-laced affine extension, here denoted $D_6^{=}$ and depicted in Fig. 5. Again, the affine root can be expressed in terms of the other roots as

$$-\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6. \tag{6}$$

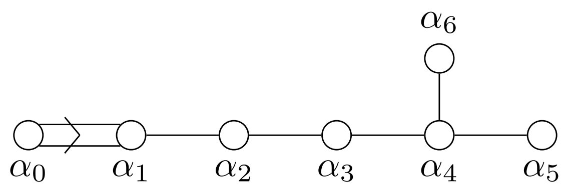
However, D_6 is unusual in that it also has two affine extensions with a different root length, one which we shall denote by $D_6^{<}$, because the new root is shorter than the others. In this case, the affine root is given by

$$-\alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \frac{1}{2}\alpha_5 + \frac{1}{2}\alpha_6, \tag{7}$$

There is, moreover, one with a longer root, which we denote by $D_6^{>}$ (both are shown in Fig. 6). Its

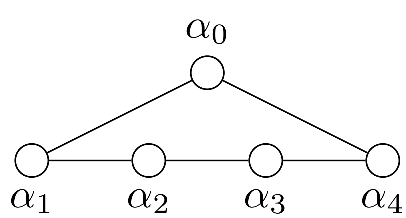


$$A\left(D_6^<\right) = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$



$$A\left(D_6^>\right) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

FIG. 6. Dynkin diagrams and Cartan matrices for the standard affine extensions of D_6 with a short affine root (here denoted $D_6^<$), and that with a long affine root, here denoted $D_6^>$. The arrow conventionally points to the shorter root.



$$A\left(A_4^=\right) = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

FIG. 7. Dynkin diagram and Cartan matrix for the simply-laced standard affine extension of A_4, here denoted $A_4^=$.

affine root is similarly expressible in terms of the other roots as

$$-\alpha_0 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6. \tag{8}$$

A_4 also has a unique standard affine extension, which is simply-laced and hence will be denoted by $A_4^=$ (see Fig. 7). The affine root is given by

$$-\alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4. \tag{9}$$

III. AFFINE EXTENSIONS OF NON-CRYSTALLOGRAPHIC ROOT SYSTEMS INDUCED BY PROJECTION

In this section, we present a novel construction of affine extensions of non-crystallographic Coxeter groups, as illustrated in Fig. 1 and indicated by the dashed arrow there. We induce affine extensions in the lower-dimensional, non-crystallographic case by applying the projection formalism from Section II B to the five affine extensions from Figs 4-7 in Section II C. We show that the induced extensions are invariant under the Dynkin diagram automorphisms of the crystallographic groups and their affine extensions (Section III B), and that in a wider class of further extensions (simply-laced, double extensions with non-trivial projection kernel), none are compatible with the projection formalism (Section III C).

A. Projecting the affine root

In the previous section, we have introduced the projection formalism, and we have presented the standard affine extensions of the relevant crystallographic groups. In particular, in each case we have given expressions for the affine roots in terms of the root vectors of the unextended group. By the linearity of the projection, one can compute the projection of the affine root. In analogy to the fact that the other roots project to generators of the groups H_i ($i = 2,3,4$), we treat the projected affine root as an additional, affine, root for the projected group H_i, thereby inducing an affine extension of H_i.

Definition III.1 (Induced affine root)**.** *For a pair of Coxeter groups (G^U, G^D) related via projection, i.e. a non-degenerate mapping π of the root system of G^U onto the root system of G^D, we call the projection of the affine root of an affine extension of G^U the* induced (affine) root *of G^D. The matrix defined as in Eq. (2) by the induced affine root and the simple roots of G^D define the* induced (affine) Cartan matrix.

Theorem III.2 (Induced Extensions)**.** *The five affine extensions $A_4^=$, $D_6^>$, $D_6^=$, $D_6^<$ and $E_8^=$ of A_4, D_6 and E_8 induce affine extensions of H_2, H_3 and H_4 via the projections linking the respective root systems. For $\pi_\parallel$, these five induced extensions shall be denoted by $H_2^=$, $H_3^>$, $H_3^=$, $H_3^<$ and $H_4^=$. Projection with $\pi_\perp$ into the other invariant subspace yields affine roots that are the Galois conjugates of those induced by $\pi_\parallel$, and the five corresponding induced affine extensions shall be denoted by $\bar{H}_2^=$, $\bar{H}_3^>$, $\bar{H}_3^=$, $\bar{H}_3^<$ and $\bar{H}_4^=$.*

Proof. We consider the five cases in turn.

1. We begin with the case of E_8. We have shown above that the root vectors can be projected onto the H_4 root vectors a_i by the projection $\pi_{\|}$ shown in Fig. 2. The projection in Eq. (3) of the affine root in Eq. (5) is therefore

$$-a_0 = \pi_{\|}(-\alpha_0) = 2(1+\tau)a_1 + (3+4\tau)a_2 + 2(2+3\tau)a_3 + (3+5\tau)a_4. \tag{10}$$

Using the inner products from the H_4 Cartan matrix $(a_1|a_2) = -\frac{1}{2}$, $(a_2|a_3) = -\frac{1}{2}$ and $(a_3|a_4) = -\frac{\tau}{2}$, the inner products of the additional root with the roots of H_4 are $(a_0|a_1) = -\frac{1}{2}$ and $(a_0|a_2) = (a_0|a_3) = (a_0|a_4) = 0$. Thus, the Cartan matrix corresponding to the simple roots of H_4 extended by the projected affine root of $E_8^=$ is found to be

$$A\left(H_4^=\right) := \begin{pmatrix} 2 & \tau-2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -\tau \\ 0 & 0 & 0 & -\tau & 2 \end{pmatrix}. \tag{11}$$

This is one of the Kac-Moody-type affine extensions of H_4 that we derived in our previous paper[1], in the context of non-crystallographic Coxeter groups. It was listed there as the first non-trivial example of affine extensions of this type and corresponds to an affine extension of length τ along the highest root α_H of H_4. We will briefly review the results from[1] in Section IV A, which we will use to classify all induced affine extensions in Section IV B.

Projecting with $\pi_\perp$ into the other H_4-invariant subspace spanned by the basis of simple roots $\bar{a}_i$ yields the Galois conjugate of the affine root in Eq. (10)

$$-\bar{a}_0 = \pi_\perp(-\alpha_0) = 2(1+\sigma)\bar{a}_1 + (3+4\sigma)\bar{a}_2 + 2(2+3\sigma)\bar{a}_3 + (3+5\sigma)\bar{a}_4. \tag{12}$$

Using the inner products $(\bar{a}_1|\bar{a}_2) = -\frac{1}{2}$, $(\bar{a}_2|\bar{a}_3) = -\frac{1}{2}$ and $(\bar{a}_3|\bar{a}_4) = -\frac{\sigma}{2}$, the inner products of the affine root with the H_4 roots are $(\bar{a}_0|\bar{a}_1) = -\frac{1}{2}$ and $(\bar{a}_0|\bar{a}_2) = (\bar{a}_0|\bar{a}_3) = (\bar{a}_0|\bar{a}_4) = 0$, and the resulting Cartan matrix $A\left(\bar{H}_4^=\right)$ is thus the Galois conjugate of that in Eq. (11) from the other invariant subspace. Since both sets of roots a_i and $\bar{a}_i$ generate the same abstract group H_4, one has a pair of Galois conjugate induced affine roots α_0 and $\bar{\alpha}_0$ parallel to the highest root α_H with Galois conjugate lengths τ and σ, respectively. Note that $\left(A\left(H_4^=\right)\right)^T$ would also generate the same translation of length σ along α_H, and was contained in the results of[1]. We will consider whether $(A(H_4^{=}))^T$ could also arise from projection in Section V.

2. Using the same procedure as above – i.e. employing linearity to project the affine root of $D_6^{=}$ and using it as an affine extension of H_3 – generates the analogue of the previous case in three dimensions

$$A(H_3^{=}) := \begin{pmatrix} 2 & 0 & \tau-2 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}. \tag{13}$$

This is also the first non-trivial example of asymmetric affine extensions of H_3 considered in our previous paper[1], corresponding to an affine extension of length τ along the highest root α_H of H_3 (i.e. along a 2-fold axis of icosahedral symmetry). One choice of simple roots for H_3 is $\alpha_1 = (0,1,0)$, $\alpha_2 = -\frac{1}{2}(-\sigma,1,\tau)$, and $\alpha_3 = (0,0,1)$, for which $\alpha_H = (1,0,0)$.

Projection into the other invariant subspace likewise generates the Galois conjugate affine root $\bar{\alpha}_0$ and the Galois conjugate Cartan matrix $A\left(\bar{H}_3^{=}\right)$, thereby giving rise to a translation of length σ along α_H.

3. When projecting $D_6^{<}$ we find

$$A\left(H_3^{<}\right) := \begin{pmatrix} 2 & \frac{4}{5}(\tau-3) & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}. \tag{14}$$

In[1], we have considered a family of matrices of this form analytically and found a similar classification as in the other cases, according to a certain Fibonacci scaling relation (c.f. Section IV A). Note, that the projection construction here naturally leads to $\mathbb{Q}[\tau]$-valued entries of the Cartan matrix, suggesting to analyse this more general class of Cartan matrices over the extended number field $\mathbb{Q}[\tau] = \{a+\tau b | a,b \in \mathbb{Q}\}$. Cartan matrices of this form correspond to affine extensions along a 5-fold axis of icosahedral symmetry T_5, where $T_5 = (\tau,-1,0)$ in our chosen basis of simple roots, and the normalisation is chosen for later convenience. The affine root of $H_3^{<}$ is then given by $\alpha_0 = \frac{1}{2}T_5$.

Projection into the other invariant subspace again yields the Galois conjugate $\bar{\alpha}_0$ of α_0, corresponding to $\bar{\alpha}_0 = -\frac{1}{2}\sigma T_5$ for our normalisation of T_5.

4. A similar result is obtained when $\pi_{\parallel}$-projecting $D_6^>$ to $H_3^>$

$$A\left(H_3^>\right) := \begin{pmatrix} 2 & \frac{2}{5}(\tau-3) & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}. \qquad (15)$$

   The respective projections again yield the Galois conjugate pair $\alpha_0 = T_5$ and $\bar{\alpha}_0 = -\sigma T_5$. We note that even though the affine extensions $D_6^<$ and $D_6^>$ are related by transposition, the correspondence between the two induced affine extensions $H_3^<$ and $H_3^>$ (and $\bar{H}_3^<$ and $\bar{H}_3^>$) is not so straightforward, i.e. the operations of transposition and projection do not commute, schematically $[T,P] \neq 0$. However, the transposed versions of these induced lower-dimensional Cartan matrices, for instance $A\left(H_3^>\right)^T$, were among the affine extensions derived in[1]. One might therefore wonder which higher-dimensional Cartan matrices could give rise to these transposed versions after projection. We will revisit these issues later.

5. The affine root of A_4 is given by Eq. (9) and upon projection with $\pi_{\parallel}$ yields an affine extension of H_2 analogous to the other simply-laced cases considered above

$$A\left(H_2^=\right) := \begin{pmatrix} 2 & \tau-2 & \tau-2 \\ -1 & 2 & -\tau \\ -1 & -\tau & 2 \end{pmatrix}. \qquad (16)$$

   This likewise corresponds to an affine root of length τ along α_H, the highest root of H_2 given by $\alpha_H = \tau(\alpha_1+\alpha_2)$, which was also found in[1], where we also visualised its action on a pentagon. Projection with $\pi_\perp$ yields an induced extension $\bar{H}_2^=$ with the Galois-conjugate length $-\sigma$ along α_H.

This completes the proof. □

As is well known[19], the affine extensions of crystallographic Coxeter groups result in a (periodic) tessellation of the fundamental domain of the unextended group in terms of copies of the fundamental domain of the affine group. In contrast, affine extended non-crystallographic groups inherit the full fundamental domain of the unextended group. The fundamental domain of these extensions however still has the interesting property of being tessellated, but in this case the tiling is *aperiodic*, and hence the fundamental domain again has a non-trivial mathematical structure.

λ	H_2	H_3	H_4
0	10	30	120
σ	40	552	5280
1	36	361	3721
τ	40	552	5280

TABLE I. Cardinalities of extended root systems/quasicrystal fragments depending on translation length. Here, we list cardinalities $|P_i(1)|$ of the point sets achieved by extending the H_i root systems by an affine reflection along the highest root $\alpha_0 = -\lambda\alpha_H$ for various values of λ. $\lambda = 1$ corresponds to the simply-laced affine extension H_i^{aff} that was considered in[20]. The induced extensions $H_i^{=}$ and $\bar{H}_i^{=}$ considered here correspond to $\lambda = \tau$ and $\lambda = -\sigma$, respectively, and yield the same cardinality. $\left(H_i^{aff}\right)^T$ is also an affine extension corresponding to $\lambda = -\sigma$, and is contained in the solutions found in[1]. We will discuss how this could be lifted to the higher-dimensional case in Section V. We note that all three translations in each case are distinguished, i.e. they give rise to less than maximal cardinality.

In order to further explore this interesting relation with quasilattices, we begin by introducing some terminology[20]. We recall that a generic affine non-crystallographic Coxeter group H_i^+ is generated by the s_js from Section II A together with the translation T that we identified in Definition II.6.

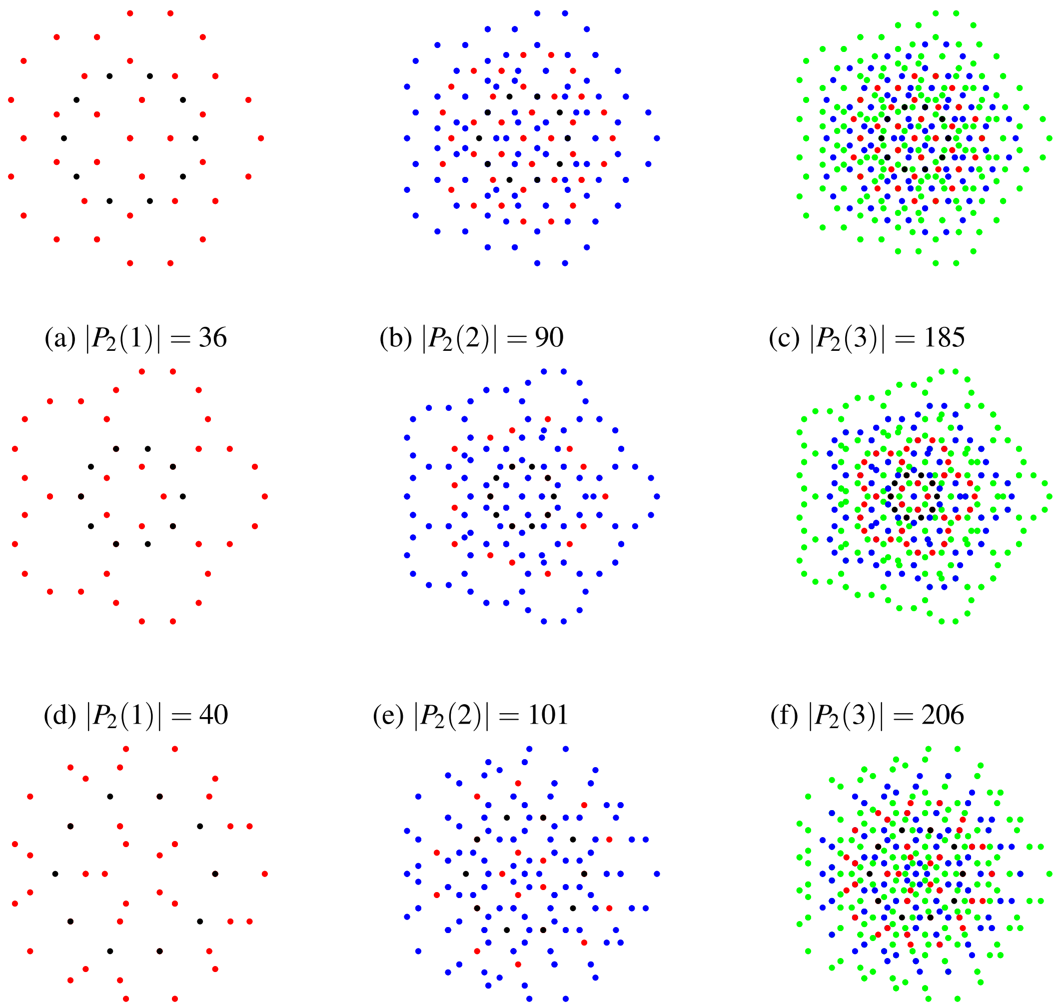
Definition III.3 (Quasicrystal fragment)**.** *Let Φ denote the root polytope of the non-crystallographic Coxeter group H_i, and let $W^m(s_j;T)$ denote the set of all* words *$w(s_j;T)$ in the* alphabet *formed from the* letters *s_j and T in which T appears precisely m times. The set of points*

$$Q_i(n) := \{W^m(s_j;T)\Phi | m \leq n\} \tag{17}$$

*is called an H_i^+-*induced quasicrystal fragment*; n is the* cut-off-parameter. *The* cardinality *of such a quasicrystal fragment will be denoted by $|Q_i(n)|$, and a generic translation yields the maximal cardinality $|Q_i^{max}(n)|$. We will say that a quasicrystal fragment with less than maximal cardinality* has coinciding/degenerate points/vertices *and we call the corresponding translation* distinguished. *This degeneracy implies* non-trivial relations *$w_1(s_j;T)v = w_2(s_j;T)v$ (for $v \in \Phi$) amongst the (words in the) generators. The set of points $P_i(n) := \{W^m(s_j;T)R | m = n\}$ will denote the* shell *of the quasicrystal fragment determined by the words that contain T precisely n times.*

The affine roots relevant here are all parallel to the respective highest root α_H but have various different lengths λ, which we write as $\alpha_0 = -\lambda\alpha_H$. Therefore, in Table I we present the cardinalities $|P_i(1)|$ of point arrays derived from the root systems of H_2, H_3 and H_4 (the decagon, the icosidodecahedron and the 600-cell, respectively) for translation lengths $\lambda = \{0, \sigma, 1, \tau\}$. $\lambda = 0$ corresponds to the unextended group, and the induced affine extensions $H_i^=$ from Eqs (16), (13) and (11) considered here correspond to $\lambda = \tau$. The simply-laced extensions H_i^{aff} considered in[20] have $\lambda = 1$. The transposes of $A(H_i^=)$ are affine extensions with length $\lambda = -\sigma$ that were also amongst those found in[1]. They are also equivalent to the induced affine extensions from the other invariant subspace, $\bar{H}_i^=$, as the compact part of the group is the same and they give rise to the same translations $\lambda = -\sigma$. This forms a subset of the extensions found in[1] that is distinguished via the projection and also through its symmetric place in the Fibonacci classification in[1], which we will discuss further in Sections IV A and IV B. All three translations belonging to the special three cases of H_i^{aff} and the induced $H_i^=$ and $\bar{H}_i^=$ are found to be *distinguished*. We also note that the Galois-conjugate translations yield the same cardinalities, i.e. the rows corresponding to σ and τ have identical entries. Investigating the corresponding cases for $H_3^<$, one finds that the Galois conjugate affine roots $\alpha_0 = \frac{1}{2}T_5$ and $\bar{\alpha}_0 = -\frac{1}{2}\sigma T_5$ yield the same cardinality of 212. For $H_3^>$, the conjugate pair $\alpha_0 = T_5$ and $\bar{\alpha}_0 = -\sigma T_5$ has the same cardinality 330. Note that these are all distinguished translations. The cardinalities are lower than for H_3^{aff}, $H_3^=$ and $\bar{H}_3^=$ where $\alpha_H = T_2 = (1,0,0)$, since there are thirty 2-fold but only twelve 5-fold axes of icosahedral symmetry.

Twarock and Patera[20] considered simply-laced affine extensions H_i^{aff} of H_2, H_3 and H_4 in the context of quasicrystals in two, three and four dimensions. As mentioned above, the affine reflections of those extensions yield translations T of length 1 along the highest root α_H. Such H_2^{aff}-induced quasicrystal fragments $Q_2(1)$, $Q_2(2)$ and $Q_2(3)$ are depicted in panels (a-c) in Fig. 8. Quasicrystals are often induced via a cut-and-project method from a projection of the root lattice, much as in the projection framework considered here. We thus consider the implications of our induced affine extensions for the quasicrystal setting. Our new projection construction yields different affine extensions from the above H_i^{aff}s, with translation lengths τ and $-\sigma$ along the highest root α_H. These are new cases with asymmetric Cartan matrices and would therefore not arise in the symmetric setting. However, following the same construction as in[20] but with the different translation lengths τ and $-\sigma$ results in a similar subset of a vertex set of a quasicrystal. For $H_2^=$, the resulting quasicrystal fragments $Q_2(1)$, $Q_2(2)$ and $Q_2(3)$ are shown in panels (d-f) in Fig. 8, and the corresponding fragments for $\bar{H}_2^=$ are shown in panels (g-i). We furthermore give the cardinalities $|P_2(1)|$, $|P_2(2)|$ and $|P_2(3)|$ in each case. Here, panel (a) corresponds to the point



(a) $|P_2(1)| = 36$ (b) $|P_2(2)| = 90$ (c) $|P_2(3)| = 185$

(d) $|P_2(1)| = 40$ (e) $|P_2(2)| = 101$ (f) $|P_2(3)| = 206$

(g) $|P_2(1)| = 40$ (h) $|P_2(2)| = 101$ (i) $|P_2(3)| = 206$

FIG. 8. Quasicrystal fragments for various affine extensions of H_2 (the black dots are the decagonal root system): Vertically, the affine root is of length 1 and parallel to the highest root for the first row of panels (a-c), which are the H_2^{aff}-quasicrystal fragments considered in[20]. The second row of panels (d-f) are the quasicrystal fragments obtained for the extension $H_2^{=}$ induced from $A_4^{=}$ considered here, where the translation length is τ. The third row of panels (g-i) are the quasicrystal fragments obtained for the extension $\bar{H}_2^{=}$ with translation length $-\sigma$, or alternatively one may think of it in terms of $(A(H_2^{=}))^T$. Horizontally, the panels show the point sets $Q_2(1)$, $Q_2(2)$ and $Q_2(3)$ derived from the root system by letting the translation operator T act once (red dots), twice (blue dots) and three times (green dots). Thus, panels (a), (d) and (g) correspond to the point sets with cardinalities 36 and 40 listed in Table I. The cardinalities of the shells $|P_2(n)|$ are also given. We note that Galois conjugate translations yield the same cardinalities.

set of cardinality 36 in Table I, and panels (d) and (g) correspond to the entries with cardinality 40. We again note that Galois conjugate affine roots yield the same cardinalities, as in the higher-dimensional cases before. Our novel construction thus leads to different types of quasicrystalline point arrays. We will later consider whether the extensions from[20] could similarly be induced from a higher-dimensional setting. We will see that they would correspond to Cartan matrices with positive and fractional (c.f. $H_3^<$ and $H_3^>$ in Eqs (14) and (15)) off-diagonal entries (Section V), making the case for a suitable generalisation of the standard approach by analysing generalised Cartan matrices over extended number fields.

The above projection procedure has thus yielded asymmetric induced Cartan matrices. In the context of Kac-Moody algebras and Coxeter groups, it is often of interest to know if an asymmetric (generalised) Cartan matrix A is symmetrisable:

Definition III.4 (Symmetrisability)**.** *An asymmetric Cartan matrix A is* symmetrisable *if there exist a diagonal matrix D with positive integer entries and a symmetric matrix S such that $A = DS$.*

We have investigated the symmetrisability of the induced non-symmetric Cartan matrices[1]. They are indeed symmetrisable, but the entries of the resulting symmetric matrices are no longer from $\mathbb{Z}[\tau]$ (see also the discussion in Section V). Given that the Cartan matrix is defined in terms of the geometry of the roots as $A_{ij} = 2(\alpha_i|\alpha_j)/(\alpha_i|\alpha_i)$, i.e. is given in terms of the angles between root vectors and their length, such matrices would imply a geometry for the root system that is no longer compatible with an (aperiodic) quasilattice, and the corresponding affine groups would therefore lose their distinctive structure. Indeed, it is that relation with quasilattices that makes these affine extended groups mathematically interesting, and distinguishes them from the free group obtained by an extension via a random translation. Therefore, we will not use these symmetric matrices in our context.

B. Invariance of the projections under Dynkin diagram automorphisms

Before we classify the induced affine extensions, we show in this section that no additional induced extensions arise from the Dynkin diagram automorphisms of the simple and affine Lie algebras considered above.

Lemma III.5 (Invariance of the induced extensions)**.** *The induced affine extensions $H_2^=$, $H_3^>$, $H_3^=$, $H_3^<$, $\bar{H}_2^=$, $\bar{H}_3^>$, $\bar{H}_3^=$ and $\bar{H}_3^<$ are invariant under the Dynkin diagram automorphisms of A_4, D_6, $A_4^=$,*

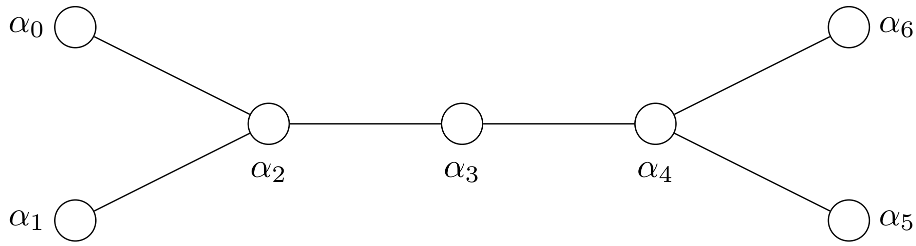


FIG. 9. A more symmetric version of the $D_6^=$ Dynkin diagram, that makes the $\mathscr{D}_4$-automorphism symmetry manifest.

$D_6^>$, $D_6^=$ and $D_6^<$.

Proof. We consider the four cases in turn.

1. The Dynkin diagram of D_6 has a $\mathbb{Z}_2$-automorphism that acts by permuting the roots α_5 and α_6 (denoted as $5 \leftrightarrow 6$ in the following). The projection displayed in Fig. 3, however, is not symmetric in α_5 and α_6. Therefore, the choice of projection could potentially alter the induced affine extension. However, as can be seen from equations (6), (7) and (8), all three possible affine roots are in fact invariant under the exchange of a_5 and a_6, so that the result of the projection is not affected.

2. Similarly, the simply-laced extension $D_6^=$ has an additional $\mathscr{D}_4$ automorphism symmetry (here $\mathscr{D}_n$ denotes the dihedral group of order n) that allows one to swap the roots labelled by $5 \leftrightarrow 6$ or $0 \leftrightarrow 1$ separately, as well as an overall left-right symmetry of the diagram obtained by swapping the pairs of terminal roots $(0,1) \leftrightarrow (5,6)$ together with $2 \leftrightarrow 4, 3 \leftrightarrow 3$[2]. This symmetry is made manifest in the diagram shown in Fig. 9. Thus, the four terminal roots are equivalent, and one could define four different projections, depending on which terminal root one considered as the affine root. Once one decides on the affine root, the rest of the diagram is fixed by the projection. However, the formula for the affine root is symmetric in $(0,1,5,6)$ as can be seen from (6). Thus, the induced affine extension is again independent of which projection one chooses.

3. Likewise, the A_4 diagram has a $\mathbb{Z}_2$-symmetry swapping left and right, that is broken by the projection. However, the affine root (9) is again invariant, so that the induced affine extension is not affected.

4. The extended diagram $A_4^=$ has an enhanced $\mathscr{D}_5$-automorphism symmetry[2], under which the affine root can be seen as invariant by rewriting Eq. (9) as

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0. \tag{18}$$

Thus, in this case, one could choose any of the roots of the extended diagram as the affine root, and the others are then fixed by the projection.

The invariance is at the level of the affine roots before projection, so it does not matter into which invariant subspace one projects. Thus, the Dynkin diagram automorphisms do not affect the induced affine extensions. □

C. Extending by two nodes

Until now, we have considered extending a diagram by a single root, and we have projected this single affine root onto the single induced affine root. However, as is shown in Fig. 2, other roots are projected in pairs, e.g. α_1 and α_7 project onto a_1 and τa_1, which results in the single H_4-root a_1. In analogy, we now consider affine extensions of the diagrams by two nodes such that the two additional roots project as a pair onto a single affine root. This can be achieved by further extending the above affine extensions by another node, or by extending the initial diagrams by two nodes at once.

We first check whether further extending the above groups $A_4^=$, $D_6^>$, $D_6^=$, $D_6^<$ and $E_8^=$ by another node leads to new induced affine extensions. We show that in such a case, the only possibilities are in fact the above diagrams with a disconnected node. Thus, this type of extension is trivial, and the additional affine root will not be a superposition of the other roots:

Lemma III.6. *The affine extensions of $A_4^=$, $D_6^>$, $D_6^=$, $D_6^<$ and $E_8^=$ by a further node are disconnected.*

Proof. The determinants of general Kac-Moody-type affine extensions of $A_4^=$, $D_6^>$, $D_6^=$, $D_6^<$ and $E_8^=$ are quadratic in the entries in the new row and column with negative coefficients. Since the entries are non-positive, the determinant is therefore also non-positive. Since the zero entries in a Cartan matrix are symmetric, the determinant vanishes if and only if all the entries in the new row and column vanish. Thus, the extended diagram has a disconnected node. □

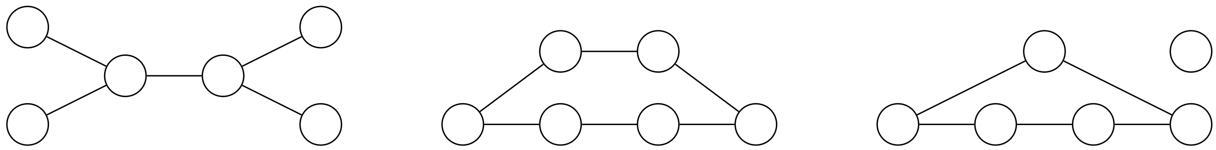


FIG. 10. Double extensions A_4^{++} of A_4.

The other possibility is to extend by two nodes at once, and to demand that the Cartan matrix of the double-extension be affine, i.e. that it has zero determinant, but that none of the principal minors has this property.

Definition III.7 (Affine double extension)**.** *An* affine double extension *is a Kac-Moody-type extension of a diagram by two nodes.*

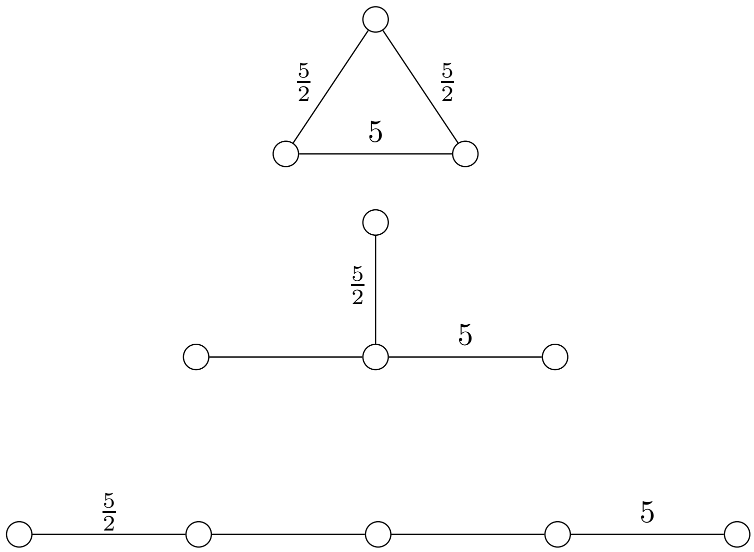
We analyse here the simply-laced double extensions with a trivial projection kernel, which give 292 such matrices for E_8, 27 for D_6, and 6 for A_4. Note, however, that the number of different Dynkin diagrams is actually lower. For instance, there are only three diagrams that occur for A_4^{++}, which are displayed in Fig. 10. The first corresponds to a single extension of the D-series, c.f. $D_6^{=}$ above. The second is a single extension of A_5, and the third a diagram with a trivial disconnected node. The diagrams for D_6 and E_8 have a richer mathematical structure. However, it shall suffice here to consider these matrices from the projection point of view – a more detailed analysis of double extensions will be relegated to future work.

Lemma III.8. *There are no simply-laced affine double extensions of* A_4*,* D_6 *and* E_8 *with trivial projection kernel.*

Proof. In all the cases mentioned above (292 for E_8, 27 for D_6 and 6 for A_4), it is not possible to express both additional roots simultaneously in terms of linear combinations of the roots of the unextended group, c.f. the diagrams for A_4 in Fig. 10. Hence, the Cartan matrices can be obtained only in terms of higher-dimensional vectors, i.e. the kernel is non-trivial. □

Corollary III.9. *Simply-laced affine double extensions of* A_4*,* D_6 *and* E_8 *with trivial projection kernel do not induce any further affine extensions of the non-crystallographic Coxeter groups.*

Earlier, we have demonstrated that the induced affine extensions do not depend on the non-trivial automorphism properties of the simple and extended diagrams. Therefore, in summary, we conclude that the ten cases considered above are actually the only cases that arise in the context of trivial projection kernels.



$$A\left(H_2^{aff}\right)=\begin{pmatrix}2 & \tau' & \tau'\\ \tau' & 2 & -\tau\\ \tau' & -\tau & 2\end{pmatrix}$$

$$A\left(H_3^{aff}\right)=\begin{pmatrix}2 & 0 & \tau' & 0\\ 0 & 2 & -1 & 0\\ \tau' & -1 & 2 & -\tau\\ 0 & 0 & -\tau & 2\end{pmatrix}$$

$$A\left(H_4^{aff}\right)=\begin{pmatrix}2 & \tau' & 0 & 0 & 0\\ \tau' & 2 & -1 & 0 & 0\\ 0 & -1 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -\tau\\ 0 & 0 & 0 & -\tau & 2\end{pmatrix}$$

FIG. 11. Coxeter-Dynkin diagrams and Cartan matrices for (from top to bottom) H_2^{aff}, H_3^{aff} and H_4^{aff}, the unique symmetric affine extensions in[20]. Note that Coxeter angles of $2\pi/5$ lead to labels $\frac{5}{2}$ (or, τ' in the notation of[20]). In[1], we have found infinite families of generalisations of these examples, which are obtained from the symmetric cases via scalings with τ and thus follow a Fibonacci recursion relation. We have also found more general examples, which likewise display this scaling property.

IV. CLASSIFICATION OF INDUCED AFFINE EXTENSIONS

The affine extensions induced via the projection in Section III (right arrow in Fig. 1) are subsets of the infinite families developed purely in a non-crystallographic framework in[1] (bottom arrow in Fig. 1). Therefore, we first summarise the relevant results from this paper in Subsection IV A, and then analyse in Subsection IV B how the induced affine extensions relate to our classification in[1].

A. Construction of affine extensions in the non-crystallographic case

In the case of non-crystallographic Coxeter groups, which have Cartan matrices given in terms of the extended integer ring $\mathbb{Z}[\tau]$, the earlier definition of affine extensions from Section II C via introducing affine hyperplanes $H_{\alpha_0,i}$ as solutions to the equations $(x|\alpha_0)=i$, where $x\in\mathscr{E}$, $\alpha_0\in\Phi$ and $i\in\mathbb{Z}$, is not possible because the crystallographic restriction[15] implies that the planes cannot be stacked periodically; however, $i\in\mathbb{Z}[\tau]$ is too general because $\mathbb{Z}[\tau]$ is dense in $\mathbb{R}$.

In contrast, the definition of affine extensions of non-crystallographic Coxeter groups via the Kac-Moody-type extensions of their Cartan matrices in Definition II.7 still works. For instance, affine extensions along a 2-fold axis of icosahedral symmetry $T_2 = (1,0,0)$ (i.e. along the simple roots) have the following general form[1]

$$A = \begin{pmatrix} 2 & 0 & x & 0 \\ 0 & 2 & -1 & 0 \\ y & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}, \tag{19}$$

since T_2 is orthogonal to two of the simple roots, and thus there is only one pair of off-diagonal entries that is non-zero. This family of matrices contains $H_3^=$ in Eq. (13) as a special case. For this type of matrix to be affine, the determinant constraint $\det A = xy - \sigma^2 = 0$ determines the product $A_{13}A_{31} = xy$ of the non-zero entries x and y as $xy = 2 - \tau = \sigma^2$. From the definition of the Cartan matrix, the products of the off-diagonal entries give the angle of the affine root with the simple roots, such that xy gives the only non-trivial angle of the affine root with the simple roots. Corresponding extensions of H_2 and H_4 satisfy the same constraint. This determinant constraint therefore includes the symmetric affine extensions H_i^{aff} $(i = 2,3,4)$ found in[20], which satisfy $x = y = \sigma$ (τ' in their notation) and are displayed in Fig. 11. Writing $x = (a + \tau b)$ and $y = (c + \tau d)$ with $a,b,c,d \in \mathbb{Z}$, and denoting (x,y) by the quadruplet $(a,b;c,d)$, the H_i^{aff} correspond to the simplest case $(a,b;c,d) = (1,-1;1,-1)$. The units in $\mathbb{Z}[\tau]$ are the powers of τ, so scaling $x \to \tau^{-k}x$ and $y \to \tau^k y$ $(k \in \mathbb{Z})$ leaves the product xy invariant, and thus one can generate a series of solutions to the determinant constraint from any particular reference solution. In terms of quadruplets $(a,b;c,d)$, this scaling of the series of solutions by (τ^{-k}, τ^k) amounts to $(a,b;c,d) \to (b-a,a;d,c+d)$, which is a Fibonacci defining relation. The case $x = y = \sigma$ is distinguished by its symmetry, and we choose to generate the whole *Fibonacci series of solutions* from this particular reference solution. Thus, under the τ-rescaling (τ^{-1}, τ), $H_3^=$ is the 'first' asymmetric example with $(x,y) = (-\sigma^2, -1) = (\tau - 2, -1)$ corresponding to the quadruplet $(-2,1;-1,0)$. Since the powers of τ are the only units in $\mathbb{Z}[\tau]$, these solutions are in fact the only solutions to the determinant constraint. The length of a root is given by $\sqrt{x/y}$, so that τ-rescalings do not change the angle but generate affine roots of different lengths. There is thus a countably infinite set $(k \in \mathbb{Z})$ of affine extensions of H_i with affine reflection hyperplanes at distances $\tau^k/2$ from the origin. For any given k, there is an infinite stack of parallel planes with separation $\tau^k/2$, including one containing the origin, which corresponds to one of the reflections in the unextended group.

By choosing an ansatz similar to Eq. (19), one can also obtain translations along 3- and 5-fold axes of icosahedral symmetry, $T_3 = (\tau, 0, \sigma)$ and $T_5 = (\tau, -1, 0)$. These cases correspond to one pair of non-zero entries (x, y) in the Cartan matrix in the fourth row and column, and in the second row and column, respectively, since these symmetry axes are again orthogonal to two of the simple roots. From now on, we will use the notation $A^{aff} = \begin{pmatrix} 2 & \underline{v}^T \\ \underline{w} & A \end{pmatrix}$ to write affine extensions succinctly in terms of the vectors $\underline{v}$ and $\underline{w}$ in the additional row and column. Thus, affine extensions along a 3-fold axis correspond to $\underline{v} = (0, 0, x)^T$ and $\underline{w} = (0, 0, y)^T$, whilst affine extensions along a 5-fold axis correspond to $\underline{v} = (x, 0, 0)^T$ and $\underline{w} = (y, 0, 0)^T$. These affine extensions lead to determinant constraints that do not have solutions in $\mathbb{Z}[\tau]$, e.g. $xy = \frac{4}{3}\sigma^2$ for an affine extension along a 3-fold axis. If one is prepared to relax the conditions on the Cartan matrix entries to $\mathbb{Q}[\tau]$ instead of $\mathbb{Z}[\tau]$, one can solve the determinant constraint in $\mathbb{Q}[\tau]$, and generate similar families of affine extensions via τ-rescalings from a particular reference solution. We introduce a pair of fractions $\gamma, \delta \in \mathbb{Q}$ and write the entries of the Cartan matrix as $\mathbb{Z}[\tau]$-integers multiplied by γ, δ as $x = \gamma(a + \tau b)$ and $y = \delta(c + \tau d)$. The solution is therefore now given by a quadruplet $(a, b; c, d)$ plus multipliers (γ, δ). The latter need to make up the fraction in the determinant constraint, e.g. $\gamma\delta = \frac{4}{3}$ for an affine extension along a 3-fold axis. For instance, $(1, -1; 1, -1)$ and $(1, \frac{4}{3})$ correspond to the solution $(\sigma, \frac{4}{3}\sigma)$ of length $\sqrt{x/y} = \frac{1}{2}\sqrt{3} = \frac{1}{2}|T_3|$.

For extensions along the 5-fold axis, the determinant constraint is proportional to $(3 - \tau)$, which cannot be solved symmetrically ($x = y$) in $\mathbb{Z}[\tau]$; one solution is, for instance, $(1, -2; 1, -1)$. This implies that swapping x and y produces a different solution to the determinant constraint. Since the length of the affine root is given by $\sqrt{x/y}$, this generates a solution of different length. These two solutions are independent and generate two Fibonacci families of affine roots by τ-multiplication. For the previous determinant constraints with symmetric solutions, transposition and τ-rescalings are equivalent, such that only one Fibonacci family arises.

In summary, one can thus label an affine extension in the following way: A solution is given in terms of an integer quadruplet $(a, b; c, d)$ that is related to a particular reference solution via rescaling with a power k of τ, together with a multiplier pair (γ, δ).

B. Identification of the induced affine extensions in the Fibonacci classification

We now identify the induced affine extensions derived in Section III within the Fibonacci families from[1] discussed in the previous Section IV A.

The affine extensions $H_i^{=}$ induced from the three simply-laced affine extensions of the crystallographic groups are all related to the H_i^{aff}s via τ-rescalings. In particular, we have seen that $H_i^{=}$ corresponds to $(-2,1;-1,0)$, which is derived from the symmetric solution $(1,-1;1,-1)$ corresponding to H_i^{aff} via rescaling (x,y) as $(\tau^{-1}x,\tau y)$. Likewise, $(A(H_i^{=}))^T$ corresponds to $(\tau x,\tau^{-1}y)$ and is equivalent to $\bar{H}_i^{=}$. The induced affine extensions found in this paper are therefore the 'first' asymmetric solutions in the Fibonacci family with the symmetric reference solution H_i^{aff} obtained by rescaling with one power of τ. Since for these examples, the determinant constraint $xy=\sigma^2$ is solved in $\mathbb{Z}[\tau]$, the multiplier pair (γ,δ) is trivially $(1,1)$.

The determinant constraint for the other two induced affine extensions of H_3 found above, $H_3^{<}$ and $H_3^{>}$, is $xy=\frac{4}{5}(3-\tau)$, which has no symmetric solution. Thus, two inequivalent series of solutions are generated by the quadruplets $(-1,0;-3,1)$ and $(-3,1;-1,0)$ or, equivalently, from their τ-multiples $(1,-1;1,-2)$ and $(1,-2;1,-1)$. In our notation, the multiplier pair $(\gamma,\delta)=(\frac{4}{5},1)$ and the quadruplet $(-3,1;-1,0)$ give $H_3^{<}$, and $(\gamma,\delta)=(\frac{2}{5},2)$ with $(-3,1;-1,0)$ gives $H_3^{>}$, such that they both belong to a Fibonacci family found in[1], represented by $(1,-2;1,-1)$ and scaled by (τ^{-1},τ).

We summarise the relation of all induced affine extensions to the Fibonacci classification[1] in Table II.

Theorem IV.1 (Classification)**.** *The only affine extensions of the non-crystallographic groups H_4, H_3 and H_2 induced via projection from at most simply-laced double extensions of E_8, D_6 and A_4 with trivial projection kernel are those in Table II, and they are a subset of the Fibonacci Classification scheme presented in*[1].

Proof. The five known affine extensions of E_8, D_6 and A_4 give the induced extensions listed in the table. By the Invariance Lemma, the projection is invariant under the (extended) Dynkin diagram automorphisms. Further extensions by a single node would be disconnected, and the simply-laced double extensions are incompatible with the projection formalism via the two Lemmas III.6 and III.8 in Section III C. Thus, no more cases arise in this setting. The classification was performed in[1]. □

group	xy	$(a,b;c,d)_{ref}$	k	(γ,δ)	$\underline{v}^T$	$\underline{w}^T$
$H_4^=$	$2-\tau$	$(1,-1;1,-1)$	-1	$(1,1)$	$(\tau-2,0,0,0)$	$(-1,0,0,0)$
$H_3^=$	$2-\tau$	$(1,-1;1,-1)$	-1	$(1,1)$	$(0,\tau-2,0)$	$(0,-1,0)$
$H_3^<$	$\frac{4}{5}(3-\tau)$	$(1,-2;1,-1)$	-1	$(\frac{4}{5},1)$	$(\frac{4}{5}(\tau-3),0,0)$	$(-1,0,0)$
$H_3^>$	$\frac{4}{5}(3-\tau)$	$(1,-2;1,-1)$	-1	$(\frac{2}{5},2)$	$(\frac{2}{5}(\tau-3),0,0)$	$(-2,0,0)$
$H_2^=$	$2-\tau$	$(1,-1;1,-1)$	-1	$(1,1)$	$(\tau-2,\tau-2)$	$(-1,-1)$

TABLE II. Identification of the induced extensions of H_2, H_3 and H_4 within the Fibonacci classification: For $x=\gamma(a+\tau b)$, $y=\delta(c+\tau d)$, the $\mathbb{Z}[\tau]$-quadruplet part $(a,b;c,d)$ of the solution (x,y) to the determinant constraint xy (second column) is given by scaling a representative reference solution within a Fibonacci family, e.g. one distinguished by its symmetry (see Section IV A) like $(1,-1;1,-1)$, (third column) by a power τ^k of τ (fourth column). The rational part of the solution is contained in the multiplier pair (γ,δ) (fifth column). This contains all the information to construct the row and column vectors $\underline{v}$ and $\underline{w}$ in the extended Cartan matrix for $H_4^=$, $H_3^=$, $H_3^<$, $H_3^>$ and $H_2^=$, which are given in the last two columns. The affine extensions induced by projection into the other invariant subspace $\bar{H}_4^=$, $\bar{H}_3^=$, $\bar{H}_3^<$, $\bar{H}_3^>$ and $\bar{H}_2^=$ have Cartan matrices that are Galois conjugate to the above five cases and that follow a similar Fibonacci classification in terms of $\sigma=1-\tau$. They are essentially equivalent to transposes of the former type since the two sets of simple roots a_i and $\bar{a}_i$ generate equivalent compact groups H_i.

V. DISCUSSION

We have shown that via affine extensions of the crystallographic root systems E_8, D_6 and A_4 (upper arrow in Fig. 1) and subsequent projection (dashed arrow), one obtains affine extensions of the non-crystallographic groups H_4, H_3 and H_2 of the type considered in[1] (lower arrow). This provides an alternative construction of affine extensions of this type, and by placing them into the broader context of the crystallographic group E_8, we open up new potential applications in Lie theory, modular form theory and high energy physics. The ten induced extensions derived here are a subset of the extensions in the Fibonacci classification scheme derived in[1]. The Fibonacci classification contains an infinity of solutions of which the ones derived here are thus a subset distinguished by the projection. For the simply-laced cases, the induced extensions $H_i^=$ and $\bar{H}_i^=$ (since this is equivalent to $A(H_i^=)^T$) can be derived from the symmetric solutions H_i^{aff} from[20] via rescaling with τ and are in that sense the 'first' asymmetric members of the corresponding

Fibonacci families of solutions. These distinguished affine extensions could thus have a special rôle in practical applications, e.g. in quasicrystal theory, virology and carbon chemistry.

The induced extensions are $\mathbb{Z}[\tau]$-valued in the simply-laced cases, and $\mathbb{Q}[\tau]$-valued for the other two non-simply-laced cases. This suggests to further generalise the Kac-Moody framework of[1] to allow extended number fields in the entries in the extended Cartan matrices of H_2, H_3 and H_4; this could be $\mathbb{Q}[\tau]$, but a milder extension might also suffice. One could therefore argue to also allow corresponding generalisations in the extended Cartan matrices of E_8, D_6 and A_4, from which the non-crystallographic cases are obtained via projection. Such a generalisation might lead to interesting mathematical structures and could open up novel applications in hyperbolic geometry and rational conformal field theory, where similar fractional values can occur[36,46,47]. Various other approaches hint at this same generalisation, which we now explore in turn.

In particular, the projection (left arrow in Fig. 1) is in fact one-to-one, since integer Cartan matrix entries in the higher-dimensional setting project onto $\mathbb{Z}[\tau]$-integers in half the number of dimensions, and the two parts in a $\mathbb{Z}[\tau]$-integer do not mix. This is due to the irrationality of the projection angle, which projects a lattice in higher dimensions to an aperiodic quasilattice in lower dimensions, without a null space of the same dimension. Thus, one can invert the projection by 'lifting' the affine roots and thereby Cartan matrices of the non-crystallographic groups considered in the Fibonacci classification in[1] to those of the crystallographic groups (i.e. by inverting the dashed arrow). We now lift (denoted by L) such extended Cartan matrices of H_4, H_3 and H_2 in order to determine what type of extensions of E_8, D_6 and A_4 could induce them via projection (denoted by P). Again, we denote generic extensions E_8^+, D_6^+ and A_4^+ by their additional row and column vector in the Cartan matrix as follows

$$A\left(E_8^+\right) = \begin{pmatrix} 2 & \underline{\mathrm{v}}^T \\ \underline{\mathrm{w}} & E_8 \end{pmatrix},\ A\left(D_6^+\right) = \begin{pmatrix} 2 & \underline{\mathrm{v}}^T \\ \underline{\mathrm{w}} & D_6 \end{pmatrix} \text{ and } A\left(A_4^+\right) = \begin{pmatrix} 2 & \underline{\mathrm{v}}^T \\ \underline{\mathrm{w}} & A_4 \end{pmatrix}.$$

We begin by lifting the affine extensions H_i^{aff} from[20], which are the symmetric special cases in the Fibonacci families of solutions. One might have thought intuitively that these H_i^{aff} would arise via projection, rather than the $H_i^=$. It is thus interesting to lift the affine roots of the H_i^{aff} to the higher-dimensional setting and to see which Cartan matrix they would therefore give rise to. For example, for H_4^{aff} the vectors giving the additional row and column in the Cartan matrix are given by

$$L\left(A\left(H_4^{aff}\right)\right) = \begin{pmatrix} 2 & \underline{v}^T \\ \underline{w} & E_8 \end{pmatrix} \text{ with } \underline{v}_4^{aff} = \underline{w}_4^{aff} = (1,0,0,0,0,0,-1,0)^T. \quad (20)$$

Similarly, the vectors corresponding to H_3^{aff} and H_2^{aff} are given by $\underline{v}_3^{aff} = \underline{w}_3^{aff} = (0,1,0,-1,0,0)^T$ and $\underline{v}_2^{aff} = \underline{w}_2^{aff} = (1,-1,-1,1)^T$, respectively. We note that the lifted versions of the symmetric extensions H_i^{aff} are also symmetric. However, the requirement of non-positivity of the off-diagonal Cartan matrix entries that is usual in the Lie algebra context is not satisfied by these matrices. This is in agreement with the fact that the only standard affine extensions of E_8, D_6 and A_4 are the five cases presented in Section II C. The lifted versions of H_i^{aff} could thus motivate to relax this requirement of non-positivity in order to arrive at an interesting more general class of Cartan matrices.

Analogously, one can consider lifting the transposes of the Cartan matrices obtained earlier that are induced from $\pi_\|$ (c.f. equations (11)-(16)), e.g. $(A(H_4^=))^T$. In particular, they are also contained in the Fibonacci classification of affine extensions in[1] (see Section IV A). They are in fact also related to the extensions induced by $\pi_\perp$, since they give rise to equivalent compact parts with the same translation lengths. For example, lifting the transpose $(A(H_4^=))^T$ of $A(H_4^=)$ in Eq. (11) gives the following matrix in 9D

$$LTP(A(E_8^=)) = L\left((A(H_4^=))^T\right) = \begin{pmatrix} 2 & \underline{v}^T \\ \underline{w} & E_8 \end{pmatrix} \text{ with } \underline{v}_4^= = \frac{1}{2}\underline{w}_4^= = (-1,0,0,0,0,0,\frac{1}{2},0)^T, \quad (21)$$

where we have denoted the combination of projecting the affine extension of E_8, transposing, and lifting again, by LTP. We note that there are again positive, but now also fractional entries – neither occurs in the context of simple Lie Theory. We also observe that the consistency conditions (the Lemma in[1]) stipulated in our previous paper are still obeyed. One may find rational entries surprising, but these actually arise naturally in the context of the affine extensions considered in the non-crystallographic setting[1], e.g. $H_3^<$. Perhaps, therefore generalising integer to rational entries also in the higher-dimensional crystallographic case could lead to interesting new mathematical structures. The other cases correponding to $D_6^<$, $D_6^>$, $D_6^=$ and $A_4^=$ are given by $\underline{v}_3^< = \frac{5}{12}\underline{w}_3^< = (-1,0,0,0,0,\frac{1}{3})^T$, $\underline{v}_3^> = \frac{5}{3}\underline{w}_3^> = (-2,0,0,0,0,\frac{2}{3})^T$, $\underline{v}_3^= = \frac{1}{2}\underline{w}_3^= = (0,-1,0,\frac{1}{2},0,0)^T$, and $\underline{v}_2^= = \frac{1}{2}\underline{w}_2^= = (-1,\frac{1}{2},\frac{1}{2},-1)^T$, respectively. In an analogous manner, one could proceed to lift all the solutions in the Fibonacci family rather than just H_i^{aff} and $A(H_i^=)^T$, but these instructive examples shall suffice to give some indication towards the generalisations that arise.

As explained in Definition III.4 in Section III A, in the Coxeter group and Lie Algebra contexts, one is often interested in symmetrisable Cartan matrices. For completeness, we therefore present here the symmetrised version of $LTP\left(A\left(E_8^{=}\right)\right)$, which we denote by $SLTP$, as an example of which type of matrix arises through symmetrisation

$$SLTP\left(A\left(E_8^{=}\right)\right) = \begin{pmatrix} 1 & \underline{v}^T \\ \underline{w} & E_8 \end{pmatrix} \text{ with } \underline{v}^T = \underline{w}^T = (-1,0,0,0,0,0,\frac{1}{2},0). \tag{22}$$

Here we have relaxed the requirement that the symmetrised matrix be integer-valued, since even before symmetrisation, the Cartan matrix has fractional values. Again, positive and fractional values arise. This matrix is positive semi-definite, which was expected as that corresponds to the affine case. Thus, even the criterion of symmetrisability suggests positive entries and extending the number field for the Cartan matrix entries to $\mathbb{Q}[\tau]$ or $\mathbb{Z}[\tau]+\frac{1}{2}\mathbb{Z}[\tau]$.

In summary, we have provided a novel, alternative construction of affine extensions of the type considered in[1] from the two familiar concepts of affine extensions of crystallographic groups and projection of root systems. This construction results in a special subset of the point arrays used in mathematical virology and carbon chemistry that is distinguished via the projection, which could therefore play a special rôle in applications. It also extends the quasicrystal framework considered in[20] to a wider class of quasicrystals. We furthermore made the case for admitting extended number fields in the Cartan matrix. These extended number fields arise in a variety of cases, namely in the lower-dimensional picture[1], via projection, via lifting the Fibonacci family of solutions (including H_i^{aff} from[20] and transposes of $H_i^{=}$) and via symmetrising. Fractional entries in Cartan matrices arise in hyperbolic geometry and rational conformal field theory. Our construction here is thus another example of such fractional entries that could open up a new type of analysis and enticing possibilities in these fields.

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