ON THE COMPUTATION OF OPTIMAL SYSTEM

ASYMPTOTIC ROOT-LOCI

by

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Abstract

This note outlines a procedure for determining the asymptotic behaviour of the optimal closed-loop poles of a time-invariant linear regulator as the weight on the input in the performance criterion approaches zero. It is based on the systematic use of dynamic input/output transformations to the relevant return-difference.

1. Introduction

It is well-known[1] that the stabilizable and detectable time-invariant linear system S(A,B,C)

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad , \quad x(t) \in R^n$$
$$y(t) = C\,x(t) \quad , \quad y(t) \in R^m \quad , \quad u(t) \in R^{\ell} \tag{1}$$

with state feedback controller minimizing the performance criterion

$$J = \int_o^{\infty} \{y^T(t)\,Qy(t) + p^{-1}\,u^T(t)\,R\,u(t)\}dt \tag{2}$$

(where both Q and R are positive definite and $p > o$) has closed-loop poles equal to the left-half plane solutions of the equation,

$$|I_{\ell} + p\,G^T(-s)\,G(s)| = 0 \tag{3}$$

where

$$G(s) = Q^{\frac{1}{2}}\,C\,(sI_n - A)^{-1}\,B\,R^{-\frac{1}{2}} \tag{4}$$

A fairly complete theoretical analysis of the unbounded solutions of equation (3) as $p \rightarrow +\infty$ has been provided by Kwakernaak[1] but computational procedures were not suggested until quite recently[2,3] However, in (2), proofs are provided for the first few orders of infinite zero and the techniques of (3) are primarily suited for systems with small numbers of inputs. This note provides a complete analysis and computational method for the case of S(A,B,C) left-invertible and hence $m \geqslant \ell$ and $|G^T(-s)\,G(s)| \not\equiv 0$. The case of S(A,B,C) right-invertible can be deduced by replacing equation (3) by the equivalent equation $|I_m + G(s)\,G^T(-s)| = 0$. The approach used is that described in references (4-6). In section 2 some basic background results on the use of dynamic transformations are outlined. The main results are described in section 3 where it is proved that the computational techniques of refs. (4) - (6) can always be applied to the determination of the orders, asymptotic directions and pivots of the infinite zeros of the optimal root-locus. It is proved that the orders are always even and that the pivots always lie on the imaginary axis of the complex plane. An unusual sensitivity problem is observed and illustrated by an example.

2. Dynamic Transformations and Multivariable Root-loci[4-6]

Consider a $\ell x \ell$ invertible strictly proper system with transfer function matrix $pQ(s)$, $p > o$, subjected to unity negative feedback. The closed-loop system poles can be computed by finding the solutions of the relation $|I_\ell + pQ(s)| = 0$. In particular the unbounded solutions as $p \to +\infty$ can be listed in the form[4-6]

$$s_{j\ell} = p^{1/\nu_j} \eta_{j\ell} + \alpha_j + \varepsilon_{j\ell}(p)$$

$$\lim_{p\to\infty} \varepsilon_{j\ell}(p) = 0 \quad , \quad 1 \leqslant \ell \leqslant \nu_j \quad , \quad 1 \leqslant j \leqslant m \tag{5}$$

where $\eta_{j\ell}$, $1 \leqslant \ell \leqslant \nu_j$, are the distinct ν_j^{th} roots of a non-zero complex number. Equation (5) is said to be an <u>infinite zero</u> of <u>order</u> ν_j with <u>asymptotic directions</u> $\eta_{j\ell}$ and <u>pivot</u> α_j.

A case of particular simplicity is that when $Q(s)$ has <u>uniform rank k</u> i.e. there exists an integer $k \geqslant 1$ such that $\lim_{s\to\infty} s^k Q(s)$ is finite and nonsingular. In this case it can be shown[4-6] that the closed-loop system has only infinite zeros of order k with asymptotic directions and pivots easily computed from the Markov parameter matrices $\tilde{P}_k$, $\tilde{P}_{k+1}$ in the expansion (valid for all large $|s|$)

$$Q(s) = \sum_{j=1}^{\infty} s^{-j} \tilde{P}_j \tag{6}$$

The simplicity of the uniform rank case has motivated[4] a computational technique based on reduction of the problem to a number of equivalent uniform rank problems, by the use of the following theorem:

<u>Theorem 1</u>[4,6]

Suppose that there exists integers $q \geqslant 1$, $k_1 < k_2 < \ldots < k_q$ and d_j , $1 \leqslant j \leqslant q$, a real nonsingular transformation T_1 and unimodular matrices (dynamic transformations) of the form

$$L(s) = \begin{pmatrix} I_{d_1} & 0 & \cdots & \cdots & 0 \\ 0(s^{-1}) & I_{d_2} & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & 0 \\ 0(s^{-1}) & \cdots & \cdots & 0(s^{-1}) & I_{d_q} \end{pmatrix}$$

$$M(s) = \begin{pmatrix} I_{d_1} & 0(s^{-1}) & \cdots & \cdots & 0(s^{-1}) \\ 0 & I_{d_2} & & & \vdots \\ & & & & \vdots \\ & & & & 0(s^{-1}) \\ 0 & & & 0 & I_{d_q} \end{pmatrix} \qquad (7)$$

such that

$$L(s)T_1^{-1}\,Q(s)T_1\,M(s) = \text{block diag}\,\{G_j(s)\}_{1\leqslant j\leqslant q} + 0(s^{-(k_q+2)}) \qquad (8)$$

where the $d_j x d_j$ transfer function matrices $G_j(s)$ have uniform rank k_j, $1 \leqslant j \leqslant q$. Then the closed-loop system has $k_j d_j$ k_j th- order infinite zeros, $1 \leqslant j \leqslant q$, and the k_j th order infinite zeros have asymptotic directions and pivots identical to those obtained by consideration of the uniform rank problem,

$$|I_{d_j} + p\,G_j(s)| = 0\,. \qquad (9)$$

In effect, the theorem states that, under the stated conditions, the asymptotic directions and pivots can be computed from the relevant Markov parameter matrices of the uniform rank systems $G_j(s)$, $1 \leq j \leq q$. A computation algorithm is described in reference (4) and (6) based on algebraic operations on the matrix

$$M_r \triangleq [\tilde{P}_1, \tilde{P}_2, \ldots\ldots, \tilde{P}_r] \tag{10}$$

for some $r \geq k_q + 1$. The conditions of the theorem are known to be generically valid[6] and it is known that it is always possible to ensure their validity by suitable choice of forward path control system. The matrices T_1, $L(s)$, $M(s)$ are, in general, non-unique.

3. Asymptotic Behaviour of the Optimal Root-locus.

Inspection of equation (3) suggests that the concept of infinite zeros can be applied to the optimal system root-locus (generated by the solutions of this equation as p varies in the interval $o \leq p < +\infty$) by letting $Q(s) = G^T(-s)\, G(s)$ and by only accepting those branches of the root-locus in the left-half complex plane[2,3]. A main result of this note is that the technique outlined in section 2 can always be applied to the calculation of the orders, asymptotic directions and pivots of the optimal system root-locus in the sense that the conditions specified in theorem 1 are always satisfied.

Theorem 2

The conditions of theorem 1 are always valid for $Q(s) = G^T(-s)\, G(s)$ if $G(s)$ is left-invertible. Moreover, it is always possible to choose T_1 to be real and orthogonal, $L(s) \equiv M^T(-s)$ and the uniform rank systems $G_j(s)$ of the form

$$G_j(s) = N_j^T(-s)\, N_j(s) \tag{11}$$

for some $m \times d_j$ left-invertible transfer function matrix $N_j(s)$ $1 \leq j \leq q$.

Proof

The para-Hermitean structure of Q(s) ensures the existence of an <u>even</u> integer $k_1 \geq 2$ such that $\lim_{|s| \to \infty} s^{k_1} Q(s)$ is finite and non zero and equal to the <u>real</u>, <u>symmetric</u> matrix $\tilde{P}_{k_1}$. If this matrix is non-singular, the result is proved with $T_1 \equiv L(s) \equiv M(s) \equiv I_\ell$. Suppose therefore that $d_1 \triangleq \text{rank } P_{k_1} < \ell$ and let T_1 be a real orthogonal eigenvector matrix such that

$$T_1^T G^T(-s) G(s) T_1 = \begin{pmatrix} G_1(s) & O(s^{-(k_1+1)}) \\ O(s^{-(k_1+1)}) & O(s^{-(k_1+1)}) \end{pmatrix} \tag{12}$$

where $G_1(s)$ is $d_1 x d_1$ and of uniform rank k_1. Noting that this matrix is para-Hermitean, it is easily verified that it is possible to construct a unimodular matrix of the form

$$M_1(s) = \begin{pmatrix} I_{d_1} & O(s^{-1}) \\ 0 & I_{\ell-d_1} \end{pmatrix} \tag{13}$$

such that

$$M_1^T(-s)\; T_1^T G^T(-s) G(s) T_1 M_1(s) = \begin{pmatrix} G_1(s) & 0 \\ 0 & H_2(s) \end{pmatrix} \tag{14}$$

However, $H_2(s)$ has a decomposition of the form $H_2(s) = V^T(-s)\, V(s)$ where $V(s)$ is the $mx(\ell - d_1)$ matrix generated by the last $\ell - d_1$ columns of $G(s)\, T_1 M_1(s)$. In particular, the assumption of left-invertibility ensures that $|H_2(s)| \not\equiv 0$ and hence that $V(s)$ is left-invertible. Applying a similar procedure to $H_2(s)$, and continuing by induction, it is possible to find q, $d_j (1 \leq j \leq q)$ and unimodular matrices of the form

$$M_j(s) \begin{pmatrix} I_{d_1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & I_{d_{j-1}} & & 0 \\ \vdots & & & I_{d_j} & 0(s^{-1}) \\ 0 & \cdots & \cdots & 0 & I_{\ell-d_1-\cdots-d_j} \end{pmatrix} \quad , 1 \leqslant j \leqslant q-1 \tag{15}$$

and orthogonal matrices

$$T_j = \begin{pmatrix} I_{d_1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & I_{d_{j-1}} & & 0 \\ 0 & \cdots & 0 & & \tilde{T}_j \end{pmatrix} \quad , 1 \leqslant j \leqslant q-1 \tag{16}$$

such that

$$M_{q-1}^T(-s)\, T_{q-1}^T M_{q-2}^T(-s)\, T_{q-2}^T \cdots M_1^T(-s) T_1^T\, G^T(-s) G(s) T_1\, M_1(s) \cdots$$

$$\cdots T_{q-1} M_{q-1}(s) = \text{block diag } \{G_j(s)\}_{1 \leqslant j \leqslant q} \tag{17}$$

where $G_j(s)$, $1 \leqslant j \leqslant q$, are of uniform rank k_j, $1 \leqslant j \leqslant q$. The existence of k_j at each stage of the decomposition is guaranteed by the left-invertibility assumption.

Noting that we can replace T_1 by $T_1\, T_2 \ldots T_{q-1}$ without changing the structure of the $M_j(s)$, we can set $T_j = I_\ell$, $2 \leqslant j \leqslant q-1$ and the conditions of theorem 1 are satisfied with

$$M(s) = M_1(s)\, M_2(s) \; . \; . \; . \; . \; . \; M_{q-1}(s)$$

$$L(s) = M_{q-1}^T(-s) \; . \; . \; . \; M_1^T(-s) \equiv M^T(-s) \tag{18}$$

Finally the decomposition of equation (11) is valid with $M_j(s)$ equal to $m \times d_j$ matrix generated by the $d_1 + d_2 + .. + d_{j-1} + 1$, $d_1 + d_2 + .. + d_{j-1} + 2, ...$ $d_1 + d_2 + ... + d_j$ th columns of $G(s)T_1M(s)$.

Q.E.D.

Although the result guarantees the viability of the algorithm outlined in section 2, it is possible to obtain more information on the orders and pivots of the root-locus:

<u>Theorem 3</u>

The optimal system root-locus has only even order infinite zeros with pivots on the imaginary axis of the complex plane.

<u>Proof</u>

Equation (11) indicates that all the integers $k_j, 1 \leqslant j \leqslant q$, are even whence a combination of theorems one and two indicates that all the infinite zeros have even order. In particular, the structure of the pivots of the k_j th order infinite zeros can be assessed by considering the return-difference determinant,

$$|I_{d_j} + p\, N_j^T(-s)\, N_j(s)| = 0 \tag{19}$$

In particular, the para-Hermitean and uniform rank structure of $G_j(s) = N_j^T(-s)\, N_j(s)$ indicates that

$$N_j^T(-s)N_j(s) = s^{-k_j} P_{k_j}^{(j)} + s^{-(k_j+1)} P_{k_j+1}^{(j)} + O(s^{-(k_j+2)}) \tag{20}$$

where $P_{k_j}^{(j)}$ is <u>real</u>, <u>symmetric</u> and <u>nonsingular</u> and $P_{k_j+1}^{(j)}$ is <u>real</u> and skew-symmetric. In particular, suitable modifications to T_1 enable us to assume that $P_{k_j}^{(j)}$ is diagonal with <u>real</u>, <u>non-zero</u> diagonal elements.

Following the techniques described in refs. (4) and (6) for uniform rank systems, the pivots can be identified as the eigenvalues of diagonal blocks of $P^{(j)}_{k_j+1}$ multiplied by a real number. Noting that these diagonal blocks must be real and skew-symmetric, it follows directly that the pivots are pure imaginary numbers.

Q.E.D.

Corollary

The pivots are 'almost always' equal to zero.

Proof

If the eigenvalues of $P^{(j)}_{k_j}$ are distinct (the generic case!) then the pivots are equal[4,6] to a real number multiplying the diagonal terms of $P^{(j)}_{k_j+1}$!

Q.E.D.

The corollary suggests that the optimal system root-loci may suffer from a sensitivity problem analogous to that noticed in more general studies[6]. This is easily illustrated by considering the case of

$$G(s) = \begin{pmatrix} \frac{(1+\varepsilon)}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{pmatrix} \tag{21}$$

when

$$G^T(-s)G(s) = \frac{1}{s^2}\begin{pmatrix} -(1+\varepsilon)^2 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{s^3}\begin{pmatrix} 0 & -(1+\varepsilon) \\ (1+\varepsilon) & 0 \end{pmatrix} + O(s^{-4}) \tag{22}$$

has uniform rank two. Application of the algorithms of references (4) and (6) yields the left-half plane infinite zeros of the asymptotic forms for $\varepsilon \neq 0$

$$s \simeq -p^{1/2} + \varepsilon_1(p) \quad , \quad s \simeq -(1+\varepsilon)p^{1/2} + \varepsilon_2(p)$$

$$\lim_{p\to\infty} \varepsilon_j(p) = 0 \quad , \quad j = 1,2 \tag{23}$$

and, for $\varepsilon = 0$,

$$s \simeq -p^{1/2} + \frac{j}{2} + \varepsilon_1(p) \quad , \quad s \simeq -p^{1/2} - \frac{j}{2} + \varepsilon_2(p)$$

$$\lim_{p\to\infty} \varepsilon_j(p) = 0 \quad , \quad j = 1,2 \tag{24}$$

Note the discontinuous behaviour of the pivots in the vicinity of $\varepsilon = 0$.

4. Summary

It has been shown that a recently derived computational method[4,6] can always be applied to the calculation of the asymptotic behaviour of the root-locus of optimal linear regulators. The analysis has also demonstrated that the optimal root-locus has only even order infinite zeros with pivots on the imaginary axis of the complex plane. In particular the pivots are almost always (but not always) equal to zero suggesting that the optimal root-locus has sensitivity characteristics similar to those noted in multivariable root-locus studies[6].

5. References

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