Monograph:

Billings, S.A. and Fakhouri, S.Y. (1979) Identification of Systems Composed of Linear Dynamic and Static Nonlinear Elements. Research Report. ACSE Research Report 83 . Department of Automatic Control and Systems Engineering

IDENTIFICATION OF SYSTEMS COMPOSED OF LINEAR DYNAMIC AND STATIC NONLINEAR ELEMENTS

by

S. A. Billings, B.Eng.,Ph.D.,M.Inst.M.C.,AMIEE

S. Y. Fakhouri, B.Sc.,M.Sc.,AMIEE

Department of Control Engineering
University of Sheffield
Mappin Street
Sheffield
England

Research Report No. 83

February 1979

IDENTIFICATION OF SYSTEMS COMPOSED OF LINEAR DYNAMIC AND STATIC NONLINEAR ELEMENTS

S. A. Billings and S. Y. Fakhouri

Department of Control Engineering, University of Sheffield,
Mappin Street, Sheffield S1 3JD, England

Abstract. Identification of nonlinear systems which can be represented by combinations of linear dynamic and static nonlinear elements are considered. Previous results by the authors based on correlation analysis are combined to provide a unified treatment for this class of systems. It is shown that systems composed of cascade, feedforward, feedback and multiplicative connections of linear dynamic and zero memory nonlinear elements can be identified in terms of the individual component subsystems from measurements of the system input and output.

Keywords. Control nonlinearities, correlation theory, identification, nonlinear systems.

1. INTRODUCTION

Various authors have studied the class of systems which can be represented by interconnections of linear dynamic and zero-memory nonlinear subsystems. A transform representation and rules for algebraic manipulation were developed by George (1959). The analysis and synthesis of cascade systems has been studied by Smets (1960) and Shanmugam and Lal (1976), and a structure theory was developed by Smith and Rugh (1974). Zames (1963) and Narayanan (1970) studied nonlinear feedback systems and numerous authors (Gardiner, 1973; Cooper and Falkner, 1975; Narendra and Gallman, 1966; Sandor and Williamson, 1978; Douce, 1976) have considered the identification of systems within this class.

In the present study previous results (Billings and Fakhouri, 1978a,b,c) derived for the general model defined as a linear system in cascade with a static nonlinear element followed by another linear system are briefly reviewed. By considering the separable class of random processes it is shown that computation of the first and second order cross-correlation functions when the input is white Gaussian effectively decouples the identification of this class of non-linear systems into two distinct steps; identification of the linear subsystems and characterisation of the nonlinear element. The relationship between the first and second order correlation functions also provides valuable information regarding the system structure; notably the position of the nonlinear element with respect to the linear systems. Although the algorithms cannot be directly applied for pseudorandom inputs an alternative procedure (Billings and Fakhouri, 1978f) based on compound pseudorandom excitation is presented and the selection of inputs is discussed.

The results are extended to include the identification of the component subsystems in nonlinear feedback systems (Billings and Fakhouri, 1979a), feedforward systems (Billings and Fakhouri, 1979b), systems containing multiplicative connections of linear dynamic elements (Billings and Fakhouri, 1978c) and other common system structures (Billings and Fakhouri, 1979c). In all these cases the identification procedure provides estimates of the individual elements of the system such that the components can be synthesised in a manner which preserves the system structure and provides valuable information for control.

2. THE OPEN-LOOP GENERAL MODEL

The general model illustrated in Fig. 1 consists of a linear system $h_1(t)$ in cascade with a zero memory nonlinear element $F[\cdot]$ followed by a second linear system $h_2(t)$. For generality it is assumed that the measured output contains an unknown additive noise component $v(t)$ and that the nonlinear element can, in theory, be represented by a polynomial $y(t) = \sum_{i=1}^{k} \gamma_i x^i(t)$.

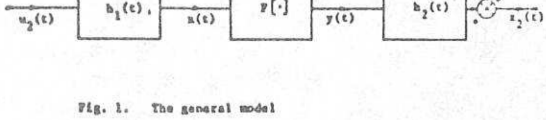

Fig. 1. The general model

The identification problem can now be defined as identification of the individual components $h_1(t)$, $h_2(t)$ and a suitable representation of the static nonlinear element $F[\cdot]$ from measurements of the input $u(t)$ and noise corrupted output $z_2(t)$.

2.1 Separable Process Inputs

The output of the general model, Fig. 1 can be expressed as

$$z_2(t) = \int\int h_2(\theta)Q(t-\theta,\tau_1)u_2(t-\theta-\tau_1)d\theta d\tau_1+v(t) \qquad (1)$$

where $Q(t,\tau_1)$ is a function of t and τ_1 only and can be readily evaluated by considering the Volterra series expansion for $z_2(t)$. The output correlation function can then be defined as

$$\phi_{z_1z_2}(\varepsilon) = \int\int h_2(\theta)\overline{Q(t-\theta,\tau_1)u_2(t-\theta-\tau_1)u_1(t-\varepsilon)}\,d\tau_1 d\theta + \overline{v(t)u_1(t-\varepsilon)} \qquad (2)$$

If $u_1(t)$ is separable with respect to $x(t)$ then the invariance property (Nuttall, 1958)

$$\phi_{u_1y}(\sigma) = C_{FG}\phi_{u_1x}(\sigma) \;\forall\; F \text{ and } \sigma \qquad (3)$$

exists across the nonlinear element where C_{FG} is Booton's equivalent gain. Expanding eqn (3)

$$\int\overline{Q(t,\tau_1)u_2(t-\tau_1)u_1(t-\sigma)}d\tau_1 = C_{FG}\int h_1(\tau_1)\overline{u_2(t-\tau_1)u_1(t-\sigma)}d\tau_1 \qquad (4)$$

and substituting in eqn (2) yields

$$\phi_{z_1z_2}(\varepsilon) = C_{FG}\int\int h_2(\theta)h_1(\tau_1)\phi_{u_1u_2}(\varepsilon-\theta-\tau_1)d\theta d\tau_1 + \phi_{u_1v}(\varepsilon) \qquad (5)$$

In a similar manner defining the second order correlation function

$$\phi_{u_1^2z_2}(\varepsilon) = \int\int h_2(\theta)\overline{Q(t-\theta,\tau_1)u_2(t-\theta-\tau_1)u_1^2(t-\varepsilon)}d\theta d\tau_1 + \overline{u_1^2(t-\varepsilon)v(t)} \qquad (6)$$

and expanding the invariance property for double nonlinear transformations

$$\phi_{u_1^2y}(\sigma) = C_{FFG}\phi_{u_1^2x^2}(\sigma) \;\forall\; F \text{ and } \sigma \qquad (7)$$

and substituting into eqn (6) gives

$$\phi_{u_1^2z_2}(\varepsilon) = C_{FFG}\int\int\int h_2(\theta)h_1(\tau_1)h_1(\tau_2)\overline{u_2(t-\theta-\tau_1)u_2(t-\theta-\tau_2)u_1^2(t-\varepsilon)}d\tau_1 d\tau_2 d\theta + \phi_{u^2v}(\varepsilon) \qquad (8)$$

Notice that eqns (3) and (7) only hold when separability is preserved under linear and double nonlinear transformation respectively. Although these properties hold only under very restrictive conditions the separability of a Gaussian process is maintained under both these transformations.

For the special case when $u_1(t) = u(t)$, $u_2(t) = u(t)+b$ where $u(t)$ is a zero mean white Gaussian process and b is a nonzero mean level eqns (5) and (8) reduce to (Billings and Fakhouri, 1978c)

$$\phi_{uz'}(\varepsilon) = C_{FG}\int h_1(\tau_1)h_2(\varepsilon-\tau_1)d\tau_1 \qquad (9)$$

$$\phi_{u^2z'}(\varepsilon) = C_{FFG}\int h_2(\tau_1)h_1^2(\varepsilon-\tau_1)d\tau_1 \qquad (10)$$

$$C_{FG} = \gamma_1+2\gamma_2 b\int h_1(\theta)d\theta+3\gamma_3\int h_1^2(\theta)d\theta+3\gamma_3 b^2\int\int h_1(\tau_1)h_1(\tau_2)d\tau_1 d\tau_2+\ldots$$

$$C_{FFG} = 2\gamma_2+6\gamma_2 b\int h_1(\theta)d\theta+\ldots$$

where, provided $h_1(t)$ is stable bounded-inputs bounded-outputs, C_{FG} and C_{FFG} are constants, $\phi_{uv}(\varepsilon)$, $\phi_{u^2v}(\varepsilon)$ tend to zero when $v(t)$ is independent of the input, and the superscript ' is used throughout to indicate a zero mean process.

The estimates of eqns (9) and (10) are quite independent of the nonlinear element $F[\cdot]$ except for the constant scale factors C_{FG}, C_{FFG}. Correlation analysis thus effectively decouples the identification problem into two distinct steps; identification of the linear subsystems and characterisation of the nonlinear element. Estimates of the individual linear subsystems $\mu_1h_1(t)$, $\mu_2h_2(t)$ can be obtained using a least squares decomposition technique (Billings and Fakhouri, 1978b). Once the linear subsystems have been identified the problem is reduced to fitting a polynomial, a series of straight line segments or any other appropriate function to the static nonlinearity by minimising the sum of squares. Because the system is identified in terms of the individual linear and nonlinear elements and not as a Volterra series even systems containing very violent nonlinearities such as saturation and dead-zone can be readily identified (Billings and Fakhouri, 1978d).

Analysis of higher order cascade connections of linear dynamic and static nonlinear systems shows that the first and second order correlation functions do not fit into the pattern of results derived above. For example, a system consisting of a nonlinear element in cascade with a linear system $h(t)$ followed by a second nonlinearity gives rise to first and second order correlation functions which are power series in $h(t)$. This problem arises because in general separability does not hold under linear transformation.

2.2 Pseudo-random Inputs

Although it can readily be shown that a binary pseudorandom sequence is a separable process, it is not separable under linear and double nonlinear transformation and hence the results of section 2.1 are not valid for these inputs. An alternative procedure must therefore be developed for this class of inputs (Billings and Fakhouri, 1978f).

When the input to the general model illustrated in Fig. 1 is a compound input $u_2(t) = x_1(t)+x_2(t)$ where $x_1(t)$ and $x_2(t)$ are pseudorandom sequences the output $z_2(t)$ can be expressed as

$$z_2(t) = \sum_{i=1}^{k} \{\gamma_i \int \ldots \int h_1(\tau_1)\ldots h_1(\tau_i)h_2(\theta) \{\prod_{j=1}^{i} (x_1(t-\tau_j-\theta)+x_2(t-\tau_j-\theta))d\tau_j\}d\theta\}+v(t) \quad (11)$$

If the correlation functions are computed directly with the measured system output eqn (11), anomalies associated with the multidimensional autocorrelations of the pseudorandom sequences (Barker and Pradisthayon, 1970; Barker and Obidegwu, 1973) are introduced and the estimates do not reduce to the form of eqns (9) and (10). This problem can be overcome by isolating the first and second order correlation functions of the outputs of the first and second order Volterra kernels respectively.

2.2.1. Multilevel testing

Consider a series of experiments with multilevel compound inputs $\alpha_i u(t)$ where $\alpha_i \neq \alpha_\ell$ $\forall\, i \neq \ell$, then the output correlation function $\phi_{x_1'z_{\alpha_i}'}(\varepsilon)$ can be expressed as

$$\phi_{x_1'z_{\alpha_i}'}(\varepsilon) = \sum_{j=1}^{n} \alpha_i^{\,j} \phi_{x_1'w_j'}(\varepsilon) \quad , \; i=1,2,\ldots n \quad (12)$$

assuming that the input signal $x_1(t)$ and noise process $v(t)$ are independent. Providing $\alpha_i \neq 0$, $\alpha_i \neq \alpha_\ell \forall\; i \neq \ell$ eqn (12) has a unique solution for $\phi_{x_1'w_j'}(\varepsilon)\;\forall\;\varepsilon$, $j=1,2..n$. If $x_1(t)$ and $x_2(t)$ are independent, $\phi_{x_1x_2}(\lambda) = 0 \;\forall\; \lambda$, zero mean, $\bar{x}_1 = \bar{x}_2 = 0$, pseudorandom sequences with autocorrelation functions

$$\phi_{x_ix_i}(\lambda) = \beta_i\delta_i(\lambda) \quad , \quad i = 1,2$$

$$\text{where} \quad \delta_i(\lambda) = \begin{cases} 1/\Delta t_i & \lambda = 0 \\ 0 & \lambda \neq 0 \end{cases} \quad (13)$$

Δt_i is the clock interval and $\int \delta_i(\lambda)d\lambda = 1.0$ then $\phi_{x_1'w_1'}(\varepsilon)$ which can be isolated using the above procedure reduces to

$$\phi_{x_1'w_1'}(\varepsilon) = \beta_1\gamma_1\int h_1(\varepsilon-\theta)h_2(\theta)d\theta \quad (14)$$

Following a similar procedure as above and isolating the second order correlation function associated with the second Volterra kernel yields

$$\phi_{x_1'x_2'w_2'}(\varepsilon) = \gamma_2 \iiint h_1(\tau_1)h_1(\tau_2)h_2(\theta) \{\prod_{j=1}^{2} (x_1(t-\tau_j-\theta)+x_2(t-\tau_j-\theta))d\tau_j - \prod_{j=1}^{2} (\bar{x}_1+\bar{x}_2)d\tau_j\}(x_1(t-\varepsilon)-\bar{x}_1)(x_2(t-\varepsilon)-\bar{x}_2)d\theta \quad (15)$$

When $x_1(t)$ and $x_2(t)$ have the properties defined in eqn (13) this reduces to

$$\phi_{x_1'x_2'w_2'}(\varepsilon) = 2\beta_1\beta_2\gamma_2\int h_1^{\,2}(\varepsilon-\theta)h_2(\theta)d\theta \quad (16)$$

Although multilevel inputs must be employed only $\phi_{x_1'w_1'}(\varepsilon)$ and $\phi_{x_1'x_2'w_2'}(\varepsilon)$ and not the individual kernel outputs $w_i(t)$ must be computed. This considerably reduces the computational burden because for stable linear subsystems the correlation functions will tend to steady-state after a small number of values.

Providing $x_1(t)$ and $x_2(t)$ are pseudorandom sequences with properties defined in eqn (13) the results of eqns (14) and (16) are exact and the errors normally associated with the identification of this class of systems using pseudorandom inputs and correlation analysis are avoided.

Since the results of eqns (14) and (16) are dependent upon $x_1(t)$ and $x_2(t)$ having a zero mean value an obvious choice of input would be a compound ternary sequence. It would however be far more convenient if pseudorandom binary inputs could be employed in this application. However, whilst the first order correlation function eqn (14) remains unbiased for a compound prbs input, the nonzero mean level of this input introduces a time varying bias (Billings and Fakhouri, 1978f) $e(\varepsilon)$ into the estimate of eqn (16)

$$e(\varepsilon) = -2\gamma_2(a_1\bar{x}_1\beta_2+a_2\bar{x}_2\beta_1+\bar{x}_1^{\,2}\beta_2+\bar{x}_2^{\,2}\beta_1) \iint h_1(\tau_1)h_1(\varepsilon-\theta)h_2(\theta)d\tau_1 d\theta \quad (17)$$

where $\bar{x}_i = a_i/N_i$, a_i is the amplitude and N_i the sequence length. This bias tends to zero as N_1 and N_2 are increased and will be negligible in most applications.

Independent prbs with the same bit interval can be generated either by multiplying by the rows of a Hadamard matrix or correlating over the product of sequence lengths and independent ternary sequences can be generated using the latter approach (Briggs and Godfrey 1966).

The results of eqns (14) and (16) are analogous to the results obtained for a separable white Gaussian input when $\gamma_1, \gamma_2 \neq 0$ and can be used directly to identify the individual linear and nonlinear elements of the system illustrated in Fig. 1. The relationship between the first and second order correlation functions also provides valuable information regarding the system structure.

2.3 Structure Testing

Consider the identification of an unknown system which has been excited by a separable white Gaussian input with mean level b. Initially the experimenter must determine the structural form of the model which best describes the system under test. This information can be obtained by inspection of the first and second order correlation functions, eqns (9) and (10), for cascade connections of linear dynamic and static nonlinear subsystems.

If the system under test is linear then $\gamma_i = 0 \quad i \neq 1$ and eqns (9) and (10) reduce to

$$\phi_{uz'}(\varepsilon) = \gamma_1 \int h_1(\tau_1) h_2(\varepsilon - \tau_1) d\tau_1 \qquad (18)$$

$$\phi_{u^2z'}(\varepsilon) = 0 \;\forall\; \varepsilon \qquad (19)$$

Thus if $\phi_{u^2z'}(\varepsilon) = 0 \;\forall\; \varepsilon$ the system must be linear and once a pulse transfer function model has been fitted to $\phi_{uz'}(\varepsilon)$ the identification is complete. The second order correlation function $\phi_{u^2z'}(\varepsilon)$ is therefore a measure of nonlinearity.

If $h_1(t) = \delta(t)$ the general model reduces to the Hammerstein model (Billings and Fakhouri, 1979b) and

$$\phi_{uz'}(\varepsilon) = C_{FH} h_2(\varepsilon) \qquad (20)$$

$$\phi_{u^2z'}(\varepsilon) = C_{FFH} h_2(\varepsilon) \qquad (21)$$

If therefore $\phi_{uz'}(\varepsilon)$ and $\phi_{u^2z'}(\varepsilon)$ are equal except for a constant of proportionality the system must have the structure of a Hammerstein model.

When $h_2(t) = \delta(t)$ the general model reduces to the Wiener model (Billings and Fakhouri, 1977) and

$$\phi_{uz'}(\varepsilon) = C_{FW} h_1(\varepsilon) \qquad (22)$$

$$\phi_{u^2z'}(\varepsilon) = C_{FFW} h_1^2(\varepsilon) \qquad (23)$$

Thus when $\{\phi_{uz'}(\varepsilon)\}^2$ is equal to $\phi_{u^2z'}(\varepsilon)$ except for a constant of proportionality the system must have the structure of a Wiener model.

Finally, if none of the above conditions hold the system may have the structure of the general model. However, this is a necesarry and not a sufficient condition which must be confirmed by parameterising the linear systems and nonlinear element and examining the mean squared error. Alternatively, an algorithm by Douce (1976) provides a very convenient test for cascade systems in this class.

Identification of cascade connections of linear dynamic and static nonlinear systems using correlation analysis thus inherently provides information regarding the structure of these systems.

The relationship between $\phi_{x_1'w_1'}(\varepsilon)$ and $\phi_{x_1'x_2'w_2'}(\varepsilon)$ for pseudorandom inputs are analogous to the above results providing these correlation functions exist.

3. NONLINEAR FEEDBACK SYSTEMS

The identification algorithms derived in previous sections can be applied to nonlinear feedback systems (Billings and Fakhouri, 1979a) if the form of the Volterra kernels can be related to the component subsystems of the original process. As in the case of cascade systems the objective is to identify the individual elements of the system such that the structure of the process is preserved and truncation errors normally associated with a finite Volterra series description are avoided.

3.1 Unity Feedback Systems

Consider the unity feedback general model illustrated in Fig. 2. Notice that in general the system output will be corrupted by noise and hence the feedback signal cannot be computed and the problem cannot be reduced to one of open-loop identification.

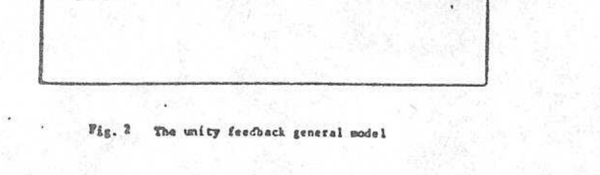

Fig. 2 The unity feedback general model

Applying the operator calculus developed by Brilliant (1958) and George (1959) it can readily be shown that the Volterra kernels of the equivalent open-loop system $\underline{G}$ can be expressed as

$$\underline{G}_1 = [\underline{I}+\gamma_1\underline{H}*\underline{C}]^{-1}*\gamma_1\underline{H}*\underline{C} \tag{24}$$

$$\underline{G}_2 = [\underline{I}+\gamma_1\underline{H}*\underline{C}]^{-1}*[\gamma_2\underline{H}o(\underline{C}^2)o((\underline{I}-\underline{G}_1)^2)] \tag{25}$$

$$\underline{G}_\ell = [\underline{I}+\gamma_1\underline{H}*\underline{C}]^{-1}* \sum_{n=2}^{\ell} \Sigma_Q \gamma_n (\underline{H}o(\underline{C}^n))o(\underline{K}_{i_1}\cdot\cdot\underline{K}_{i_n})] \tag{26}$$

where $\underline{K}_1 = \underline{I}-\underline{G}_1$, $\underline{K}_\ell = -\underline{G}_\ell$ for $\ell \geq 2$.

Although the series is an infinite operator series the structural form of the first two kernels can be exploited to provide estimates of $\underline{C}$, $F[\cdot]$ and $\underline{H}$. The outputs $w_1(t)$ and $w_2(t)$ of the first two Volterra kernels $\underline{G}_1$ and $\underline{G}_2$ can be isolated using the algorithm described in section 2.2.1. When the input is a separable white Gaussian process u(t) with mean level b, inspection of eqns (9) and (24) show that

$$\phi_{uw_1'}(\varepsilon) = G_1(\varepsilon) \tag{27}$$

Taking the Z-transform of eqn (27), a pulse transfer function model can be fitted to $\phi_{uw_1'}(\varepsilon)$ to yield

$$Z\{\phi_{uw_1'}(\varepsilon)\} = \frac{\hat{N}g_1(z^{-1})}{\hat{D}g_1(z^{-1})} = \frac{\gamma_1 H(z^{-1})C(z^{-1})}{1+\gamma_1 H(z^{-1})C(z^{-1})} \tag{28}$$

and estimates of the numerator and denominator can be obtained from

$$\gamma_1 H(z^{-1})C(z^{-1}) = \frac{\hat{N}g_1(z^{-1})}{\hat{D}g_1(z^{-1})-\hat{N}g_1(z^{-1})} \tag{29}$$

$$1+\gamma_1 H(z^{-1})C(z^{-1}) = \frac{\hat{D}g_1(z^{-1})}{\hat{D}g_1(z^{-1})-\hat{N}g_1(z^{-1})} \tag{30}$$

The output data z(t) can now be filtered using the estimate of eqn (30), such that the kernels of the equivalent open loop system reduce to

$$\underline{G}_1 = \gamma_1\underline{H}*\underline{C}$$

$$\underline{G}_2 = [\gamma_2\underline{H}\ o\ (\underline{C}^2)o((\underline{I}-\underline{G}_1)^2)] \tag{31}$$

The second order correlation function can then be evaluated using the results of eqns (10), (31) and section 2.2.1 to yield

$$\phi_{u^2w_{r2}'}(\varepsilon) = 2\gamma_2\int H(\theta)T_1^{\,2}(\varepsilon-\theta)d\theta \tag{32}$$

where w_{r2} is the filtered output of the second order kernel. Taking the Z-transform of eqn (32) a pulse transfer function can be fitted to $\phi_{u^2w_{r2}'}(\varepsilon)$

$$Z\{\phi_{u^2w_{r2}'}(\varepsilon)\} = 2\gamma_2 H(z^{-1})TT(z^{-1}) \tag{33}$$

The results of eqns (29) and (23) can be decomposed using a multistage least squares algorithm (Billings and Fakhouri, 1978b) to provide estimates of the pulse transfer functions $\mu_1 H(z^{-1})$, $\mu_2 C(z^{-1})$ and a suitable function can be fitted to the nonlinear element by minimising the sum of squares of error using an algorithm by Peckham (1970).

Because the unity feedback Wiener and Hammerstein models are subclasses of the unity feedback general model the identification procedure is applicable to systems with these structures.

3.2 Precascaded Feedback Systems

The first two Volterra kernels for the precascaded feedback system illustrated in Fig. 3 can be expressed as

$$\underline{G}_1 = \underline{V}_1*\underline{P} = \{[\underline{I}+\lambda_1\underline{A}_1]^{-1}*\underline{A}_1\}*\underline{P} \tag{34}$$

$$\underline{G}_2 = \underline{V}_2*\underline{P} = \{-\underline{V}_1\ o\ \lambda_2(\underline{V}_1^{\,2})\}*\underline{P} \tag{35}$$

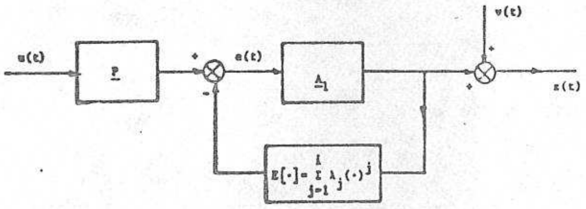

Fig. 3 Precascaded nonlinear feedback system

Following the procedure of the previous section it can readily be shown that

$$\phi_{uw_1'}(\varepsilon) = \int V_1(\varepsilon-\theta)P(\theta)d\theta \tag{36}$$

$$\phi_{u^2w_2'}(\varepsilon) = 2\lambda_2\int G_1^{\,2}(\varepsilon-\theta)V_1(\theta)d\theta \tag{37}$$

Estimates of $\mu_1 V_1(z^{-1})$ and $\mu_2 P(z^{-1})$ can be obtained by decomposing the pulse transfer functions $Z\{\phi_{uw_1'}(\varepsilon)\}$, $Z\{\phi_{u^2w_2'}(\varepsilon)\}$, and a suitable function can be fitted to the nonlinear element by minimising the sum of squared errors.

Although all the results for feedback systems have been derived for separable white Gaussian inputs analogous results can be obtained for a compound pseudorandom input by computing $\phi_{x_1'w_1'}(\varepsilon)$ and $\phi_{x_1'x_2'w_2'}(\varepsilon)$

(Billings and Fakhouri, 1978f). The selection of pseudorandom inputs and the error analysis for binary sequences is exactly the same as the open-loop case section 2.2.

4. MULTIPLICATIVE SYSTEMS

Consider the multiplicative system illustrated in Fig. 4 and commonly referred to as the factorable Volterra system where the factorable kernel of order k can be realised as a system composed of k linear dynamic subsystems connected in parallel with outputs multiplied in the time domain. Concepts of reachability and observability for this class of systems were studied by Harper and Rugh (1976) who developed an identification scheme based on the system response to two-tone sinusoidal inputs.

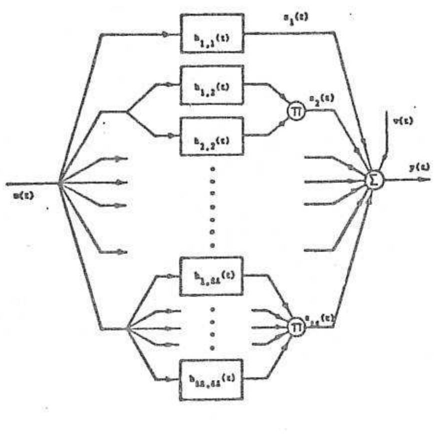

Fig. 4. A non-linear factorable Volterra system

Identification algorithms based on both white Gaussian and pseudorandom excitation have been developed by the authors (Billings and Fakhouri, 1978e) but only pseudorandom inputs will be considered in the present analysis.

4.1 Identification of Factorable Kernels

Although the outputs of the factorable kernels $z_i(t)$, $i = 1,\ldots \ell\ell$ in Fig. 4 can be isolated using multilevel testing, section 2.2.1, this may involve a long experimentation time and can be avoided by implementing the sequential algorithm outlined below.

Consider a factorable Volterra system which is composed of factorable kernels up to order $\ell\ell$. When the system is excited by the compound input $u(t) = \sum_{j=1}^{\ell\ell} x_j(t)$ the system response can be expressed as

$$y(t) = \sum_{j=1}^{\ell\ell} z_j(t) = \sum_{j=1}^{\ell\ell} \int \ldots \int h_{1,\ell\ell}(t_1) \ldots h_{j,\ell\ell}(t_j) \Big(\sum_{i=1}^{j} \sum_{k}^{\ell\ell} x_k(t-t_i) dt_1 \ldots dt_j + v(t) \tag{38}$$

If the individual inputs $x_j(t)$, $j = 1,2..\ell\ell$ are zero mean independent processes with autocorrelation functions $\phi_{x_j x_j}(\tau) = \beta_j \delta(\tau)$, $j = 1,2..\ell\ell$, then the system output correlation function defined as

$$\overline{y'(t)x_1(t-\sigma_1) \prod_{i=2}^{\ell\ell} x_i(t-\sigma)} = \sum_{j=1}^{\ell\ell} \int \ldots \int \overline{h_{1,j}(t_1) \ldots h_{j,j}(t_j) \Big(\prod_{i=1}^{j} \sum_{k=1}^{\ell\ell} x_k(t-t_j) \Big) . x_1(t-\sigma_1) \prod_{m=2}^{\ell\ell} x_m(t-\sigma)} dt_1 \ldots dt_j + \overline{v'(t)x_1(t-\sigma_1) \prod_{j=2}^{\ell\ell} x_1(t-\sigma)} \tag{39}$$

reduces to the output correlation function for the $\ell\ell$'th kernel

$$\phi_{x_1 \ldots x_{\ell\ell} y'}(\sigma_1, \sigma \ldots \sigma) = \phi_{x_1 .. x_{\ell\ell} z'}(\sigma_1, \sigma .. \sigma) \tag{40}$$

Thus by computing $\phi_{x_1 .. x_{\ell\ell} y'}(\sigma_1, \sigma .. \sigma)$ the correlation function associated with the $\ell\ell$'th kernel has been automatically isolated. This result holds exactly even for a compound prbs input.

When the inputs $x_j(t)$ have the properties defined above $\phi_{x_1 .. x_{\ell\ell} z'_{\ell\ell}}(\sigma_1, \sigma .. \sigma)$ reduces to

$$\phi_{x_1 .. x_{\ell\ell} z'_{\ell\ell}}(\sigma_1, \sigma .. \sigma) = (\ell\ell - 1)! \Big(\prod_{n=1}^{\ell\ell} \beta_n \Big) \sum_{i=1}^{\ell\ell} \{ h_{i,\ell\ell}(\sigma_i) \prod_{\substack{j=1 \\ j \neq i}}^{\ell\ell} h_{j,\ell\ell}(\sigma) \} \tag{41}$$

and the function $\psi_{\ell\ell}(\sigma_1, \sigma)$ can be defined as

$$\psi_{\ell\ell}(\sigma_1, \sigma) = \phi_{x_1 .. x_{\ell\ell} z'_{\ell\ell}}(\sigma_1, \sigma .. \sigma) . \frac{1}{(\ell\ell - 1)! \Big(\prod_{n=1}^{\ell\ell} \beta_n \Big)}$$

$$= \sum_{i=1}^{\ell\ell} \{ h_{i,\ell\ell}(\sigma_i) \prod_{\substack{j=1 \\ j \neq i}}^{\ell\ell} h_{j,\ell\ell}(\sigma) \} \tag{42}$$

The above results can be realised exactly using independent white Gaussian inputs $x_j(t)$ or independent ternary sequences. If prbs inputs are employed the errors introduced in the estimates of the first and second order factorable kernels have the same form as the errors for the cascade general model section 2.2.1 which tend to zero as the sequence lengths become large.

Once $\psi_{\ell\ell}(\sigma_1, \sigma)$ has been computed estimates of the individual linear subsystems $h_{i,\ell\ell}(t)$ can be readily obtained by decomposing eqn

(42) using a nonlinear Marquardt algorithm.

When the linear subsystems associated with the $\ell\ell$'th kernel have been estimated using the algorithm outlined above the predicted output $\hat{z}_{\ell\ell}(t)$ can be computed

$$\hat{z}_{\ell\ell}(t) = \int..\int \hat{h}_{1,\ell\ell}(t_1)..\hat{h}_{\ell\ell,\ell\ell}(t_{\ell\ell})\left(\prod_{j=1}^{\ell\ell}\sum_{i=1}^{\ell\ell} x_i(t-t_j)\right)dt_1..dt_{\ell\ell} \qquad (43)$$

and a reduced system output $\hat{yy}'_{\ell\ell-1}(t) = y'(t)-\hat{z}'_{\ell\ell}(t)$ can be defined.

Continuing the above procedure the $(\ell\ell-1)$'th kernel can be identified by computing the $(\ell\ell-1)$'th system output correlation function

$$\phi_{x_1..x_{\ell\ell-1}yy'_{(\ell\ell-1)}}(\sigma_1,\sigma..\sigma)=\phi_{x_1..x_{\ell\ell-1}z'_{(\ell\ell-1)}}(\sigma_1,\sigma..\sigma) \qquad (44)$$

to provide an estimate of $\psi_{\ell\ell-1}(\sigma_1,\sigma)$ which can be decomposed using the Marquardt algorithm to yield $h_{i,\ell\ell-1}(t)$, $i = 1,2..\ell\ell-1$. The linear systems associated with the remaining kernels can be identified by continuing this procedure.

It can readily be shown that providing any noise corrupting the system output $y(t)$ is independent of the input process this tends to zero in the analysis and unbiased estimates are obtained.

5. S_m SYSTEMS

The S_m model illustrated in Fig. 5 consists of a series of general models with reduced nonlinear elements connected in parallel with outputs summated. This class of systems was originally studied by Baumgartner and Rugh (1975) and later by Wysocki and Rugh (1976) and Sandor and Williamson (1978). Identification algorithms based on steady-state sinusoidal measurements were developed by these authors.

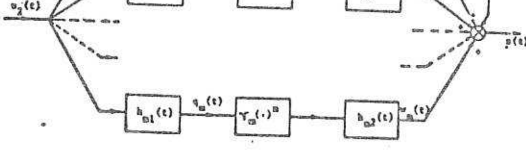

Fig. 5 THE S_m MODEL

Applying the results of section 2.1 it can readily be shown that if the output correlation functions of each branch are isolated using the multilevel testing algorithm of section 2.2.1, these reduce to

$$\phi_{uw_k'}(\varepsilon) = C_{Fk}\int h_{k1}(\tau_1)h_{k2}(\varepsilon-\tau_1)d\tau_1 \qquad (45)$$

$$\phi_{u^2w_k'}(\varepsilon) = C_{FFk}\int h_{k2}(\tau_1)h_{k1}^2(\varepsilon-\tau_1)d\tau_1 \qquad (46)$$

These estimates are identical to the results for the general model eqns (9), (10) and estimates of the linear subsystems $h_{ki}(t)$, $i = 1,2$ and the nonlinear coefficients γ_i can be obtained using the procedure outlined in earlier sections.

The algorithms can be readily extended to include the identification of other nonlinear systems within this class including feedforward systems (Billings and Fakhouri, 1979b) and other common structures.

6. CONCLUSIONS

A unified approach to the identification of nonlinear systems which can be represented by interconnections of linear dynamic and static nonlinear elements has been presented. Although the algorithms utilize the structural properties of the first two kernels in the Volterra series expansion characterization in terms of these kernels is avoided and truncation errors are not incurred. Thus even systems with very violent nonlinearities can be identified.

Cascade and multiplicative systems prove to be particularly tractable and estimates of the individual component subsystems can be readily obtained from single test experiment. The information regarding system structure which is inherent in the results for cascade systems should be particularly valuable. Although multilevel testing is necessary in the identification of feedback and S_m systems this is often necessary in nonlinear system identification although several authors avoid this problem by considering systems which are defined by a single kernel. This constraint can be avoided by using the technique of Lee and Schetzen (1965) but this involves the computation of multidimensional correlation functions even for simple systems. Whilst the algorithms presented are based upon the calculation of first and second order correlation functions both these are defined as functions of a single argument and estimates of the component subsystems can be obtained by using simple extensions of established linear techniques.

All the algorithms can be implemented for Gaussian white inputs but the convenience of pseudorandom sequences suggests that the compound input method would be more appropriate in many applications.

Unfortunately lack of space prohibits the inclusion of simulated examples but these are contained in the original publications referenced in the text.

Identification of nonlinear systems in terms of the individual elements preserves the system structure and provides valuable information for control. This approach overcomes many of the disadvantages associated with black-box identification and provides a very concise description of the process.

REFERENCES

Barker, H.A., Obidegwu, S.N. (1973). Effects of nonlinearities on the measurement of weighting functions by cross correlation using pseudorandom signals. Proc. IEE, 120, 1293-1300.

Barker, H.A., Pradisthayon, T. (1976). High order autocorrelation functions of pseudorandom signals based on m sequences. Proc.IEE, 117, 1857-1863.

Baumgartner, S.L., Rugh, W.S. (1975). Complete identification of a class of nonlinear systems from steady-state frequency response. IEEE Trans. Cts. Syst., CAS-22, 753-758.

Billings, S.A., Fakhouri, S.Y. (1977). Identification of nonlinear systems using the Wiener model. Elect. Lett., 13, 502-504.

Billings, S.A., Fakhouri, S.Y. (1978a). Identification of a class of nonlinear systems. IEE Colloquium, Digest 1978/31.

Billings, S.A., Fakhouri, S.Y. (1978b). Identification of nonlinear systems using correlation analysis. Proc.IEE, 125, 691-697.

Billings, S.A., Fakhouri, S.Y. (1978c). The theory of separable processes with applications to the identification of nonlinear systems. Proc.IEE, 125, 1051-1058.

Billings, S.A., Fakhouri, S.Y. (1978d). Identification of a class of nonlinear systems using correlation analysis. Proc.IEE, Corres., 125, 1307-1308.

Billings, S.A., Fakhouri, S.Y. (1978e). Identification of factorable Volterra systems. Res. Rep. 71, Sheffield Univ.

Billings, S.A., Fakhouri, S.Y. (1978f). Identification of nonlinear systems using correlation analysis and pseudorandom inputs. Res. Rep. 78, Sheffield Univ.

Billings, S.A., Fakhouri, S.Y. (1979a). Identification of nonlinear unity feedback systems. 17th IEEE Conf. Dec. and Control, San Diego, 255-260.

Billings, S.A., Fakhouri, S.Y. (1979b). Nonlinear system identification using the Hammerstein model. Int. J. Systems Sci. (to appear).

Billings, S.A., Fakhouri, S.Y. (1979c). Identification of nonlinear S_m systems. Res. Rep. 81, Sheffield Univ.

Briggs, P.A.N., Godfrey, K.R. (1966). Pseudorandom signals for the dynamic analysis of multivariable systems. Proc.IEE, 113, 1259-1267.

Brilliant, M.B. (1958). Theory of the analysis of nonlinear systems. Tech. Rep. 304, MIT.

Cooper, B., Falkner, A.H. (1975). Identification of simple nonlinear systems. Proc.IEE, 122, 753-755.

Douce, J.L. (1976). Identification of a class of nonlinear systems. IFAC Symp., Tbilisi.

Gardner, A.B. (1973). Identification of processes containing single valued nonlinearities. Int. J. Control, 18, 1029-1039.

George, D.A. (1959). Continuous nonlinear systems. Tech. Rep. 355, MIT.

Harper, T.R., Rugh, W.J. (1976). Structural features of factorable Volterra systems. IEEE Trans.Auto.Cont., AC-21, 822-832.

Lee, Y.W., Schetzen, M. (1965). Measurement of the Wiener kernels of a nonlinear system by cross correlation. Int.J.Cont., 2, 237-259.

Narayanan, S. (1970). Application of Volterra series to intermodulation distortion analysis of transfer feedback amplifiers. IEEE Trans. Ct. Th., CT-17, 518-527.

Narendra, K.S., Gallman, P. (1966). An iterative method for the identification of nonlinear systems using the Hammerstein model. IEEE Trans.Auto.Cont., AC-11, 546-550.

Nuttall, A.H. (1958). Theory and application of the separable class of random processes. Tech. Rep. 343, MIT.

Peckham, G. (1970). A new method for minimising a sum of squares without calculating gradients. Computer J., 13,418-420.

Sandor, J., Williamson, D. (1978). Identification and analysis of nonlinear systems by tensor techniques. Int.J.Cont., 27, 853-878.

Shanmugan, K.S., Lal, M. (1976). Analysis and synthesis of a class of nonlinear systems. IEEE Trans. Cts. Syst., CAS-23, 17-25.

Smets, H.B. (1960). Analysis and synthesis of nonlinear systems. IRE Tran. Ct. Th., 459-469.

Smith, W.W., Rugh, W.J. (1974). On the structure of a class of nonlinear systems. IEEE Trans.Auto.Cont., AC-19, 701-706.

Wysocki, E.M., Rugh, W.J. (1976). Further results on the identification problem for the class of nonlinear systems S_m. IEEE Trans.Cts.Syst., CAS-23, 664-670.

Zames, G. (1963). Functional analysis applied to nonlinear feedback systems. IEEE Trans. Ct. Th., CT-10, 392-404.