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COMPENSATION THEORY FOR MULTIVARIABLE ROOT-LOCI

by

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Abstract

Regarding system compensation as the manipulation of the asymptotic directions and pivots of the root-locus of the closed-loop multivariable system, this paper describes a general framework for the construction of forward path and minor loop compensation elements for a square, invertible plant with transfer function matrix G(s). The approach is based on the use of dynamic transformations of G and the properties of the inverse plant G^{-1}.

1. Introduction

The concept of the root-locus of an m-input/m-output, square, invertible, strictly proper system $S(A,B,C)$ subjected to unity negative feedback with scalar gain $p \geq 0$ is now well-established[1-10] and significant progress has been made in the asymptotic analysis of the behaviour of the closed-loop poles as $p \to +\infty$. In essence[1], the results state that, as $p \to +\infty$,

(a) If $S(A,B,C)$ has n_z zeros, then n_z closed-loop poles move asymptotically to these points of the complex plane, and

(b) the remaining $n-n_z$ poles are unbounded and take the form,

$$s_{j\ell}(p) = p^{1/\nu_j}\eta_{j\ell} + \alpha_j + \varepsilon_{j\ell}(p)$$

$$\lim_{p\to\infty} \varepsilon_{j\ell}(p) = 0 \quad , \quad 1 \leq \ell \leq \nu_j \quad , \quad 1 \leq j \leq m \qquad \ldots(1)$$

where p^{1/ν_j} is the positive, real ν_j^{th} root of p and $\eta_{j\ell}$, $1 \leq \ell \leq \nu_j$, are the distinct ν_j^{th} roots of a non-zero (complex) number $-\lambda_j$.

In this paper, attention is focussed on the unbounded pole $s_{j\ell}(p)$, termed an <u>infinite zero of order ν_j</u> with <u>asymptotic direction</u> $\eta_{j\ell}$ and <u>pivot</u> α_j. Although the orders may be non-integer[1,2], it has been shown[3] that they are almost always equal to certain integer invariants of the triple (A,B,C) under a group of transformations[4]. More precisely[1,3], it is <u>always</u> possible to choose a proportional controller to ensure that the resulting system only has integer order infinite zeros and, in fact, a random choice of such a controller will achieve this objective with probability one. For these reasons, non-integer orders are not considered in this paper.

The orders, asymptotic directions and pivots of the infinite zeros have interpretations in terms of stability, response speed and oscillatory behaviour at high gain. These directly parallel the well-known interpretations for single-input/single-output systems and suggest that the manipulation of asymptotic directions and pivots is an important design consideration. For the purposes of analysis, compensation of system dynamics will be regarded in this paper solely as the systematic and independent manipulation of each asymptotic direction and pivot of the root-locus. Although a rather restricted viewpoint, it will be noted that the desired compensation can always be achieved by quite simple lead-lag dynamic structures with a large number of degrees of freedom remaining to aid the inclusion of other design constraints.

Three feedback configurations are considered (see Fig.1) for the control of the mxm, strictly proper, invertible plant G(s). In all cases, $K_1(s)$ and $K_2(s)$ are assumed to be proper, invertible and minimum-phase and H(s) is a minor loop/dynamic state feedback element. The concepts of uniform rank systems and dynamic transformations are reviewed in section 2 and shown to be natural tools in compensation studies. These results are extended in section 3 to the analysis of minor loop compensation schemes. Finally, in section 4, these results and others are discussed.

2. Forward Path Compensation

2.1 Uniform Rank Systems: A Significant Special Case

The techniques described throughout this paper rely on reduction of the problem to a number of problems of the type discussed below. Considering, for simplicity, the configuration of Fig.1(a), it is well known[1,5,6] that the properties of the infinite zeros can be deduced from the series expansion

$$G(s)K_1(s) \quad = \quad s^{-1}Q_1 + s^{-2}Q_2 + \ldots \qquad \ldots(2)$$

of GK_1 about the point at infinity. A simple basis for analysis and computation is obtained by considering the case when the <u>Markov parameters</u> Q_j, $j \geq 1$, satisfy

$$Q_j = 0 \quad , \quad 1 \leq j \leq k-1 \quad , \qquad |Q_k| \neq 0 \qquad \ldots(3)$$

when GK_1 is said to have <u>uniform rank k</u>[1,5,7]. Equivalently $\lim_{s\to\infty} s^k GK_1$ exists and is nonsingular.

Assuming, for simplicity, that Q_k has a nonsingular eigenvector matrix T_o and ℓ distinct eigenvalues η_j, $1 \leq j \leq \ell$, of multiplicity d_j, $1 \leq j \leq \ell$, and write

$$T_o^{-1}Q_kT_o \quad = \quad \text{block diag } \{\eta_j I_{d_j}\}_{1 \leq j \leq \ell} \qquad \ldots(4)$$

together with

$$T_o^{-1}Q_{k+1}T_o \quad = \quad \begin{pmatrix} N_{11} & \cdot\cdot\cdot\cdot\cdot & N_{1\ell} \\ \vdots & & \vdots \\ N_{\ell 1} & \cdot\cdot\cdot\cdot\cdot & N_{\ell\ell} \end{pmatrix} \qquad \ldots(5)$$

where N_{ij} has dimension $d_i \times d_j$ and N_{jj} has eigenvalues α_{jr}, $1 \leq r \leq d_j$, $1 \leq j \leq \ell$. A simple technique for the calculation of the asymptotic directions and pivots is provided by the following result[1,5].

Theorem 1[1,5]

With the above notation, the system of Fig.1(a) has km k^{th}-order infinite zeros of the form,

$$s_{ijr}(p) = p^{1/k}\,\eta_{ij} + \frac{\alpha_{jr}}{k\eta_j} + \varepsilon_{ijr}(p)$$

$$\lim_{p\to\infty} \varepsilon_{ijr}(p) = 0 \quad , \quad 1\le i\le k \; , \; 1\le r\le d_j \; , \; 1\le j\le \ell \qquad \ldots(6)$$

where $p^{1/k}$ is the positive-real k^{th} root of p and η_{ij}, $1\le i\le k$, are the distinct k^{th} roots of $-\eta_j$, $1\le j\le \ell$. The system has no other infinite zeros.

The results are easily implemented on a digital computer and only require knowledge of the two Markov parameters Q_k and Q_{k+1}. If the asymptotic directions and pivots have unsatisfactory numerical values, suitable compensation elements can be included as indicated in Fig.1(b) and discussed below.

It is self-evident from equation (2) that the asymptotic directions and T_o (which depend only upon the eigenstructure of Q_k) can be manipulated by choice of $K_1(s)$ ($K_1(s)$ could also be chosen to achieve other objectives such as diagonal dominance etc.). Suppose therefore that the asymptotic directions and T_o are as required and consider the choice of the compensation element

$$K_2(s) \triangleq T_o \text{ block diag } \{K_2^{(j)}(s)\}_{1\le j\le \ell}\; T_o^{-1}$$

$$K_2^{(j)}(s) = \text{diag}\{\frac{s+a_{jr}}{s+b_{jr}}\}_{1\le r\le d_j} \quad , \quad 1\le j\le \ell \qquad \ldots(7)$$

where it is assumed that T_o is such that $\{N_{jj}\}$ are in diagonal, Jordan or triangular form.

Theorem 2[1,8]

The system of Fig.1(b) with K_2 as in equation (7) has km k^{th}-order infinite zeros of the form

$$s_{ijr}(p) = p^{1/k}\eta_{ij} + k^{-1}\{ \frac{\alpha_{jr}}{\eta_j} + a_{jr} - b_{jr}\} + \varepsilon_{ijr}(p)$$

$$\lim_{p\to\infty} \varepsilon_{ijr}(p) = 0 \quad , \quad 1\le i\le k \ , \quad 1\le r\le d_j \ , \quad 1\le j\le \ell \qquad \ldots(8)$$

Comparing this result with theorem 1, it is seen that the simple dyadic[11] compensator of equation (7) leaves the asymptotic directions unchanged but enables the systematic and independent manipulation of each pivot by suitable choice of parameters a_{jr} and b_{jr}.

2.2 The General Case:[1,7]

Consider now the configuration of Fig.1(a) and the series expansion of equation (2) without the assumption that GK_1 has uniform rank. The calculation of the orders, asymptotic directions and pivots of the root-locus is based on decomposition of the structure of GK_1 into a number of uniform rank subsystems by the use of dynamic transformations. More precisely, under very weak assumptions, there exists integers q, d_j $(1\le j\le q)$ and k_j $(1\le j\le q)$ such that

$$1 \le k_1 < k_2 < \ldots < k_q \le n \qquad \ldots(9)$$

and a constant mxm nonsingular transformation T_1 together with unimodular dynamic transformations

$$L(s) = \begin{bmatrix} I_{d_1} & 0 & \cdots & \cdots & 0 \\ O(s^{-1}) & I_{d_2} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ O(s^{-1}) & \cdots & \cdots & O(s^{-1}) & I_{d_q} \end{bmatrix}$$

$$M(s) = \begin{bmatrix} I_{d_1} & O(s^{-1}) & \cdots & \cdots & O(s^{-1}) \\ 0 & I_{d_2} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & O(s^{-1}) \\ 0 & \cdots & \cdots & 0 & I_{d_q} \end{bmatrix} \qquad \ldots(10)$$

such that

$$L(s)T_1^{-1}G(s)K_1(s)T_1M(s) = \text{block diag}\{G_j(s)\}_{1\leq j\leq q} + O(s^{-(k_q+2)}) \qquad \ldots(11)$$

where the $d_j x d_j$ TFMs $G_j(s)$ are of uniform rank k_j, $1\leq j\leq q$.

Theorem 3[1,5]

The system of Fig.1(a) has $k_j d_j$ k_j^{th} order infinite zeros, $1\leq j\leq q$, whose orders, asymptotic directions and pivots are identical to those of the uniform rank systems $G_j(s)$, $1\leq j\leq q$, subject to unity negative feedback with scalar gain $p\geq 0$ (see Fig.2(a)).

An algorithm for the calculation of the relevant Markov parameter matrices of the systems $G_j(s)$, $1\leq j\leq q$, and the matrix T_1 has been derived elsewhere[1,5] based on algebraic operations on the truncated output controllability matrix $[Q_1, Q_2, \ldots, Q_r]$ for any $r \geq k_q+1$. The existence of the decomposition (11) is known[1] to be generic and can be arranged by suitable choice of K_1.

Given that it is possible to replace GK_1 by the uniform rank systems G_j, $1\leq j\leq q$, for the basis of computation, it is natural to ask how compensators $K_2^{(j)}(s)$, $1\leq j\leq q$, designed individually (Fig.2(b)) to produce the desired manipulation of the root-locus of the $G_j(s)$, $1\leq j\leq q$, can be combined to produce the same effect on the root-locus of GK_1. Consider the configuration of Fig.1(b), with equation (11) satisfied and K_2 of the form

$$K_2(s) = T_1 \text{ block diag } \{ K_2^{(j)}(s)\}_{1\leq j\leq q} T_1^{-1} \qquad \ldots(12)$$

where the $d_j \times d_j$ TFM's $K_2^{(j)}(s)$ are proper, minimum-phase and $\lim_{s\to\infty} K_2^{(j)}(s)$ exists and is nonsingular, $1\leq j\leq q$. Then,

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<u>Theorem 4</u>[1,8]

The system of Fig.1(b) has $k_j d_j$ k_j^{th} order infinite zeros, $1\leq j\leq q$, whose orders, asymptotic directions and pivots are identical to those of the <u>uniform rank</u> systems $G_j(s)K_2^{(j)}(s)$, $1\leq j\leq q$, subjected to unity negative feedback with scalar gain $p\geq 0$ (Fig.2(b)).

In effect, compensation of the root-locus of GK_1 can be undertaken by separate compensation of the uniform rank systems G_j, $1\leq j\leq q$, using $K_2^{(j)}$, $1\leq j\leq q$, followed by construction of K_2 as in equation (12). The compensation of each G_j could be undertaken using the techniques of section 2.1.

2.3 Use of the Inverse System[1,7]

The results of the previous sections can be rederived in terms of the inverse system

$$(G(s)K_1(s))^{-1} \equiv s^k A_o + s^{k-1} A_1 + \dots + A_k + H_o(s) \qquad \dots(12)$$

where $A_o \neq 0$ and $H_o(s)$ is strictly proper. The composite plant GK_1 has uniform rank k if, and only if, $|A_o| \neq 0$, when it is readily verified that

$$Q_k = A_o^{-1} \quad , \qquad Q_{k+1} = -A_o^{-1} A_1 A_o^{-1} \qquad \dots(13)$$

The computation and compensation of the asymptotes could now proceed using the analyses of section 2.1

In the more general case of GK_1 of non-uniform rank, a parallel analysis[1,7] to that of section 2.2 indicates that $(GK_1)^{-1}$ has a decomposition of the form

$$L(s)T_1^{-1}(G(s)K_1(s))^{-1}T_1M(s) = \text{block diag}\{G_{q+1-j}^{-1}(s)\}_{1 \le j \le q} + O(s^{k_1-2}) \qquad \dots(14)$$

by suitable choice of T_1 and

$$L(s) = \begin{pmatrix} I_{d_q} & 0 & \cdots & 0 \\ O(s^{-1}) & I_{d_{q-1}} & & \vdots \\ \vdots & & \ddots & 0 \\ O(s^{-1}) & \cdots & O(s^{-1}) & I_{d_1} \end{pmatrix}$$

$$M(s) = \begin{pmatrix} I_{d_q} & O(s^{-1}) & \cdots & O(s^{-1}) \\ 0 & I_{d_{q-1}} & & \vdots \\ \vdots & & & O(s^{-1}) \\ 0 & \cdots & 0 & I_{d_1} \end{pmatrix} \qquad \ldots(15)$$

It can be shown[1,7] that theorem 3 is still valid and that

$$1 \leq k^* = k_1 < k_2 < \ldots . < k_q = k \qquad \ldots(16)$$

where k* is the index of the first nonzero Markov parameter in equation (2). It is also true[1] that theorem 4 is still valid provided that $K_2(s)$ takes the form (c.f. equation (12))

$$K_2(s) = T_1 \text{ block diag } \{ K_2^{(q+1-j)}(s) \}_{1 \leq j \leq q} T_1^{-1} \qquad \ldots(17)$$

A major use of the inverse system is described in the next section.

3. Minor Loop Compensation

Consider now the configuration of Fig.1(c). Algebraically this configuration can be regarded as Fig.1(b) with GK_1 replaced by $(I+GK_1H)^{-1}GK_1$. The advantage of the inverse system for analysis is suggested by noting that H appears linearly in the inverse $(GK_1)^{-1}+H$.

3.1 Uniform Rank Systems:

Assuming that GK_1 has uniform rank k with inverse of the form of equation (12), write

$$H(s) = s^{k-1}H_1 + O(s^{k-2}) \qquad \ldots(18)$$

Consideration of the inverse $(GK_1)^{-1}+H$ indicates that the effect of H(s) on the asymptotic directions and pivots is described by replacing A_1 by A_1+H_1. More precisely, $(I+GK_1H)^{-1}GK_1$ has uniform rank k and Markov parameters (see equation (13))

$$\tilde{Q}_k = A_o^{-1} \quad , \quad \tilde{Q}_{k+1} = Q_{k+1} - A_o^{-1}H_1A_o^{-1} \qquad \text{...(19)}$$

In particular, the numerical magnitude of the <u>pivots</u> (as represented by the structure of $\tilde{Q}_{k+1}$) can be manipulated by choice of H_1 (and hence by minor loop feedback of the (k-1)th output derivative). For example, choose

$$H_1 = A_oT_o \text{ block diag}\{H_1^{(j)}\}_{1\leq j\leq \ell} T_o^{-1}$$

$$H_1^{(j)} = \text{diag}\{ h_r^{(j)}\}_{1\leq r\leq d_j} \quad , \quad 1\leq j\leq \ell \qquad \text{...(20)}$$

with T_o, ℓ, d_j as defined in section 2 and $K_2(s)$ as in equation (7). Application of the algorithm of section 2.1 verifies that the closed-loop system has km k^{th}-order infinite zeros of the form

$$s_{ijr}(p) = p^{1/k}\eta_{ij} + k^{-1}\{\frac{\alpha_{jr}}{\eta_j} + a_{jr} - b_{jr} - h_{jr}\} + \varepsilon_{ijr}(p)$$

$$\lim_{p\to\infty} \varepsilon_{ijr}(p) = 0 \quad , \quad 1\leq i\leq k \quad , \quad 1\leq r\leq d_j \quad , \quad 1\leq j\leq \ell \qquad \text{...(21)}$$

and hence that the minor loop feedback provides an additional or alternative approach to the independent manipulation of the pivots of the root-locus.

Typical examples of minor loop elements are, in the case of $k = 1$, $H(s) \equiv H_1$ corresponds to minor loop constant output feedback and, in the case of k = 2, $H(s) \equiv sH_1+H_2$ represents the inclusion of minor loop rate feedback.

3.2 The General Case:

Suppose now that GK_1 is not of uniform rank, but that it has the decomposition given in equation (14). Noting that G_j, $1\leq j\leq q$, all have uniform rank, suppose that the minor loop compensation schemes illustrated in Fig.2(c) with $K_2^{(j)}(s)$ proper, minimum phase, $\lim_{s\to\infty} K_2^{(j)}(s)$ nonsingular and

$$H^{(j)}(s) = s^{k_j-1} H_1^{(j)} + O(s^{k_j-2}) , \quad 1\leq j\leq q \qquad \ldots(22)$$

have been constructed to produce the desired asymptotic directions and pivots for each of the orders of infinite zero. This compensation scheme can be converted into one of the type shown in Fig.1(c) by choosing $K_2(s)$ as in equation (17), defining

$$\tilde{H}(s) \triangleq T_1L^{-1}(s) \text{ block diag } \{ H^{(q+1-j)}(s)\}_{1\leq j\leq q}M^{-1}(s)T_1^{-1} \qquad \ldots(23)$$

and using the following result:

Theorem 5

The system of Fig.1(c) with GK_1 satisfying equation (14), $K_2(s)$ given by equation (17) and

$$G(s)K_1(s) \{H(s) - \tilde{H}(s)\} = O(s^{-2}) \qquad \ldots(24)$$

has k_jd_j k_j^{th}-order infinite zeros, $1\leq j\leq q$, whose orders, asymptotic directions and pivots are identical to those of the uniform rank configurations in Fig.2(c). Moreover it is always possible to choose H(s) to be a polynomial matrix.

Proof

Noting that

$$L(s)T_1^{-1}((GK_1)^{-1}+\tilde{H})T_1M = \text{block diag}\{G_{q+1-j}^{-1}(s)+H^{(q+1-j)}(s)\}_{1\leq j\leq q} + O(s^{k_1-2}) \quad \ldots(25)$$

the result follows directly from theorem 4 and the following lemma:

Lemma[1,7]: the orders, asymptotic directions and pivots of the infinite zeros of the system of Fig.1(c) are independent of H(s) if $G(s)K_1(s)H(s) = O(s^{-2})$.

Equivalently, any H and $\tilde{H}$ satisfying equation (24) are identical for compensation purposes. The final part of the result follows by writing $\tilde{H}(s) = P(s)+O(s^{k_1-2})$ where P(s) is a polynomial matrix and noting that $H(s) \equiv P(s)$ satisfies equation (24) (see equation (16)) Q.E.D.

The availability of any H(s) satisfying (24) is important for practical applications as $\tilde{H}(s)$ may have a highly complex dynamic structure due to the presence of $L^{-1}(s)$ and $M^{-1}(s)$ in equation (23). It is particularly significant that it is always possible to choose H(s) to be a polynomial matrix ie the minor loop feedback can be realized by feedback of outputs, rates, accelerations etc. Finally, the inversion of L and M does not present numerical problems as they are easily deduced from previous numerical algorithms[1,5] as the product of elementary operations.

4. Discussion and Conclusions

Given the plant G and the 'precompensator' K_1 the paper has described existence and algebraic synthesis results for the construction of forward path compensators K_2 and minor loop elements H allocating the asymptotic directions and pivots of the root-locus to desired numerical values. The minor loop element can always be taken to be a polynomial matrix representing output, rate and acceleration feedbacks etc. and the forward path element can always be taken to be a dyadic system of elementary lead-lag networks. The technique revolves around the replacement of GK_1 by q uniform rank systems G_j, $1 \leq j \leq q$, and systematic analysis of the q configurations represented by Fig.2(c) and subsequent construction of suitable K_2 and H from the designed $K_2^{(j)}$ and $H^{(j)}$, $1 \leq j \leq q$.

The developed approach based on dynamic transformation of GK_1 or its inverse is, a priori, only one of many possible compensation procedures. It does, however, have a natural simplicity for computation and analysis and suggests design procedures achieving the desired objectives whilst retaining many degrees of freedom to achieve other design objectives. In particular the precompensator could be used to produce diagonal dominance or high frequency eigenvector alignment and could be used to manipulate T_1. The choice of eigenvector matrix T_o in the analysis of the uniform rank systems is also a free design parameter and although the differences $a_{jr}-b_{jr}$ may be specified by the desired pivot allocation, the absolute numerical values of a_{jr}, b_{jr} are not specified. It is not clear at the present time, how these free parameters can be used to shape the closed-loop transient performance nor, indeed, is the impact of any given choice of asymptotic directions and pivots on transient performance fully understood. This area is, in the authors opinion, a fruitful area for further work. In particular it appears that the sensitivity problem[1,10] associated with the pivots and the associated slow movement to the asymptote may have profound significance and, by analogy with classical root-locus techniques, it should be expected that the analysis of the necessary role and impact of system zeros in design and choice of compensation elements is a vital unsolved problem.

A particularly interesting observation[1,7] can be deduced from the lemma of section 3.2, namely that, for the purposes of compensation studies, the composite system GK_1 can be replaced by the inverse $\{s^k A_o + s^{k-1} A_1 + .. + A_k\}^{-1}$ of the polynomial component of its inverse (see equation (12)). An immediate consequence of this result is that the various design tools[1,10] available for plants with polynomial inverses can be brought to bear on the analysis of even large-scale systems. In certain cases[12], such considerations can lead to complete and highly successful design techniques.

5. References

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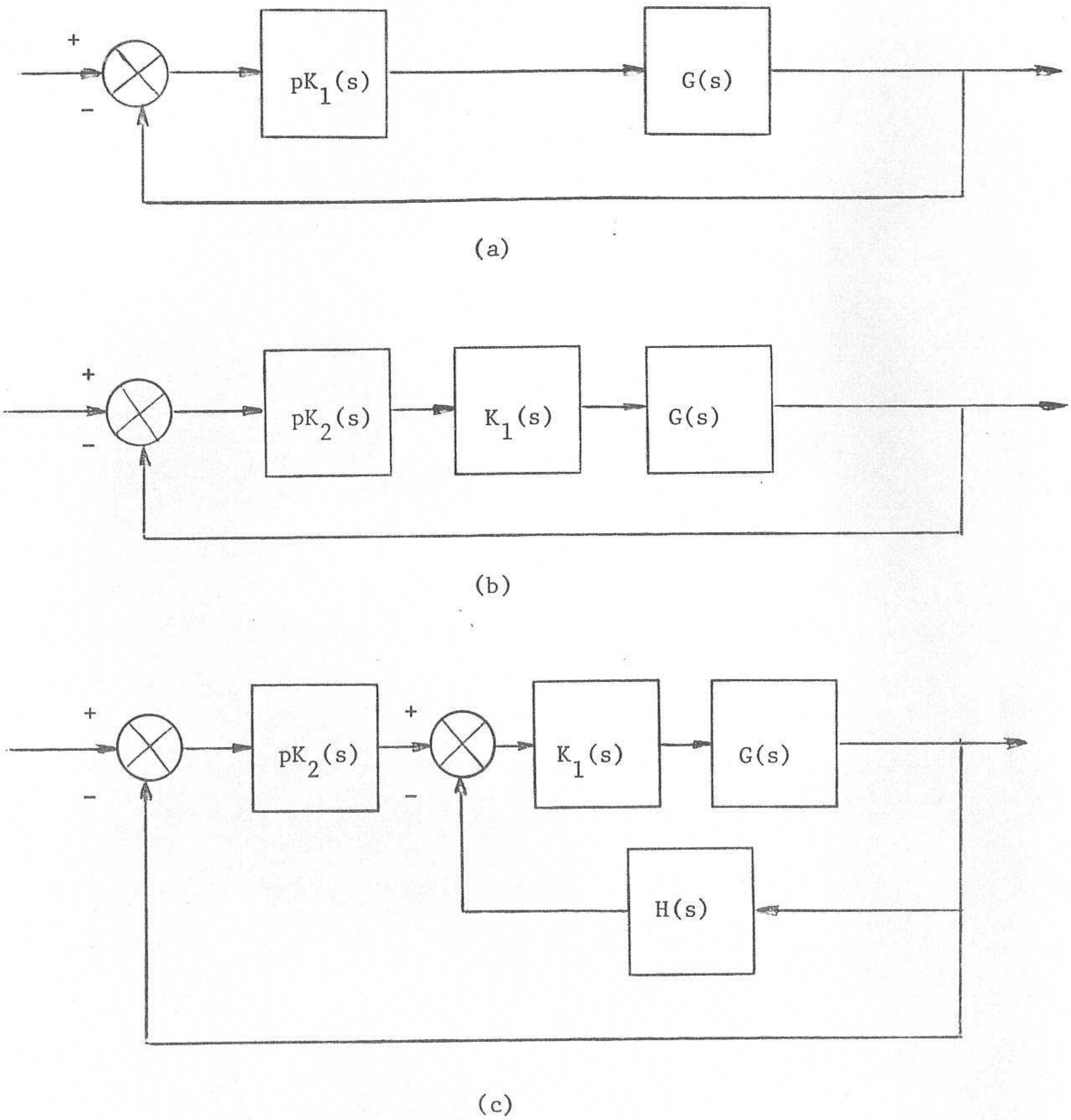

Fig. 1 Multivariable Feedback Configurations

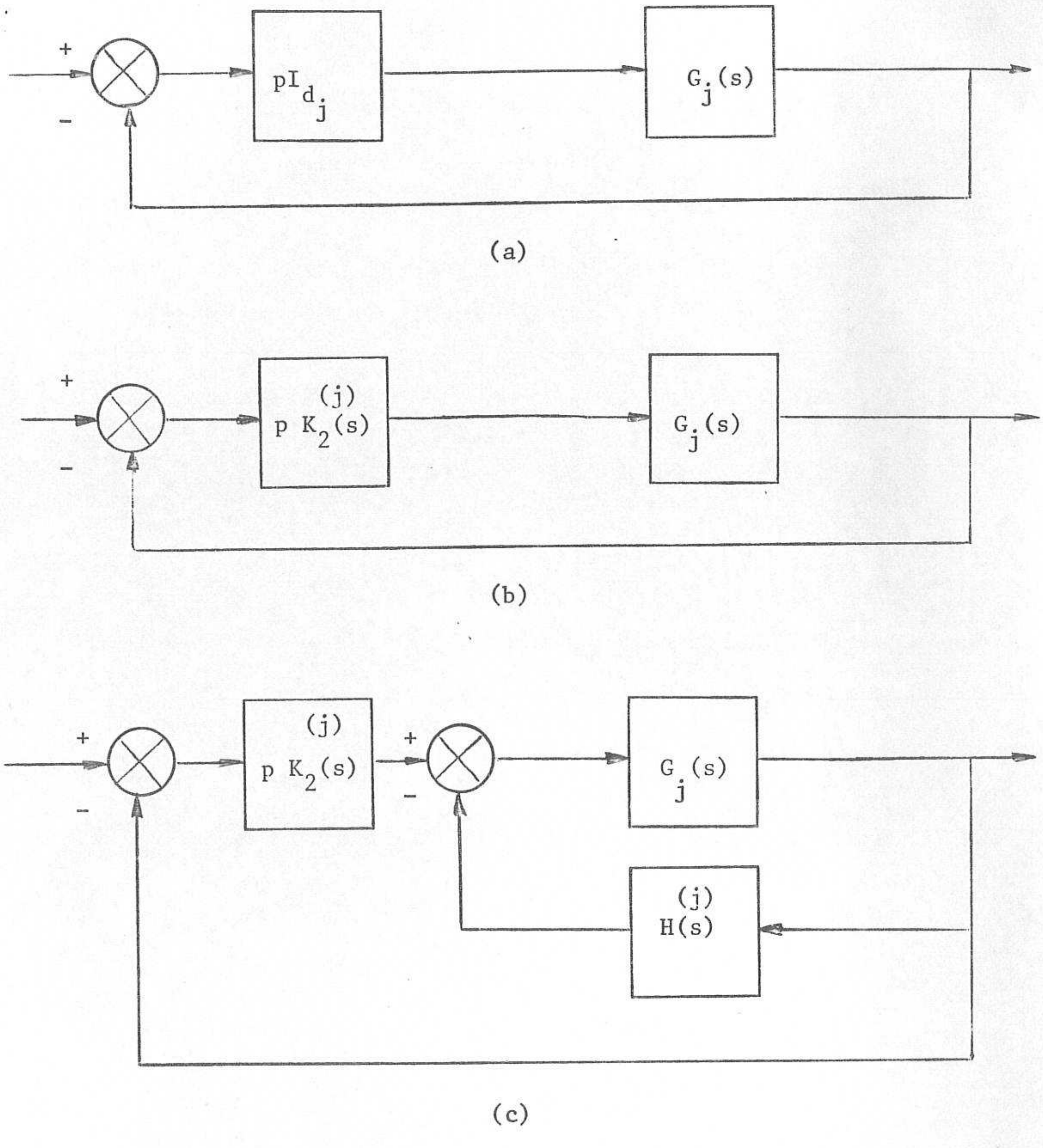

Fig. 2 Uniform rank configurations used in compensation studies