Article:

Applebaum, D. and van Neerven, J. (2015) Second quantisation for skew convolution products of infinitely divisible measures. Infinite Dimensional Analysis, Quantum Probability and Related Topics (idaqp), 18 (1). ISSN 0219-0257

https://doi.org/10.1142/S0219025715500034

SECOND QUANTISATION FOR SKEW CONVOLUTION PRODUCTS OF INFINITELY DIVISIBLE MEASURES

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Abstract.

1. Introduction

Let $E_i, i = 1, 2$ be Banach spaces equipped with Radon probability measures μ_1 and μ_2, respectively. A Borel measurable mapping $T : E_1 \to E_2$ is called a *skew map* for the pair (μ_1, μ_2) if there exists a Radon probability measure ρ on E_2 so that μ_2 is the convolution of ρ with the image of μ_1 under the action of T. In this case we obtain a linear contraction $P_T : L^p(E_2, \mu_2) \to L^p(E_1, \mu_1)$ given by

$$P_T f(x) = \int_{E_2} f(T(x) + y)\rho(dy).$$

Such constructions arise naturally in the study of Mehler semigroups, linear stochastic partial differential equations driven by additive Lévy noise and operator self-decomposable measures (see [2]). In this context, the problem of "second quantisation" is to find a functorial manner of expressing P_T in terms of T. The reason for this name is that the first work on this subject [3], within the context of Gaussian measures, exploited constructions that were similar to those that are encountered in the construction of the free quantum field from one-particle space (see e.g. [7]) wherein the nth chaos spanned by multiple Wiener-Itô integrals corresponds to the n-particle space within the Fock space decomposition. In our previous paper [2] we implemented this programme and constructed P_T as the second quantisation of T in the two cases where for $i = 1, 2, \mu_i$ are Gaussian (generalising [3] and [6]), and are infinitely divisible measures of pure jump type (generalising [8]). In this article, we complete the programme by dealing with the case where the μ_i's are general infinitely divisible measures, and so are convolutions of the cases previously considered.

2. Background

Let μ be an infinitely divisible Radon probability measure defined on a (separable) Banach space E. It is well-known that the generic such measure may be written as the convolution $\mu = \mu_G * \mu_P$ where μ_G is a Gaussian measure (see e.g. [5, 4]). In fact, it follows from the Lévy-Itô decomposition of [9] that μ may always be realised as the law of an E-valued random variable X defined on some probability space $(\Omega, \mathscr{F}, P)$ for which $X = X_1 + X_2$, where the summands X_1 and X_2 are independent. Here X_1 is Gaussian and has law μ_G, while X_2 is controlled by a Poisson random measure on $\mathbb{E}$ whose intensity measure is a Lévy measure ν, and X_2 has law μ_P. From [2], we know that we can effectively realise the second quantisation of twist maps of μ_G in the symmetric Fock space $\Gamma(H)$ of the reproducing kernel space H of μ_G which is naturally isomorphic to $L^2(E,\mu_G)$. To second quantise twist maps of μ_P, we use $L^2(E,\mu_P) \simeq \Gamma(L^2(E,\nu))$. To unify these two approaches we make use of the following:

$$\begin{aligned} L^2(E,\mu) &= L^2(E,\mu_G * \mu_P) \hookrightarrow L^2(E,\mu_G) \otimes L^2(E,\mu_P) \\ &\simeq \Gamma(H) \otimes \Gamma(L^2(E,\nu)) \simeq \Gamma(H \oplus L^2(E,\nu)). \end{aligned}$$

We give a more detailed account of these embeddings and isomorphisms in the sequel.

3. Main result

Suppose μ is an infinitely divisible measure, say

$$\mu = \gamma * \Pi$$

with γ centred Gaussian and Π as in [2] [1]. For a function $f \in L^2(\mu)$ let

$$F_f(x,y) := f(x+y).$$

Using the fact that $L^2(\gamma)\widehat{\otimes}L^2(\Pi) = L^2(\gamma\times\Pi)$ isometrically (with $\widehat{\otimes}$ indicating the Hilbert space tensor product) it is immediate to verify that

$$\|f\|^2_{L^2(\mu)} = \int_E\int_E |f(x+y)|^2\,d\gamma(x)\,d\Pi(y) = \|F_f\|^2_{L^2(\gamma)\widehat{\otimes}L^2(\Pi)}.$$

As a result the mapping $f \mapsto F_f$ is an isometry from $L^2(\mu)$ into $L^2(\gamma)\widehat{\otimes}L^2(\Pi)$. This brings us to the setting with independence structure as discussed in [1]. Following that reference, on the algebraic tensor product $L^2(\gamma)\otimes L^2(\Pi)$ we define

$$D := D_\gamma \otimes I + I \otimes D_\Pi,$$

where we denote the 'Gaussian' and the 'pure jump' derivatives with subscripts γ and Π, respectively.

Consider the Hilbert spaces

$$\mathscr{H}_n := \bigoplus_{\substack{j,k\ge 0\\ j+k=n}} H^{\circledS j}\widehat{\otimes}L^2(\nu)^{\circledS k}.$$

Then,

$$\begin{aligned} L^2(\gamma\times\Pi) = L^2(\gamma)\widehat{\otimes}L^2(\Pi) &= \Big(\bigoplus_{j=0}^{\infty} H^{\circledS j}\Big)\widehat{\otimes}\Big(\bigoplus_{k=0}^{\infty} L^2(\nu)^{\circledS k}\Big) \\ &= \bigoplus_{n=0}^{\infty}\Big(\bigoplus_{\substack{j,k\ge 0\\ j+k=n}} H^{\circledS j}\widehat{\otimes}L^2(\nu)^{\circledS k}\Big) = \bigoplus_{n=0}^{\infty}\mathscr{H}_n \end{aligned}$$

may be viewed as the associated Wiener-Itô decomposition. We define the n-fold stochastic integral on $I_n : \mathscr{H}_n \to L^2(\Omega)$ by

$$I_n(f\otimes g) := I_{j,\gamma}f \otimes I_{k,\Pi}g$$

for $f \in H^{\circledS j}$ and $g \in L^2(\nu)^{\circledS k}$ with $j+k=n$, where we denote the 'Gaussian' and the 'pure jump' integrals with subscripts γ and Π, respectively.

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[1] Be more precise

In what follows, in order to tidy up the notation we will refrain from writing subscripts γ and Π; expectations taken in the the left and right sides of tensor products refer to γ and Π, respectively.

Proposition 3.1. *For all $F \in L^2(\gamma \times \Pi)$,*

$$F = \sum_{m=0}^{\infty} \frac{1}{m!} I_m(\mathbb{E}(D^m F)).$$

Proof. Let $F = f \otimes g$ with $f \in H^{\circledS j}$ and $g \in L^2(\nu)^{\circledS k}$. By Leibniz's rule,

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{1}{m!} I_m \mathbb{E} D^m F &= \sum_{m=0}^{\infty} \frac{1}{m!} I_m \Big(\mathbb{E} \sum_{\ell=0}^{m} \binom{m}{\ell} D^\ell f \otimes D^{m-\ell} g \Big) \\
&= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{1}{\ell!(m-\ell)!} I_m \Big(\mathbb{E} \big(D^\ell f \otimes D^{m-\ell} g \big) \Big) \\
&= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{1}{\ell!(m-\ell)!} I_{\ell,\gamma}(\mathbb{E} D^\ell f) \otimes I_{m-\ell,\Pi}(D^{m-\ell} g) \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} I_{j,\gamma}(\mathbb{E} D^j f) \otimes \sum_{k=0}^{\infty} \frac{1}{k!} I_{k,\Pi}(\mathbb{E} D^k g), \\
&= f \otimes g \\
&= F
\end{aligned}$$

using the Last-Penrose type decompositions for γ and Π in the second last identity. □

Suppose now that two measures μ_1 and μ_2 are given as above, on Banach spaces E_1 and E_2, respectively, say $\mu_i = \gamma_i * \Pi_i$ for $i = 1, 2$. Let $T : E_1 \to E_2$ be a linear skew mapping with respect to both (γ_1, γ_2) and (Π_1, Π_2) with skew factors ρ_γ and ρ_Π. Recall that this means that $T\gamma_1 * \rho_\gamma = \gamma_2$ and $T\Pi_1 * \rho_\Pi = \Pi_2$.

Set $\rho := \rho_\gamma * \rho_\Pi$.

Lemma 3.2. *Under these assumptions, T is skew with respect to (μ_1, μ_2) with skew factor ρ.*

Proof. Since for any two measures on E_1 one has $T(\nu_1 * \nu_2) = (T\nu_1) * (T\nu_2)$, this follows from

$$T\mu_1 * (\rho_\gamma * \rho_\Pi) = (T\gamma_1 * T\Pi_1) * (\rho_\gamma * \rho_\Pi) = (T\gamma_1 * \rho_\gamma) * (T\Pi_1 * \rho_\Pi) = \gamma_2 * \Pi_2 = \mu_2.$$

□

It follows from the lemma that we may define $P_T : L^2(E_2, \mu_2) \to L^2(E_1, \mu_1)$ by

$$P_T f(x) := \int_{E_2} f(Tx + y)\, d\rho(y), \quad x \in E_1,$$

where ρ is the skew factor on E_2, i.e., $T\mu_1 * \rho = \mu_2$. Similarly we can define an operator $P_T \otimes P_T : L^2(\gamma_2) \otimes L^2(\Pi_2) \to L^2(\gamma_1) \otimes L^2(\Pi_1)$ in the obvious way (with an apology for the abuse of notation) and we then have:

Lemma 3.3. *Under the above assumptions, $F_{P_T f} = (P_T \otimes P_T) F_f$.*

Proof. For $(\gamma \times \Pi)$-almost all $x, y \in E_2$ we have

$$
\begin{aligned}
(P_T \otimes P_T)(\phi \otimes \psi)(x, y) = (P_T\phi \otimes P_T\psi)(x, y) \\
\int_{E_2} \phi(Tx + z)\, d\rho_\gamma(z) \int_{E_2} \psi(Ty + z)\, d\rho_\Pi(z) \\
\int_{E_2}\int_{E_2} (\phi \otimes \psi)(Tx + z_1, Ty + z_2)\, d\rho_\gamma(z_1)\, d\rho_\Pi(z_2).
\end{aligned}
$$

Now suppose that $G_n = F_f$ in $L^2(\gamma \times \Pi)$, where each f_n belongs to the algebraic tensor product $L^2(\gamma) \otimes L^2(\Pi)$. By the above identity and linearity it follows, after passing to a subsequence if necessary, that for $(\gamma \times \Pi)$-almost all $x, y \in E_2$ we have

$$
\begin{aligned}
(P_T \otimes P_T)F_f(x, y) &= \lim_{n\to\infty} (P_T \otimes P_T)G_n(x, y) \\
&= \lim_{n\to\infty} \int_{E_2}\int_{E_2} G_n(Tx + z_1, Ty + z_2)\, d\rho_\gamma(z_1)\, d\rho_\Pi(z_2) \\
&= \int_{E_2}\int_{E_2} F_f(Tx + z_1, Ty + z_2)\, d\rho_\gamma(z_1)\, d\rho_\Pi(z_2) \\
&= \int_{E_2}\int_{E_2} f(Tx + Ty + z_1 + z_2)\, d\rho_\gamma(z_1)\, d\rho_\Pi(z_2) \\
&= \int_{E_2} f(Tx + Ty + z)\, d(\rho_\gamma * \rho_\Pi)(z) \\
&= \int_{E_2} f(Tx + Ty + z)\, d\rho(z) \\
&= P_T f(x + y) \\
&= F_{P_T}(x, y).
\end{aligned}
$$

□

For $h \in H$ and $y_1, \dots, y_n \in E$ and $h \in H$ we define

$$D_{h;y_1,\dots,y_n} := D_h \otimes I + I \otimes D_{y_1,\dots,y_n},$$

Lemma 3.4. *For all $f \in L^2(E_2, \mu_2)$, $h \in H$, and $y_1, \dots, y_n \in E_1$,*

$$\mathbb{E}_{\gamma_1\times\Pi_1} D^n_{h;y_1,\dots,y_n} F_{P_T f} = \mathbb{E}_{\gamma_2\times\Pi_2} D^n_{Th;Ty_1,\dots,Ty_n} F_f. \tag{3.1}$$

Proof. We approximate F_f by finite sums of elementary tensors as in the proof of the previous lemma. For such functions G_n the identity follows from the results in [2] for the Gaussian and Poissonian case.

`Take care of details, closedness argument needed?`
`Our` D`'s are unbounded.` □

`Can we define a derivative` D `in` $L^2(\mu)$ `satisfying the requirement`

$$E_\mu D^n f = E_{\gamma_1\times\Pi_1} D^n F_f \ ?$$

`(On the right, this is the` D `defined previously on` $L^2(\gamma) \otimes L^2(\Pi)$`, extended (by closbility? check) to a closed operator on` $L^2(\gamma\times\Pi)$`). That would clean up the lemma as well as the commuting diagram.`

For Hilbert spaces H and $\underline{H}$ we note that

$$\Gamma(H, \oplus \underline{H}) = \bigoplus_{n=0}^{\infty} \Big(\bigoplus_{\substack{j,k \geq 0 \\ j+k=n}} H^{\circledS j} \widehat{\otimes} \underline{H}^{\circledS k} \Big).$$

Theorem 3.5. *Putting everything togehter, under the above assumptions the following diagram commutes:*

$$\begin{array}{ccc}
L^2(E_2, \mu_2) & \xrightarrow{P_T} & L^2(E_1, \mu_1) \\
{\scriptstyle f \mapsto F_f} \Big\downarrow & & \Big\downarrow {\scriptstyle f \mapsto F_f} \\
L^2(\gamma_2 \times \Pi_2) & \xrightarrow{P_T \otimes P_T} & L^2(\gamma_1 \times \Pi_1) \\
{\scriptstyle \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathbb{E}_{\gamma_2 \times \Pi_2} \tilde{D}^n} \Big\downarrow & & \Big\downarrow {\scriptstyle \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathbb{E}_{\gamma_1 \times \Pi_1} \tilde{D}^n} \\
\Gamma(L^2(E_2, \nu_2) \oplus H_2) & \xrightarrow{\bigoplus_{n=0}^{\infty} (T^*)^{\circledS n}} & \Gamma(L^2(E_1, \nu_1) \oplus H_1)
\end{array}$$

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